МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

Національний аерокосмічний університет

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«Харківський авіаційний інститут»

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ІНТЕГРАЛЬНЕ ЧИСЛЕННЯ ФУНКЦІЇ ОДНІЄЇ ЗМІННОЇ

(англійською мовою)

Навчальний посібник

Харків «ХАІ» 2001

УДК 517

Інтегральне числення функції однієї змінної / Г.І. Кошовий. – Навч. посібник (англійською мовою).- Харків: Нац. аерокосмічний ун-т "Харк. авіац. ін-т", 2001.- 46 с.

Посібник складається з п'яти частин: основні методи знаходження невизначеного інтегралу, стандартні методи інтегрування, визначений інтеграл, геометричні застосування та невласні інтеграли. Теоретичний матеріал добре проілюстрований розв'язаними прикладами. Кожний розділ містить достатню кількість прикладів для самостійної роботи. Загальна кількість прикладів близько 300, в тому числі 40 розв'язаних.

Посібник відповідає програмі курсу "Вища математика" для втузів з поглибленою математичною підготовкою.

Для студентів 1 курсу вищих технічних навчальних закладів. Може бути також корисний для дипломників та аспірантів.

Іл. 7. Бібліогр.: 5 назв

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Content

Chapter 1.Basic methods of computing the indefinite integral

1.1.Definition of indefinite integral

A function $F(x)$ is called ***an antiderivative (a primitive)*** of the function $f(x)$ defined on some set X if $F'(x)=f(x)$ for all $x \in X$. If $\Phi(x)$ and $F(x)$ are two antiderivatives (two primitive functions) of one and the same function $f(x)$, then $\Phi(x)=F(x)+C$, where C is a constant. Conversely, if $F(x)$ is an antiderivative of the function $f(x)$, the set $\{F(x)+C|C \in R\}$ is the totality of all of its primitive function $f(x)$ and denoted by the symbol $\int f(x)dx$ which is read: indefinite integral of function. Thus, by definition, the ***Indefinite Integral***

$$\int f(x)dx=\{F(x)+C\}, \tag{1}$$

where F (x) is one of the antiderivatives of the function f (x) and the constant C takes on real values.

By virtue of the established tradition, equality (1) is written without denoting explicitly the set on the right, i. e. in the form $\int f(x)dx=F(x)+C$, where C is called an arbitrary constant.

Properties of the indefinite integral.

1. $$\left(\int f(x)dx\right)' = \left(F(x)+C\right)' = f(x);$$
2. $$\int af(x)dx = a\int f(x)dx, a \neq 0;$$
3. $$\int f'(x)dx = f(x)+C;$$
4. $$\int (f_1(x)+f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx$$

The table of basic indefinite integral:

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1}+C$, $(n \neq -1)$; 2. $\int \dfrac{dx}{x} = \ln|x|+C$;

3. $\int a^x dx = \dfrac{a^x}{\ln a}+C$ $(a \neq 1)$; $\int e^x dx = e^x + C$;

4. $\int \sin x dx = -\cos x + C$; 5. $\int \cos x dx = \sin x + C$;

6. $\int \frac{dx}{\cos^2 x} = \tan x + C$; 7. $\int \frac{dx}{\sin^2 x} = -\cot x + C$;

8. $\int \frac{dx}{\sin x} = \ln|\tan \frac{x}{2}| + C$; 9. $\int \frac{dx}{\cos x} = \ln|\tan(\frac{x}{2} + \frac{\pi}{4})|$;

10. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$, $(a \neq 0)$;

11. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C$;

12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C, |x| < |a|$;

13. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C, |x| > |a|$;

14. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$;

15. $\int \sinh x dx = \cosh x + C$; 16. $\int \cosh x dx = \sinh x + C$;

17. $\int \frac{dx}{\cosh^2 x} = -\tanh x + C$; 18. $\int \frac{dx}{\sinh^2 x} = -\coth x + C$.

1.2.Direct Integration

The finding of the indefinite integral with the aid of the table of integrals and identical transformations is called direct integration.

Example 1. Compute $\int \frac{dx}{x^2 + x^4}$.

Solution: The integrand we transform:

$$\frac{1}{x^2 + x^4} = \frac{1}{x^2(1 + x^2)} = \frac{1 + x^2 - x^2}{(1 + x^2)x^2} = \frac{1}{x^2} - \frac{1}{1 + x^2}$$

Thus, we have:

$$\int\frac{dx}{x^2+x^4}=\int\frac{dx}{x^2}-\int\frac{dx}{1-x^2}=-\frac{1}{x}+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+C$$

Find the indicated integrals, making use of the table of basic integrals.

1. $\int\sqrt{mx}\,dx$;
2. $\int\frac{dx}{\sqrt[n]{x}}$;
3. $\int\left(\frac{1}{\sqrt[3]{x^2}}-\frac{x+1}{\sqrt[4]{x^3}}\right)dx$;
4. $\int\frac{(\sqrt{a}+\sqrt{x})^2}{\sqrt{ax}}dx$;
5. $\int\frac{x^3+2}{x}dx$;
6. $\int 2^x e^x dx$;
7. $\int 2^x(1+3x^2 2^{-x})dx$;
8. $\int(2x+3\cos x)dx$;
9. $\int\frac{2-\sin x}{\sin^2 x}dx$;
10. $\int\frac{3-2\cot^2 x}{\cos^2 x}dx$;
11. $\int\sin^2\frac{x}{2}dx$;
12. $^*(a)\int\tan^2 x dx,\ (b)\int\tanh^2 x dx$;
13. $\int\frac{dx}{\cos 2x+\sin^2 x}$;
14. $\int(\arcsin x+\arccos x)dx$;
15. $\int(\sin\frac{x}{2}-\cos\frac{x}{2})^2 dx$;
16. $\int\frac{dx}{x^2+4}$;
17. $\int\frac{dx}{5-x^2}$;
18. $\int\frac{dx}{\sqrt{3-x^2}}$;
19. $\int\frac{\sqrt{x^2-3}-\sqrt{x^2+3}}{\sqrt{x^2-9}}dx$;
20. $\int\frac{(1+x)^2}{x(1+x^2)}dx$;
21. $\int(x+a)(x+b)dx$;
22. $\int\left(a^{\frac{1}{3}}+x^{\frac{1}{3}}\right)^3 dx$;
23. $\int\frac{\cos^2 x+3\cos x-2}{\cos^2 x}dx$;
24. $(a)\int\cot^2 x dx,\quad (b)\int\coth^2 x dx$;

25. $\int \frac{dx}{\sqrt{x^2 - 7}}$; 26. $\int \frac{x^2-9}{x^2-8} dx.$

There are two variants of the technique ***Integration by change of variable***

1.3.The method of placing under the differential sign

Let it be required to complete the integral $\int f(x)dx$. Let us assume that there exists a differentiable function $u = \phi(x)$ and a function g (x) such that the integrand f(x)dx can be written in the form $f(x)dx = g(\phi(x))\ \phi'(x)dx = g(u)du$ (this transformation is known as placing $u = \phi(x)$ under the differential sign). Note that the following relation is fulfilled:

$$\int f(x)dx = \int g(\phi(x))\phi'(x)dx = \int g(u)du\Big|_{u=\phi(x)}$$

Therefore the computation of the integral $\int f(x)dx$ is reduced to computing the integral $\int g(u)du$ (which may turn out to be simpler than the original one) followed by the substitution $u = \phi(x)$.

Example 2. Compute the integral $\int \sin^3 x \cos x dx$.

Solution:

The integrand can be written in the form: $\sin^3 x \cos x dx = \sin^3 x d \sin x$.

Thus, we have:

$$\int \sin^3 x d \sin x = \left| u = \sin x \right| = \int u^3 du = \frac{u^4}{4} + C \ \left| = \frac{\sin^4 x}{4} + C \right. .$$

Example 3. Compute the integral $\int \frac{2x+1}{x^2+x-3} dx$

Solution:

We have: $\int \frac{2x+1}{x^2+x-3} dx = \int \frac{d(x^2+x-3)}{x^2+x-3} = \int \frac{du}{u} =$

$$= \ln|u|\,\big|_{u=x^3+x-3} + C = \ln|x^2+x-3| + C$$

The operation of placing the function $\phi(x)$ under differential sign is equivalent to replacing the variable x by a new variable $u=\phi(x)$.

Example 4. Compute the integral $\int\frac{dx}{\sqrt[3]{(3x+1)^2}}$.

Solution:

Let us change variable by the formulas $u = 3x + 1$.

Then $du = 3dx$, i.e. $dx = \frac{1}{3}du$

and

$$\int\frac{dx}{\sqrt[3]{(3x+1)}^2}=\frac{1}{2}\int\frac{du}{u^{2/3}}=u^{\frac{1}{3}}\Big|_{u=3x+1}+C=\sqrt[3]{3x+1}+C.$$

The transformation performed is equivalent to placing the function u=3x+1 under the differential sign.

Compute the given integral with the aid of a suitable substitution.

27. $\int\sqrt{3+x}dx$; 28. $\int(3-4\sin x)^{\frac{1}{3}}\cos dx$; 29. $\int\cosh x\sinh xdx$;

30. $\int\frac{\sec^2 x}{\tan^4 x}dx$; 31. $\int\frac{dx}{a+bx}$; 32. $\int\frac{dx}{x\ln^2 x}$;

33. $\int\frac{\sec^2 x}{a-b\tan x}$; 34. $\int\frac{\cos\frac{x}{\sqrt{2}}}{2-3\sin\frac{x}{\sqrt{2}}}$; 35. $\int\cot xdx$;

36. $\int 3^{4x}dx$; 37. $\int\cos(ax+b)dx$; 38. $\int\sin(\ln x)\frac{dx}{x}$;

39. $\sin\sqrt{x}\frac{dx}{\sqrt{x}}$; 40. $\int\frac{dx}{\cos(x-\frac{\pi}{4})}$; 41. $\int\frac{x}{\sqrt[3]{x^2-1}}$;

42. $\int x\cdot 5^{-x^2}dx$; 43. $\int\frac{dx}{\sinh^2 3x}$; 44. $\int\frac{dx}{1-4x^2}$;

45. $\int \frac{e^{-ax}}{1+e^{-ax}}dx$; 46. $\int \frac{dx}{\sqrt{5-3x^2}}$; 47. $\int \frac{dx}{\sqrt{9x^2-1}}$;

48. $\int \frac{\sin x dx}{\sqrt{\cos^2 x+4}}$; 49. $\int \frac{x^3 dx}{x^8+1}$; 50. $\int \frac{x dx}{\sqrt{x^4+1}}$;

51. $\int \frac{dx}{a^2+b^2x}$; 52. $\int \frac{\sin ax}{\cos^3 ax}dx$; 53. $\int \cosh^2 x \sinh x dx$;

54. $\int \frac{e^x}{(7-e^x)}dx$; 55. $\int \tan s dx$; 56. $\int \coth 4x dx$;

57. $\int \frac{a^{1/2}}{x^2}dx$; 58. $\int \frac{x dx}{\cosh^2(x^2+1)}$; 59. $\int \frac{a^x}{\sqrt{a^{2x}-1}}dx$;

60. $\int \frac{dx}{4x^2+7}$; 61. $\int \frac{x dx}{4x^2+7}$; 62. $\int \frac{a^x}{\sqrt{a^{2x}-1}}dx$.

63. $\int \frac{dx}{(a-b)x^2-(a+b)}\ (0<b<a)$

1.4 The method of substitution.

Let it be required to compute the integral $\int f(x)dx$, where the function $f(x)$ is defined on a set X.

We introduce a new variable u by the formula $x=\varphi(u): U \to X$, where the function $\varphi(u)$ is differentiable on some set U and maps U one-to-one onto X, i. e. has the inverse function $u=\varphi^{-1}(x): X \to U$.

Substituting $x=\varphi(u)$ into the original integrand we obtain

$$f(x)dx = f(\varphi(u))\varphi'(u)du = g(u)du$$

The following equality holds true

$$\int f(x)dx = \int f(\varphi(u))\varphi'(u)du = \int g(u)du\,\Big|_{u=\varphi^{-1}(x)},$$

I.e. the computation of the integral $\int f(x)dx$ is reduced to finding the integral $\int g(u)du$ (which may turn out to be simpler than the initial one) followed by the substitution $u=\varphi^{-1}(x)$.

Example 5. Compute the integral $\int\frac{1+x}{1+\sqrt{x}}dx.$

Solution:

In the present case the domain of definition of the integrand $X=(0,+\infty)$. let us make the substitution $x=\varphi(u)=u^2, u\in[0,+\infty)$.

Then $dx=2udu, u=\varphi^{-1}(x)=\sqrt{x}$, whence

$$\int\frac{1+x}{1+\sqrt{x}}dx=2\int\frac{u^3+u}{u+1}dx=2\int(u^2-u+2)du-4\int\frac{du}{u+1}=2\left(\frac{1}{2}u^3-\frac{1}{3}u^2+2u\right)-=$$

$$-4\ln(u+1)\Big|_{u=\sqrt{x}}=2\left(\frac{1}{3}x^{2/3}-\frac{1}{2}x+2x^{1/2}\right)-4\ln(\sqrt{x}+1)+C$$

Find the indefinite integrals making the indicated substitutions.

64. $\int\frac{\sin x dx}{4+\cos^2 x}, t=\cos x;$

65. $\int\frac{dx}{x(1-\ln^2 x)}, t=\ln x;$

66. $\int\frac{e^{2x}}{1+3e^{2x}}dx, t=e^{2x};$

67. $\int x^2e^{4x^3}dx, t=4x^3;$

68. $\int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx, t=\sqrt{x};$

69. $\int\frac{x^3dx}{\sqrt{x-1}}, t^2=x-1;$

70. $\int\frac{dx}{1+\sqrt[3]{x+1}}, x+1=t^3;$

71. $\int\sqrt{4x-1}\cdot dx,\quad 4x-1=t^2;$

72. $\int\sqrt[3]{5+6x}\cdot dx,\quad 5+6x=t^3;$

73. $\int\sin(a+bx)dx,\ a+bx=t;$

74. $\int\frac{dx}{x\sqrt{1-x^3}}, x=(1-t^2)^{1/3};$

75. $\int\frac{dx}{x\sqrt{4-x^2}}, x=\frac{2}{t};$

76. $\int\frac{dx}{x+\sqrt{x}}, x=t^2;$

77. $\int\frac{e^{2x}}{e^x+1}dx,\ x=\ln t.$

1.5.The method of integration by parts

If $u\ (x)$ and $v\ (x)$ are differentiable functions, the following formula for integration by parts is valid

$$\int udv\ =\ uv\ -\ \int vdu \tag{2}$$

This formula is used when the integrand $f(x)dx$ can be represented in the form udv so that, with a proper choice of expressing u and dv, the integral standing on the right of (2) may turn out to be simpler than the original integral. One should be in mind that u must be supplied with such factors, which get simplified during differentiation. For instance, if the integrand is a product of a polynominal by a trigonometric or exponential function, the polynominal should be distributed to u and the remaining expression to dv. Here formula (2) may be used repeatedly.

Example 6. Find $\int x^2 \cos x dx$.

Solution: We set $u = x^2$ and $dv = \cos x dx$.

Then $du = 2xdx$ and $v = \int \cos x dx = \sin x$ (the constant $\mathbb{C}$ is assumed here to be equal to zero, i.e. we take as v one of the antiderivatives).

By formula (2), we have $\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx$.

We apply the formula for integration by parts once again to the integral standing on the right, equating u, as before, to the polynominal (i.e. to 2x).

We have: $u = 2x, dv = \sin x dx$. Hence, $du = 2dx$ and $v = \int \sin x dx = -\cos x$.

Applying formula (2), we finally obtain:

$$\int x^2 \cos x dx = x^2 \sin x - (-2x\cos x - \int(-\cos x)2dx) = x^2 \sin x + 2x\cos x - 2\sin x + C$$

If a logarithmic or inverse trigonometric function is contained (as a factor) in the integrand, then it should be taken for u since these functions are simplified during differentiation.

Example 7. Find $\int \ln x dx$.

Solution: We set $u = \ln x, dv = dx$.

Then $du = \frac{dx}{x}$ and $v = \int dx^2 x$. Substituting into formula (2), we find

$$\int \ln x dx = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C$$

After repeated application of the formula for integration by parts, we sometimes arrive in the right-hand side at an expression containing the original integral; i.e. we obtain an equation with the sought for integral as an unknown.

Example 8. . Find $\int e^{2x}\sin x dx$.

Solution:

We set $u = e^{2x}, dv = \sin x dx$.

then $du = e^{2x} 2dx$ and $v = -\cos x$ substituting into formula (2), we find

$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2\int e^{2x}\cos x dx$$

Applying the formula for integration by parts once again to the integral standing on the right, equating u, as before, to the exponential function, we have

$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2(e^{2x}\sin x - 2\int e^{2x}\sin x dx)$$

Thus, we obtain the equation for unknown integral $I = \int e^{2x}\sin x dx$.

$$I = -e^{2x}\cos x + 2e^{2x}\sin x - 4I,$$

$$5I = -e^{2x}\cos x + 2e^{2x}\sin x,$$

Finally we have:

$$\int e^{2x}\sin x dx = \frac{1}{5}e^{2x}(-\cos x + 2\sin x).$$

Find the indicated integrals, applying the formula for integration by parts.

78. $\int \arccos x dx$; 79. $\int x\cos x dx$; 80. $\int x \ln x dx$

81. $\int \frac{\ln x}{\sqrt[3]{x}} dx$; 82. $\int (x^2 - x + 1)\ln x dx$;

83. $\int x^2 \sin x dx$; 84. $\int x^2 e^{-x} dx$; 85. $\int x^3 e^x dx$;

86. $\int x^3 e^{-x^2} dx$; 87. $\int \frac{\ln^2 x}{x^2} dx$; 88. $\int x \arctan x dx$;

89. $\int \frac{x\sin x}{\cos^3 x} dx$; 90. $\int e^{ax}\cos bx dx$; 91. $\int e^{\arccos x} dx$;

92. $\int \ln(x + \sqrt{1+x^2}) dx$; 93. $\int x^3 \ln x dx$; 94. $\int x 3^x dx$;

95. $\int (x^2 - 2x + 3)\cos x dx$; 96. $\int \frac{x dx}{\cos^2 x}$; 97. $\int \cos(\ln x) dx$.

Chapter 2. Standard Methods of Integration

Here we shall present some classes of functions which can he integrated by means of certain standard methods. It should be noted that in some cases those standard methods may not be the simplest. It is often advisable to perform certain preliminary transformations. But the reader will be able to find the simplest technique leading to the desired result only after necessary experience in computing integrals will be acquired.

2.1 Integration of Rational Functions.

Any rational function (rational fraction) can be represented in the form of a sum of an entire rational function (a polynomial), in case the fraction in question is improper, and partial rational fractions. A polynomial can be integrated termwise by means of the simplest methods. Partial fractions of the form $\frac{A}{(x-a)^{\alpha}}$ can also be integrated quite easily.

Example 9. Integrate function $\frac{x^3-2x+3}{x(x-1)(x+2)^2}$.

Solution: If it is necessary to integrate function $\frac{x^3-2x+3}{x(x-1)(x+2)^2}$ which is a proper fraction, so that it can be presented in the form of partial rational fractions. Then, we obtain

$$\int\frac{x^3-2x+3}{x(x-1)(x+2)^2}dx=\int\left[-\frac{3}{4}\frac{1}{x}+\frac{2}{9}\frac{1}{(x-1)}-\frac{1}{6}\frac{1}{(x+2)^2}+\frac{55}{36}\frac{1}{(x+2)}\right]dx=$$

$$=-\frac{3}{4}\ln|x|+\frac{2}{9}\ln|x-1|+\frac{1}{6}\frac{1}{(x+2)}+\frac{55}{36}\ln|x+2|+C$$

Hence, now we must consider partial fractions of the form

$$\frac{Mx+N}{(x^2+px+q)\beta}\quad (p^2-4q<0).$$

We begin the integration with a simplification of the numerator. Namely, taking into account that $(x^2+px+q)'=2\left(x+\frac{p}{2}\right)$ we replace x in the numerator by $\left(x+\frac{p}{2}\right)-\frac{p}{2}$ and then combine similar terms without removing the parentheses. After that we break up the integral into two integrals.

The first integral is of the form

$$\int \frac{(x+\frac{p}{2})dx}{(x^2+px+q)^\beta} = \frac{1}{2}\int \frac{d(x^2+px+q)}{(x^2+px+q)^\beta}$$

and we therefore find it immediately.

The second integral is of the form

$$\int \frac{dx}{(x^2+px+q)^\beta}.$$

To compute it we complete the square in the denominator which results is $x^2+px+q=(x+a)^2+b$ where a and b are constants. Now, if we put we arrive at the integral

$$I_\beta = \int \frac{1}{(y^2+b)^\beta}dy. \qquad (3)$$

which can be easily found in the case $\beta = 1$ (how can we do it?). To find integral (3) for $\beta = 1,2,3...$ we shall deduce a recurrence formula which will enable us to pass from I_β to the simpler integral $I_{\beta-1}$ etc. The formula is obtained by means of integration by parts.

We have

$$I_\beta = \frac{1}{b}\int \frac{b}{(y^2+b)^\beta}dy = \frac{1}{b}\int \frac{(b+y^2)-y^2}{(y^2+b)^\beta}dy = \frac{1}{b}I_{\beta-1} - \frac{1}{b}\int y\frac{y}{(y^2+b)^\beta}dy$$

Here we put $\qquad u=y, dv = \frac{y}{(y^2+b)}dy,$ i.e. $\quad du = dy,$

$$v = \int \frac{ydy}{(y^2+b)^\beta} = \frac{1}{2}\int \frac{d(y^2+b)}{(y^2+b)^\beta} = \frac{-1}{2(\beta-1)}\frac{1}{(y^2+b)^{\beta-1}}$$

Hence,

$$I_\beta = \frac{1}{b}I_{\beta-1} + \frac{y}{2b(\beta-1)(y^2+b)^{\beta-1}} - \frac{1}{b}\int \frac{1}{2(\beta-1)}\frac{y}{(y^2+b)^{\beta-1}}dy = \qquad (4)$$

$$= \frac{y}{2b(\beta-1)(y^2+b)^{\beta-1}} + \frac{2\beta-3}{2b(\beta-1)}I_{\beta-1}$$

(let the reader verify all the calculations!).

As we have already mentioned, formulas of this type are called *recurrence formulas.* Such formulas express an unknown quantity dependent on a number (this is the quantity I_β with the number β in our case) in terms of similar quantities with lower numbers (this is the quantity $I_{\beta-1}$ with the number $\beta-1$ in our case). These formulas may not yield the solution immediately but they enable us to obtain the solution after several successive reductions of the number. Thus, formula (4)expresses I_β in terms of $I_{\beta-1}$. If we repeatedly apply the formula to $I_{\beta-1}$, that is if we substitute $\beta-1$ for β into formula (4),we obtain the expression of $I_{\beta-1}$ in terms of $I_{\beta-2}$ etc. Finally, we arrive at the integral I_1 which is immediately found, as has already been indicated. For instance for $\beta = 2$ and $\beta = 3$ we have:

Example 10.

$$I_2 = \int\frac{dx}{(1+x^2)^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\int\frac{dx}{1+x^2} = \frac{x}{2(1+x^2)} + \frac{1}{2}\,arctgx + C;$$

Example 11.

$$I_3 = \int\frac{dx}{(1+x^2)^3} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}I_2 = \frac{x}{4(x^2+1)^2} + \frac{3x}{8(1+x^2)} + \frac{3}{8}arctgx + C.$$

Example 12. Compute $\int\frac{x+2}{(x^2+2x+3)^2}dx$

Solution:

At first replace $x+2$ in the numerator by $\frac{1}{2}(2x+2)+1$ and then break up the integral into two:

$$\frac{1}{2}\int\frac{2x+2}{(x^2+2x+3)^2}dx + \int\frac{dx}{(x^2+2x+3)^2}$$

The first integral we transform:

$$\frac{1}{2}\int\frac{d(x^2+2x+3)}{(x^2+2x+3)^2} = \frac{1}{2}\int\frac{du}{U^2} = -\frac{1}{2}\frac{1}{U} + C_1 = -\frac{1}{2}\frac{1}{x^2+2x+3} + C_1,$$

To find the second integral we complete the square in the denominator $x^2+2x+3 = (x+1)^2+2$ and then change of variable by the formula $x+1=y$.

$$\int\frac{dx}{(x^2+2x+3)^2} = \int\frac{dy}{(y^2+2)^2} = I_2 = \frac{y}{4(y^2+2)} + \text{ (see Example 10)}$$

$$+\frac{1}{4}\int\frac{dy}{1+y^2} = \frac{x+1}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}arctg\frac{x+1}{\sqrt{2}} + C$$

Thus, we have

$$\int\frac{x+2}{(x^2+2x+3)^2}dx = -\frac{1}{2}\frac{1}{x^2+2x+3} + \frac{x+1}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}arctg\frac{x+1}{\sqrt{2}} + C =$$

$$= \frac{x-1}{4(x^2+2x+3)} + \frac{1}{4\sqrt{2}}arctg\frac{x+1}{\sqrt{2}} + C$$

Example 13. Compute $\int\frac{dx}{x(x^2+1)^2}$

Solution: The fraction $\frac{1}{x(x^2+1)^2}$ is a proper one. It's decomposition into partial fraction is

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

So we have

$$1\equiv A\left(x^2+1\right)^2+(Bx+C)x\left(x^2+1\right)+(Dx+E)x.$$

For x = 0 we obtain A = 1.

For $x=i$ we get $1=(Di+E)i \Leftrightarrow 1=-D+Ei \Leftrightarrow D=-1,\ E=0$.

The coefficients of x^4 are $0=A+B \Leftrightarrow B=-A=-1$.

The coefficients of x^3 are $0=C$.

Thus

$$\int\frac{dx}{x\left(x^2+1\right)^2}=\int\left(\frac{1}{x}-\frac{x}{x^2+1}-\frac{x}{\left(x^2+1\right)^2}\right)dx=\ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{2\left(x^2+1\right)}+C$$

Hence, the integral of a rational fraction is always expressible in terms of elementary functions, and this can be achieved by means of the above standard methods. The elementary functions in terms of which an integral of this type is expressed are rational functions, the logarithmic function and the arc tangent. The most difficult thing in the integration is the factorization of the denominator.

Compute the five integrals.

98. $\int\frac{dx}{x^2+4x-5}$; 99. $\int\frac{dx}{2x^2-4x+5}$; 100. $\int\frac{dx}{x^2-6x}$;

101. $\int\frac{4x-3}{x^2-2x+6}$; 102. $\int\frac{xdx}{x^4+6x^2+13}$; 103. $\int\frac{x^3+2}{x^3-4x}dx$;

104. $\int\frac{dx}{x\left(x^2+2\right)}$; 105. $\int\frac{dx}{x^4+1}$; 106. $\int\frac{dx}{x^3+8}$;

107. $\int\frac{x^2-x+4}{(x+1)(x-2)(x-3)}dx$; 108. $\int\frac{5x-13}{\left(x^2-5x+6\right)^2}dx$.

Methods of computing many integrals of other types which we are going to study here are essentially based on the transition from a given integral to an integral of a rational function by means of suitable substitutions. This is the so-called rationalization of the integral which reduces the computation to the above standard methods.

2.2 Integration of Irrational Functions Involving Linear and Linear-Fractional Expressions.

First we take an integral of the form

$$\int R(x,\sqrt[n]{ax+b})dx\ \ (n=2,3,\ldots)$$

where a and b are constants and $R(x, y)$ is a rational function of its two arguments x and y. The integrand is an irrational function here because it contains the radical. To rationalize the integral let us use the substitution $ax + b = t^n$, $adx = nt^{n-1}dt$ which yields

$$\int R(x, \sqrt[n]{ax+b})dx = R\left(\frac{t^n - b}{a}, t\right)nt^{n-1}dt$$

The integrand in the last integral is a rational function (why?). Similarly, an integral of the form $\int R(x, \sqrt[n]{ax+b}, \sqrt[m]{ax+b}, ...)\,dx \quad (n, m = 2,3,4,...)$

where $R(x, y, z, ...)$ is a rational function of its arguments $x, y, z, \ldots$ goes into an integral of a rational function after the substitution $ax + b = t^p$ with p suitably chosen (how must we choose p in the general case?).

Example 14. Find $\int \frac{dx}{\sqrt{2x+3} - 2\sqrt[3]{2x+3}}$

Solution:

The substitution $2x + 3 = t^6$, $2dx = 6t^5 dx$ yields

$$\int \frac{dx}{\sqrt{2x+3} - 2\sqrt[3]{2x+3}} = \int \frac{3t^5\,dt}{t^3 - 2t^2} = 3\int \frac{t^3}{t-2}dt$$

Performing the division of t^3 by $t - 2$ we find

$$\frac{t^3}{t-2} = t^2 + 2t + 4 + \frac{8}{t-2}$$

and hence we finally obtain

$$\int \frac{dx}{\sqrt{2x+3} - 2\sqrt[3]{2x+3}} = 3\int\left(t^2 + 2t + 4 + \frac{8}{t-2}\right)dt = t^3 + 3t^2 + 12t + 24\ln|t-2| + C =$$
$$= \sqrt{2x+3} + 3\sqrt[3]{2x+3} + 12\sqrt[6]{2x+3} + 24\ln\left|\sqrt[6]{2x+3} - 2\right| + C$$

The rationalization of an integral of the form

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx \quad (n = 2,3,...)$$

where $R(x, y)$ is a rational function is carried out by means of the substitution

$$\frac{ax+b}{cx+d} = t^n, \quad ax + b = cxt^n + d\cdot t^n, \quad x = \frac{d\cdot t^n - b}{a - ct^n}$$

Thus, integrals in which R is a rational function of its arguments are always expressible in terms of elementary functions.

Compute the five integrals.

109. $\int \frac{dx}{(5+x)\sqrt{1+x}}$; 110. $\int \frac{xdx}{\sqrt[3]{2x-3}}$; 111. $\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$

112. $\int\sqrt[3]{\frac{x+1}{x-1}}\frac{dx}{(x-1)^3}$; 113. $\int\frac{dx}{(\sqrt[3]{x}+4)\sqrt{x}}$; 114. $\int\frac{dx}{(x-1)\sqrt[4]{x^3}}$;

115. $\int\frac{1}{x}\sqrt{\frac{x-1}{x+1}}dx$; 116. $\int\frac{\sqrt[6]{x+5}-1}{(x+5)(1+\sqrt[3]{x+5})}dx$

2.3. Integration of Irrational Expressions Containing Quadratic Trinomials.

Here we mean integrals of the form

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$$

where $R(x, y)$ is a rational function of its arguments. Such an integral can also be expressed in terms of elementary functions in all cases. In computing these integrals we apply ***trigonometric substitutions***. In order to do this we first complete the square and pass to a new integral:

$$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx=\int R\left(x,\sqrt{\pm(kx+l)^2\pm m^2}\right)dx$$

where k, l and m are constants. After that we use one of the following substitutions:

$$kx+l=m\tan t \text{ for the radical } \sqrt{(kx+l)^2+m^2}$$

$$kx+l=m\sin t \text{ for the radical } \sqrt{-(kx+l)^2+m^2} \text{ and}$$

$$kx+l=\frac{m}{\cos t} \text{ for the radical } \sqrt{(kx+l)^2-m^2}$$

(of course, we cannot have the case $\sqrt{-(kx+l)^2-m^2}$ for real integrals). The substitutions enable us to extract the roots (check it up!) and thus we come to an integral of the form $\int R_1(\cos t,\sin t)dt$

where $R_1(x,y)$ is another rational function of its arguments. Now we shall describe methods of computing an integral of the form

Example 15. Find $\int\frac{dx}{\sqrt{(x^2+4x+7)^3}}$.

Solution:

Using the substitution $u=x+2$ we get $\int\frac{du}{\sqrt{(u^2+3)^3}}$, then we make the second substitution

$$u=\sqrt{3}\tan t,\ du=\frac{\sqrt{3}}{\cos^2 t}dt,\ \sqrt{u^2+3}=\frac{\sqrt{3}}{\cos t}.$$

Thus,

$$\int\frac{dx}{\sqrt{(...)^3}}=\frac{1}{3}\int\cos t dt=\frac{1}{3}\sin t+C=\frac{1}{3}\frac{u}{\sqrt{u^2+3}}+C=\frac{1}{3}\frac{x+2}{\sqrt{x^2+4x+7}}+C.$$

2.4 . Integration of Functions Rationally Involving Trigonometric Functions.

Here we shall deal with an integral of the form

$$\int R(\sin x,\cos x)dx \qquad (5)$$

where $R\,(u,\,v)$ is a rational function in u and v. Such an integral is always expressible in terms of elementary functions. To prove this let us make the so-called ***universal substitution*** $\tan\frac{x}{2}=t$. Then we have

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2},\quad dx=\frac{2dt}{1+t^2}$$

(verify the calculations!). Hence, integral (5) reduces to the integral

$$\int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt$$

where the integrand is a rational function of t. The last integral can he found by means of the method of Sec. 2.1.

The universal substitution often leads to very complicated expressions containing rational fractions and it is therefore preferable to avoid it in problem-solving practice. In certain particular cases it is better to use some other substitutions which we are going to consider here.

1. Let the integrand in integral $\int R(\sin x,\cos x)dx$ be an odd function with respect to $\sin x$, that is let $R(-\sin x,\cos x)\equiv -R(\sin x,\cos x)$.
Then we can write

$$I=\int R(\sin x,\cos x)dx=\int\frac{R(\sin x,\cos x)}{\sin x}\sin x dx=\int R_1(\sin x,\cos x)\sin x dx$$

where R_1 is an even function with respect to $\sin x$. R_1 being a rational function, we can easily express it in terms of $\sin^2 x$ and $\cos x$.
It follows that

$$I=\int R_2(\sin^2 x,\cos x)\sin x dx=-\int R^2(1-\cos^2 x,\cos x)d\cos x$$

and therefore if we put $\cos x=t$ we arrive at an integral of a rational function.

2. Similarly, if the integrand in $\int R(\sin x, \cos x)dx$ is an odd function with respect to $\cos x$ then the substitution $\sin x = t$ rationalizes the integral.

Example 16. Find I = $\int \frac{\sin^2 x dx}{\cos^3 x - 2\sin x \cos x} = \int \frac{\sin^2 x \cos x dx}{\cos^2 x(\cos^2 x - 2\sin x)}$

Solution: We have putting $\sin x = t, \cos x dx = dt$ we derive

$$I = \int \frac{t^2 dt}{(1-t^2)(1-t^2-2t)}$$

The last integral is readily found if we decompose the integrand into partial fractions or if we take advantage of the equality

$$t^2 = \frac{1}{4}\left[(1-t^2) - (1-t^2-2t)\right].$$

3. If the integrand does not change its value when we simultaneously change the signs of $\sin x$ and $\cos x$, that is if

$$R(-\sin x, -\cos x) \equiv R(\sin x, \cos x)$$

then we can apply the substitution $\tan x = t$ (or $\cot x = t$). We can easily verify that this yields the rationalization of the integral in the general case but we are not going to do this hero because in every concrete example the advisability of the substitution is confirmed by the results of the calculations.

Example 17. . Find I = $\int \frac{dx}{\sin^2 x \cos^4 x}$

Solution: We have $I = \int \frac{dx}{\tan^2 x \cos^6 x}$

Putting $\tan x = t, \cos^2 x = \frac{1}{1+t^2}, \frac{dx}{\cos^2 x} = dt$, we complete the integration:

$$I = \int \frac{(1+t^2)^2}{t^2} dt = \int \left(t^2 + 2 + \frac{1}{t^2}\right) dt = \frac{t^3}{3} + 2t - \frac{1}{t} + C =$$

$$= \tan^3 x + 2\tan x - \cot x + C$$

The same integral can be computed if we represent it as

$$\int \frac{(\sin^2 x + \cos^2 x)^2}{\sin^2 x \cos^4 x} dx$$

and remove the brackets in the numerator (check it up!).

Let us separately consider integrals of the form

$$\int \sin^m x \cos^n x dx$$

where m and n are arbitrary integers of any sign. In case m is odd the integral belongs to case 1 considered above, and thus it can be found by means of the substitution $\cos x = t$. If n is odd the integral belongs to case 2. Finally, if both m and

n are even we have case 3. But the calculations can sometimes be simplified. For instance, if $m \ge 0$ and $n \ge 0$ and if both m and n are even we can apply the formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \sin x \cos x = \frac{1}{2}\sin 2x \text{ and } \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

The same result can be obtained if we express the trigonometric functions in terms of exponential functions by using Euler's formulas.

We sometimes perform integration by parts in computing integrals in order to reduce the positive exponents and increase the negative exponents in powers of $\sin x$ and $\cos x$.

Example 18. Find $\int \frac{\cos^2 x}{\sin^3 x}dx$,

Solution:

We can put $\cos x = u$ and $\int \frac{\cos^2 x}{\sin^3 x}dx$, $dv = \frac{\cos x}{\sin^3 x}dx$ (that is $du = -\sin x dx$ and $v = -\frac{1}{2\sin^2 x}$) when integrating the function $\frac{\cos^2 x}{\sin^3 x}$:

$$\int \frac{\cos^2 x}{\sin^3 x}dx = -\frac{\cos x}{2\sin^2 x} - \int \frac{dx}{2\sin x}$$

Now, making the change of variable $\tan\frac{x}{2} = t$ we obtain

$$\int \frac{\cos^2 x}{\sin^3 x}dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\int \frac{2dt\left(1+t^2\right)^{-1}}{2t\left(1+t^2\right)} = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln|t| + C =$$

$$= -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$$

Find the given integrals:

117. $\int \sin^3 x dx$; 118. $\int \frac{\sin^3 x}{\cos^8 x}dx$; 119. $\int \cos^4 \frac{x}{2}dx$;

120. $\int \frac{dx}{\sin^6 x}$; 121. $\int \frac{\sin^2 x}{\cos^6 x}dx$; 122. $\int \frac{dx}{\sin^4 x \cos^2 x}$;

123. $\int \frac{dx}{4\cos x + 3\sin x + 5}$; 124. $\int \frac{dx}{1-5\sin^2 x}$; 125. $\int \frac{dx}{3\cos x + 2}$;

126. $\int \frac{dx}{4\sin^2 x - 7\cos^2 x}$; 127. $\int \frac{\sin 2x dx}{1+4\cos^2 x}$; 128. $\int \frac{dx}{2-\sin x}$;

129. $\int \frac{1+ctgx}{1-ctgx}dx$; 130. $\int \frac{dx}{\sin^2 x + 8\sin x \cos x + 12\cos^2 x}$.

2.5 General Remarks.

Since integration is a much more complicates procedure compared to differentiation the reader must carefully study the basic methods of integration. But, on the other hand, it is inexpedient to carry out complicated calculations every time when it is necessary to compute an integral. It is therefore advisable to use reference books in which the most widely encountered integrals are collected in orderly way.

Many important integrals are not elementary functions, that is they cannot be expressed in terms of finite combinations of the simplest elementary functions which are studied in elementary mathematical courses.

The integrals

$$\left.\begin{array}{l}\int \sin x \cdot x^{\alpha} dx \\ \int e^{\pm x} \cdot x^{\alpha} dx \\ \int \cos x \cdot x^{\alpha} dx\end{array}\right\} (\alpha \neq 0,1,2,...)$$

are not expressible in terms of elementary functions, and therefore all the integrals that can be reduced to these integrals cannot be expressed in terms of elementary functions either.

Such integrals are:

Example 19.

$$\int e^{-x^2} dx = \frac{1}{2} \int e^{-u} u^{-\frac{1}{2}} du, \text{ where } u = x^2.$$

Example 20.

$$\int \frac{dx}{\ln x} = \left| x = e^u, dx = e^u du \right| = \int e^u u^{-1} du;$$

Example 21.

$$\int \sin x^2 dx = \left| x^2 = u \right| = \frac{1}{2} \int \sin u \cdot u^{-\frac{1}{2}} du.$$

Chapter 3. The definite integral and methods of its computation.

3.1 The definite integral as the limit of integral sum.

If a function *f(x)* is defined on the interval *[a,b]* and $a=x_0<x_1<x_2...<x_n=b$ is an arbitrary division of this interval into n parts, then the integral sum of the function *f(x)* on *[a,b]* is defined as a sum of the form $S_n=\sum_{k=1}^{n} f(\xi_k)\Delta x_k$, where $\xi_k\in[x_{k-1},x_k]$, $\Delta x_k=x_k-x_{k-1}$ (See Fig. 1)

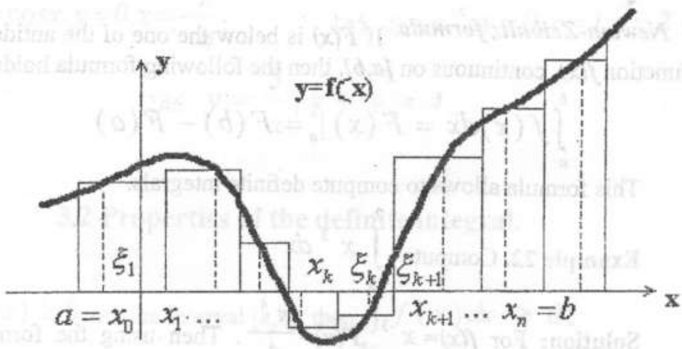

Fig. 1

If a function *f(x)*, defined on an interval *[a,b]*, is such that there exists a finite limit of the sequence of integral sums S_n on condition that the greatest of the differences Δx_k tends to zero and this limit depends neither on the way in which the interval *[a,b]* is divided into the subintervals $[x_{k-1},x_k]$, not on the choice of the points ξ_k is said to be integrable on the interval *[a,b]* and the limit is called the definite integral of the function *f(x)* in the limit from ***a*** to ***b*** and is denoted by the symbol $\int_a^b f(x)\,dx$,which is read: the integral from ***a*** to ***b*** of *f(x)*. Thus $\int_a^b f(x)dx=\lim_{n\to\infty} S_n$, $\max\Delta x_k\to 0$.

The theorem on the existence of a definite integral: If the function *y=f(x)* is continuous on segment *[a,b]* , then the limit of the integral sums exists and does not depend on the manner of partitioning the segment into subsegments or on selection of the points ξ_k

To put it in another way, ***a function continuous on an interval [a,b] is integrable on this interval.***

Geometrical sense of the definite integral: if $f(x)\geq 0$ on an interval $[a,b]$, then the definite integral is equal to the area of the figure bounded above by the graph of the function $y=(x)$, on the side by two strait lines $x=a$, $x=b$, and below by segment.

In general case the definite integral represents an algebraic sum of areas of the figures bounded by the graph of the function $y = f(x)$, the x-axis and the strait lines $x=a$, $x=b$, while areas above the x-axis enter into this sum with the plus sign, and those under the x-axis with the minus sign.

Newton-Zeibnitz formula. If $F(x)$ is below the one of the antiderivatives of a function $f(x)$, continuous on $[a,b]$, then the following formula holds true:

$$\int_a^b f(x)dx = F(x)\,|_a^b = F(b) - F(a) \qquad (6)$$

This formula allows to compute definite integrals.

Example 22. Compute $\int_{-1}^{2} x^3 dx$.

Solution: For $f(x)= x^3$,$F(x)=\frac{x^4}{4}$. Then using the formula (6) we have:

$$\int_{-1}^{2} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4}.$$

Using the Newton-Leibnitz formula, compute the indicated integrals:

131. $\int_0^1 (\sqrt{x} + \sqrt[3]{x^2})dx$;

132. $\int_1^8 \frac{2 + 5\sqrt[3]{x}}{x^3} dx$;

133. $\int_0^3 2^x dx$;

134. $\int_2^5 \frac{dx}{x}$;

135. $\int_1^2 \frac{dx}{2x - 1}$;

136. $\int_0^1 \frac{x^2 dx}{1 + x^6}$;

137. $\int_0^2 sh^3 x dx$;

138. $\int_0^1 \frac{dx}{4x^2 + 4x + 5}$;

139. $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}}$; 140. $\int_1^e \frac{\cos(\ln x)}{x}dx$;

141. $\int_0^2 \frac{2x-1}{2x+1}dx$.

Compute the areas of the figures bounded by the indicated lines.

142. $y=6-x-2x^2, y=x+2$; 143. $y=\frac{y^2}{4}, y=2\sqrt{x}$;

144. $y=\cos x, y=0, x=-\frac{\pi}{4}$; 145. $y=e^{-x}, y=0, x=1, x=2$;

146. $y=\frac{3}{x}, x+y=4$.

3.2 Properties of the definite integral.

1. If $f(x)\geq 0$ on the interval [a,b], then $\int_a^b f(x)dx \geq 0$;
2. If $f(x)\leq g(x)$ on [a,b] then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$;
3. $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$;
4. If $f(x)$ is continuous on [a,b], m is the least and M is the greatest value of $f(x)$ on [a,b], then

$$m(b-a)\leq \int_a^b f(x)dx \leq M(b-a)$$

(the theorem of estimation of the definite integral)

5. If $f(x)$ continuous on [a,b], then there exists a point $c\in(a,b)$ such that the following equality is valid:

$$\int_a^b f(x)dx = f(c)(b-a)$$ ***(the mean-value theorem).***

6. Integration of even and odd functions over symmetric limits.

If *f(x)* is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

And if *f(x)* is an odd function, then $\int_{-a}^{a} f(x)dx = 0$;

7. If a function *f(x)* is continuous on [a,b], then the integral with a variable upper limit

$$D'(x) = \left(\int_{a}^{x} f(t)dt\right)' = f(x)$$

3.3. Change of variable

If the function *f(x)* is continuous on the integral *[a,b]*, and the function $x = \varphi(t)$ is continuously differentiable on the interval *[α,β]*, and $a = \varphi(\alpha), b = \varphi(\beta)$, then

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt = \int_{\alpha}^{\beta} \psi(t)dt \quad . \quad (7)$$

This formula allows to reduce the integral $\int_{a}^{b} f(x)dx$ to the integral $\int_{\alpha}^{\beta} \psi(t)dt$ which may be simpler than the initial one.

Example 23. Compute $\int_{\sqrt{2}/2}^{1} \frac{\sqrt{1-x^2}}{x^2}dx$.

Solution: Let us apply the substitution *x=sint*. Then *dx=costdt*, *t=arcsinx*, $t_1 = \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ and $t_2 = \arcsin 1 = \frac{\pi}{2}$. Consequently

$$\int_{\sqrt{2}/2}^{1} \frac{\sqrt{1-x^2}}{x^2}dx = \int_{\pi/4}^{\pi/2} \frac{\sqrt{1-\sin^2 t}}{\sin^2 x}\cos t dt = \int_{\pi/4}^{\pi/2} \frac{\cos^2 t}{\sin^2 t} = \int_{\pi/4}^{\pi/2} \frac{1-\sin^2 t}{\sin^2 t}dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dt}{\sin^2 x} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dt = -\cot\frac{\pi}{2} + \cot\frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}.$$

Evaluate the given integrals with the aid of the indicated substitutions

147. $\int_{1}^{6} \frac{dx}{1+\sqrt{3x-2}}, 3x-2=t^2$; 148. $\int_{\ln 3}^{\ln 8} \frac{dx}{\sqrt{e^x+1}}, e^x+1=t^2$;

149. $\int_{0}^{sh\,1} \sqrt{x^2+1}\,dx$, $x = sh\,t$; 150. $\int_{0}^{\pi/2} \frac{dx}{3+2\cos x}, tg\,\frac{x}{2}=t$;

151. $\int_{0}^{\pi/4} \frac{dx}{1-2\sin^2 x}, tgx=t$; 152. $\int_{-1}^{1} \sqrt{3-2x-x^2}\,dx, x+1=2\sin t$;

Evaluate the indicated integrals by changing the variable

153. $\int_{2/\sqrt{3}}^{2} \frac{dx}{x\sqrt{x^2-1}}$; 154. $\int_{2}^{4/\sqrt{3}} \frac{\sqrt{x^2-4}}{x}dx$;

155. $\int_{-2}^{2} \frac{dx}{(4+x^2)^2}$; 156. $\int_{-2}^{0} \frac{dx}{\sqrt{x+3}+\sqrt{(x+3)^3}}$;

157. $\int_{1}^{5} \frac{dx}{x+\sqrt{2x-1}}$; 158. $\int_{\ln 2}^{\ln 6} \frac{e^x\sqrt{e^x-2}}{e^x+2}dx$;

159. $\int_{0}^{3} x^2\sqrt{9-x^2}\,dx$; 160. Show that $\int_{e}^{e^2} \frac{dx}{\ln x} = \int_{1}^{2} \frac{e^x}{x}dx$;

3.4 Integration by parts.

If two functions $u = u(x)$, $v = v(x)$ and their derivatives $u'(x)$ and $v'(x)$ are continuous on [a,b] then

$$\int_{a}^{b} u\,dv = uv\,|_a^b - \int_{a}^{b} v\,du \tag{8}$$

(the formula for integration by parts)

This formula is used when the integrand $f(x)dx$ can be represented in the form udv so that the integral standing on the right of (8) may be simpler than the original integral.

Example 3. Compute $\int_1^e \ln\, xdx$.

Solution: Let us set $u=\ln x$, $dv=dx$, then $du=\frac{dx}{x}, v=x$.

We have $\int_1^e \ln xdx = x\ln x\Big|_1^e - \int_1^e x\frac{1}{x}dx = e - x\Big|_1^e = 1$

Evaluate the given integrals using the method of integration by parts.

161. $\int_0^1 xe^x dx$; 162. $\int_0^1 \frac{\arcsin x}{\sqrt{1+x}}dx$;

163. $\int_1^e \ln^2 xdx$; 164. $\int_{\pi/6}^{\pi/3} \frac{xdx}{\cos^2 x}$;

165. $\int_1^e x\ln xdx$; 166. $\int_0^{\pi/4} e^{3x}\sin 4xdx$;

167. $\int_0^1 xarctgxdx$; 168. $\int_0^{\pi/4} x^2\cos 2xdx$;

169. $\int_0^{\pi/2} e^x\cos xdx$;

170. Show that for the integral $I_n = \int_0^{\pi/2} \cos^n xdx$ the recurrence formula $I_n = \frac{n-1}{n}I_{n-2}$ is valid. Evaluate I_7.

Chapter 4. Geometrical applications of the definite integral.

4.1 Area of the plane figure.

If the figure bounded by the graph of a continuous function $y=f(x)\geq 0$, two straight lines ***x=a, x=b*** and the *x*-axis (a curvilinear trapezoid), the area is computed by the formula:

$$S = \int_a^b f(x)\,dx \quad . \tag{9}$$

Example 25. Find the area of the figure lying in the right-hand half-plane and bounded by the circle $x^2+y^2=8$ and the parabola $y^2=2x$.

Solution:

Let us first find the points of intersection of the given curves by solving the system of equations

$$\begin{cases} x^2+y^2=8 \\ y^2=2x \end{cases}, \quad \begin{array}{l} x^2+2x-8=0 \\ x_{1,2}=-1\pm\sqrt{9} \end{array} \quad x_1=2 \quad (x_2=-4<0)$$

We obtain the points (2,2) and (2,-2).

Making use of the symmetry about the *x*-axis, we find the desired area ***S*** as the doubled sum of the areas of the curvilinear trapezoid bounded, respectively, by the arcs of parabola $y=\sqrt{2x}, 0\leq x\leq 2$ and the circle $y=\sqrt{8-x^2}$, $2\leq x\leq\sqrt{8}$.

$$S=2\left(\int_0^2\sqrt{2x}dx+\int_2^{\sqrt{8}}\sqrt{8-x^2}\,dx\right)=$$

$$=2\left(\sqrt{2}\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2+\frac{x}{2}\sqrt{8-x^2}+4\arcsin\frac{x}{\sqrt{8}}\right)\Big|_2^{\sqrt{8}}=2\pi+\frac{4}{3}.$$

The area of the figure bounded by the graphs of two continuous functions $y=f_1(x)$ and $y=f_2(x)$, $y=f_2(x)\geq y=f_1(x)$ and two straight line ***x=a, x=b*** is determined by the formula (See Fig. 2):

$$S=\int_a^b\left[f_2(x)-f_1(x)\right]dx \tag{10}$$

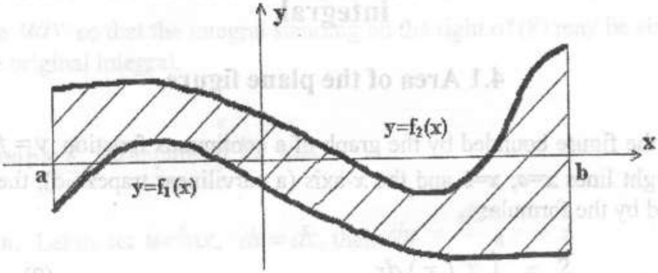

Fig. 2

If a figure bounded by a curve, having parametric equations $x=x(t)$, $y=y(t)$, straight lines $x=a$, $x=b$, and the x-axis, then it's area is computed by the formula:

$$S = \int_{t_1}^{t_2} y(t)\,x'(t)\,dt \ , \qquad (11)$$

where the limits of integration or found from the equations $a=x(t_1)$, $b=x(t_2)$ ($y(t)\geq 0$ on the interval $[t_1,t_2]$).

This formula is also applicable for calculating the area of the figure bounded by a close curve.

Example 26. Find the area of the loop made by the curve $x=a(t^2-1)$, $y=b(4t-t^3)$ $(a>0,\ b>0)$.

Solution:

Let us first find the points of intersection of the given curve and the coordinate axes. We have: $x=0$ for $t=\pm 1$; $y=0$ for $t=0,\ \pm 2$. Consequently, we obtain the following points: $(0,36)$ for $t=1$, $(0,-36)$ for $t=-1$; $(-a,0)$ for $t=0$; $(3a,0)$ for $t=\pm 2$. The point $(3a,0)$ is a point of self-intersection of the curve. For $0\leq t\leq 2 \quad y\geq 0$, for $-2\leq t\leq 0,\ y\geq 0$. The desired area is found as the doubled area of upper half of the loop by the formula(11)

$$S = 2\int_{-a}^{3a} y\,dx = 2\int_0^2 y(t)x'(t)\,dt = 2\int_0^2 b(4t-t^3)a2t\,dt = 4ab\int_0^2 (4t^2-t^4)\,dt =$$

$$= 4ab\left(\frac{4}{3}t^3 - \frac{1}{5}t^5\right)\Big|_0^2 = \frac{256}{15}ab$$

The area of a figure bounded by the graph of a continuous function $r=r(\varphi)$ and two rays $\varphi=\alpha, \varphi=\beta$ where φ and r are polar coordinates (a curvilinear sector) is computed by the formula (See Fig. 3):

$$S = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\varphi . \qquad (12)$$

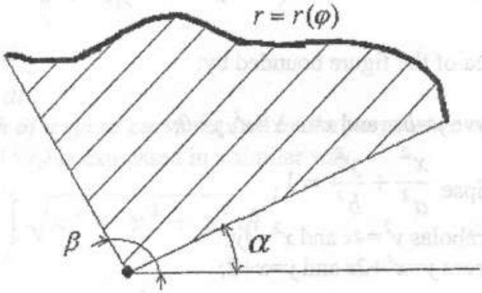

Fig. 3

Example 27. Find the area of circular line bounded by arcs of circles $r = 2a\cos\varphi$, $r = 2a\sin\varphi, 0 \le \varphi \le \frac{\pi}{2}$ (See Fig. 4).

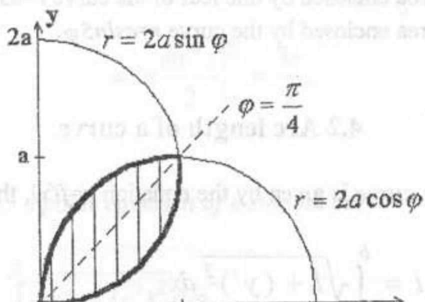

Fig. 4

Solution: The circles intersect for $\varphi = \frac{\pi}{4}$; the figure under consideration is symmetric about the ray $\varphi = \frac{\pi}{4}$. Consequently, its area can be computed in the following way:

$$S = 2\frac{1}{2}\int_0^{\pi/4} 4a^2 \sin^2 \varphi d\varphi =$$

$$= 2a^2 \int_0^{\pi/4} (1-\cos 2\varphi)d\varphi = 2a^2\left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Bigg|_0^{\pi/4} = (\frac{\pi}{2} - 1)a^2.$$

Find the area of the figure bounded by:

171. the curve $y = \ln x$ and $x = e$, $x = e^2$, $y = 0$;

172. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

173. the parabolas $y^2 = 4x$ and $x^2 = 4y$;

174. the curves $y = x^2 + 2x$ and $y = x + 2$;

175. the curves $y = \frac{27}{x^2 + 9}$ and $y = \frac{x^2}{6}$.

176. Find the area enclosed by the astroid $x = a\cos^3 t,\ y = a\sin^3 t$.

177. Find the area of the figure bounded by the arch of the cycloid $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ and the x-axis.

178. Find the area enclosed by the cordioida $r = a(1 + \sin\varphi)$.

179. Find the area enclosed by one leaf of the curve $r = a\sin 2\varphi$.

180. Find the area enclosed by the curve $r = a\sin 5\varphi$.

4.2 Arc length of a curve.

If ***a smooth curve*** is given by the equation $y = f(x)$, then ***the length*** l ***of its arc*** is

$$l = \int_a^b \sqrt{1 + (y')^2}\, dx\ , \qquad (13)$$

where a and b are the abscissas of the end points of the arc.

Example 28. Find the arc length of the semicubical parabola $y^2 = x^3$ from the origin to the point (4,8).

Solution: We have $y = x^{\frac{3}{2}}, y' = \frac{3}{2}x^{\frac{1}{2}}$. Using the formula (13), we find:

$$l = \int_0^4 \sqrt{1 + \frac{9}{4}x}dx = \frac{4}{9}\cdot\frac{2}{3}(1 + \frac{9}{4}x)^{\frac{3}{2}}\Bigg|_0^4 = \frac{8}{27}(10\sqrt{10} - 1).$$

If a *curve* is represented ***by parametric*** equations $x=x(t), y=y(t)$ $(t_1 \le t \le t_2)$ then

$$l = \int_{t_1}^{t_2} \sqrt{x_t^2 + y_t^2}\, dt \, , \qquad (14)$$

where $x_t = \dfrac{dx}{dt}$

The arc length of a space curve defined ***by the parametric*** equations $x=x(t)$, $y=y(t)$, $z=z(t)$ $(t_1 \le t \le t_2)$ is expressed in a similar way

$$l = \int_{t_1}^{t_2} \sqrt{x_t^2 + y_t^2 + z_t^2}\, dt \, , \qquad (15)$$

Example 29. Find the arc length of astroid $x = a \cdot \cos^3 t, y = a \cdot \sin^3 t$

Solution:

We have $x_t = (a\cos^3 t)_t = -3a\cos^2 t \sin t$, $\quad y_t = 3a\sin^2 t\cos t$.

Let us consider quarter of the astroid and using the formula (15) we find:

$$\frac{1}{4} l = \int_0^{\pi/2} \sqrt{9a^2\cos^4 \sin^2 t + 9a^2 \sin^4 t \cos^2 t dt} = 3a \int_0^{\pi/2} \cos t \sin t dt =$$

$$= 3a \frac{\sin^2 t}{2}\Bigg|_0^{\pi/2} = \frac{3a}{2},$$

whence ***l=6a.***

If we are given ***a polar equation of a smooth curve*** $r = r(\varphi), \alpha \le \varphi \le \beta$, then

$$l = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\varphi \, . \qquad (16)$$

Example 30. Find the arc length of the cardioid $r = a(1 - \cos\varphi)$.

Solution:

We have $r' = a\sin\varphi$. Let us consider half of the cardioid and using the formula (16) we find:

$$\frac{1}{2} l = \int_0^{\pi} \sqrt{a^2(1-\cos\varphi)^2 + a^2\sin^2\varphi d\varphi} = a\int_0^{\pi}\sqrt{2(1-\cos\varphi) d\varphi} = 2a\int_0^{\pi}\sin\frac{\varphi}{2} d\varphi = 4a$$

whence ***l=8a***

Find the arc length of:

181. the parabola $y = x^2$ from $x=0$ to $x=1$.

182. the curve $y = (1 - \frac{x}{3})\sqrt{x}$ between the points of its intersection with the x-axis.

183. the closed curve $8a^2y^2=x^2(a^2-x^2)$

184. the catenary $y = \frac{1}{2}ch2x$ from $x=0$ to $x=3$.

185. the curve $x=a(3\cos t-3\cos 3t)$, $y=a(3\sin t-\sin 3t)$ from $t=0$ to $t=\frac{\pi}{2}$.

186. the loop of the curve $x = t^2$, $y = t(\frac{1}{3} - t^2)$.

187. the entire curve $r = a\sin^4\frac{\varphi}{4}$.

188. the logarithmical spiral $r = e^{a\varphi}$ found inside the circle.

189. the cardioid $r = 2(1-\cos\varphi)$ found inside $r=1$.

190. the space curve $x = at^2$, $y=a(t+\frac{1}{3}t^3)$, $z=a(t-\frac{t^3}{3})$ form $t=0$ to $t=\sqrt{3}$.

191. . the space curve $x=e^t\cos t, y=e^t\sin t, z=e^t$ between the planes $z=1$, $z=e$.

4.3. The area of a surface of revolution.

The area of a surface generated by revolving on arc of a curve defined by the function $y=f(x), x\in[a,b]$, about the *x*-axis is computed by the formula:

$$Q_x = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx. \qquad (17)$$

If an arc is represented by ***parametric*** equations $x=x(t), y=y(t),\ t_1\le t\le t_2$ then

$$Q_x = 2\pi\int_{t_1}^{t_2} y(t)\sqrt{x_t^2+y_t^2}\,dt. \qquad (18)$$

If an arc is given in ***polar*** coordinates $r = r(\varphi)$, $\alpha \le \varphi \le \beta$, then

$$Q_x = 2\pi \int_{\alpha}^{\beta} r \sin\varphi \sqrt{r^2 + (r')^2}\, d\varphi. \qquad (19)$$

Example 31. Find the area of the surface generated by revolving the cardioid $r = 2(1+\cos\varphi)$ about the polar axis.

Solution:

We have $r' = -2\sin\varphi$,

$$\sqrt{r^2 + (r')^2} = \sqrt{4(1+\cos\varphi)^2 + 4\sin^2\varphi} = 4\cos\frac{\varphi}{2},$$

and, further, by the formula (19):

$$Q_x = 2\pi \int_0^{\pi} 2(1+\cos\varphi)\sin 4\cos\frac{\varphi}{2}\, d\varphi = 64\pi \int_0^{\pi} \cos^4\frac{\varphi}{2}\sin\frac{\varphi}{2}\, d\varphi = 128\,\pi/5$$

Find the area of the surface generated by rotating

192. an arc of the catenary $y = \dfrac{ch\,2x}{2}$, $0 \le x \le 3$ about the x-axis.

193. the arc of the curve $y = \dfrac{x^3}{3}$ from $x=-1$ to $x=1$ about the x-axis.

194. the ellipse $4x^2+y^2=4$ about a) the x-axis.

195. b) the y-axis.

196. . the arc of the curve $y = \dfrac{1}{6}\sqrt{x}(x-12)$ between the points of its intersection with the x-axis.

197. the loop of the curve $9ay^2=x(-3ax)^2$ about a) the x-axis,

b) the y-axis.

198. the arc of the curve $x=(3\cos t - \cos 3t)$ $y=(3\sin t - \sin 3t)$, $0 \le t \le \dfrac{\pi}{2}$ about the polar axis.

199. the circle $r = 2a\sin\varphi$ about the polar axis.

200. the arc of the curve $r = a\sec^2\dfrac{\varphi}{2}$, $0 \le t \le \dfrac{\pi}{2}$ about the polar axis.

4.4. Volume of body.

If the area $S(x)$ of a section of a body by a plane perpendicular to the x-axis is a continuous function on the interval [a,b], then the volume of the body is computed by the formula:

$$V = \int_a^b S(x)dx \quad (20)$$

Example 32. The plane of an isosceles triangle moves in a perpendicular manner towards a fixed diameter of a circle of radius ***a***. The base of the triangle is a chord of the circle and its vertex lies on the straight line, parallel to the fixed diameter of a distance ***h*** from the plane containing the circle. Find the volume of the body formed by the plane of the triangle during its motion from one end of the diameter to the other (See Fig. 5)

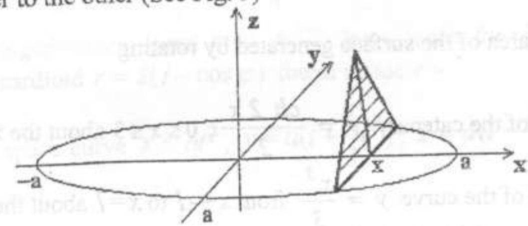

Fig. 5

Solution:

Choosing the coordinate system so that the center of the circle turns out to lie at the origin and the fixed diameter on the x-axis we obtain the equation of the circle in the form $x^2+y^2=a^2$.

The section of the body by a plane perpendicular to the x-axis is an isosceles triangle with base $2y=2\sqrt{a^2-x^2}$ and altitude ***h***.

We have:

$$S(x)=\frac{1}{2}2\sqrt{a^2-x^2}h=h\sqrt{a^2-x^2}\ (-a\le x\le a),$$

then by (20):

$$V=h\int_{-a}^{a}\sqrt{a^2-x^2}dx=2h\int_0^a\sqrt{a^2-x^2}dx=2h\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}\right)\Big|_0^a=2h\times$$

$$\times\frac{a^2}{2}\frac{\pi}{2}=\frac{1}{2}\pi a^2h$$

In case of solids of revolution we obtain a rather simple expression for the function ***S(x)***. For instance, if a curvilinear trapexoid, bounded by a curve $y=f(x),\ a \le x \le b$, rotates about the ***x*** or ***y***-axis, then ***the volumes of solids of revolution*** are computed by the respective formulas:

$$V_x = \pi \int_a^b f^2(x)dx \ , \tag{21}$$

$$V_y = 2\pi \int_a^b x|f(x)|dx, a \ge 0 \ . \tag{22}$$

Example 33. The figure bounded by the curve: $x = a \cdot \cos t, y = a \cdot \sin 2t$ $(0 \le t \le \pi/2)$ and the *x*-axis rotates about the *y*-axis. Find the volume of the solid of revolution.

Solution:

It is evident, that $0 \le x \le a$ and $0 \le y \le a$, as well as $y=0$ for $t=0$ and $t=\pi/2$, considered figure is curvilinear trapezoid.

Further, for $t=0, x=a$ and for $t=\pi/2$, $x=0$. So, desired volume is determed by the formula (22). We have:

$$V_y = 2\pi \int_a^b x(t) \cdot y(t)dt = 2\pi \int_{\pi/2}^0 a\cos t \cdot a \sin 2t \cdot (-a \sin t)dt =$$

$$= \pi a^3 \int_0^{\pi/2} \sin^2 2t dt = \frac{\pi a^3}{2} \int_0^{\pi/2} (1 - \cos 4t)dt = \frac{\pi^2 a^3}{4}.$$

If a curvilinear sector, bounded by the curve $r = r(\varphi)$ and the rays $\varphi = \alpha, \varphi = \beta$, rotates about the polar axis, then ***the volume of the solid of revolution*** are computed by the formulas:

$$\mathrm{V} = \frac{2}{3}\pi \int_\alpha^\beta r^3 \sin\ \alpha\, d\varphi \ . \tag{23}$$

Example 34. The cardioid $r = a(1 - \cos\varphi)$ rotates about the polar axis. Find the volume of the solid of revolution thus generated.

Solution:

Using the formula (23), we obtaine:

$$V=\frac{2}{3}\pi\int_0^{\pi}a^3(1-\cos\varphi)^3\sin\varphi d\varphi=\frac{2}{3}\pi a^3\left.\frac{(1-\cos\varphi)^4}{4}\right|_0^{\pi}=\frac{8}{3}\pi a^3.$$

Find the volume of the solid obtained by rotating about the x-axis the figure enclosed by

201. the lines $y=e^{-2x}-1, y=e^{-x}+1, x=0$;

202. the lines $2y=x^2,\ 2x+2y=3$;

203. the curve $x=at^2,\ y=a\ln t$ (a>0);

204. the astroid $x=2\cos^3 t, y=2\sin^3 t$

Find the volume of the solid generated by revolving the figure enclosed by the lines:

205. $y=x, y=x+\sin^2 x\ (0\le x\le\pi)$ about the y-axis;

206. $y=\frac{x^2}{2}+2x+2, y=2$ about the y-axis;

207. $x=at^2, y=a\ln t$, (a>0) about the y-axis;

208. $r=a\sin^2\varphi$ about the polar axis;

209. $r^2=a^2\cos 2\varphi$ (the lemniscate) about the polar axis.

Chapter 5. Improper integrals.

5.1. Integrals with infinite limits.

If a function $f(x)$ is continuous for $a \le x \le +\infty$, then, by definition,

$$\int_a^{+\infty} f(x)dx = \lim_{N \to +\infty} \int_a^N f(x)dx . \qquad (24)$$

If there exists a finite limit in the right-hand side of formula, then the improper integral is said to be ***convergent***, and if this limit does not exist, then it is said to be ***divergent***.

Example 35. Evaluate $\int_0^{+\infty} e^{-3x} dx$.

Solution: We have

$$\int_0^{+\infty} e^{-3x} dx = \lim_{N \to +\infty} \int_0^N e^{-3x} dx = \lim_{N \to +\infty} (-\frac{1}{3} e^{-3x} \Big|_0^N) =$$

$$= \lim_{N \to +\infty} \frac{1}{3}(1 - e^{-3N}) = \frac{1}{3}$$

The integrals $\int_{-\infty}^{b} f(x)dx$ and

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{\infty} f(x)dx$$

are determined in a similar way.

Example 36. Investigate $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ for convergence.

Solution:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1} = \lim_{N \to -\infty} \int_{N}^{0} \frac{dx}{x^2+1} + \lim_{M \to +\infty} \int_{0}^{M} \frac{dx}{x^2+1} =$$

$$= -\lim_{N \to -\infty} (arctgN) + \lim_{M \to +\infty} ((arctgM) = \pi.$$

Hence, this improper integral is convergent.

Evaluate the given improper integral (or prove that it is divergent).

210. $\int\limits_{e}^{+\infty} \frac{dx}{x \ln^3 x}$; 211. $\int\limits_{e}^{+\infty} \frac{dx}{x\sqrt{\ln x}}$;

212. $\int\limits_{0}^{+\infty} \frac{xdx}{x^2+4}$; 213. $\int\limits_{-\infty}^{+\infty} \frac{dx}{x^2+6x+11}$;

214. $\int\limits_{0}^{+\infty} e^{-2x} \cos xdx$; 215. $\int\limits_{1}^{+\infty} \frac{1+2x}{x^2(1+x)}dx$;

216. $\int\limits_{2}^{+\infty} \frac{xdx}{\sqrt{(x^2+5)^3}}$; 217. $\int\limits_{0}^{+\infty} xe^{-x^2}dx$;

218. $\int\limits_{0}^{+\infty} x \cos xdx$.

Geometrically improper integral (24) for $f(x)>0$ is ***the area*** of the figure bounded by the graph of the function $y=f(x)$, the straight line $x=a$, and the x-axis (asymptote) (See Fig. 6).

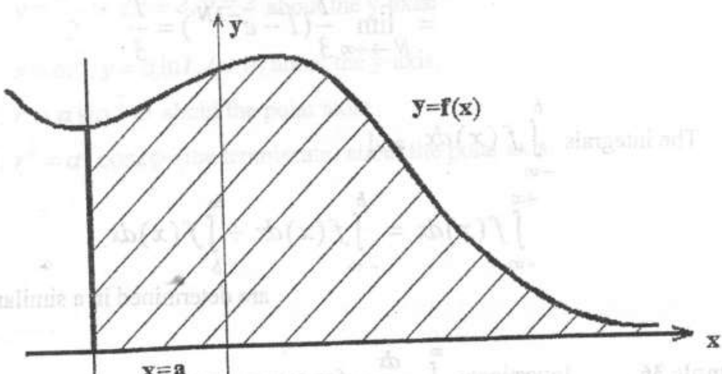

Fig. 6

Example 37.

Find the area of the figure bounded by the curve $y = x^2$, the straight line $x=1$ and the x-axis (asymptote).

Solution:

The desired area is presented by the integral:

$$\int\limits_{1}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Bigg|_1^{\infty} = 1,$$

thus, the area equals 1.

5.2.The Test for Convergent

We shall consider the tests for convergent only for integrals of form $\int_a^{+\infty} f(x)\,dx$.

a) If $F(x)$ is an antiderivative for $f(x)$ and there exists a finite limit

$$\lim_{x\to+\infty} F(x) = F(+\infty),$$

then integral converges and is equal to $F(+\infty) - F(a)$; and if $\lim_{x\to+\infty} F(x)$ does not exist, then integral is divergent.

b) Yet for $a \le x \le +\infty$, $0 \le f(x) \le g(x)$.

If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges as well, and

$$\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} g(x)dx.$$

If $\int_a^{+\infty} f(x)dx$ diverges, then $\int_a^{+\infty} g(x)dx$ also diverges (***comparison tests***).

b) If for $a \le x \le +\infty$, $f(x) > 0, g(x) > 0$ and there exists a finite limit

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} \ne 0,$$

then the integral $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ either converge or diverge simultaneously (***limiting comparison test***).

d) If $\int_a^{+\infty} |f(x)|dx$ converges, then $\int_a^{+\infty} f(x)dx$ converges as well (in this case the latter integral is called absolutely convergent).

e) If for $x \to +\infty$ the function $f(x)>0$ is an infinitesimal of order α as compared with $\frac{1}{x}$, then the integral $\int_a^{+\infty} f(x)dx$ is convergent for α >1 and divergent $\alpha \le 1$.

Example 38.

Investigate $\int_{1}^{+\infty} \frac{x+1}{\sqrt{x^3}}\,dx$ for convergence.

Solution:

We have $\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}};\ \alpha = \frac{1}{2} < 1.$

The given integral is divergent, since $\int_{1}^{+\infty} \frac{dx}{\sqrt{x}}$ is divergent.

Investigate the given integrals for convergence.

219. $\int_{0}^{+\infty} \frac{xdx}{x^3+4}$; 220. $\int_{1}^{\infty} \frac{x^2+1}{x^3}dx$; 221. $\int_{0}^{\infty} \frac{dx}{\sqrt{1+x^3}}$;

222. $\int_{2}^{+\infty} \frac{dx}{\sqrt[3]{x^3-1}}$; 223. $\int_{1}^{+\infty} \frac{dx}{3+2x^2+5x^4}$; 224. $\int_{0}^{\infty} e^{-x}dx$.

5.3.Integrals of unbounded functions

If a function *f(x)* is continuous for $a \le x < b$ and $f(b) = \infty$, then by definition,

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon \to 0} \int_{a}^{b-\varepsilon} f(x)dx \qquad (25)$$

If a finite limit exists in the right-hand side, then we say that the improper integral is ***convergent.*** If this limit does not exist, then the equality becomes meaningless, and the improper integral written on the left is said to be ***divergent.***

If $f(a) = \infty$, the improper integral is defined in a similar way.

If $f(c) = \infty,\ c \in (a,b)$, then

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon_1 \to 0} \int_{a}^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2 \to 0} \int_{c+\varepsilon_2}^{b} f(x)dx\ .$$

Example 39. Investigate the improper integral:

$$\int_0^1 \frac{1}{x^{\alpha}}dx \quad (\alpha>0) \text{ for convergence.}$$

Solution:

If $\alpha<1$ then

$$\int_0^1 \frac{1}{x^{\alpha}}dx = \lim_{\varepsilon\to 0}\int_{\varepsilon}^1 \frac{dx}{x^{\alpha}} = \lim_{\varepsilon\to 0}\left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_{\varepsilon}^{1} = \frac{1}{1-\alpha}.$$

If $\alpha=1$ then

$$\int_0^1 \frac{1}{x}dx = \lim_{\varepsilon\to 0}\int_{\varepsilon}^1 \frac{dx}{x} = \lim_{\varepsilon\to 0}\ln x\Big|_{\varepsilon}^{1} = +\infty.$$

If $\alpha>1$, then $\alpha-1>0$ and

$$\int_0^1 \frac{1}{x^{\alpha}}dx = \lim_{\varepsilon\to 0}\left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_{\varepsilon}^{1} = \frac{1}{1-\alpha}\lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon^{\alpha-1}}\right) = +\infty.$$

Hence, this improper integral is convergent for $\alpha<1$ and divergent for $\alpha\geq1$.

Evaluate the given integral (or prove that it is divergent.

225. $\displaystyle\int_1^1 \frac{dx}{x^2+x^4}$; 226. $\displaystyle\int_0^2 \frac{xdx}{(x^2-1)^{4/5}}$; 227. $\displaystyle\int_1^e \frac{dx}{x\ln^3 x}$.

228. $\displaystyle\int_0^2 \frac{x^3dx}{\sqrt{4-x^2}}$; 229. $\displaystyle\int_0^{\sqrt{2/\pi}} \cos\frac{1}{x^2}\frac{dx}{x^3}$; 230. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$.

Geometrically, if $f(x)>0$ improper integral (25) is ***the area*** of the figure bounded by the graph of the function $y=f(x)$, the straight line $x=a$, and the vertical asymptote $x=b$ (See Fig. 7)

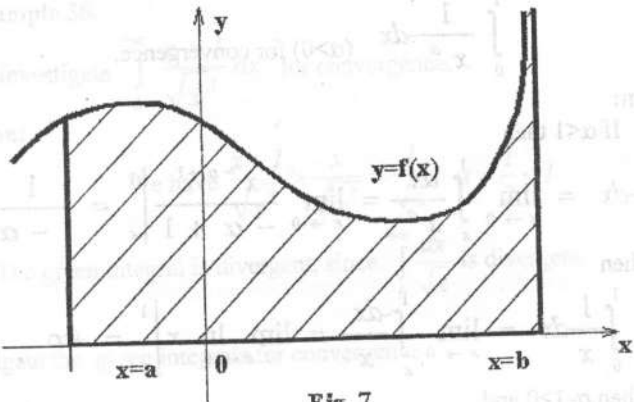

Fig. 7

Example 40.

Find the area of the figure bounded by the curve $y = \frac{1}{\sqrt{x}}$, the straight line $x=4$ and the y-axis (asymptote).

Solution:

The desired area is presented by the integral: $\int_0^4 \frac{dx}{\sqrt{x}}$.

$$\int_0^4 \frac{1}{\sqrt{x}}dx = \lim_{\varepsilon \to 0} \int_\varepsilon^4 x^{-\frac{1}{2}}dx = \lim_{\varepsilon \to 0} 2\sqrt{x}\Big|_\varepsilon^4 = 4.$$

Hence the area is equal to 4squre units.

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Appendix

Reading of the main Mathematical symbols and formulas.

$a+b=c$ a plus b is equal to (or equals) c

15-5=10 fifteen minus five is equal to 10

$a \times b$ a multiplied by b (or a times b)

$a:b$ a divided by b

$a>b$ a is greater than b

$a<b$ a is less than b

a' a prime

a'' a second prime

b^2 b square (or squared)

a_1 a sub one (or a first)

10^{-11} ten to the minus eleventh (power)

[ ] brackets, square brackets

( ) round brackets, parentheses

{ } braces

$\sqrt{a}$ the square root of a

$\sqrt[n]{a}$ the n-th root of a

$\frac{dy}{dx}$ dy over dx (or the first derivative of y with respect to x)

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(англійською мовою)

Редактор Л. І. Волчанська

Зв. план, 2001
Підписано до друку 21.03.2001
Формат 60Х 84 1/16. Папір офс. № 2. Офс. друк.
Умовн. - друк. арк. 2,5. Облік. – вид. арк. 2,87. Т. 100 прим.
Замовлення 152. Ціна вільна

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