МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

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(англійською мовою)

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Посібник складається з двох частин: вступ до аналізу та диференціальне числення. Теоретичний матеріал добре проілюстрований розв'язаними прикладами. Кожний розділ містить достатню кількість прикладів для самостійної роботи. Загальна кількість прикладів-близько 350, в тому числі 26 розв'язаних.

Посібник відповідає програмі курсу "Вища математика" для втузів з поглибленою математичною підготовкою.

Для студентів 1 курсу вищих технічних навчальних закладів. Може бути також корисним для дипломників та аспірантів.

This textbook consists of two chapters and includes the topics which, as a rule, are studied during the first semester of the course.

Each sections in all chapters is supplied with a brief introduction containing both the relevant theoretical material (definitions, formulas and theorems) and a plenty thoroughly worked examples.

There are more then 300 examples.

Іл. 3. Бібліогр.: 5 назв

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Contents:

Chapter 1. Introduction to Analysis

1.1. Real numbers

The absolute value or the modulus of real number x is defined as the nonnegative number

$$|x| = \begin{cases} x, & \text{if } x \ge 0; \\ -x, & \text{if } x < 0; \end{cases}$$

It is supposed that the rules for comparing real numbers as well as arithmetical operations on them are already known to the reader.

Solve the given equations:

1. $|3x-4| = \frac{1}{2}$; 2. $\sqrt{x^2} + x^3 = 0$; 3. $|-x^2 + 2x - 3| = 1$;

4. $\left|\frac{2x-1}{x+1}\right| = 1$; 5. $\sqrt{(x-2)^2} = -x+2$;

Solve the given inequalities:

6. $|x-2| \ge 1$; 7. $|x^2 - 7x + 12| > x^2 - 7x + 12$;

8. $x^2 + 2\sqrt{(x+3)^2} - 10 \le 0$; 9. $\frac{1}{|x-1|} < 4 - x$;

10. $\sqrt{(x+1)^2} \le -x-1$;

A real number X is rational, i.e. representable in the form of the ratio $\frac{m}{n}, m, n \in Z$, if and only if decimal $[x].x_1, x_2 \ldots$ is periodic. Otherwise the number X is irrational.

Example 1.

Prove that the number $\log_{10} 5$ is irrational.

Solution: Let us assume that lg5 is a rational number, i.e. $\lg 5 = \frac{m}{n}$. Then $10^{\frac{m}{n}} = 5$; $10^m = 5^n$; $2^m \times 5^m = 5^n$. But the last equality is impossible, since the number 2 enters into the factorization of the left-hand member in simple factors, but does not enter into a similar factorization of the right-hand member which contradicts the uniqueness of presenting whole numbers in the form of prime factors. Therefore our assumption is false and, consequently, the number lg5 is irrational.

Prove that the given numbers are irrational:

11. $\sqrt{3}$; 12. $2+\sqrt{2}$; 13. $\log_3 7$; 14. $\log_2 5$;

Example 2.

Compare the given numbers $\sqrt{2}-\sqrt{5}$ and $\sqrt{3}-2$.

Solution:

Suppose the below inequality is correct: $\sqrt{2}-\sqrt{5}<\sqrt{3}-2$;

Then

$\sqrt{2}+2<\sqrt{5}+\sqrt{3}$, $6+4\sqrt{2}<8+2\sqrt{15}$, $2\sqrt{2}<1+\sqrt{15}$, $8<16+2\sqrt{15}$.

Since the last inequality is true, by virtue of the equivalence of the transformations performed, the initial inequality is also true.

Compare the given numbers

15. $\log_{\frac{1}{2}}\frac{1}{3}$ and $\log_{\frac{1}{3}}\frac{1}{2}$; 16. $\left(\frac{1}{5}\right)^{\lg\frac{1}{7}}$ and $\left(\frac{1}{7}\right)^{\lg\frac{1}{5}}$ 17. $\log_{\log_8 2}\frac{1}{2}$ and 1.

1.2. Set and Set Operations

A set is understood as any well-defined collection of objects called the elements or members of the set.

The notation $a\in A$ means that the objects a is an element of set A (belongs to the set A), otherwise we write $a\notin A$. A special set that plays an important part in Set theory is the ***empty set*** sometimes called ***Null set***, which contains no elements. The notation used to denote the empty set is the symbol $\varnothing$. The notation $A\subset B$ (read: "A is contained in B", or, equivalently, "B contained A") means that every element in a set A is also an element of a set B, in this case A is called a subset of B. The sets A and B are ***equal*** (written A=B) if $A\subset B$ and $B\subset A$.

There are two basic ways of defining (describing) a set:

a)The set A is defined by a direct enumeration of all its elements $a_1,a_2,...,a_n$, i.e. written in the form :

$$A=\{a_1,a_2,...,a_n\};$$

b)The set A is defined as the totality of those and only those elements belonging to a certain basic set T which possess to the general property α. In this case we use a shorter notation:

$$A = \{x \in T \,|\, \alpha(x)\};$$

where $\alpha(x)$ means that the element x possesses the property α.

Example 3.

Describe the set:

$$A = \{x \in Z \,|\, (x-3)\times(x^2-1)=0 and x\geq 0\};$$

by enumerating (or listing) its elements.

Solution:

A is the set of all integer nonnegative roots of the equation $(x-3)\times(x^2-1)=0$. Consequently, $A=\{1,3\}$;

The *union* of sets A and B is defined as the set:

$$A \cup B = \{x \,|\, x \in A or x \in B\};$$

the *intersection* of the sets A and B is defined as the set:

$$A \cap B = \{x \,|\, x \in A and x \in B\};$$

the *difference* between the sets A and B is understood as the set:

$$A \setminus B = \{x \,|\, x \in A and x \notin B\}.$$

In particular, if A is a subset of some universal set T, then the difference set T\A is denoted by the symbol $\overline{A}$ and is called the *complement* of the set A (relative to T).

Describe the given set by listing all its elements:

18. $A = \{x \in R \,|\, x^3 - 3x^2 + 2x = 0\}$; 19. $A = \left\{x \in R \,\middle|\, x + \frac{1}{x} \leq 2 and x > 0\right\}$;

20. $A = \{x \in \mathrm{N} \,|\, x^2 - 3x - 4 \leq 0\}$; 21. $A = \left\{x \in Z \,\middle|\, \frac{1}{4} \leq 2^x < 5\right\}$;

22. $A = \left\{x \in \mathrm{N} \,\middle|\, \log_{\frac{1}{2}} \frac{1}{x} < 2\right\}$; 23. $A = \{x \in R \,|\, \cos^2 2x = 1 and 0 < x \leq 2\pi\}$;

Represent the indicated sets on the coordinate plane:

24. $\{(x,y) \in R^2 \,|\, x + y = 2\}$; 25. $\{(x,y) \in R^2 \,|\, (x^2-1)\times(y+2) = 0\}$;

26. $\{(x,y) \in R^2 \,|\, y > \sqrt{2x+1} and 2x+1 \geq 0\}$; 27. $\{(x,y) \in R^2 \,|\, y^2 > 2x+1\}$;

28. $\{(x,y)\in R^2 | x^2+y^2\le 4\}$; 29. $\{(x,y)\in R^2 | x^2<2y+1\}$;

30. $\{(x,y)\in R^2 | x^2+2x+y^2>3\}$;

A set X is said to be countable if there can be established one-to one correspondence between the elements of this set and those of the set N of all natural numbers.

Example 4.

Show that set Z of all integers is countable.

Solution:

Let us establish one-to-one correspondence between the elements of this set and natural numbers, for instance, by ordering the set Z in the following way: 0,1, -1,2, -2, … , and then associating each integer with its ordinal number in this sequence.

Prove that the given sets are countable.

31. $\{n\in N | n=2k, k\in N\}$; 32. $\{n\in N | n=2^k, k\in N\}$

1.3. Logic symbolism

When presenting mathematical considerations it is expedient to use a short-notation with specific symbols utilized in logic. Here we shall introduce only several simplest and most widely used symbols.

Let α, β, …be some ***sentences*** or ***statements***, i.e. narrative sentences each of which can be identified as true or false.

The notation $\overline{\alpha}$ means "not α", i.e. the ***negation*** of the statement α.

The notation α⇒β means:" the statement α implies the statement β"(⇒ is the ***implication*** symbol).

The notation α⇔β means: "the statement α is equivalent to the statement β"i.e. "if α, then β, and if β, then α" (⇔ is the ***equivalence*** symbol).

The symbolic notation α ∧ β means "α and β"(∧ is the symbol of ***conjunction***).

The notation α ∨ β means "α or β"(∨ is the symbol of ***disjunction***).

The logic notation: ∀x∈X α(x) means: "for any element x∈X the statement α(x) is true" (∀ is ***the generality*** or ***universal quantifier***).

The notation ∃ x∈X α(x) means: "there exists an element $x\in X$ such that the statement $\alpha(x)$ is true for it" (∃ is the ***existential quantifier***).

If an element $x\in X$, for which the statement $\alpha(x)$ is true, not only exists, but is also unique, then we write:

$$\exists!\ x \in X\alpha(x);$$

Example 5.

Using logic symbolism, formulate the principle of mathematical induction.

Solution:

Let α be some statement making sense for all $n \in N$. Let us introduce the set $A = \{n \in N | \alpha(n)\}$, i.e. the set of all those natural numbers for which the statement α is true. Then the principle of mathematical induction can be formulated in the following way:

$$((1 \in A) \wedge (n \in A) \Rightarrow (n+1) \in A) \Rightarrow A = N; \quad (1)$$

Since the notation $\alpha(n)$ means that the statement α is true for the number $n \in N$ statement (1) can be written differently:

$$(\alpha(1) \wedge \alpha(n) \Rightarrow \alpha(n+1)) \Rightarrow \forall n \in N\alpha(n);$$

1.4. The Notion of Function

Let D be an arbitrary set of real numbers. If each number $x \in D$ is associated with exactly one definite real number $f(x)$, then we say that ***numerical function*** is defined on the set D. The set D is called the ***domain of definition*** and the set: $E = \{y \in R | y = f(x), x \in D\}$, the ***set of values*** of the numerical function f, the symbolical notation being:

$$f : D \rightarrow E \quad or \quad y = f(x);$$

The most widespread method of specifying a function is its analytical representation. The analytical representation of a function is given by a formula, which shows how the value of the ***dependent variable y*** for any value of the ***independent variable x*** can be determined. When a function is specified analytically its domain (of definition) is usually understood (provided that there are no additional conditions) as the maximum set of values of x for which the formula representing the function makes sense. This means that the application of the formula to these values of x should result in definite real values of y (***natural*** domain of definition of function).

Example 6.

Find the domain (of definition) and the set of values of the function

$$f(x) = \frac{1}{\sqrt{1-x^2}}.$$

Solution:

The natural domain of definition of this function is the set $D = \{x \| x| < 1\} = (-1,1)$, and the set of values is the set $E = \{y | y \geq 1\} = [1, \infty)$.

Let the function $f : D \to E$ be so that for any $x_1, x_2 \in D$ from the condition $x_1 \neq x_2$ it follows that $f(x_1) \neq f(x_2)$. In this case any number $y \in E$ can be associated with quite a definite number $x \in D$ so that $f(x) = y$, in this way a new function is defined $f^{-1} : E \to D$ called the ***inverse*** of the given function f.

Here are two functions: $f : X \to Y$ and $g : Y \to Z$. Their ***composition*** (or the ***composite function*** obtained by combining the functions f and g) is defined as the function: $h = g \circ f : X \to Z$, specified by the equality: $h(x) = g(f(x))$, $x \in X$.

Find the natural domain of definition and the set of values E for the given function:

33. $y = \lg(x+3)$; 34. $y = \sqrt{5-2x}$; 35. $y = \arccos\frac{1-2x}{4}$;

36. $y = \log_2(1 - 2\cos x)$; 37. $y = \sqrt{1-|x|}$; 38. $y = \log_3(5x - x^2 - 6)$;

Find the set of zeros $D_0 = \{x | f(x) = 0\}$, the domain of positiveness $D_+ = \{x | f(x) > 0\}$ and the domain of negativeness $D_- = \{x | f(x) < 0\}$ for the given function:

39. $f(x) = 1 + x$; 40. $f(x) = 2 + x - x^2$; 41. $f(x) = \sin\frac{\pi}{x}$;

A function $f(x)$ is said to be ***even (odd)*** if its domain is symmetric about the point x=0 and $f(-x) = f(x)(f(-x) = -f(x))$;

Which of the indicated functions are even, which are odd, and which are neither even nor odd:

42. $f(x) = x^4 + 5x^2$; 43. $f(x) = x + \sin 2x$; 44. $f(x) = \sin x - \cos x$;

45. $f(x) = \lg\frac{1+x}{1-x}$;

A function $f(x)$ is called ***periodic*** if there is a positive number T (period of the function) such that $\forall x \in D(f(x+T)=f(x))$.

Which of the functions given below are periodic?
Determine the least period T.

46. $f(x)=5\cos 7x$; 47. $f(x)=\cos^2 2x$; 48. $f(x)=x\sin x$;
49. $f(x)=tg\frac{x}{2}-2tg\frac{x}{3}$;

Find the inverse function and domain of its definition if the given function is defined on the indicated interval:

50. $y=x^2-1$; a) $x\in\left(-\infty,-\frac{1}{2}\right]$; b) $x\in\left[\frac{1}{2},+\infty\right)$;

51. $y=\sin x$; a) $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; b) $x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.

Find the compositions $f\circ g$ and $g\circ f$ of the indicated functions:

52. $f(x)=x^2, g(x)=\sqrt{x}$; 53. $f(x)=1-x, g(x)=x^2$;
54. $f(x)=2^x, g(x)=\log_2 x$.

1. 5. Elementary Functions and Their Graphs

Listed below are the ***basic elementary*** functions:

1.Power function: $y=x^a, a\in R$.

2.Exponential function: $y=a^x, a>0, a\neq 1$.

3.Logarithmic function: $y=\log_a x, a>0, a\neq 1$.

4.Trigonometric functions: $y=\sin x, y=\cos x, y=\tan x, y=\cot x$.

5.Inverse trigonometric (circular) functions:
$y=\arcsin x, y=\arccos x, y=\arctan x, y=\operatorname{arccot} x$;

Any function which can be constructed of a finite number of basic elementary functions with the aid of arithmetical operations and operations of forming a function of a function is called an ***elementary function***.

The ***graph*** of the function $y=f(x)$ is defined as the set

$$\Gamma=\{(x,y)\in R^2 \mid x\in D, y=f(x)\},$$

where R^2 is the set of points in the plane.

On a plane with a rectangular Cartesian system of coordinates Oxy the graph of a function is represented by a set of points $M(x,y)$ which coordinates satisfy the relation $y=f(x)$ (graphical representation of a function).

Construct the graphs of the given elementary function:

55. $y=x-\frac{1}{3}$ 56. $y=-1+x$

57. $y=|2-x|+|2+x|$ 58. $y=-\frac{1}{2}tg(2x+\frac{3\pi}{2})$

59. $y=\cos x+|\sin x|$ 60. $y=4\arcsin(x-1)$

61. $y=x sign(\cos x)$ 62. $y=2^{-x+1}$

63. $y=\sqrt{\cos x-1}+\frac{x}{2}$ 64. $y=2^{|x|}-1$

65. $y=\log_{\frac{1}{2}}|x-3|$ 66. $y=|\log_2(x+1)|$

67. $y=\arcsin(\sin(x+\frac{\pi}{4}))$ 68. $y=\arccos(\cos 3x)$

69. $y=|arctg(x-1)|$ 70. $y=\sin^2\frac{x}{2}$

1. 6. The concept of a Sequence

The ***sequence*** of real number is understood as a function $f:N\to R$ defined on the set of all natural numbers.

The number $f(n)$ is called ***the n^{th} term*** of the sequence and is denoted by the symbol x_n, and $x_n=f(n)$ is called the ***formula of the general term*** of the sequence $(x_n)_{n\in N}$.

Limit of the sequence. A number a is said to be the ***limit*** of the sequence $(x_n)_{n\in N}$, i.e.

$$\lim_{n\to\infty}x_n=a,$$

if for any $\varepsilon>0$ there is an ordinal number $N(\varepsilon)$ such that for $n>N(\varepsilon)$ the inequality $|x_n-a|<\varepsilon$ is fulfilled. In this case the sequence itself is called ***convergent***.

Compute the indicated limits:

71. $\lim_{n\to\infty}\frac{n-1}{3n}$ 72. $\lim_{n\to\infty}\frac{n+1}{2n^3}$ 73. $\lim_{n\to\infty}\frac{5n+1}{7-9n}$

74. $\lim_{n\to\infty}\frac{(n+2)^3-(n-2)^3}{95n^3+39n}$ 75. $\lim_{n\to\infty}\frac{\sqrt[3]{n^4+3n+1}}{n-1}$

76. $\lim_{n\to\infty}\frac{3n^2-7n+1}{2-5n-6n^2}$. 77. $\lim_{n\to\infty}(\frac{2n-1}{5n+7}-\frac{1+2n^3}{2+5n^3})$.

78. $\lim_{n\to\infty}(\sqrt{2+n}-\sqrt{n})$. 79. $\lim_{n\to\infty}n^{\frac{3}{2}}(\sqrt{n^3+1}-\sqrt{n^3-2})$.

80. $\lim_{n\to\infty}(\frac{1}{n^2}+\frac{2}{n^2}+...+\frac{n-1}{n^2})$. 81 $\lim_{n\to\infty}\frac{2^n+3^n}{2^n-3^n}$.

82. $\lim_{n\to\infty}\frac{1^2+2^2+...+n^2}{n^3}$ 83. $\lim_{n\to\infty}\frac{\sqrt[3]{n^2}\sin(n^2)}{n-1}$

84. $\lim_{n\to\infty}(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n(n+1)})$.

85. $\lim_{n\to\infty}(\frac{1}{5}-\frac{1}{25}+...+(-1)^{n-1}\frac{1}{5^n})$.

A sequence $(x_n)_{n\in N}$ is said to be ***infinitely small*** if $\lim_{n\to\infty}x_n=0$.

A sequence $(x_n)_{n\in N}$ is said to be ***infinitely large*** (converging to infinity) (written $\lim_{n\to\infty}x_n=\infty$) if for any $\varepsilon>0$ there exists an ordinal number $N(\varepsilon)$ such that for $n>N(\varepsilon)$ the inequality $|x_n|>\varepsilon$ is fulfilled.

A number a is called ***a limit point of a sequence*** $(x_n)_{n\in N}$, if for any $\varepsilon>0$ there can be found an infinite number of terms in this sequence satisfying the condition $|x_n-a|<\varepsilon$.

Bolzano-Weierstrass principle: any bounded sequence has at least one limit point.

The largest (smallest) of the limit points of a sequence is called ***the limit superior (the limit inferior)*** of this sequence and is denoted by the symbol

$$\overline{\lim}x_n(\underline{\lim}x_n).$$

Find all the limit points of the given sequence:

86. $x_n=\frac{2+(-1)^n}{2-(-1)^n}$ 87. $x_n=\cos(\frac{n\pi}{4})$

For each of the following sequences find $\overline{\lim}$ and $\underline{\lim}$:

88. $x_n = 1+\frac{1}{n}$ 89. $\frac{n+1}{n}\cos^2\frac{\pi n}{4}$ 90. $\frac{2+(-1)^n}{2}-\frac{1}{n}$

1. 7. Limit of a Function

Let a function $y = f(x)$ be defined on a set D.

A number a is called the ***limit of the function*** $y = f(x)$ at a point x_0 written $\lim\limits_{x\to x_0} f(x) = a$ if for any $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ such that for any $x \in D$ the condition $0 < |x - x_0| < \delta(\varepsilon)$ implies the inequality $|f(x) - a| < \varepsilon$.

We shall use following ***remarkable limits***:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad (2)$$

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e, \qquad (3)$$

where e =2.71828... is the base of natural logarithms.

Evaluate the given limits:

91. $\lim\limits_{x\to\infty}\frac{3x+1}{5x+\sqrt[3]{x}}$. 92. $\lim\limits_{x\to 10}\frac{\sqrt{x-1}-3}{x-10}$. 94. $\lim\limits_{n\to 0}\frac{\sqrt{x+n}-\sqrt{x}}{n}, x>0$.

95. $\lim\limits_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$. 96. $\lim\limits_{x\to 0}\frac{|\sqrt{1+x}-1|}{x^2}$. 97. $\lim\limits_{x\to\infty}\frac{\sqrt{2x}}{\sqrt{3x+\sqrt{3x+\sqrt{3x}}}}$.

100. $\lim\limits_{x\to 0}\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3}$. 101. $\lim\limits_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2-x}}{\sqrt[3]{2+x}-\sqrt[3]{2-x}}$.

102. $\lim\limits_{x\to a}(\sqrt{x-a}-\sqrt{x})$. 104. $\lim\limits_{x\to\infty}(\sqrt{4x^2-7x+4}-2x)$.

Compute the given limits, using remarkable limit (2):

106. $\lim\limits_{x\to 0}\frac{\sin 3x}{x}$. 107. $\lim\limits_{x\to\pi}\frac{\sin 7x}{\tan 3x}$ 108. $\lim\limits_{\alpha\to 0}\frac{\sin(\alpha^n)}{(\sin\alpha)^m}$, n, m $\in$N.

109. $\lim\limits_{x\to 0} x\cot\pi x$. 110. $\lim\limits_{x\to 0}\frac{3\arcsin x}{4x}$. 111. $\lim\limits_{x\to 0}\frac{1-\cos 2x}{x^2}$.

112. $\lim\limits_{\alpha\to 0}\frac{\sin 2\alpha-\tan 2\alpha}{\alpha^3}$. 113. $\lim\limits_{x\to 0}\frac{\cos\alpha x-\cos\beta x}{x^2}, \alpha\neq\beta$.

114. $\lim\limits_{x\to 0}(\frac{1}{\sin x}-\cot x)$. 115. $\lim\limits_{x\to\alpha}\tan\frac{\pi x}{2\alpha}\sin\frac{x-\alpha}{2}$.

116. $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sqrt{2}-2\cos x}{\pi-4x}$. 117. $\lim\limits_{x\to\frac{\pi}{2}}(\frac{\pi}{2}-x)\tan x$.

118. $\lim\limits_{\alpha\to 0}\frac{(1-\cos\alpha)^2}{\tan^2\alpha-\sin^2\alpha}$. 119. $\lim\limits_{x\to\pi}\frac{1+\cos 5x}{1-\cos 4x}$

When computing limits of the form $\lim\limits_{x\to x_0}U(x)^{V(x)}$, where $\lim\limits_{x\to x_0}U(x)=1$, $\lim\limits_{x\to x_0}V(x)=\infty$, remarkable limit (3) is used.

Example 7.

Compute $\lim\limits_{x\to\infty}(\frac{x}{2+x})^{3x}$.

Solution:

We have $(\frac{x}{2+x})^{3x}=(1+\frac{-2}{2+x})^{3x}=(1+\frac{-2}{2+x})^{\frac{2+x}{-2}(\frac{-2}{2+x}\cdot 3x)}$

Since $\lim\limits_{x\to\infty}(1+\frac{-2}{2+x})^{\frac{2+x}{-2}}=\lim\limits_{t=\frac{-2}{2+x}\to 0}(1+t)^{\frac{1}{t}}=e$, and $\lim\limits_{x\to\infty}\frac{-2}{2+x}\cdot 3x=-6$

we obtain $\lim\limits_{x\to\infty}(\frac{x}{2+x})^{3x}=e^{-6}$

(here the continuity of a composition of continuous functions has been used).

Using remarkable limit (3) compute the following limits:

120. $\lim\limits_{x\to\infty}(\frac{x+3}{x-2})^{2x+1}$ 121. $\lim\limits_{x\to\infty}(\frac{x^2+5}{x^2-5})^{x^2}$. 122. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

123. $\lim\limits_{x\to 0}(1+\tan^2\sqrt{x})^{\frac{2}{x}}$. 124. $\lim\limits_{x\to\infty}x(\ln(2+x)-\ln x)$. 125. $\lim\limits_{x\to 0}\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}}$.

126. $\lim\limits_{x\to 1}\frac{a^x-a}{x-1}$. 127. $\lim\limits_{x\to a}\frac{\log_a x-1}{x-a}$. 128. $\lim\limits_{x\to 0}\frac{e^{\alpha x}-e^{bx}}{x}$.

Let us also introduce the following notion of a ***one-side (unilateral) limit***. The number a is termed the ***right-hand (left-hand) limit*** of the function $y=f(x)$ at the point x_0 (written $\lim\limits_{x\to x_0+0}f(x)=a(\lim\limits_{x\to x_0-0}f(x)=a)$) if for any

$\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that the condition $0 < x - x_0 < \delta$ $(-\delta < x - x_0 < 0)$ implies $|f(x) - a| < \varepsilon$. The notion of a one-side limit at infinitely ($\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$) is introduced in a similar way.

Compute the indicated one-side limits:

129. $\lim\limits_{x \to 3 \pm 0} \dfrac{x-3}{|x-3|}$. 130. $\lim\limits_{x \to 2 \pm 0} \dfrac{2+x}{4-x^2}$. 131. $\lim\limits_{x \to 2 \pm 0} 7^{\frac{1}{2-x}}$

132. $\lim\limits_{x \to 0 \pm 0} (2+x)^{\frac{1}{x}}$. 133. $\lim\limits_{x \to \pm\infty} arctgx$.

1. 8. Infinitesimals and Infinities
(Infinitely Small and Infinitely Large Quantities)

A function $\alpha(x)$ is called ***infinitesimal*** as $x \to x_0$, if $\lim\limits_{x \to x_0} \alpha(x) = 0$.

Infinitesimals $\alpha(x)$ and $\beta(x)$ are said to be ***comparable*** if there exists at least one of the limits $\lim\limits_{x \to x_0} \dfrac{\beta(x)}{\alpha(x)}$ or $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)}$.

Let $\alpha(x)$ and $\beta(x)$ be comparable infinitesimals as $x \to x_0$ and let for the sake of definiteness, there exist $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{\beta(x)} = C$.

Then:

a) If $C \neq 0$, then $\alpha(x)$ and $\beta(x)$ are infinitesimals of the ***same order***.

b) In particular, for $C = 1$ the infinitesimals $\alpha(x)$ and $\beta(x)$ are said to be ***equivalent***, written $\alpha \sim \beta$.

c) If $C = 0$, then $\alpha(x)$ is an infinitesimal of ***higher order*** than $\beta(x)$; in this case we write $\alpha = o(\beta)$.

d) If there exists a real number $r > 0$ such that $\lim\limits_{x \to x_0} \dfrac{\alpha(x)}{(\beta(x))^r} \neq 0$, then $\alpha(x)$ is said to be infinitesimal of ***order r*** relative to $\beta(x)$.

A function $\alpha(x)$ is called ***infinitely large*** as $x \to x_0$, if $\lim\limits_{x \to x_0} \alpha(x) = \infty$. The notion of comparable infinities and their classification is introduced much in the same way as it was done for infinitesimals.

If $\alpha(x) \sim \alpha_1(x)$, $\beta(x) \sim \beta_1(x)$ as $x \to x_0$, then

$$\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to x_0}\frac{\alpha(x)}{\alpha_1(x)}\cdot\frac{\alpha_1(x)}{\beta_1(x)}\cdot\frac{\beta_1(x)}{\beta(x)} = \lim_{x\to x_0}\frac{\alpha_1(x)}{\beta_1(x)}$$

Example 8.

Compute $\lim\limits_{x\to 0}\dfrac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)}$

Solution: We have $\lim\limits_{x\to 0}\dfrac{\arcsin\frac{x}{\sqrt{1-x^2}}}{\ln(1-x)} = \lim\limits_{x\to 0}\dfrac{\frac{x}{\sqrt{1-x^2}}}{-x} = -1$,

because $\arcsin\alpha \sim \alpha$ $\ln(1-x) \sim -x$.

Compute the given limits:

134. $\lim\limits_{x\to 1}\dfrac{1-x}{\lg x}$
135. $\lim\limits_{x\to 0}\dfrac{\cos x - \cos 2x}{1-\cos x}$
136. $\lim\limits_{x\to \frac{1}{2}}\dfrac{4x^2-1}{\arcsin(1-2x)}$
137. $\lim\limits_{x\to 0}\dfrac{arctgx^2}{\arcsin 3x\cdot\sin\frac{x}{2}}$
138. $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{2\sin^2 x + xtg7x)}$
139. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin 2x}$

1.9. Continuity of a Function

Continuity of a Function at a Point. A function $y = f(x)$ with the domain of definition D is said to be ***continuous*** at a point x_0, when the following three conditions are fulfilled:

(a) the function $y = f(x)$ is defined at the point x_0, i.e. $x_0 \in D$

(b) there exists $\lim\limits_{x\to x_0} f(x)$;

(c) $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

If condition (a) is fulfilled, then condition (b) and (c) are equivalent to the following:

$$\lim_{\Delta x\to 0}\Delta f(x_0,\Delta x) = 0$$

where $$\Delta f(x_0,\Delta x) = f(x_0+\Delta x) - f(x_0)$$

is the ***increment of the function*** $y = f(x)$ at the point x_0 corresponding to the ***increment of the argument*** $\Delta x = x - x_0$. If at least one of the three conditions is violated at the point x_0, then x_0 is called a ***point of discontinuity*** (or simply, a *discontinuity*) of the function $y = f(x)$.

Classification of points of discontinuity

The following three cases are distinguished:

(a) $\lim\limits_{x \to x_0} f(x)$ exists but the function is not defined at the point x_0 or the condition $\lim\limits_{x \to x_0} f(x) = f(x_0)$ is violated. In this case x_0 is called a ***removable discontinuity***.

(b) $\lim\limits_{x \to x_0} f(x)$ does not exists.

If both one-sided limits $\lim\limits_{x \to x_0+0} f(x)$ and $\lim\limits_{x \to x_0-0} f(x)$ exist (obviously not equal to each other), then x_0 is termed a ***discontinuity of the first kind.***

(c) In all other cases x_0 is called a ***discontinuity of the second kind.***

Find the point of discontinuity of the given function, investigate their nature, and if a discontinuity is removable, redefine the given function so as to make it continuous.

141. $f(x) = \dfrac{1}{x^2(x-1)}$.	142. $f(x) = \dfrac{\lvert 3x-5 \rvert}{3x-5}$.
143. $f(x) = \dfrac{(1+n)^n - 1}{x}$.	144. $f(x) = \dfrac{1}{x}\sin x$.
145. $f(x) = 1 - x\sin\dfrac{1}{x}$.	146. $f(x) = \dfrac{x}{34 - x^2}$.
147. $f(x) = (x+1)\arctan\dfrac{1}{x}$.	148. $f(x) = \dfrac{\lvert x+2 \rvert}{\arctan(x+2)}$.

Continuity on a Set. Uniform Continuity.

A function $y = f(x)$ is said to be ***continuous on a set D*** if it is continuous at each point $x \in D$. It is ***called uniformly continuous*** on D if for any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any $x', x'' \in D$ the inequality $\lvert x' - x'' \rvert < \delta(\varepsilon)$ implies $\lvert f(x') - f(x'') \rvert < \varepsilon$.

Cantor's theorem.

If a function $y = f(x)$ is continuous on the interval *[a; b]*, then it is uniformly continuous on this interval.

Chapter 2. Differential calculus: function of one variable derivatives

2.1. The Derivative Defined

Differentiation of Explicitly Represented Functions.

Let $\Delta f(x_0,\Delta x)=f(x_0+\Delta x)-f(x_0)$ be ***the increment of the function*** $y=f(x)$ at the point x_0 corresponding to the ***increment of the argument*** Δx. ***The derivative*** of the first order (or ***the first derivative***) of a function $y=f(x)$ at a given point x_0 is defined as the limit

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta f(x_0,\Delta x)}{\Delta x} \qquad (2.1)$$

The derivative of function $f(x)$, considered on the set of those points where it exists, is a function itself. The process of finding the derivative is also called differentiation.

Table of derivatives of basic elementary functions:

1. $(x^a)'=ax^{a-1}, a\neq 0$	2. $(a^x)'=a^x\ln a, a>0; (e^x)'=e^x$
3. $(\log_a x)'=\log_a e\cdot\frac{1}{x}, a>0, a\neq 1;$	4. $(\ln x)'=\frac{1}{x}$
5. $(\sin x)'=\cos x$	6. $(\cos x)'=-\sin x$
7. $(\tan x)'=\frac{1}{\cos^2 x}$	8. $(\cot x)'=-\frac{1}{\sin^2 x}$
9. $(\arcsin x)'=-(\arccos x)'=\frac{1}{\sqrt{1-x^2}}$	10. $(\arctan x)'=\ (arc\cot x)'=\frac{1}{1+x^2}$

Rules for differentiation of functions:

1. Let C be a constant and let $f(x), g(x)$, be differentiable functions.

Then:

1. $(C)'=0$ 2. $(f+g)'=f'+g'$ 3. $(Cf)'=Cf'$

4. $(fg)'=f'g+fg'$ 5. $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}, g\neq 0$.

Find the derivatives of the given functions:

149. $y=3-2x+\frac{2}{3}x^4$;

150. $y=-\frac{5x^5}{a^2}$;

151. $y=\frac{x-1}{x+1}$;

152. $y=\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}$;

153. $y=(x^2-1)(x^2-4)(x^2+9)$;

154. $y=\frac{x^2+1}{x^3-x}$;

155. $y=\frac{1+3x^2}{\sqrt{2\pi}}$;

156. $y=\frac{1}{x^3+3x-1}$;

157. $y=\frac{a}{\sqrt[5]{x^3}}+\frac{\sqrt[3]{x^2}}{b}$;

158. $y=\frac{a+bx}{c+dx}$;

159. $y=\frac{2}{2x-1}-\frac{1}{x}$;

160. $y=\frac{2+\sqrt{x}}{2-\sqrt{x}}$;

161. $y=(\sqrt{x}-1)(\frac{1}{\sqrt{x}}+1)$;

162. $y=3\sqrt[3]{x^2}-2\sqrt[4]{x^3}$;

163. $y=(3\sqrt[3]{x^2}+6\sqrt[3]{x})\sqrt[3]{x^4}$;

164. $y=\frac{4}{\sqrt[4]{x^3}}-\frac{3}{\sqrt[3]{x^2}}$;

165. $y=x^3\cot x$;

166. $y=\frac{\tan x}{\sqrt[3]{x^2}}$;

167. $y=\frac{\cos x}{1+\sin x}$;

168. $y=\sqrt{x}\sin x$;

169. $y=3x^3\log_2 x+\frac{x^2}{e^x}$;

170. $y=\frac{\sin x-\cos x}{\sin x+\cos x}$.

2.2. Differentiation of a Composite function

Let the function $y = f(x)$ have a derivative at the point x_0 and let the function $z = g(y)$ have a derivative at the point $y_0 = f(x_0)$. Then the composite function $z = g(f(x))$ has a derivative at the point x_0 equal to

$$z'(x_0) = g'(y_0) f'(x_0) \qquad (2.\,2)$$

(the rule for differentiation of a composite function).

Example 9. Find the derivative of the function $z = \log_3(\arcsin x)$.

Solution:

Setting $z = \log_3 y$ and $y = \arcsin x$, we have

$$z'(y) = \log_3 e \cdot \frac{1}{y} \text{ and } y'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Hence, according to (2. 2), we obtain

$$z'(x) = \frac{\log_3 e}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}.$$

Find the derivatives of the given functions:

171. $y = x^{\frac{3}{2}} \sqrt[3]{x^5 + a}$;

172. $y = \sqrt{\frac{1-x^2}{1+x^2}}$;

173. $y = \sin\frac{3x}{2}$; $y = 6\cos\frac{2x}{3}$

174. $y = (1+4x^2)^3$;

175. $y = \sqrt[4]{(1+3x^2)^3}$;

176. $y = \sin^2\frac{x}{2}$;

177. $y = x^2 e^{-2x}$;

178. $y = \sqrt{1+\sin 4x} - \sqrt{1-\sin 4x}$;

179. $y = x \arcsin \ln x$;

180. $y = \cos^2(\frac{\pi}{4} - \frac{x}{2})$;

186. $y = \sqrt{arc\cot\frac{x}{2}}$;

187. $y = \sqrt{1+\tan(x+\frac{1}{x})}$;

188. $y = \cos^2(\sin\frac{x}{3})$;

189. $y = \sqrt{\sin\sqrt{x}}$;

190. $y = \arctan(x - \sqrt{1+x^2})$;

191. $y = \arccos\frac{b + a\cos x}{a + b\cos x}$;

192. $y = \sqrt{x} e^{\frac{x}{2}}$;

193. $y = \frac{e^{-x^2}}{2^x}$;

194. $y = 2^{\frac{x}{\ln x}}$;

195. $y = 2^{\sqrt{\sin^2 x}}$;

181. $y=\sqrt[4]{(1+\sin^2 x)^3}$; 196. $y=3^{2^x}$;

182. $y=e^{\frac{x}{3}}\cos^2\frac{x}{3}$; 197. $y=\log_2 \ln 2x$;

183. $y=\frac{x}{2}\sqrt{x^2+a}+\frac{a}{2}\ln(x+\sqrt{x^2+a})$; 198. $y=\ln x\log_a x-\ln a\log_a x$;

184. $y=\ln\tan(\frac{\pi}{4}+\frac{x}{2})$; 199. $y=e^{\sqrt{\ln(ax^2+bx+c)}}$;

185. $y=\ln\sqrt{\frac{1+x}{1-x^2}}$; 200. $y=\ln\arctan\sqrt{1+x^2}$;

201. $y=\ln(x+\sqrt{a^2+x^2})$.

Find the derivatives of the given hyperbolic functions:

202. $y=\sinh x=\frac{e^x-e^{-x}}{2}$ (hyperbolic sine),

203. $y=\cosh x=\frac{e^x+e^{-x}}{2}$ (*hyperbolic* cosine),

204. $y=\tanh x=\frac{\sinh x}{\cosh x}$ (hyperbolic tangent),

205. $y=\coth x=\frac{\cosh x}{\sinh x}$ (hyperbolic cotangent).

The logarithmic derivative of a function $y=f(x)$ is defined as the derivative of the logarithm of this function, i.e.

$$(\ln y)'=\frac{y'}{y}$$

Taking the logarithm prior to finding the derivative often simplifies the computation of the derivative.

Example 10. Find the derivative of the function $y=\sqrt{\frac{x(x-1)}{x-2}}$.

Solution:

We have

$$\ln y=\frac{1}{2}(\ln(x)+\ln(x-1)-\ln(x-2)),$$

$$(\ln y)'=\frac{y'}{y}=\frac{1}{2}(\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x-2}),$$

hence

$$y' = (\ln y)'y = \frac{x^2 - 4x + 2}{2\sqrt{(x-1)(x-2)^3}}.$$

Example 11. Find the derivative of an ***exponential composite function***: $y = (1 + \frac{1}{x})^x$

Solution:

Taking the logarithm, we obtain

$$\ln y = x \ln(1 + \frac{1}{x})$$

Then we find the derivatives of the left-hand and right-hand members

$$(\ln y)' = \frac{y'}{y} = \ln(1 + \frac{1}{x}) - \frac{1}{1+x}.$$

Consequently,

$$y' = (\ln y)'y = (1 + \frac{1}{x})^x (\ln(1 + \frac{1}{x}) - \frac{1}{1-x}).$$

Taking first the logarithm of the given function, find its derivative:

206. $y = \frac{(x-3)^2(2x-1)}{(x+1)^3}$; 207. $y = \sqrt{\frac{(x+2)(x-1)^2}{x^5}}$;

208. $y = \frac{\sqrt{x+2}}{\sqrt[3]{(x-1)^2(2x+1)}}$; 209. $y = x^3\sqrt{\frac{x-1}{(x+2)\sqrt{x-2}}}$;

210. $y = x^x$; 211. $y = x^{2^x}$; 212. $y = \sqrt{x}^{\sqrt[3]{x}}$;

213. $y = (\ln x)^{\frac{1}{x}}$; 214. $y = (\sin x)^{\arcsin x}$; 215. $y = x^{x^x}$.

2.3. Differentiation of Functions Represented Implicitly or Parametrically

The function $y = f(x), x \in (a,b)$ is said to be represented implicitly by the equation $F(x,y) = 0$ if for all $x \in (a,b)$

$$F(x, f(x)) \equiv 0 \qquad (2.3)$$

To compute the derivative of the function $y = f(x)$, one should differentiate identity (2. 3) with respect to x (regarding the left-hand member as a composite function of x) and then solve the obtained equation with respect to $f'(x)$.

Example 12. The equation $x^2 + y^2 = 1$ defines implicitly on the interval (-1;1) two functions:

$$y_1(x) = \sqrt{1-x^2}$$
$$y_2(x) = -\sqrt{1-x^2} \tag{2.4}$$

Find their derivatives without using explicit expression (2. 4).

Solution:

Let $y(x)$ be any of these functions. Then, differentiating the identity $x^2 + y^2(x) \equiv 1$ with respect to x, we obtain

$$2x + 2y(x)y'(x) = 0$$

Hence

$$y'(x) = -\frac{x}{y(x)}$$

i.e.

$$y_1'(x) = -\frac{x}{y_1(x)} = -\frac{x}{\sqrt{1-x^2}}$$

and

$$y_2'(x) = -\frac{x}{y_2(x)} = \frac{x}{\sqrt{1-x^2}}.$$

Example 13. Deduce the rule for differentiation of an inverse function.

Solution:

If $x = f^{-1}(y), y \in E,$ is the inverse of the function $y = f(x), x \in D,$ then for all the following equality is fulfilled:

$$f(f^{-1}(y)) - y = 0$$

In other words, the inverse function $x = f^{-1}(y)$ is a function represented implicitly by the equation

$$f(x) - y = 0 \tag{2. 5}$$

To compute the derivative of the function $x = f^{-1}(y)$, we differentiate (2. 5) with respect to y:

$$f_x'(x(y))x_y'(y) - 1 = 0$$

whence

$$x_y'(y) = \frac{1}{f_x'(x(y))}$$

For implicitly represented functions as well as for composite functions we shall also use symbols of type $y_x^{\prime}$ to denote their derivatives whenever it is necessary to indicate with respect to which variable differentiation is carried out.

Find $y_x^{\prime}$ for the indicated implicitly represented functions:

216. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; 217. $x^4+y^4=x^2y^2$; 218. $2y\ln y=x$;

219. $e^x\sin y-e^y\cos x=0$; 220. $\sin(xy)+\cos(xy)=0$;

221. $2^x+2^y=2^{x+y}$; 222. $x-y=\arcsin x-\arcsin y$;

223. $\sqrt{x}+\sqrt{y}=\sqrt{a}, a>0$.

Let there be given two functions:

$$x=\phi(t), y=\psi(t), t\in(\alpha,\beta) \tag{2. 6}$$

If the function $x=\phi(t)$ has an inverse $t=\phi^{-1}(x)$ on the interval (α,β) then a new function is defined:

$$y(x)=\psi(\phi^{-1}(x)) \tag{2. 7}$$

which is said to be represented parametrically by relations (2. 6). Differentiating (2. 7) with respect to x and applying the rule for differentiation of an inverse function (Example 13), we obtain

$$y_x^{\prime}=\psi_t^{\prime}\cdot t_x^{\prime}=\frac{\psi_t^{\prime}}{\phi_t^{\prime}}=\frac{y_t^{\prime}}{x_t^{\prime}} \tag{2. 8}$$

Example 14. Find $y_x^{\prime}$ if

$$x=\cos^2 t, y=\sin t, t\in(0,\frac{\pi}{2})$$

Solution:

Since $\psi_t^{\prime}=\cos t, \phi_t^{\prime}=-2\cos t\sin t$, by formula (2. 8), we find

$$y_x^{\prime}=\frac{1}{2\sin t}, x=\cos^2 t.$$

Find $y_x^{\prime}$ for the function represented parametrically:

224. $x=2t, y=3t^2-5t, t\in(-\infty,+\infty)$;
225. $x=t^3+2, y=0.5t^2, t\in(-\infty,+\infty)$;
226. $x=\frac{1}{t+1}, y=(\frac{t}{t+1})^2, t\neq-1$;
227. $x=2^{-t}, y=2^{2t}, t\in(-\infty,+\infty)$;
228. $x=a\cos\varphi, y=b\sin\varphi, \varphi\in(0,\pi)$;

229. $x = \tan t, y = \sin 2t + 2\cos 2t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$;

230. $x = \arccos\frac{1}{\sqrt{1+t^2}}, y = \arcsin\frac{1}{\sqrt{1+t^2}}, t \in (0,+\infty)$;

231. $x = \ln(1+t^2), y = t - \arctan t, t \in (0,+\infty)$;

232. $x = 3\log_2 \cot t, y = \tan t + \cot t, t \in (0, \frac{\pi}{2})$;

233. $x = \arcsin(t^2-1), y = \arccos 2t, t \in (0, \sqrt{2})$.

Find y_x' at the indicated points.

234. $x = t\ln t, y = \frac{\ln t}{t}, t = 1$;

235. $x = t(t\cos t - 2\sin t), y = t(t\sin t + 2\cos t), t = \frac{\pi}{4}$;

236. $x = e^t \cos t, y = e^t \sin t, t = \frac{\pi}{6}$;

237. $x = \frac{3at}{1+t^2}, y = \frac{3at^2}{1+t^2}, t = 2$.

2.4. Derivatives of Higher Orders

The ***derivative of the second order*** of the function $y = f(x)$ is defined as the derivative of its first derivative, i.e.

$$y''(x) = (y'(x))'. \tag{2. 9}$$

In general, the ***derivative of order*** n ***(or the*** n***th derivative)*** is defined as the derivative of the derivative of order $(n-1)$, i.e.

$$y^{(n)}(x) = (y^{(n-1)}(x))', n = 2,3,... \tag{2. 10}$$

For the derivative of order n the notation $\frac{d^n y}{dx^n}$ is also used.

Example 15. Find y'' if $y = \ln(x + \sqrt{1+x^2})$.

Solution:

We have $y' = \frac{1}{\sqrt{1+x^2}}$. Consequently

$$y'' = (\frac{1}{\sqrt{1+x^2}})' = -\frac{x}{(1+x^2)^{3/2}}.$$

Find the derivatives of the second order of the given functions.

238. $y = \cos^2 x$; 239. $y = e^{-x^2}$;

240. $y = \arctan x^2$; 241. $y = \frac{\arcsin x}{\sqrt{1-x^2}}$.

Example 16. Find the second derivative of the function represented implicitly.

$$\sqrt{x^2 + y^2} = ae^{\arctan\frac{y}{x}}, a > 0.$$

Solution:

Differentiating the equation defining the function $y(x)$, we obtain

$$\frac{x + yy'}{\sqrt{x^2+y^2}} = ae^{\arctan\frac{y}{x}} \cdot \frac{y'x - y}{x^2 + y^2} = \frac{y'x - y}{\sqrt{x^2+y^2}}$$

Hence, $$x + yy' = xy' - y$$

and, consequently, $$y' = \frac{x+y}{x-y}\ . \tag{2. 11}$$

Differentiating (2. 11) and making use of the expression (2. 9), found for y', we get

$$y'' = \frac{2(x^2+y^2)}{(x-y)^3}.$$

Find the second derivative of the function represented implicitly.

242. $y^2 = 2px$; 243. $y = \tan(x+y)$;

244. $y = 1 + xe^y$; 245. $e^{x-y} = xy$.

Example 17. Find the second derivative of the function represented parametrically $x=\ln t, y=t^2, t\in(0,+\infty)$

Solution:

We have $y_x' = \dfrac{y_t'}{x_t'} = 3t^3$

and $y_{xx}'' = (y_x')_x' = (y_x')_t' \, t_x' = \dfrac{(y_x')_t'}{x_t'} = \dfrac{9t^2}{1/t} = 9t^3.$

Note that in the present case the parameter t is readily eliminated from the given equations by putting $t=e^x$. Consequently, the expression for y_{xx}'' as a function of x has the form $y_{xx}'' = 9e^{3x}$.

In the general case, if $x=\varphi(t), y=\psi(t)$, then y_{xx}'' is computed by the formula

$$y_{xx} = \frac{\psi''(t)\varphi'(t)-\varphi''(t)\psi'(t)}{(\varphi'(t))^3} = \frac{\begin{vmatrix}\varphi'(t) & \psi'(t)\\ \varphi''(t) & \psi''(t)\end{vmatrix}}{(\varphi'(t))^3} \qquad (2.\,12)$$

Find the second derivative of the function represented parametrically.

246. $x=\sec t, y=\tan t, t\in(0,\frac{\pi}{2})$;

247. $x=\arctan t, y=\ln(1+t^2), t\in(-\infty,+\infty)$;

248. $x=\arcsin t, y=\ln(1-t^2), t\in(-1,1)$;

249. $x=a\cos^3 t, y=a\sin^3 t, t\in(0,\frac{\pi}{2})$.

2.5. Geometrical Applications of the Derivative

The magnitude of the derivative $f'(x_0)$ of the function $y = f(x)$ at the point x_0 is equal to the slope $k = \tan\varphi$ of the tangent line TT' to the graph of this function drawn through the point $M_0(x_0, y_0)$ where $y_0 = f(x_0)$ (Fig. 2. 1). The equation of the tangent line TT' to graph of the function $y = f(x)$ at its point $M_0(x_0, y_0)$ has the form

$$y - y_0 = f'(x_0)(x - x_0)$$

The straight line NN' passing through the point of tangency M_0 perpendicular to the tangent (line) is called the normal to the graph of the function $y = f(x)$ at this point.

The equation of the normal is:

$$(x - x_0) + f'(x_0)(y - y_0) = 0.$$

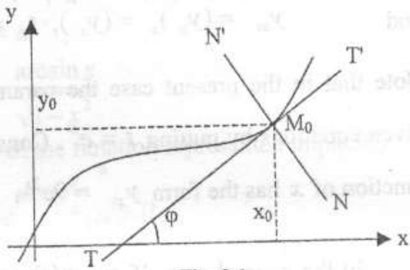

Fig. 2.1

Write the equations of the tangent and the normal to the graph of the given function $y = f(x)$ at the indicated point if:

250. $y = x^2 - 5x + 4, x_0 = -1$;
251. $y = \sqrt{x}, x_0 = 4$;
252. $y = x^3 + 2x^2 - 4x - 3, x_0 = -2$;
253. $y = \tan 2x, x_0 = 0$;
254. $y = \ln x, x_0 = 1$;
255. $y = e^{1-x^2}, x_0 = -1$.

2.6. Differential of First Order

A function $y = f(x)$ is called ***differentiable*** at a point x_0 if its increment $\Delta y(x_0, \Delta x)$ can be represented in the form:

$$\Delta y(x_0, \Delta x) = A\Delta x + 0(\Delta x)$$

The principal linear part A Δx of the increment Δy is called the ***differential*** of this function at the point x_0, corresponding to the increment Δx and in denoted by the symbol $dy(x_0, \Delta x)$.

For a function $y = f(x)$ to be differentiable at a point x_0, it is necessary and sufficient that the derivative $f'(x_0)$ exists; the equality $A = f'(x_0)$ being valid.

This assertion allows any function having a derivative to be called differentiable. It is in this sense that we have used this expression in the preceding section.

The expression for the differential has the form:

$$dy(x_0, dx) = f'(x_0)dx,$$

where $dx = \Delta x$.

Example 18. Find approximately the volume V of the sphere of radius *r=1,02m.*

Solution:

Since $V(r) = \frac{4}{3}\pi r^3$, setting $r_0 = 1, \Delta r = 0{,}02$ and using formula $(\Delta y \approx dy)$, we obtain:

$$V(1.02) = V(1) + \Delta V(1.0.02) \approx V(1) + V'(1) \times 0{,}02 = \frac{4}{3}\pi + 4\pi \times 0.02 \approx 4{,}43m^3.$$

Geometrical meaning of the differential. The differential $dy(x_0, \Delta x)$ is equal to the increment of the ordinate of the tangent line TT' to the graph of the function $y = f(x)$ at the point $M_0(x_0, y_0)$ with the increment of the argument equal to Δx (Fig. 2).

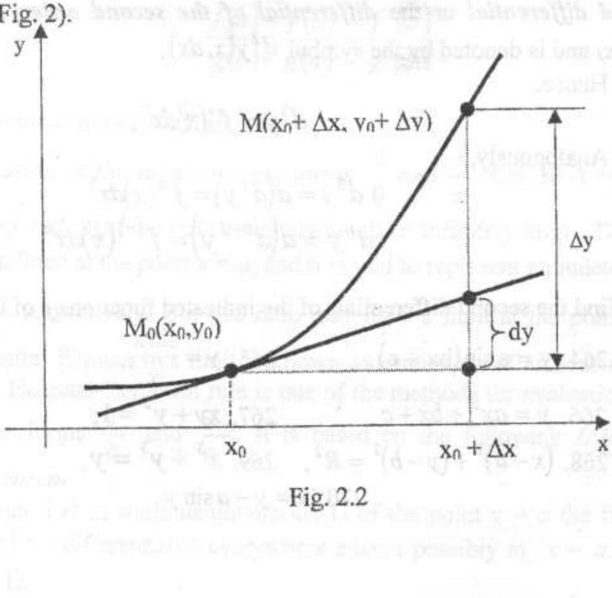

Fig. 2.2

Find the differentials of the indicated functions for arbitrary values of the argument x and its arbitrary increment $\Delta x = dx$.

256. $x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}-5$. 259. $x\ln x-x+1$. 121 $y^5+y-x^2=1$.

257. $\sin x-x\cos x+4$. 260. $x\arcsin x+\sqrt{1-x^2}-3$.

258. $x\arctan x-\ln\sqrt{1+x^2}$. 261. $\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}$.

262. $e^y=x+y$.

263. Compute approximately:

arcsin 0,05; b) arctan 1,04; c) ln 1,2.

2.7. Differentials of Higher Orders

Consider the differential $dy(x,\Delta_1 x)=f'(x)\Delta_1 x$ as a function x with a fixed $\Delta x=\Delta_1 x$. Assuming that the function $y=f(x)$ is twice differentiable at the point x we find the differential of $dy(x,\Delta_1 x)$ *for* $\Delta x=\Delta_2 x$:

$$d(dy(x,\Delta_1 x))\big|_{x,\Delta x=\Delta_2 x}=f''(x)\Delta_1 x\Delta_2 x$$

The value of the obtained expression for $\Delta_1 x=\Delta_2 x=dx$ is known as the ***second differential*** or the ***differential of the second order*** of the function $y=f(x)$ and is denoted by the symbol $d^2y(x,dx)$.

Hence:

$$d^2y=f''(x)dx^2. \qquad (2.13)$$

Analogously,

$$d^3y=d(d^2y)=f'''(x)dx^3$$
$$d^ny=d(d^{n-1}y)=f^{(n)}(x)dx^n \qquad (2.14)$$

Find the second differentials of the indicated functions y of the argument x.

264. $y=a\sin(bx+c)$. 265. $y=\frac{\sin x}{x}$.

266. $y=ax^2+bx+c$. 267. $xy+y^2=1$.

268. $(x-a)^2+(y-b)^2=R^2$. 269. $x^3+y^3=y$.

270. $x=y-a\sin y$.

2.8. Theorems on differentiable functions

Mean value Theorems.

Rolle's theorem. If a function *f(x)* is continuous on the closed interval *[a;b]* differentiable for $x \in (a,b)$ *and* $f(a)=f(b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c)=0$.

Points at which $f'(x)=0$ are called the ***stationary points*** of the function *f(x)*.

Lagrange's theorem. If a function *f(x)* is continuous on the closed interval *[a,b]* and differentiable for $x \in (a,b)$, then there exists at least one point $c \in (a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$ ***(Lagrange's formula).***

Cauchy's theorem. If two functions *f(x)* and g*(x)* are continuous on the closet interval *[a,b]* differentiable for $x \in (a,b)$ *and* $g'(x) \neq 0$ for all $x \in (a,b)$ then there exists at least one point $c \in (a,b)$ such that:

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\psi)}{g'(\psi)}.$$

L'Hospital-Bernoulli Rule.

Evaluation of the indeterminate forms. $\frac{0}{0}$ ***and*** $\frac{\infty}{\infty}$. Let, as $x \to a$, the functions *f(x)* and $\varphi(x)$ be both infinitely small or infinitely large. Then their ratio is not defined at the point *x* = *a*, and it is said to represent an indeterminate form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$, respectively. But this ratio may have a limit at the point *x* = *a,* finite or infinite. Finding this limit is known as evaluation of an indeterminate form. The L'Hospital-Bernoulli rule is one of the methods for evaluation of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is based on the following ***L'Hospital-Bernoulli theorem.***

Theorem. Let in some neighborhood U of the point *x* = *a* the functions *f(x)* and $\varphi(x)$ be differentiable everywhere except possibly at *x* = *a,* and let $\varphi'(x) \neq 0$ in U.

If the functions $f(x)$ and $\varphi(x)$ are both either infinitesimals, or infinitely large as $x \to a$ and $\frac{f'(x)}{\varphi'(x)}$ approaches a limit as x approaches a, then $\frac{f(x)}{\varphi(x)}$ approaches the same limit, i.e.

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)}. \qquad (2.15)$$

The rule is also applicable when $a = \infty$.

Example 19. Find $\lim\limits_{x \to 0} \frac{e^{2x} - 1}{\arctan 5x}$ (i.e. evaluate the indeterminate form $\frac{0}{0}$).

Solution:

Using formula (2. 15), we obtain:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{\arctan 5x} = \lim \frac{2e^{2x}}{\frac{1}{1+25x^2} \times 5} = \frac{2}{5},$$

since $e^{2x} \to 1$ *and* $\frac{1}{1+25x^2 \times 5} \to 1$ *as* $x \to 0$.

In some cases evaluation of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may require repeated application of the L'Hospital-Bernoulli rule.

Example 20. Find $\lim\limits_{x \to +\infty} \frac{\ln^2 x}{x^3}$ (i.e. evaluate the indeterminate form $\frac{\infty}{\infty}$).

Solution:

Applying formula (2. 15) twice, we obtain:

$$\lim_{x \to +\infty} \frac{\ln^2 x}{x^3} = \lim \frac{\frac{2 \ln x}{x}}{3x^2} = \frac{2}{3} \lim_{x \to +\infty} \frac{\ln x}{x^3} = \frac{2}{3} \lim_{x \to +\infty} \frac{\frac{1}{x}}{3x^2} = 0.$$

At each step of applying the L'Hospital-Bernoulli rule it is advisable to use various identical transformations to simplify the ratio, and also to combine this rule with any other methods for evaluation of limits.

Example 21. Find $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$, (i.e. evaluate the indeterminate form $\frac{0}{0}$).

Solution:

Using formula (2. 15), we obtain:

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{\cos^2 x} - \cos x}{3x^2} = \frac{1}{3} \lim_{x \to 0} \frac{1 - \cos^3 x}{x^2 \cos^2 x}.$$

We then eliminate the factor $\cos^2 x$ ro the denominator, since it has the limit 1 as $x \to 0$. Expanding the difference of cubes standing in the numerator, we eliminate then the factor $(1+\cos x+\cos^2 x)$ having the limit 3 as $x \to 0$. After these simplifications we get:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{x^2}.$$

Applying (2. 15) once again, we obtain:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}.$$

Using the first remarkable limit, we obtain the final result ½ without resorting to the L'Hospital-Bernoulli rule once again.

Evaluate the given indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$.

271. $\lim\limits_{x\to 0}\frac{\ln\cos 2x}{\sin 2x}$.

272. $\lim\limits_{x\to 0}\frac{x-\arctan x}{x^3}$.

273. $\lim\limits_{x\to 0}\frac{x^m-a^m}{x^n-a^n}, m\neq n, a\neq 0.$

274. $\lim\limits_{x\to 0}\frac{e^{2x}-1}{\arcsin 3x}$.

275. $\lim\limits_{x\to 0}\frac{\ln\sin ax}{\ln\sin bx}$.

276. $\lim\limits_{x\to 0}\frac{a^x-b^x}{c^x-d^x}, a\neq b, c\neq d.$

277. $\lim\limits_{x\to 5}\frac{\sqrt[3]{x}-\sqrt[3]{5}}{\sqrt{x}-\sqrt{5}}$.

278. $\lim\limits_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}$.

279. $\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$.

280. $\lim\limits_{x\to +\infty}\frac{\pi-2\arctan x}{e^{\frac{3}{x}}-1}$.

281. $\lim\limits_{x\to 0}\frac{x-\sin x}{x-\tan x}$.

282. $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}$.

283. $\lim\limits_{x\to 0}\frac{e^{3x}-3x-1}{\sin^2 5x}$.

284. $\lim\limits_{x\to \frac{\pi}{4}}\frac{\cot x-1}{\sin 4x}$.

285. $\lim\limits_{x\to 1}\frac{x^3-4x^2+5x-2}{x^3-5x^2+7x-3}$.

2.9. The indeterminate form $0\times\infty$, $\infty - \infty$, 0°, ∞°, 1^{∞}

Evaluation of the indeterminate forms $0\times\infty$ ***and*** $\infty - \infty$. To evaluate $\lim\limits_{x\to a} f(x)\varphi(x)$, where *f(x)* is an infinitesimal, and $\varphi(x)$ is an infinite as $x\to a$ (evaluation of the indeterminate form $0\times\infty$), it is necessary to transform the product to the form $\dfrac{f(x)}{\dfrac{1}{\varphi(x)}}$ (the indeterminate form $\dfrac{\infty}{\infty}$) and then apply the L'Hospital-Bernoulli rule.

Example 22. Find $\lim\limits_{x\to 1}\sin(x-1)\times\tan\dfrac{\pi x}{2}$, (i.e. evaluate the indeterminate form $0\times\infty$).

Solution:

We have:

$$\lim_{x\to 1}\sin(x-1)\times\tan\frac{\pi x}{x}=\lim_{x\to 1}\frac{\sin(x-1)}{\cot\dfrac{\pi x}{2}}=\lim_{x\to 1}\frac{\cos(x-1)}{-\dfrac{\pi}{2}\dfrac{1}{\sin^2\dfrac{\pi}{2}}}=$$

$$=\frac{2}{\pi}\lim_{x\to 1}\cos(x-1)\times\sin^2\frac{\pi x}{2}=-\frac{2}{\pi}.$$

To evaluate $\lim\limits_{x\to a}(f(x)-\varphi(x))$, where *f(x)* and $\varphi(x)$ are infinitely large quantities as $x\to a$ (evaluation of the indeterminate form $\infty-\infty$), it is necessary to transform the difference to the form $f(x)\times\left(1-\dfrac{\varphi(x)}{f(x)}\right)$ and then evaluate the indeterminate form $\dfrac{\varphi(x)}{f(x)}$ $(i.e. \dfrac{\infty}{\infty})$. If $\lim\dfrac{\varphi(x)}{f(x)}\neq 1$, *then* $\lim\limits_{x\to a}(f(x)-\varphi(x))=\infty$. And if $\lim\limits_{x\to a}\dfrac{\varphi(x)}{f(x)}=1$, we obtain the above considered indeterminate form $(\infty\times 0)$.

Example 23. Find $\lim\limits_{x\to\infty}\left(x-\ln^3 x\right)$. (i.e. evaluate the indeterminate form $\infty-\infty$).

Solution:

We have: $\lim\limits_{x\to\infty}\left(x-\ln^3 x\right)=\lim\limits_{x\to+\infty} x\left(1-\dfrac{\ln^3 x}{x}\right)$.

Since

$$\lim_{x\to\infty}\frac{\ln^3 x}{x}=\lim_{x\to+\infty}\frac{3\ln^2 x\times\frac{1}{x}}{1}=3\lim\frac{\ln^2 x}{x}=3\lim_{x\to+\infty}\frac{2\ln x\times\frac{1}{x}}{1}=$$

$$=6\lim_{x\to+\infty}\frac{\ln x}{x}=6\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=6\lim_{x\to+\infty}\frac{1}{x}=0.$$

$$\lim_{x\to+\infty}\left(x-\ln^3 x\right)=+\infty.$$

Evaluate the indicated form $0\times\infty$ or $\infty-\infty$

286. $\lim\limits_{x\to\infty}x\left(e^{\frac{1}{x}}-1\right)$.
287. $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)$.
288. $\lim\limits_{x\leftarrow\infty}x^n\times e^{-x}$.
289. $\lim\limits_{x\to 0}x\ln^3 x$.
290. $\lim\limits_{x\to\pi}(\pi-x)\tan\frac{x}{2}$.
291. $\lim\limits_{x\to 0}\left(e^x-e^{-x}-2\right)\cot x$.
292. $\lim\limits_{x\to 0}\left(\frac{1}{\arctan x}-\frac{1}{x}\right)$.
293. $\lim\limits_{x\to 1}(x-1)\cot\pi(x-1)$.
294. $\lim\limits_{x\to\infty}x\sin\frac{a}{x}$.
295. $\lim\limits_{x\to 0}x^2e^{\frac{1}{x^2}}$.
296. $\lim\limits_{x\to 1}\ln x\times\ln(x-1)$.
297. $\lim\limits_{x\to 1}\left(\frac{1}{2(1-\sqrt{x})}-\frac{1}{3(1-\sqrt[3]{x})}\right)$.
298. $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{x}{\cot x}-\frac{\pi}{2\cos x}\right)$.
299. $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$.

Evaluation of the forms $0^0,\infty^0,1^\infty$.In all the three cases we have to evaluate the limit of the expression $\left(f(x)\right)^{\phi(x)}$, where *f(x)* is an infinitesimal in the first case, an infinite in the second, and a function having the limit equal to unity in the third. As to the function *φ(x),* it is an infinitesimal in the first two cases and an infinite in the third.

We proceed as follows: taking first the logarithm of $y=\left(f(x)\right)^{\phi(x)}$.

$$\ln y=\phi(x)\ln f(x). \qquad (2.16)$$

and find the limit of lny, whereupon the limit of y is found. In all the three cases, by virtue of (2. 16), lny is the determine form $0\times\infty$ (check this!) which is evaluated by the above discussed method.

Example 24. Find $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^{2x}$. (in other words, evaluate the indeterminate form 1^{∞}).

Solution:

Let us put $y=\left(1+\frac{1}{x}\right)^{2x}$. Then $\ln y = 2x\ln\left(1+\frac{1}{x}\right)$ is the indeterminate form $\infty\times 0$. Transforming the expression lny to the form $\ln y = 2\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$ and applying the L'Hospital-Bernoulli rule, we find:

$$\lim_{x\to\infty}\ln y = 2\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = 2\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 2.$$

Consequently, $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = e^2$.

Evaluate the indeterminate forms $0^0, \infty^0, 1^{\infty}$:

300. $\lim\limits_{x\to 0} x^{\sin x}$.

301. $\lim\limits_{x\to 0}(\arcsin x)^{\tan x}$.

302. $\lim\limits_{x\to\frac{\pi}{2}}(\pi-2x)^{\cos x}$.

303. $\lim\limits_{x\to 0}\frac{1}{x^{\ln}\left(e^x-1\right)}$.

304. $\lim\limits_{x\to+\infty}\left(x+2^x\right)^{\frac{1}{x}}$.

305. $\lim\limits_{x\to 0} x^{\frac{1}{x}}$

306. $\lim\limits_{x\to 0}(\cot x)^{\frac{1}{\ln x}}$.

307. $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{2x-\pi}$.

308. $\lim\limits_{x\to 1} x^{\frac{1}{x(1-x)}}$.

309. $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x^2}\right)^{x}$.

310. $\lim\limits_{x\to 0}(\cos 2x)^{\frac{3}{x^2}}$.

311. $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$.

2.10. Taylor's formula

If the function $y = f(x)$ has the derivative of the order n+1 at the point x_0, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x),$$

where:

$$R_{n+1}(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}, \ \theta \in (0,1).$$

For $x_0 = 0$ we have

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

(Macloren's formula).

Example 25. Expand polynomial $Q_3(x) = x^3$ into powers of $(x-2)$.
Result: $x^3 = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$.

Example 26. Expand function $f(x) = \frac{x}{1-x}$ into powers of $(x-2)$ to the term which has the third degree of $(x-2)$.
Result: $\frac{x}{x-1} = -2 + (x-2) - (x-2)^2 + (x-2)^3 + \overline{\overline{o}}\left((x-2)^3\right)$.

Write Taylor's formula for the function:

318. $y = 2x^3 - 3x^2 + 5x + 1, x_0 = -1.$

319. $y = e^x, x_0 = 0.$

320. $y = \cos x, x_0 = 0.$

321. $y = \sin^2 x, x_0 = 0.$

322. $y = \ln(4 + x^2), x_0 = 0.$

Usage of Taylor's formula.

a) Taylor's (Macloren's) formula gives a possibility to find a function's value.

Example 27. Compute $\sqrt[3]{30}$.
Solution:

As $\sqrt[3]{30}=\sqrt[3]{27\left(1+\frac{1}{3}\right)}=3\sqrt[3]{1+\frac{1}{3}}$, it is obvious that during computation we should use expansion of function $f(x)=\sqrt[3]{1+x}$ into degrees of x, i. e. Macloren's formula:

$$\sqrt[3]{1+x}=1+\sum_{k=1}^{n}\frac{1}{3}\left(\frac{1}{3}-1\right)*\ldots*\left(\frac{1}{3}-k+1\right)\frac{x^k}{k!}+R_{n+1}(x).$$

Thus,

$$\sqrt[3]{30}=3\left(1+\frac{1}{3}*\frac{1}{3}-\frac{1}{3}*\frac{2}{3}*\frac{1}{2!}\left(\frac{1}{3}\right)^2+\frac{1}{3}*\frac{2}{3}*\frac{5}{3}*\frac{1}{3!}\left(\frac{1}{3}\right)^3\right)+R_4\left(\frac{1}{3}\right)\approx$$

$$\approx 3*1{,}03574=3{,}10722.$$

b) Macloren's formula is also used to compute limits of function.

Example 28. Find $\lim\limits_{x\to 0}\dfrac{e^{\sin x\ln\cos x}-\sqrt[4]{(1+4x)^{\frac{1}{4}}}+x-\frac{3}{2}x^2}{x\sin x^2}$.

Solution:

As $\quad x\sin x^2=x^3+o\left(x^3\right)$,

$$\sqrt{1+4x}=(1+4x)^{\frac{1}{4}}=1+x-\frac{3}{2}x^2+\frac{7}{2}x^3+o\left(x^3\right),$$

$$\sin x*\ln\cos x=(x+o(x))\left(-\frac{x^2}{2}+o\left(x^2\right)\right)=-\frac{x^3}{2}+o\left(x^3\right),$$

$$e^{\sin x\ln\cos x}-\sqrt[4]{1+4x}+x-\frac{3}{2}x^2=-4x^3+o\left(x^3\right).$$

Thus, the sought limit

$$\lim_{x\to 0}\ldots=\lim_{x\to 0}\frac{-4x^3+o\left(x^3\right)}{x^3+o\left(x^3\right)}=-4.$$

2.11. The general investigation of functions and tracing of curves

The general investigation of functions may be divided into two parts: elementary and differential.

Let us start from elementary investigation.

1. Find the natural domain of definition and the set of values of given function.

2. Find the points where the curve meets the coordinate axes. These points are determined by putting $y=0$ and $x=0$ successively and thus obtaining the values of x and y respectively.
3. Find the intervals where values of function are positive or negative.
4. Symmetry. If the equation remains unaltered when x is replaced by $-x$, the graph of function is symmetrical with respect to y axis. The curve is symmetrical with respect to x axis if the equation remains unaltered when y is replaced by $-y$.

If interchanging x and y or x and $-y$ the equation remains the same the graph is symmetrical about the line $y=x$ or the line $y=-x$ respectively.

For example, the graph of function, that is represented implicitly by the equation $x^3+y^3=6xy$, is symmetrical about the line y=x.

Differential investigation.

1. Find the first derivative. The intervals where the value of derivative is positive or negative are the intervals where the function is increasing or decreasing. Also the turning points (the stationary values where $y'_x=0$ or it is not exist) are the points where the function has maximum or minimum values.
2. Find the second derivative. The intervals where the value of the second derivative is positive or negative are the intervals where the graph of function is concave upwards or is concave downwards. The points of inflexion are the points where the sign of the second derivative is changed.
3. Asymptotes. A straight line is said to be an asymptote to a curve if the perpendicular distance of the straight line from a point on the curve tends to zero as the point moves to infinity along the curve.
   For example, for function

Example 29. Trace the curve $y=\sqrt[3]{x^3-2x^2}$.

Solution:

The natural domain of definition is $(-\infty,\infty)$; the points where the graph of this function meets the coordinate axes are (0,0) and (2,0).

The first derivative $y'=\dfrac{3x^2-4x}{3\sqrt[3]{\left(x^3-2x^2\right)^2}}=\dfrac{3x-4}{3\sqrt[3]{(x-2)^2x}}$. It is equal to zero when $x=\dfrac{4}{3}$ and it is not exist when $x=0, x=2$. We have two intervals $(-\infty,0),(\frac{4}{3},\infty)$ where the function is increasing and interval $(0,\frac{4}{3})$ where the function is decreasing.

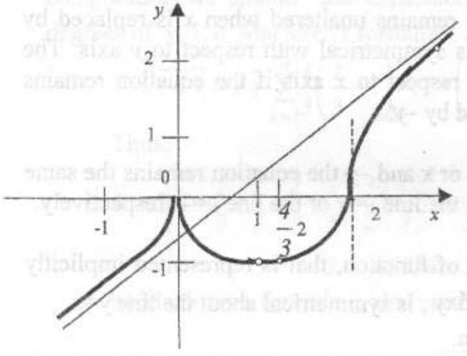

Fig.2.3

Thus (0,0) – maximum, $(\frac{4}{3},-2\sqrt[3]{\frac{4}{3}})$ - minimum.

The second derivative $y''=\frac{-8}{9\sqrt[3]{(x-2)^5x^4}}$. It is not equal to zero but it is not exist when $x=0, x=2$. Its sign is changed when $x=2$, so that the point (2,0) is the point of inflexion.

We have asymptote $y=x-\frac{2}{3}$.

Trace the curves:

321. $y=\frac{x^4}{x^3-1}$. 322. $y=\sqrt[3]{x+1}-\sqrt[3]{x-1}$.

323. $y=\sqrt[3]{x^2-2x}$. 324. $y=x^2e^{-x}$.

325. $y=x+\frac{\ln x}{x}$.

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