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КРАТНІ ІНТЕГРАЛИ І ЕЛЕМЕНТИ ТЕОРІЇ ПОЛЯ

Навчальний посібник

MULTIPLE INTEGRALS AND ELEMENTS OF THE FIELD THEORY

Tutorial

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Розглянуто теоретичні та практичні питання, пов'язані з такими важливими розділами вищої математики, як кратні інтеграли і векторний аналіз. Матеріал цих розділів викладено згідно з програмою курсу "Вища математика", який викладається у вищих технічних навчальних закладах. Теоретичний матеріал подано у перших двох розділах, а третій розділ містить практичні задачі та приклади, значну частину яких наведено з розв'язками. Для решти наведено відповіді, що є досить важливими для самостійної роботи студентів.

Теоретичний матеріал методично узгоджено з практичною частиною, до якої віднесено деякі більш громіздкі питання практичного характеру.

Для студентів другого курсу вищих технічних навчальних закладів, а також може бути корисним для дипломників та аспірантів.

Theoretical and practical questions connected with such important parts of the Higher Mathematics as Multiple Integrals and Vector Analysis are considered. The Material of these parts corresponds with the program of the "Higher Mathematics" course, which is taught in Technical Institutes. Theoretical material is shown in the first and the second chapters, and the third chapter consists of problems and exercises, the most part of which is shown with solutions. The other part has answers, which is rather important to students who solve problems by themselves.

Theoretical material methodologically corresponds with the practical part, which contains some large practical problems.

For second year students of High Technical Educational Establishments, also could be useful for graduates and postgraduates.

Іл. 26. Бібліогр.: 5 назв.

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Chapter 1.

Multiple Integrals.

1.1. Some examples leading to the notion of a multiple integral.

Let us consider a solid (Ω) with the density of mass distribution ρ. The density ρ can be variable, that is different at different points of the solid. Let the function $\rho = \rho(M)$ (where M is a point of Ω) be known and let it be necessary to determine the whole mass m of the solid.

Let us mentally divide the solid (Ω) into n parts (subregions) (Ω_k), $k = 1,2,...,n$, denote $\Delta\Omega_k$ as the volume of (Ω_k) then choose arbitrary points M_k, $k = 1,2,...,n$ in each of the subregions (Ω_k). If the parts (Ω_k) are sufficiently small we can regard the density as being constant within each of the parts without an essential error. Then the mass $m_{(\Delta\Omega_k)}$ of the part (Ω_k) can be computed as the product of the density by the volume.

Thus we obtain

$$m \approx \sum_{k=1}^{n} \rho(M_k) \cdot \Delta\Omega_k .$$

This is an approximate equality since the densities of the parts are nevertheless variable. But the smaller parts the greater accuracy. Hence, passing to the limit, as $\Delta\Omega_k \to 0$, we obtain the exact equality

$$m = \lim \sum_{k=1}^{n} \rho(M_k) \cdot \Delta\Omega_k$$

The limit is taken here in a process in which not only the volumes but also all the linear sizes of the parts tend to zero. Besides, it is supposed that the limit does not depend on the way of the partitioning (Ω) into subregions.

Reasoning in a similar way we can conclude that if an electric charge is distributed over a solid (Ω) with density σ the magnitude q of the charge is found by means of the formula

$$q = \lim \sum_{k=1}^{n} \sigma(M_k) \cdot \Delta\Omega_k .$$

1.2. Definition and basic properties.

For definiteness, let us consider integrals over three-dimensional regions. Suppose we are given a bounded (finite) region (Ω) in space. Let a function $u=f(\Omega)$ be defined over (Ω) and let the value $f(\Omega)$ of the function be finite at each point M of the region. To compose an integral sum we arbitrarily break up the region (Ω) into subregions (Ω_1) (Ω_2), ... , $(\Omega_n)$and take an arbitrary point. M_k $(k=1,2,\ldots,n)$ in each of them. Then we write down the integral sum:

$$\sum_{k=1}^{n} f(M_k)\cdot\Delta\Omega_k\ ,$$

where $\Delta\Omega_k$ denotes the volume of the subregion (Ω_k).

The limit of the integral sum taken in a process in which all the linear sizes of the subregions entering into the partitions of the region (Ω) are unlimitedly decreased is called the integral of the function f over the region (Ω). Denoting the integral by the symbol $\int\limits_{(\Omega)} f d\Omega$ we can write

$$\int\limits_{(\Omega)} f(M)d\Omega=\lim\sum_{k=1}^{n} f(M_k)\Delta\Omega_k$$

(Ω) is called the region (domain) of integration. Compare this with the basic definition of the definite integral.

The basic properties of a definite integral are implied by the definition of an integral as the limit of the integral sum. Therefore we can easily extend these properties to multiple integrals.

We enumerate them here.

1) The integral of a sum equals the sum of the integrals of the summands (the same is true for the difference):

$$\int\limits_{(\Omega)}(f_1\pm f_2)d\Omega=\int\limits_{(\Omega)} f_1 d\Omega\pm\int\limits_{(\Omega)} f_2 d\Omega\,.$$

2) A constant factor can be taken outside the sign of integration:

$$\int\limits_{(\Omega)} cf d\Omega=c\int\limits_{(\Omega)} f d\Omega\,,(c=const)\,.$$

3) The theorem of a partition of the region of integration: for any partition of the region (Ω) into parts the integral over the whole region is equal to the sum of the integrals over the parts. For definiteness, if (Ω) is divided into the parts (Ω_1) and (Ω_2) we have:

$$\int\limits_{(\Omega)} f d\Omega=\int\limits_{(\Omega_1)} f d\Omega+\int\limits_{(\Omega_2)} f d\Omega\,.$$

4) The integral of unity is equal to the measure of the region of integration

$$\int\limits_{(\Omega)} f d\Omega=V_{\Omega}\,.$$

5) It is allowable to integrate inequalities:

if $f_1\le f_2$ then $\int\limits_{(\Omega)} f_1 d\Omega\le\int\limits_{(\Omega)} f_2 d\Omega\,.$

6) An integral satisfies the inequalities:

$$f_{\min} \cdot V \le \int\limits_{(\Omega)} f d\Omega \le f_{\max} \cdot V .$$

7) They are connected with the notion of the mean value $\bar{f}$ of a function f over a region (Ω).

$$\bar{f} \cdot V = \int\limits_{(\Omega)} f \cdot d\Omega .$$

8) There is an inequality of the form

$$\left| \int\limits_{(\Omega)} f \cdot d\Omega \right| \le \int\limits_{(\Omega)} |f| \cdot d\Omega .$$

1.3. Geometrical meaning of an integral over a plane region.

Such an integral, unlike other multiple integrals, can be directly interpreted geometrically. Its geometric meaning is similar to that of an ordinary define integral.

Let us be given an integral of the form $\int\limits_{(S)} u \cdot dS$, where (S) is a domain

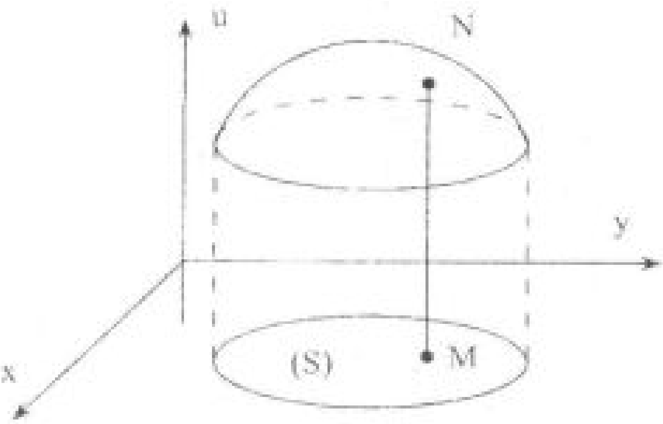

Fig. 1

lying in a plane. (See Fig. 1). Let us draw the u-axis perpendicularly to the plane and construct a line segment of length $f(M)$ parallel to the u-axis and passing through a point M belonging to the domain (S).

For simplicity's sake we now consider positive values of f; then the segment is drawn in the positive direction of the u-axis and the end-point N of the segment lies above the plane P. When the point M runs throughout the domain (S) the corresponding point N describe a surface, which is the graph of the integrand. The surface together with the plane figure (S) and the cylindrical surface formed by the line segment parallel to the u-axis and drawn through each point of the contour bordering the domain (S) bound a cylindrical body.

The geometric meaning of integral $\int\limits_{(S)} f dS$ lies in the fact that it is equal to the volume of the cylindrical body. Indeed, the element of volume corresponding to a plane element dS of the domain (S) containing a point M can be regarded as a right cylinder with base dS and height $f(M)$ to within infinitesimals of higher order. Hence, this volume is approximately equal to $dV = f(M)dS$. Summing up these elements of volume we arrive at the formula

$$V = \int\limits_{(S)} f \cdot dS = I$$

which is what we set up to prove.

1.4. Integrals over a rectangle.

We now consider an integral $I = \int\limits_{(\Omega)} u\, d\Omega$ where (Ω) is a rectangle bounded by coordinate lines of the Cartesian coordinate system arbitrary chosen in a plane (see Fig. 2). The rectangle is described by inequalities $a \le x \le b$ and $c \le y \le d$ where a, b, c, d are some constants. When forming an integral sum it is natural to break up (Ω) into parts by means of straight lines parallel to the coordinate axes which divide the integral $a \le x \le b$ into parts Δx_i and the integral $c \le y \le d$ into parts Δy_i.

Let us denote by u_{ik} the value of the integrand $u = u(x,y)$ at a point belonging to the subregion adjoining the intersection of the i-th vertical line with the k-th horizontal line (see Fig. 2).

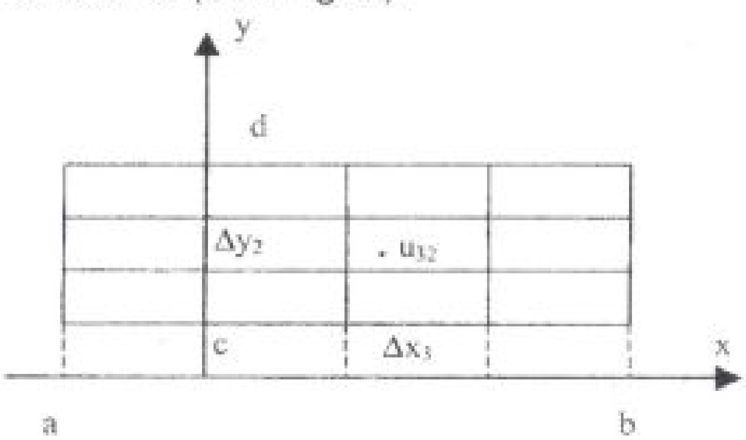

Fig. 2

We then approximately have $I = \sum_{i,k} u_{ik} \Delta x_i \Delta y_k$ where the summation is extended over all the subregions. It is a two-dimensional integral sum:

$$S \approx \sum_{i=1}^{m} \left(\sum_{k=1}^{n} u_{ik} \Delta x_i \Delta y_k \right) = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} u_{ik} \Delta y_k \right) \Delta x_i .$$

If the divisions along the y-axis are sufficiently small the sum inside the brackets is close to the corresponding integral:

$$\sum_{k=1}^{n} u_{ik} \Delta y_k \approx \left(\int_c^d u dy \right)_i .$$

It follows that

$$S \approx \sum_{i=1}^{m} \left(\int_c^d u dy \right)_i \Delta x_i . \tag{1}$$

But this is also an integral sum for function, which depends on x. Hence, if the divisions along the x-axis are also sufficiently small we can write:

$$S \approx \int_a^b \left(\int_c^d u(x,y) dy \right) dx . \tag{2}$$

In the process of decreasing the subregions of the partitions equalities (1) and (2) become more and still more accurate and turn into the precise relations in the limit. Consequently,

$$I = \int_{(\Omega)} u d\Omega = \int_a^b \left(\int_c^d u(x,y) dy \right) dx .$$

Thus, to compute an integral taken over a rectangle with sides parallel to the coordinate axes we can first perform the integration with respect to y, for a fixed x (the inner integration) and then integrate the result (the outer integration).

The reverse order of passing from sum to an iterated two-fold sum (see above) would yield:

$$I = \int_c^d \left(\int_a^b u(x,y) dx \right) dy$$

Hence, when computing a double integral in Cartesian coordinates we have two ways of passing to a repeated two-fold integral.

The transition from one of these ways to the other is referred to as the inversion of the order of integration.

1.5. Integral over an arbitrary plane region.

Let the domain of integration be an arbitrary plane figure lying in the x, y-plane. For instance, take the domain depicted in Fig. 3.

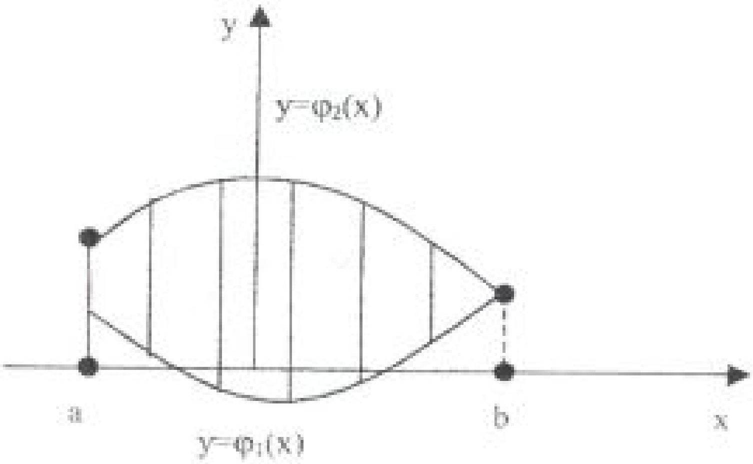

Fig. 3

The considerations given above can be transferred to this case with some slight changes. Namely, instead of the integral (1) we arrive at an integral of the form

$$\int_{y_1}^{y_2} u(x,y)\cdot dy = \int_{\varphi_1(x)}^{\varphi_2(x)} u(x,y)\cdot dy,$$

where $y = y_1 = \varphi_1(x)$ and $y = y_2 = \varphi_2(x)$ are, respectively, the equations of the upper and lower parts of the boundary of the domain. Accordingly, the final result will be of the form:

$$I = \int_{(\Omega)} u \cdot d\Omega = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} u(x,y)\cdot dy \right) dx.$$

Consequently, the limits of integration in the inner integral are variable in the general case; they depend on the variable of integration in the outer integral.

We can also invert the order of integration, that is perform the first integration with respect to x and the second – with respect to y. Then we arrive at a formula of the form

$$I = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} u(x,y)\cdot dx \right) dy.$$

It is sometimes necessary to break the domain of integration into several parts before setting up the limits of integration.

For example, $\int_0^1 dx \int_{x^2}^{2x} f(x,y)\cdot dy$.

1.6. Integral over a three-dimensional region.

Let us now consider an integral $I = \int_{(\Omega)} u \cdot d\Omega$, where (Ω) is a solid, that is a domain in space.

We compute it following the procedure, which was developed for an integral over a plane figure. The corresponding integral sum is now represented as an iterated three-fold sum. In the simplest case when (Ω) is a rectangular parallelepiped defined by the inequalities $a \le x \le b$, $c \le y \le d$ and $e \le z \le f$ we obtain, after passing to the limit in the integral sum, the formula

$$I = \int_a^b dx \int_c^d dy \int_e^f u(x,y,z)\cdot dz = \int_a^b \left(\int_c^d \left(\int_e^f u \cdot dz \right) dy \right) dx .$$

By the way, it is possible to perform here the integration by inverting the order of integration in five different ways because there are six different combinations (permutations) of the differentials dx, dy, dz.

If the domain of the integration is of more general form the determination of the limits of the integration could be more complicated.

Let the domain of integration be of the form shown in Fig. 4.

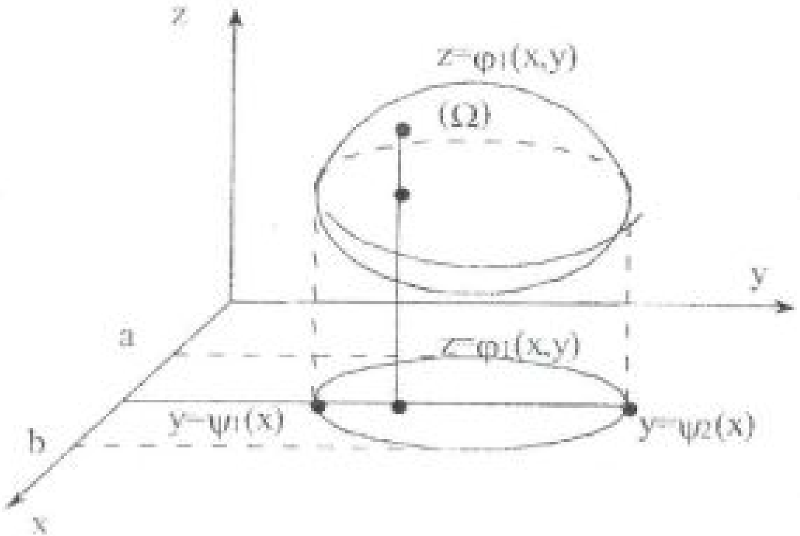

Fig. 4

Then we can put down integral I in the form:

$$\int\limits_{(\Omega)} u \cdot d\Omega = \int\limits_a^b dx \int\limits_{\psi_1(x)}^{\psi_1(x)} dy \int\limits_{\varphi_1(x,y)}^{\varphi_2(x,y)} u(x,y,z) \cdot dz\,.$$

1.7. Passing to polar coordinates in plane.

As in the case of one-dimensional integral, we can introduce different variables of integration while computing a double integral. Here we shall consider a typical example of computing a double integral in the polar coordinates.

Let us take an integral of the form: $I = \int\limits_{(\Omega)} u d\Omega$, where (Ω) is a region in the x, y – plane, which is depicted in Fig. 5. It is necessary to perform the integration in polar coordinates. We must divide the domain into parts by means of the coordinate curves of the polar coordinate system, i.e. by the lines $r=const$ and $\varphi=const$, as it is shown in Fig 5.

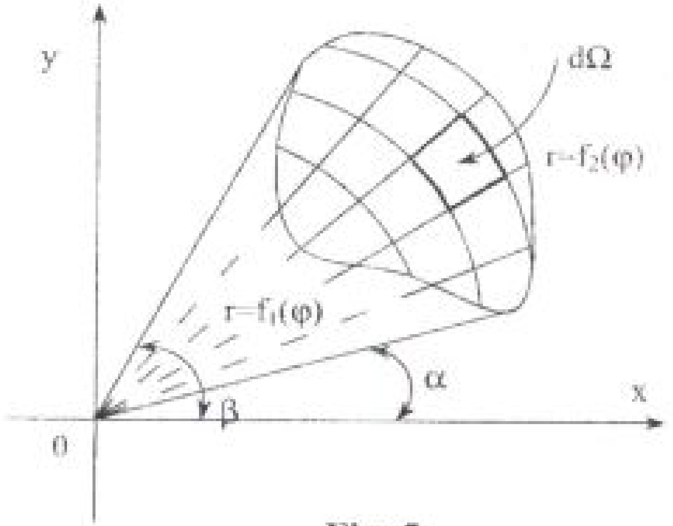

Fig. 5

Each of the elementary areas thus obtained can be regarded as being equal to a rectangle with sides dr and $rd\varphi$ to within infinitesimals of higher order. Hence, we have $d\Omega=rdrd\varphi$. Performing the summation over all the elementary areas we obtain $I = \iint\limits_{(\Omega)} urdrd\varphi$, where the integrand must of course be expressed as a function of r and φ. By analogy with 2.2., we set up the limits of integration and thus receive $\int\limits_{\alpha}^{\beta}\left(\int\limits_{f_1(\varphi)}^{f_2(\varphi)} urdr\right)d\varphi$. The geometrical meaning of the limits of the integration is illustrated in Fig. 5.

Polar coordinates are particularly convenient for regions whose boundary consists of coordinate curves of the polar coordinate system.

1.8. Passing to cylindrical and spherical coordinates.

Let us take an integral $I = \int\limits_{(\Omega)} ud\Omega$ where (Ω) is a domain in space. It is necessary to perform the integration in cylindrical coordinates. We have to divide the domain into parts by means of the coordinate surfaces of the cylindrical coordinate system, i.e. the surfaces $r=const$, $\varphi=const$ and $z=const$.

Then each of the elements of volume (see Fig. 6) can be regarded as being equal to the volume of the rectangular parallelepiped with dimensions dr, $rd\varphi$ and dz to within infinitesimals of higher order of smallness (relative to the element of volume). Consequently we have $d\Omega = rdrd\varphi dz$. Therefore the integral takes the form $\iiint\limits_{(\Omega)} urdrd\varphi dz$ where the limits of integration are still to be set up as in 1.6., where we set the limits in Cartesian coordinates.

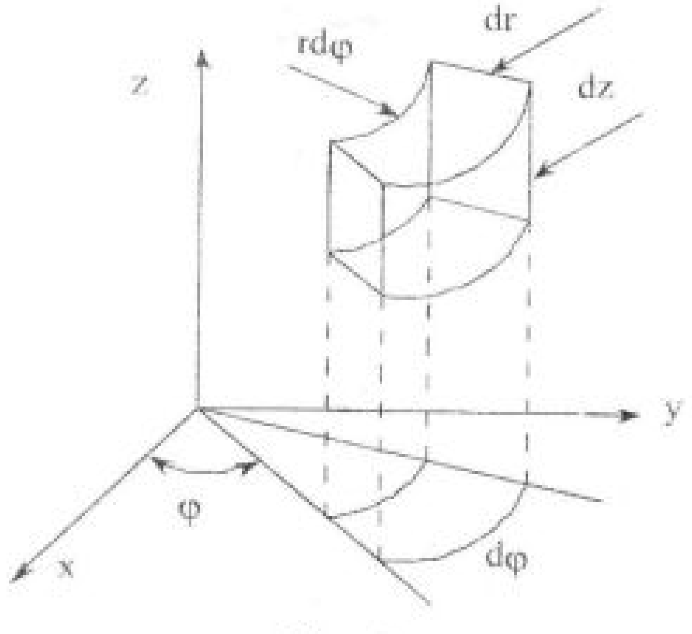

Fig. 6

Let we use spherical coordinates.

The element of volume can be again regarded as being approximately equal to the volume of the corresponding rectangular parallelepiped (see Fig. 7). In this case the rectangular parallelepiped has the sides $d\rho$, $\rho d\theta$ and $\rho\sin\theta d\varphi$. Thus we have $d\Omega=\rho^2\sin\theta d\rho d\theta d\varphi$ and $I=\iiint\limits_{(\Omega)} u\rho^2\sin\theta d\rho d\theta d\varphi$. (*)

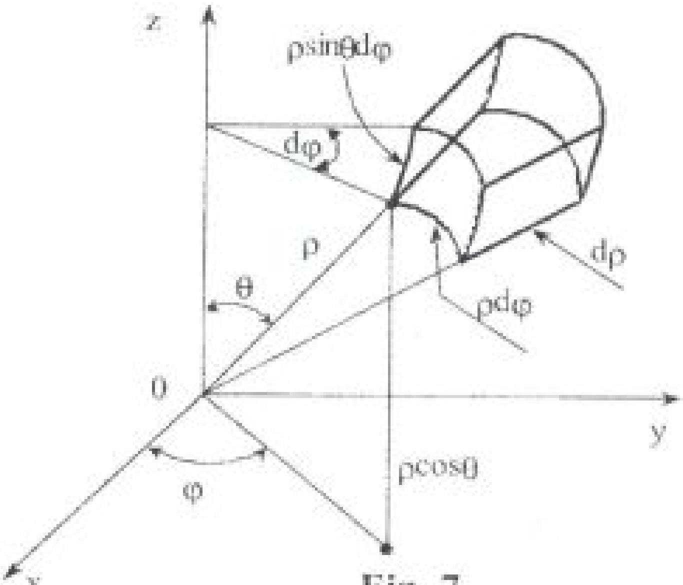

Fig. 7

The limits of the integration are set up in a particularly simple manner in this coordinate system (and also in other systems) when (Ω) consists of coordinate surfaces because in such a case not only the limits of the outer integration are constant but the first and second integrations as well.

As an example, let us consider the problem of determining the

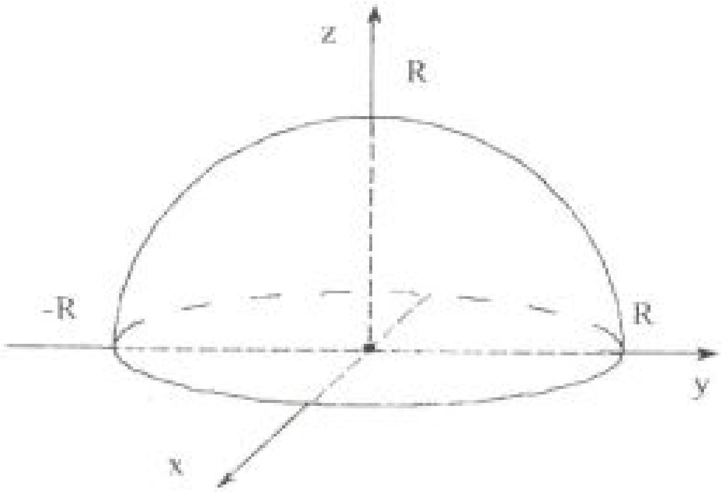

Fig. 8

position of the geometrical center of the gravity of a solid having the form of a hemisphere of radius R. To do this we place the hemisphere as it is shown in Fig.8.

Then the symmetry implies that the center of gravity will lie on the z-axis. Taking advantage of formula $z_c = \frac{1}{\frac{2}{3}\pi R^3}\int\limits_{(\Omega)} z d\Omega$, passing to spherical coordinates by means of formula (*) and taking into account that $z = \rho\cos\theta$ we obtain the following expression:

$$z_c = \frac{1}{\frac{2}{3}\pi R^3}\iiint\limits_{(\Omega)} \rho\cos\theta\rho^2\sin\theta d\rho d\theta d\varphi = \frac{3}{2\pi R^3}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int\limits_0^R \rho^3 d\rho = \frac{3}{8}R$$

1.9. Integral over an arbitrary surface.

Let us consider the integral $\int\limits_{(\sigma)} u d\sigma$ taken over an arbitrary surface (σ), which can be curvilinear in the general case (see Fig. 9). To compute it in Cartesian coordinates we must consider the projection of the surface (σ) on the coordinate planes. For definiteness, let us take the projection of (σ) on the x, y – plane which we denote by (S).

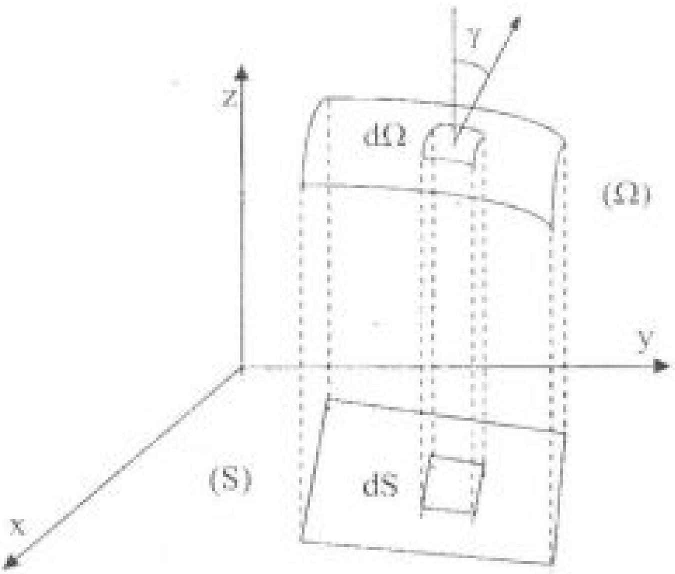

Fig. 9

Since the element (the area of an infinitesimal part) of a curvilinear surface can be regarded as being plane to within infinitesimal of higher order of smallness relative to the area, we have $dS = d\sigma|\cos\gamma|$ where $\gamma = (\vec{n} \wedge \vec{k})$ and $\vec{n}$ is a normal vector. It follows that

$$\int\limits_{(\sigma)} u\,d\sigma = \iint\limits_{(S)} u\frac{dxdy}{|\cos\gamma|}$$

Let the surface in question be represented by an equation of the form $z = f(x, y)$. Then the vector $\vec{n} = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}$ is directed along the normal to the surface at every point belonging to the surface. Hence we have:

$$\cos(\vec{n} \wedge \vec{k}) = \frac{\vec{n}\vec{k}}{|\vec{n}||\vec{k}|} = \frac{1}{\sqrt{1 + (f'_x)^2 + (f'_y)^2}}.$$

Therefore,

$$I = \iint\limits_{(S)} u(x, y, f(x, y))\sqrt{1 + (f'_x)^2 + (f'_y)^2}\,dxdy.$$

In particular we derive the formula for the area

$$S_\sigma = \iint\limits_{(S)} \sqrt{1 + (f'_x)^2 + (f'_y)^2}\,dxdy.$$

Let reader deduce the formula of $\cos(\vec{n} \wedge \vec{k})$ for a surface represented by an equation of the form $F(x, y, z) = 0$.

Chapter 2.

Elements of the Field Theory.

2.1. Scalar field. Directional derivative. Gradient.

We say that there is a scalar field u defined in space if the value of the quantity u is specified at each point M of space, i.e. $u=u(M)$. Let the Cartesian coordinate system x, y, z in space be given. Then a stationary scalar field can be regarded as a function $u=u(x,y,z)$.

Suppose that a curve (L) starts at the point M in the direction l (see Fig. 10).

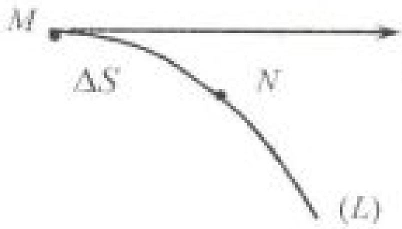

Fig. 10

Then the rate of change of the field in this direction (related to unit length) is called the derivative of u along the direction l:

$$\frac{\partial u}{\partial l}=\lim_{\Delta S\to 0}\frac{u(N)-u(M)}{\Delta S}\ ,\ \Delta S=\cup MN$$

To compute the directional derivative let us suppose that the curve (L) is represented in parametrical form by the equation $\vec{r}=\vec{r}(s)$ where the parameter s is the arc length reckoned along (L). Then the values of u taken along (L) form a composite function $u(s)=u(x(s),y(s),z(s))$. Therefore, by the rule of differentiating a composite function, we have:

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}.$$

The right-hand side can be represented as a scalar product of two vectors ($\overline{grad\,u}\cdot\frac{d\vec{r}}{ds}$). The first vector is called the gradient of the field. It is designed as $\overline{grad\,u}=\frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}$.

The second vector $\frac{\partial\vec{r}}{\partial s}=\frac{\partial x}{\partial s}\vec{i}+\frac{\partial y}{\partial s}\vec{j}+\frac{\partial z}{\partial s}\vec{k}=\vec{\tau}$ is the unit vector in the direction l.

Thus $\frac{\partial u}{\partial l}=\overline{grad\,u}\cdot\vec{\tau}=grad_l u$ (*) ($grad_l u$ denotes the projection of the gradient on the axis passing the direction l).

Note that the derivatives u'_x, u'_y and u'_z are also directional derivatives, for instance u'_x is the derivative in the direction of the x-axis.

Let us put down one more useful formula containing the gradient which is based on the definition of the total differential:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = \overline{grad}\, u \cdot d\bar{r}\ .$$

Let the field u and a point M be given. Let us set the following problem: in what direction is the derivative $\frac{\partial u}{\partial l}$ maximal? We see that on the basis of the formula (*) the problem reduces to the following question: in which direction is the projection of the vector $\overline{grad}\, u$ maximal? Evidently, the maximal projection of any vector is obtained when we take its own direction, the maximal projection being equal to the modulus of the vector.

Thus, the vector $\overline{grad}\, u$ at a point M indicates the direction of the maximal rate of increase of the field u; this maximal rate (related to the unit length) is being equal to $|\overline{grad}\, u|$.

2.2. Level surface.

Level surface of a field $u(M)$ are the surfaces on which the field assumes constant value, that is the surfaces represented by equations of the form: $u(M)=\text{const}$.

Depending on the physical meaning of the field these surfaces may be called isothermic surfaces (for the temperature field), isobaric surfaces and the like.

There is a simple relationship between these surfaces and the gradient of the field: at each point M the gradient is normal to the level surface passing through the point M.

Actually, as it is seen on Fig. 11, the surfaces $u=C$ and $u=C+\Delta C$ can be regarded as being almost plane near the point M if ΔC is sufficiently small, and besides $\frac{\partial u}{\partial l} = \frac{\Delta u}{\Delta S} = \frac{\Delta C}{\Delta S}$.

Fig. 11

But it is clear that if l directly along the normal to the surface the quantity ΔS will assume its least value, and $\frac{\partial u}{\partial l}$ will therefore assume its maximal value. This implies our assertion.

In particular, we see that the assertion enables us to solve the following problem: to find the equation of the tangent plane passing through a point $M_0(x_0, y_0, z_0)$ of a surface (L) having an equation of the form $F(x,y,z)=0$. To solve the problem let us introduce a scalar field in space by means of the equation $u=F(x,y,z)$. Then (L) becomes one of the level surfaces of the field because we have $u=F(x,y,z)=0$ on the surface.

Then the vector $(\overline{grad u})_{M_0}=(F'_x)_0\vec{i}+(F'_y)_0\vec{j}+(F'_z)_0\vec{k}$ (the subscript "zero" indicates that the corresponding derivatives are taken at the point M_0) is perpendicular to the sought-for tangent plane.

Hence, we obtain the equation of the plane:

$$(F'_x)_0(x-x_0)+(F'_y)_0(y-y_0)+(F'_z)_0(z-z_0)=0.$$

The last equation can be put down as $dF=0$.

A surface for which the tangent plane is to be constructed can be represented by an equation of the form $z=f(x,y)$. Here we can rewrite the equation as $z-f(x,y)=0$ and denote its left-hand side by $F(x,y,z)$. Then the last formula for a plane is directly applicable, and thus we have

$$-\left(\frac{\partial f}{\partial x}\right)_0(x-x_0)-\left(\frac{\partial f}{\partial y}\right)_0(y-y_0)+(z-z_0)=0$$

i.e.

$$z-z_0=\left(\frac{\partial f}{\partial x}\right)_0(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_0(y-y_0).$$

The right-hand side is equal to the total differential df, we thus obtain the geometrical meaning of the total differential of a function of two independent variables. Namely, the differential is equal to the increment of the third coordinate of the point in the tangent plane.

Take an example. Let us compute the gradient of a centrally symmetric field $u=f(r)$ where $r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$. In this case the level surfaces are concentric spheres with centre at the origin of coordinates (why is it so?). If we take two spheres for which the difference of their radii is equal to dr then the difference of the corresponding values of the function f which are taken on these surfaces will be equal to df.

Therefore, the change rate of the function in a direction, which is transversal to the level surfaces, (that is along a radius) is equal to $\frac{df}{dr}$. Hence,

$$\overline{grad}\, f(r) = \frac{df}{dr}\bar{r}^{\circ} = \frac{1}{r}\frac{df}{dr}\bar{r},$$

where $\bar{r}^{\circ} = \bar{r}/r$ is the unit vector in the direction of the vector $\bar{r}$.

Let the reader obtain this result on the basis of the definition

$$\overline{grad}\, f = f'_x i + f'_y j + f'_z k$$

2.3. Vector fields. Vector lines.

We say that there is a vector field $\vec{A}$ (field of vector $\vec{A}$) defined in space if the value of the vector quantity $\vec{A}$ is specified at each point M of the space, i.e. $\vec{A} = \vec{A}(M)$. We shall deal with a stationary field, which does not change as time passes. If such a variation takes place we shall consider the field at a fixed moment of time and thus reduce our consideration to a stationary field. As examples of vector fields, we can consider the field of velocity $\vec{v}$, the field of force $\vec{F}$, the electric field $\vec{E}$ (where $\vec{E}$ is the electric field strength) etc.

A curve *(L)*, which is tangent to the vector $\vec{A}$ at each point, is called a vector line. In other words this is a curve whose direction (i.e. the direction of its tangent) coincides with the direction of the field at each point belonging to the curve. Depending on the physical meaning of the field we speak about a streamline (flow line) of a field of velocity, a line of force of a field of force and so on.

If we take Cartesian coordinates x, y, z, the vector $\vec{A}$ can be resolved according to the formula:

$$\vec{A} = A_x(x,y,z)\vec{i} + A_y(x,y,z)\vec{j} + A_z(x,y,z)\vec{k}.$$

On the basis of Vector Algebra and Differential Calculus, we can put down the symmetric system of differential equations for the vector lines of the field $\vec{A}$:

$$\frac{dx}{A_x(x,y,z)} = \frac{dy}{A_y(x,y,z)} = \frac{dz}{A_z(x,y,z)}.$$

In the case of a plane field the system turns into the equation:

$$\frac{dx}{A_x(x,y,z)} = \frac{dy}{A_y(x,y,z)}.$$

From the geometrical point of view the problem of constructing vector lines of a given vector field is equivalent to that of constructing integral curves for the given directional field. As it was shown, when the theory of differential equations was studied, there is only one vector line passing through a non-singular point. Thus, the whole region in which a vector field is defined is filled with vector lines of the field. In a sufficiently small domain containing a non-singular point the totality of the vector lines resembles the set of parallel segments, which can be curved a little. In the vicinity of a singular point the family of vector lines can have a very complicated structure.

2.4. The flux of a vector through a surface.

Let a vector field be defined in a domain of space and let an oriented surface (σ) lie in the domain. We remind reader that orienting a surface is equivalent to indicating its outer and inner sides. The flux of a vector field $\vec{A}$ through a surface (σ):

$$Q = \int\limits_{(\sigma)} A_n d\sigma ,$$

where A_n is the projection of the vector $\vec{A}$ on the unit outer normal $\vec{n}$ to (σ).

The flux is a scalar quantity. Since it is a particular case of a surface integral it possesses all the properties of this integral. Here we point out a characteristic property of a flux: it is multiplied by (-1) when the orientation of the surface is changed because this yields the change of the sign of A_n.

The value of a flux is essentially dependent on the mutual disposition of the surface (ω) and the vector lines of the field. Indeed, if the surface (ω) is everywhere intersected by the vector lines from its inner side to the outer side (the direction of a vector line at a point is indicated by the vector of the field at this point) we have $Q > 0$; if otherwise we have $Q < 0$; finally if some of the vector lines intersect the surface in one direction and some in the opposite direction the flux is equal to the sum of a positive and a negative quantity and thus it can be positive or negative or equal to zero.

The flux is always equal to zero in the case the surface is totally covered by the arcs of the vector lines because the vector $\vec{A}$ is tangent to such a surface at each point and hence $A_n = 0$.

The physical meaning of a flux depends on the type of the field. For instance, let the velocity $\vec{v}$ of a gas flow be considered. Then the quantity $dQ = \vec{v} \cdot d\vec{\omega}$ is equal to the volume of an elementary gas cylinder passing through the area $(d\omega)$ in unit time (See Fig. 12).

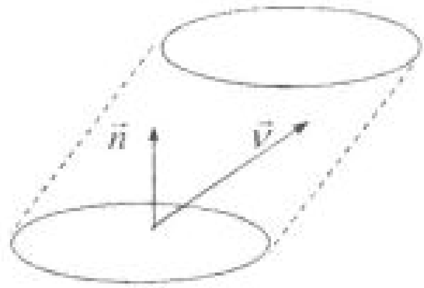

Fig. 12

2.5. Divergence.

Let us take a volume (V) bounded by a surface (σ) and lying in a domain of space where a vector field $\vec{A}$ is defined. The flux of the field through the surface $Q = \oint\limits_{(\sigma)} \vec{A} \cdot d\vec{\sigma}$ (the symbol $\oint$ indicates that the integral is taken over a closed surface). If the flux is positive this means that the number of vector lines passing through (σ) from the interior of the domain (V) exceeds the number of lines passing in the opposite direction. In this case we say that there is a source of vector lines in (V). (The flux of $\vec{A}$ through (σ) is sometimes referred to as the number of vector lines of $\vec{A}$ intersecting (σ) from its inner side to the other side).

The quantity Q characterizes the source strength. If $Q<0$ we say that the source is a sink in (V). A sink is usually termed as a negative source.

The source of a vector field can be concentrated at separate points or distributed over some surfaces or curves. They can also be distributed in space. We first turn to the latter case. Here we can introduce not only the average density $\frac{Q}{V}$ of the source in (V) but also the density of the field sources at any point M of space which is defined as $\lim\limits_{\Delta V \to M} = \lim \left[\int\limits_{(\Delta\sigma)} \vec{A} \cdot \overrightarrow{d\sigma} : \Delta V \right]$, where (ΔV) is a small volume enveloping the point M and $(\Delta\sigma)$ is the surface which bounds (ΔV). This density is called the divergence of the vector field $\vec{A}$ and is designated as $div\vec{A}$.

Let the Cartesian coordinate system be given in space. Then the vector field $\vec{A}$ can be represented in the form:

$$\vec{A} = A_x(x,y,z)\vec{i} + A_y(x,y,z)\vec{j} + A_z(x,y,z)\vec{k},$$

in this case the divergence (in Cartesian coordinates) is defined by the formula:

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

2.6. Ostrogradsky's formula.

Rewrite the formula for divergence in the form $div\vec{A}=\frac{dQ}{dV}$, i.e. $dQ=div\vec{A}dV$. The last expression represents the number of vector lines issued from the element of volume (dV). Summing together these expressions over a domain (V) we arrive at the formula for the number of vector lines coming out of the finite volume (V) (that is for the flux of the vector field $\vec{A}$):

$$\oint_{(\sigma)} A_n d\sigma = \iiint_{(V)} div\vec{A}dV,$$

where (V) is any finite domain and (σ) is its boundary surface. This is Ostrogradsky's formula, which plays an important role in the vector field theory. It was discovered by M. Ostrogradsky in 1826. The formula holds in all cases when the field $\vec{A}$ and its divergence $div\vec{A}$ do not approach infinity in (V).

2.7. Line integral and circulation.

Let an oriented curve (L) (i.e. such a direction of describing this curve is indicated) be given in the domain of space where a vector field $\vec{A}$ is defined. Then we can form the line integral

$$I = \int_L \vec{A}d\vec{r} = \int_L A_x dx + A_y dy + A_z dz$$

To define the integrals we write the formulas:

$$\int_L U \left.\begin{matrix} dy \\ dx \\ dz \end{matrix}\right\} = \lim \sum_{k=1}^{n} U(M_i) \begin{matrix} \Delta y_k \\ \Delta x_k \\ \Delta z_k \end{matrix},$$

where Δx_k is the increment of the abscissa x along the k-th elementary arc etc.; we pass to the limit in the process when all the lengths of the elementary areas decrease unlimitedly and $n \to \infty$.

Integrals of this type arc readily reduced to ordinary definite integrals. For example, if the curve (L) is represented in a parametric form we have

$$\int_L U dx = \int_{\alpha}^{\beta} U(x(t), y(t), z(t)) \cdot \dot{x}(t) dt$$

where the values $t=\alpha$ and $t=\beta$ correspond to the ends of the curve (L).

A line integral has an obvious physical meaning when $\vec{A}$ is a field of the force. In this case the integral is equal to the work performed by the field when the point upon which the force acts describes the curve (L).

If the curve (L) is a closed curve line integral is called the circulation (in this case we can write $\oint_L \vec{A}d\vec{r}$)

2.8. Rotation.

Let us consider a circulation taken along an infinitesimal closed loop (ΔL). Let the contour (ΔL) be placed near a point M of space and (ΔS) is the area of a surface bounded by the curve (ΔL) and $\vec{n}$ is the unit outer normal to the surface.

Then $\lim\limits_{\Delta(L)\to M} \dfrac{\oint_{(\Delta L)} \vec{A}\cdot d\vec{r}}{\Delta S} = \left(rot_n \vec{A}\right)_M$, where $rot\vec{A}$ is the rotation (or curl) of the vector field $\vec{A}$. Thus, the projection of the field rotation on any direction at any point M of space is equal to the circulation of the field over an infinitesimal loop bounding a surface perpendicular to $\vec{n}$ to the area of the surface.

The rotation (in Cartesian coordinates) is defined by the formula (see 3.8.):

$$rot\vec{A} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

The vector $rot\vec{A}$ form a new vector field in those parts of space where the original field $\vec{A}$ is defined.

The rotation of a plane field has a particularly simple expression. Indeed, if

$\vec{A} = A_x(x,y)\vec{i} + A_y(x,y)\vec{j}$, then we have $rot\vec{A} = \vec{k}\left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)$.

2.9. Green's formula.

Let us consider the circulation of a plane field $\vec{A} = P(x,y)\vec{i} + Q(x,y)\vec{j}$ over a closed loop (L), which is oriented in the positive direction. Bounded by the loop the finite domain will be denoted by S (See Fig.13). The circulation can be written as

$$\Gamma = \oint_{(L)} P(x,y)dx + \oint_{(L)} Q(x,y)dy$$

For the first integral we obtain

$$\int_a^b P(x,y_1)dx + \int_a^b P(x,y_2)dx =$$

$= -\int_a^b [P(x,y_2) - P(x,y_1)]\cdot dx$ (see Fig.13).

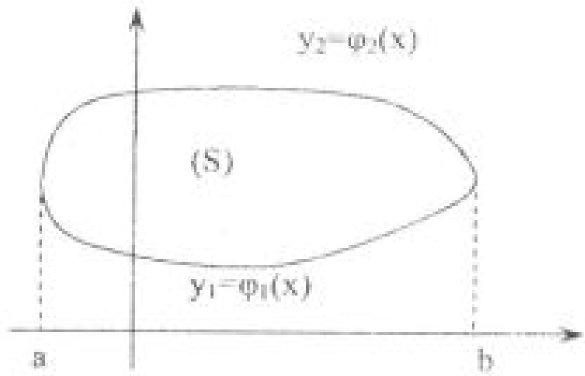

Fig.13

The expression under the sign of integration is a partial increment of P with respect to y, which can be represented in the form of an integral of the derivative:

$$P(x,y_2)-P(x,y_1)=\int_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial P}{\partial y}dy$$

Substituting this expression we obtain

$$\oint_{(L)} P(x,y)dx=-\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}\frac{\partial P}{\partial y}dy\right)dx=-\iint_{(S)}\frac{\partial P}{\partial y}dxdy$$

The second integral $\oint_{(L)} Q(x,y)dy$ is transformed similarly (we leave the calculations to the readers). Adding together the results we arrive at the Green's formula.

$$\oint_{(L)} Pdx+Qdy=\int_{(S)}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dS$$

The formula is applicable if all the functions $P,Q,\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}$ are finite everywhere in (S).

2.10. Stokes' formula.

The formula was discovered in 1854 by the English physicist and mathematician G.G. Stokes (1819-1903) is widely applied to the theory of vector field. Let a finite oriented loop (L) bounding a finite oriented surface (S) be given. Let the orientations of the (L) and (S) be coherent as it is shown in Fig.14. Divide (S) into small surfaces (ΔS_1), $(\Delta S_2),\ldots,(\Delta S_m)$ bounded by the loops (ΔL_1), $(\Delta L_2),\ldots,(\Delta L_m)$. Then we conclude that

$$\oint_{(L)}\vec{A}\cdot d\vec{r}=\sum_{i=1}^{m}\oint_{(\Delta L_i)}\vec{A}\cdot d\vec{r}$$

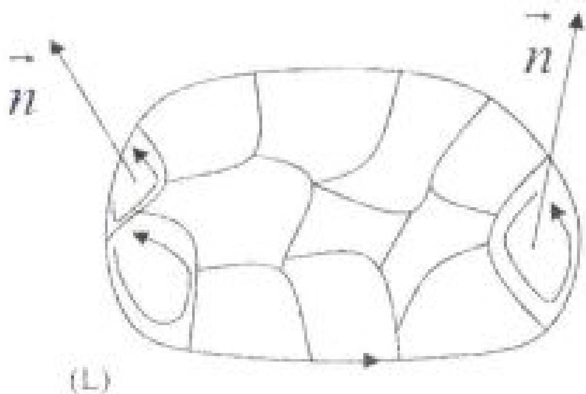

Fig.14

Because the integrals taken over the arcs entirely lying inside (L) and which enter into the right-hand side mutually cancel out and the sum of the remaining integrals just equals the left-hand side. Regarding (ΔS_i) as being infinitesimal we can apply formula:

$$\oint_{(\Delta L_i)} \vec{A}\cdot d\vec{r} = (rot_n A)_i \Delta S_i + ...$$

and obtain:

$$\oint_{(L)} \vec{A}\cdot d\vec{r} = \sum_{i=1}^{m} (rot_n A)_i \Delta S_i + ...$$

The sum on the right-hand side is an integral sum and therefore passing to the limit in the process when the linear sizes of all the sub domains are decreased unlimitedly we obtain:

$$\oint_{(L)} \vec{A}\cdot d\vec{r} = \iint_{(S)} rot_n A \cdot dS$$

Thus, the circulation of a vector field over a closed loop is equal to the flux of the rotation of the field through a surface bounded by the loop. This statement is called Stokes' theorem.

2.11. Special sorts of the vector fields.

A. Potential field. A vector field is called potential (or conservative) field if $\vec{A} = gradU$, where U is a scalar field. It is called the potential of the vector field $\vec{A}$. The condition $rot\vec{A} = 0$ is necessary and sufficient for the field $\vec{A}$ to be potential.

Properties of the potential field.

If a vector field $\vec{A}$ is potential then:

1. $\int_A^B \vec{A}d\vec{r} = \int_A^B (\vec{\nabla}U \cdot d\vec{r}) = \int_A^B dU = U(B) - U(A)$, here $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ is Hamilton's differential operator. It is used to denote differential operations of the first order: $grad\,U = \vec{\nabla}U$, $div\vec{A} = \vec{\nabla}\cdot\vec{A}$, $rot\vec{A} = \vec{\nabla}\times\vec{A}$. Also we have five main differential operations of the second order, and they as well can be denoted in terms of Hamilton's differential operator:
   a) $\vec{\nabla}\cdot\vec{\nabla}U = \Delta U$
   b) $\vec{\nabla}\times\vec{\nabla}U \equiv 0$
   c) $\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A})$
   d) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0$
   e) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A}$
2. $\oint_Z \vec{A}d\vec{r} = 0$.
3. $U(M) = \int_{M_0}^{M} \vec{A}d\vec{r} + U(M_0)$.

B. Solenoidal field. A vector field $\vec{A}$ is called solenoidal field if $div\vec{A}=0$. In this case due to Ostrogradsky's formula the flux of the vector field $\vec{A}$ through a closed surface is equal to zero: $\oiint A_n d\sigma = 0$. As an example of solenoidal field we can consider a vector field of a Vortexes $\vec{A}=rot\vec{B}$, because $divrot\vec{B}=\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)\equiv 0$.

C. Harmonic field. A vector field $\vec{A}$, which is both potential and solenoidal, is called harmonious (or Laplace's) field. We have two conditions: $rot\vec{A}=0$ and $div\vec{A}=0$. The potential U of this field is harmonic function, i.e. $\Delta U=0$. To prove it we consider above conditions $rot\vec{A}=0$,
$A=gradU, divA=divgradU=\vec{\nabla}\cdot\vec{\nabla}U=\Delta U=0$.

Chapter 3.

Problems and exercises.

3.1. Computing double integrals in Cartesian coordinates.

When computing a double integral in Cartesian coordinates we have two ways of passing to an iterated integral.

Example 1. Find $I=\iint_G \frac{x^2}{y^2}dxdy$, where G is bounded by the lines $y=x$, $y=\frac{1}{x}$, $x=2$.

Solution. Let us first draw the region (domain) of integration (see Fig. 15). It allows to use the formula $I=\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{x^2}{y^2}dy$, where $\varphi_1(x)=\frac{1}{x}$, $\varphi_2(x)=x$, $a=1$, $b=2$.

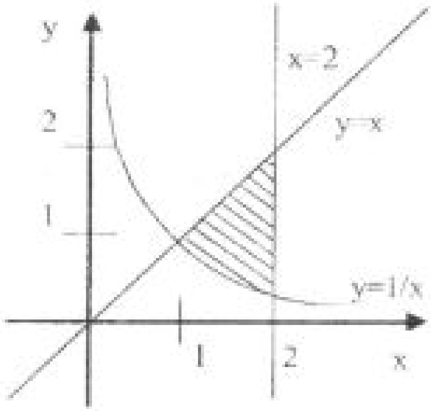

Fig. 15

We can perform the integration with respect to y (the inner int.) for a fixed x:

$$\int_{\frac{1}{x}}^{x} \frac{x^2}{y^2}dy = x^2\int_{\frac{1}{x}}^{x}\frac{dy}{y^2} = x^2\left(-\frac{1}{y}\Bigg|_{y=\frac{1}{x}}^{y=x}\right) = x^2\left(-\frac{1}{x}+x\right) = -x+x^3.$$

Then integrating the result of the first integration (which depends only on x) with respect to x within the limits of its variation we have:

$$I=\int_1^2\left(-x+x^3\right)dx = \left(-\frac{x^2}{2}+\frac{x^4}{4}\right)\Bigg|_1^2 = 2\frac{1}{4}.$$

Example 2. Invert the order of integration in the iterated integral:

$$\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx.$$

Solution. Draw the domain of integration by using the limits of integration: $\psi_1(y) = -\sqrt{1-y^2}$, $\psi_2(y) = 1-y$, $y_1 = 0$, $y_2 = 1$ (see Fig. 16).

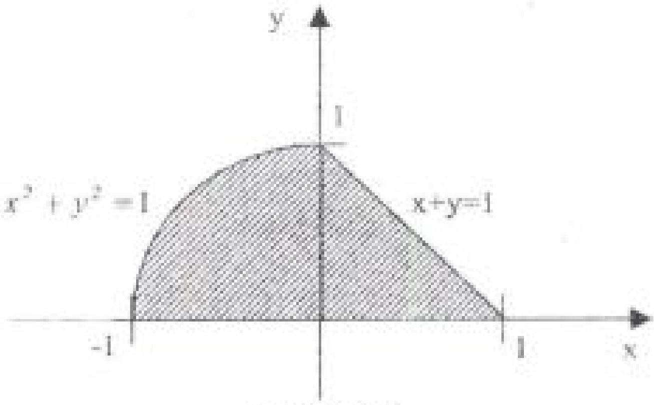

Fig. 16

The upper part of the boundary of the domain $\varphi_2(x) = \begin{cases} \sqrt{1-x^2}, -1 \le x \le 0, \\ 1-x, 0 < x \le 1; \end{cases}$ and the lower part is the straight line $y = 0$.

Thus we have:

$$\int_{-1}^{0} dx \int_{0}^{\sqrt{1-x^2}} f(x,y)dy + \int_{0}^{1} dx \int_{0}^{1-x} f(x,y)dy.$$

3.2. Computing triple integrals in Cartesian coordinates.

To compute a triple integral we ought to set up the limits of integration.

Example 3. Compute $\iiint\limits_{(V)} z\,dxdydz$, where (V) is bounded by the planes $x+y+z=1$, $z=0$, $y=0$, $x=0$.

Solution. Suppose that we want to set up the limits of integration when integrating in the following order: $\int dx \int dy \int z dz$.

Here the first (inner) integration is performed with respect to z, for fixed x and y. Therefore the limits of this integration are $z=0$ and $z=1-x-y$ (the equations of the upper and lower parts of the surface bordering the solid (V), (see Fig. 17).

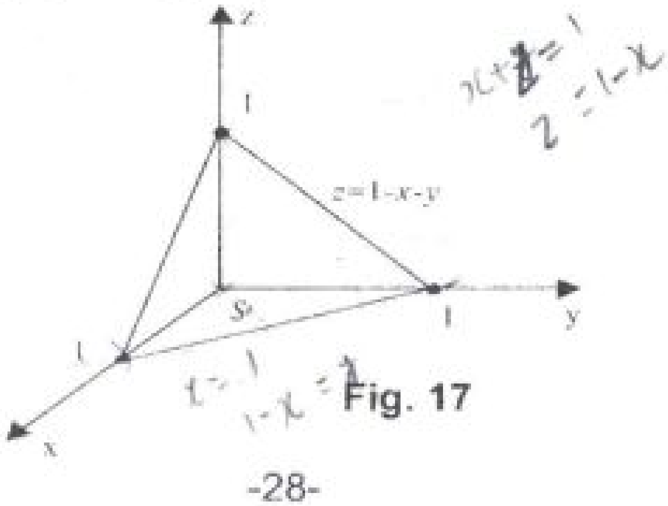

Fig. 17

$$\int_0^{1-x-y} z\,dz = \frac{1}{2}(1-x-y)^2 .$$

After the integration with respect to z and the substitution of the limits we obtain that the result of the first integration depends only on x and y. Now we pass to the projection (S) of the solid (V) on the x,y - plane and perform the second integration:

$$\int_0^{1-x} \frac{1}{2}(1-x-y)^2\,dy = \frac{1}{6}(1-x)^3 .$$

The result of the second integration will depend only on x. It should be integrated with respect to x over the maximal range of x. After the third integration we obtain:

$$\int_0^1 \frac{1}{6}(1-x)^3\,dx = -\frac{1}{6}\frac{(1-x)^4}{4}\Bigg|_0^1 = \frac{1}{24} .$$

Hometask.

Evaluate the given integrals:

1. $\int_0^1 dx \int_0^2 (x^2+y)dy$;
2. $\int_0^2 dx \int_x^{x\sqrt{3}} \frac{xdy}{x^2+y^2}$;
3. $\iint_G (x^2+y^2)dxdy$, where G is bounded by the straight lines : $y=x, x+y=2a, x=0$;
4. $\iint_G xydxdy$, G is bounded by the lines : $x+y=2,\ x^2+y^2=2y\ (x>0)$;
5. $\iint_G (x+2y)dxdy$, G is bounded by the lines: $y=x^2, y=\sqrt{x}$;
6. $\int_0^1 dx \int_0^x dy \int_0^{\sqrt{x^2+y^2}} zdz$;
7. $\int_0^a dx \int_0^{\sqrt{ax}} ydy \int_{a-x}^{2(a-x)} dz$;
8. $\iiint_V xyzdv$, where V is bounded by the surfaces: $y=x^2,\ x=y^2,\ z=xy,\ z=0$;
9. $\iiint_V (x^2+y^2)dv$, V is bounded by the surfaces $z=y^2-x^2,\ z=0,\ y=1$.

3.3. Computing double integrals in polar coordinates.

These coordinates are particularly convenient for regions whose boundary consists of coordinate curves of the polar coordinate system because in such cases when setting up the limits of integration we obtain constant limits not only in the outer integral but also in the inner one.

Example 4. Find $\iint_G (x^2+y^2)dxdy$, where G is bounded by the lines:

$y=x,\ y=0,\ y=\sqrt{4-x^2}\ (x\geq 0)$

Solution. Let us first draw the domain of integration: we have $\alpha=0, \beta=\frac{\pi}{4}, f_1(\varphi)\equiv 0, f_2(\varphi)=2$ (see Fig. 18)

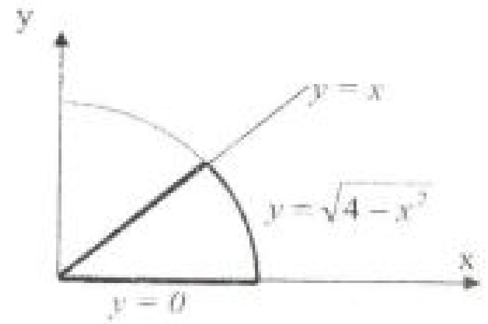

Fig. 18

Thus $\iint_G (x^2+y^2)dxdy = \int_0^{\frac{\pi}{4}} d\varphi \int_0^2 r^2 r dr = \int_0^{\frac{\pi}{4}} (\frac{r^4}{4}\Big|_0^2)d\varphi = 4\frac{\pi}{4} = \pi$

Example 5. Passing to polar coordinates find $\iint_G \sqrt{x^2+y^2}\,dxdy$, where G is bounded by the circle $x^2+y^2=2ax$

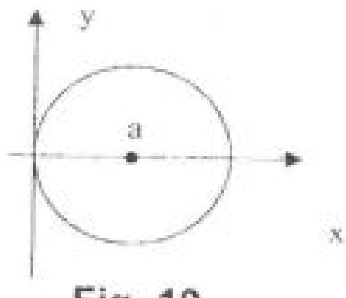

Fig. 19

Solution. Draw the circle $(x-a)^2+y^2=a^2$ (see Fig. 19); we have

$$-\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2},\ 0\leq r\leq 2a\cos\varphi$$

Thus

$$\iint\limits_G \sqrt{x^2+y^2}\,dxdy = \iint\limits_G r^2\,d\varphi dr = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{2a\cos\varphi} r^2 dr = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\Bigg|_0^{2a\cos\varphi} d\varphi =$$

$$\frac{8a^3}{3}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi d\varphi = \frac{16a^3}{3}\int\limits_0^{\frac{\pi}{2}} (1-\sin^2\varphi)\varphi d\sin\varphi = \frac{16a^3}{3}\frac{2}{3} = \frac{32}{9}a^3.$$

3.4. Triple integrals in cylindrical coordinates.

Cylindrical coordinates are particularly convenient for solids whose boundary consists of coordinate surfaces of the cylindrical coordinate system i. e. $r = const$ (cylinder), $\varphi = const$ (half-plane) and $z = const$ (plane).

$$I = \iiint\limits_V f(r\cos\varphi, r\sin\varphi, z)\,r dr d\varphi dz,$$

where the limits of integration are still to be set up as in previous lesson, where we set the limits in Cartesian coordinates.

Example 6. Passing to cylindrical coordinates compute $\iiint\limits_V z\sqrt{x^2+y^2}\,dv$,

where V is defined by inequalities $0 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2},\ 0 \le z \le a$

Solution. The equality $y = \sqrt{2x-x^2}$ in cylindrical coordinates takes the form (see Fig. 20): $r = 2\cos\varphi$ $(0 \le \varphi \le \frac{\pi}{2})$, so that

$$\iiint\limits_V z\sqrt{x^2+y^2}\,dv = \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{2\cos\varphi} r^2 dr \int\limits_0^a z dz = \frac{a^2}{2}\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{2\cos\varphi} r^2 dr =$$

$$= \frac{4a^2}{3}\int\limits_0^{\frac{\pi}{2}} \cos^3\varphi d\varphi = \frac{8}{9}a^2$$

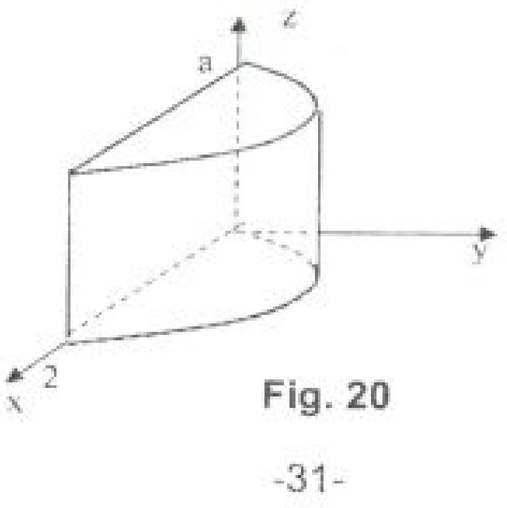

Fig. 20

Hometask:
Passing to cylindrical coordinates compute

16. $\iiint\limits_V y\,dv,\ V: x^2+y^2\le 4,\ 0\le z\le 5;$

17. $\iiint\limits_V z\,dv,\ V: x^2+y^2\le z^2,\ 0\le z\le a$

3.5. Triple integrals in spherical coordinates.

We use spherical coordinates when the region of integration is bounded by coordinate surfaces $\rho = const$ (sphere), $\theta = const$ (cone), $\varphi = const$ (half-plane). In such cases, when setting up the limits of integration, we obtain constant limits not only in the outer integral but also in the inner integrals.

Example 7. Passing to spherical coordinates find $\iint\limits_V (x^2+y^2)dv$, if$(V)$ is defined by inequalities: $x^2+y^2+z^2\le R,\ z\ge 0$.

Solution. The boundary of the region (V) consists of coordinate surfaces : $\rho = R, z=0$. In this case we have the following limits of integration:

$0\le\varphi\le 2\pi,\ \theta\in(0,\frac{\pi}{2}),\ \rho\in(0,R)$ (See Fig. 21)

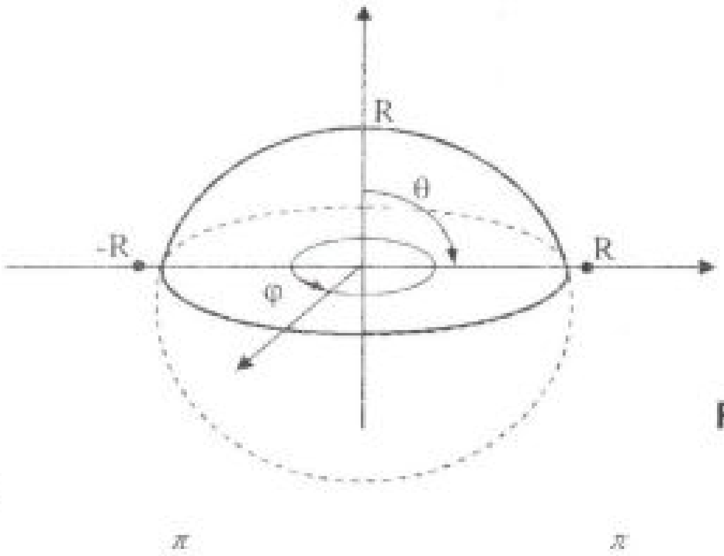

Fig. 21

So we have

$$\iiint\limits_V (x^2+y^2)dv = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} \sin^3\theta d\theta \int\limits_0^R r^4 dr = -\frac{2\pi R^5}{5}\int\limits_0^{\frac{\pi}{2}}(1-\cos^2\theta)d(\cos\theta) =$$

$$= -\frac{2\pi R^5}{5}(\cos\theta - \frac{\cos^3\theta}{3})\Big|_0^{\frac{\pi}{2}} = \frac{4}{15}\pi R^5.$$

Hometask:
Passing to spherical coordinates compute:

1. $\iiint\limits_V xyz^2\,dxdydz$, where (V) is bounded by the sphere $x^2+y^2+z^2=R^2$ and coordinate planes $x=0, y=0, z=0$ $(x\ge 0, y\ge 0, z\ge 0)$;

2. $\iiint\limits_V \frac{dxdydz}{\sqrt{x^2+y^2+z^2}}$, where (V) is a spherical region between two spheres: $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=4a^2$;

3. $\int\limits_0^{\frac{R}{\sqrt{2}}} dx \int\limits_0^{\sqrt{\frac{R^2}{2}-x^2}} dy \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{R^2-x^2-y^2}} dz$;

4. $\int\limits_0^{a} dx \int\limits_0^{\sqrt{a^2-x^2}} dy \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} z\,dz$.

3.6. Computation of scalar field's characteristics and vector lines of the vector field.

The main formulas:

$$grad\,u=\nabla u=\vec{i}\frac{\partial u}{\partial x}+\vec{j}\frac{\partial u}{\partial y}+\vec{k}\frac{\partial u}{\partial z},$$

$$\frac{\partial u}{\partial l}=(\nabla u,\vec{e}_l),\quad \vec{e}_l=\frac{\vec{l}}{|\vec{l}|}.$$

Example 8. Check up the orthogonality of level surfaces of the scalar fields $u=x^2+y^2-z^2$, $v=xz+yz$.

Solution. At each point M the gradient is normal to the level surface passing through the point M .So we ought to check up the orthogonality of gradients:

$$\nabla u=2x\vec{i}+2y\vec{j}-2z\vec{k},\quad \nabla v=z\vec{i}+z\vec{j}+(x+y)\vec{k}, \text{ so}$$
$$(\nabla u,\nabla v)=2xz+2yz-2z(x+y)=0.$$

A curve (L) which is tangent to the vector $\vec{a}$ at each point is called a vector line. We have the system of differential equations for the vector lines of the field $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$:

$$\frac{dx}{a_x}=\frac{dy}{a_y}=\frac{dz}{a_z}.$$

Example 9. Find the vector lines of the vector field $\vec{a}$: $\vec{a} = x^2\vec{i} + y^2\vec{j}$.

Solution. $\vec{a}$ is a plane field , thus the differential equations for the vector lines are $\frac{dx}{x^2} = \frac{dy}{y^2}$, $-\frac{1}{x} = -\frac{1}{y} + c_1$, or $-y = -x + c_1xy$, $y(1 + c_1x) = x$

Answer: $y = \frac{x}{1 - c_1x}$

Example 10. Find the vector line of the field $\vec{a} = -y\vec{i} + x\vec{j} + b\vec{k}$ passing through $(1,0,0)$

Solution. The system of the differential equations for the vector lines is

$$-\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{b}$$

The first equation $xdx = -ydy$, $x^2 + y^2 = c^2$. If $x = 1, y = 0$, then $c^2 = 1$. Thus $x = \cos t$, $y = \sin t$.

Answer: $x = \cos t$, $y = \sin t$, $z = bt$

Hometask:

Find gradient of given scalar field u at the point M :

1. $u = \sqrt{x^2 + y^2} - z$, $M(3,4,1)$. Answer: $\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} - \vec{k}$.

2. $u = ze^{x^2+y^2+z^2}$, $M(2,-1,1)$. Answer: $e^6(4i - 2j + 3k)$.

3. $u = \ln(3 - x^2) + xy^2z$, $M(1,-1,1)$. Answer: $2\vec{i} - 2\vec{j} + \vec{k}$.

Find derivative of given scalar field u at the point M in the direction $\vec{l}$:

4. $u = x^3 + \sqrt{y^2 + z^2}$, $M(1,-3,4)$, $\vec{l} = \vec{j} - \vec{k}$. Answer: $-\frac{7}{5\sqrt{2}}$.

5. $u = x + \ln(y^2 + z^2)$, $M(2,1,1)$, $\vec{l} = -2\vec{i} + \vec{j} - \vec{k}$. Answer: $-\sqrt{\frac{2}{3}}$.

6. $u = \ln(\sqrt{x^2 + z^2} + z)$, $M(4,3,1)$, $\vec{l} = -\vec{i} + 2\vec{j} + 2\vec{k}$. Answer: $\frac{2}{15}$.

7. $u = x^3 - 3x^2y + 3xy^2 + 1$, $M(3,1)$, $\vec{l} = \overrightarrow{MN}$, $N(6,5)$. Answer: 0.

Find the direction of the maximal rate of increase of the field u at the point M and this maximal rate related to the unit length:

8. $u = x^2y + y^2z + z^2x$, $M(1,0,0)$. Answer: $\vec{j}$, 1.

9. $u = xy + yz + xz$, $M(1,1,1)$. Answer: $i + j + k$, $2\sqrt{3}$.

Find the vector lines of the vector field $\vec{a}$:

8. $\vec{a} = z\vec{j} - y\vec{k}$, Answer: $x^2 + z^2 = c^2$

9. $\vec{a}=(z-y)\vec{i}+(x-z)\vec{j}+(y-x)\vec{k}$, Answer: $\begin{cases} x+y+z=c_2 \\ x^2+y^2+z^2=c_1 \end{cases}$

3.7. Calculating of a flux.

Example 11. Compute the flux of the field $\vec{A}=z\vec{i}+2yz\vec{j}+(2-x)\vec{k}$ through the part of the sphere $x^2+y^2+z^2=8$ bounded by the cone $z=\sqrt{x^2+y^2}$.

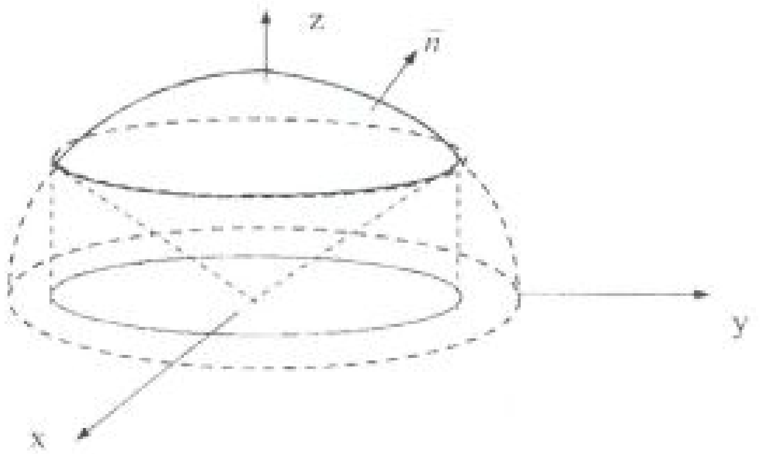

Fig. 22

Solution. Let us first draw the domain of integration in $\iint A_n d\sigma$ (see Fig. 22) and define the integrand :

$$\vec{n}=\frac{x\vec{i}+y\vec{j}+z\vec{k}}{2\sqrt{2}} \qquad A_n=\frac{2z(y^2+1)}{2\sqrt{2}} \qquad d\sigma=dxdy:\frac{z}{2\sqrt{2}}$$

Thus we have

$$\iint\limits_{(\sigma)} A_n d\sigma=\iint\limits_{(S_{xy})}\frac{2z(y^2+1)}{2\sqrt{2}}\cdot\frac{2\sqrt{2}}{z}dxdy,$$

where (S_{xy}) is the projection of (σ) on the xOy-plane. (S_{xy}) is bounded by the circle $x^2+y^2=4$. Passing to the polar coordinates we obtain

$$\iint\limits_{(\sigma)} A_n d\sigma=\int_0^{2\pi}d\varphi\int_0^2 2(r^2\sin^2\varphi+1)rdr=\int_0^{2\pi}\left[\frac{r^4}{2}\sin^2\varphi+r^2\right]_{r=0}^{2}d\varphi=$$

$$=\int_0^{2\pi}[4(1-\cos 2\varphi)+4]d\varphi=16\pi$$

Hometask:

Compute the flux of the fields:

1. $\bar{A} = y\bar{i} + z\bar{j} + x\bar{k}$ through the triangle with vertices (2,0,0),(0,2,0),(0,0,2).
Answer: 4.

2. $\bar{A} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ through the outer side of cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $z = 5$.
Answer: 60π.

3. $\bar{A} = x\bar{i} + y\bar{j} - 3z\bar{k}$ through the outer side of the surface $z = x^2 + y^2$ bounded by the plane $z = 4$.
Answer: 40π.

4. $\bar{A} = (x+z)\bar{i} + y\bar{j} + (z-x)\bar{k}$ through the part of sphere $x^2 + y^2 + z^2 = 1$ bounded by the plane $z = 0$ $(z \geq 0)$. Answer: 2π.

Using the Ostrogradsky's formula compute the flux of the field $\bar{A}$ through the surface (S):

Example 12. $\bar{A} = -x\bar{i} + 2y\bar{j} - y\bar{k}$, $S: x^2 + y^2 = 6 - z, z = 2$.

Solution. We have $divA = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(-y) = 1$. Then using the formula of Ostrogradsky we obtain:

$$\oiint\limits_{(S)} A_n d\sigma = \iiint\limits_{(V)} divA dv = V$$

To compute the volume let us draw the surfaces: $x^2 + y^2 = 6 - z, z = 2$ (see Fig. 23) and set up the limits of integration

$$V = \iint\limits_{S_{xy}} dxdy \int\limits_{2}^{6-x^2-y^2} dz = \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{2} r[6 - r^2 - 2]dr = 2\pi[2r^2 - \frac{r^4}{4}]\Big|_0^2 = 8\pi$$

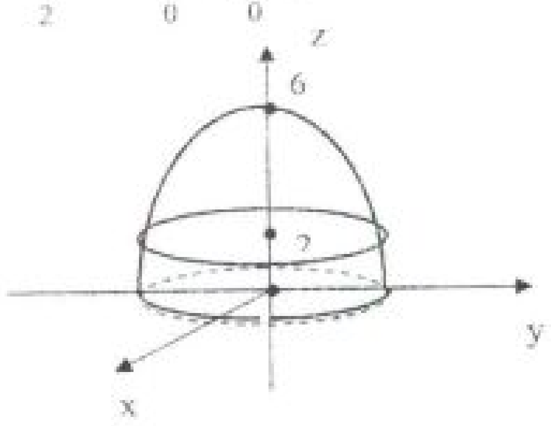

Fig. 23

Hometask:

5. $\bar{A} = (2x - \sin z)\bar{i} + (e^x + z)\bar{j} + (x + 2z)\bar{k}$
$S: x^2 + y^2 = 1, z = 4 - x - y, z = 0$ Answer: 8π

6. $\bar{A} = grad r^4$, $S: x^2 + y^2 + z^2 = R^2$. Answer: $16\pi R^2$

Example 13. Find the flux of the field $\vec{A} = yz\vec{i} + y\vec{j} - z\vec{k}$ through the part of the surface $S: x^2 + y^2 = z^2$, where $0 \le z \le 1$.

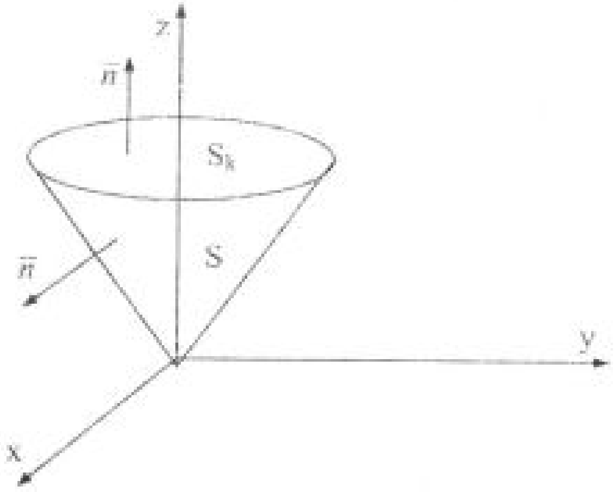

Fig. 24

Solution. Since $divA = 0$, from the Ostrogradsky's formula follows

$$\iint_{(S)} A_n d\sigma + \iint_{(S_k)} A_n d\sigma = 0 ,$$

where $S_k : x^2 + y^2 \le 1, z = 1$ (see Fig. 24), $\vec{n} = \vec{k}$, $A_k = -z$.

Thus $\iint_{(S)} A_n d\sigma = -\int_0^{2\pi} d\varphi \int_0^1 (-z)\Big|_{z=1} r dr = -2\pi \frac{1}{2} r^2 \Big|_0^1 (-1) = \pi$.

Answer: π.

3.8. Calculating line integrals and circulation.

Compute the curvilinear integrals $\int_L \vec{A} d\vec{r}$, where:

Example 14. $\vec{A} = x^3\vec{i} - y^3\vec{j}$, $L: \vec{r} = 2\cos t\vec{i} + 2\sin t\vec{j}$, $t \in (0, \frac{\pi}{2})$.

Solution. At first let us consider the integrand:

$\vec{A} d\vec{r} = (8\cos^3 t\vec{i} - 8\sin^2 t\vec{j}) d(2\cos t\vec{i} + 2\sin t\vec{j}) = 16\cos t \sin t(-\cos^2 t - \sin^2 t)dt$;

then

$$\int_L \vec{A} d\vec{r} = -\int_0^{\frac{\pi}{2}} 8\cos 2t dt = 4\cos 2t \Big|_0^{\frac{\pi}{2}} = -8.$$ Answer: -8.

Example 15. $\vec{A} = (x + 2z)\vec{i} + (x + 3y)\vec{j} + (5z + y)\vec{k}$, $L: \Delta ABC$, $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$.

Solution. Let us divide the triangle into three parts: $\Delta ABC = AB + BC + CD$, so

$\int\limits_{\Delta ABC} \vec{A}\,d\vec{r} = \int\limits_A^B \vec{A}\,d\vec{r} + \int\limits_B^C \vec{A}\,d\vec{r} + \int\limits_C^A \vec{A}\,d\vec{r}$, and calculate three integrals separately.

For instance, the second integral (BC: $z=1-y,\ x=0$):

$$\int\limits_B^C \vec{A}\,d\vec{r} = \int\limits_1^0 [(0+2z)0 + (0+3y)dy + (5(1-y)+y)(-dy)] = \int\limits_1^0 (3y-5+4y)dy =$$

$$= \int\limits_0^1 (5-7y)dy = 5 - \frac{7}{2} = \frac{3}{2}$$ Answer: 2.

Hometask:

1. $\vec{A} = \vec{r}$, $L: x = 3\cos t$, $y = 3\sin t$, $z = t$, $t \in [0, 2\pi]$. Answer: $2\pi^2$.
2. $\vec{A} = 2xy\vec{i} + y^2\vec{j} - x^2\vec{k}$, $L: x^2 + y^2 - 2z^2 = 18$, $y = x$ from the point $(3,3,0)$ to the point $(3\sqrt{2}, 3\sqrt{2}, 3)$. Answer: $9(6\sqrt{2} - 7)$.

Using the Stokes' formula compute the circulation of the field A over the closed loop L:

Example 16. $\vec{A} = z\vec{i} + x\vec{j} + y\vec{k}$, $L: x^2 + y^2 + z^2 = 9$, $x + y + z = 3$.

Solution. To use the Stokes' formula $\oint\limits_L \vec{A}\,d\vec{r} = \iint\limits_{(S)} (rot\,\vec{A})_n\,d\sigma$ ($S: x + y + z = 3$) we ought to find $rot\,\vec{A}$ and $\vec{n}$. We have

$$rot\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} \quad \text{and} \quad \vec{n} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}.$$

Thus

$\oint\limits_L \vec{A}\,d\vec{r} = \iint\limits_S \sqrt{3}\,ds = \sqrt{3}S$, where $S = \pi R^2$, $R = \sqrt{6}$. Answer: $6\sqrt{3}\pi$.

Hometask:

3. $\vec{A} = (x + 2z)\vec{i} + (x + 3y)\vec{j} + (5z + y)\vec{k}$, $L: \Delta ABC$, $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. Answer: 2.
4. $\vec{A} = (y + \dfrac{\cos xy}{x})\vec{i} + (\dfrac{\cos xy}{y} - 2x)\vec{j} + 3e^z\vec{k}$, $L: x^2 + y^2 = z^2$, $z = 3$.

Answer: -27π.

Example 17. Find the flux of the rotation of the field $\vec{A} = y\vec{i} + z\vec{j} + x\vec{k}$ through the surface $z = 2(1 - x^2 - y^2)$ bounded by the plain $z = 0$.

Solution. Inversing the Stokes' formula we have:

$\iint\limits_{(s)} (rot\vec{A})_n ds = \oint\limits_L ydx + zdy + xdz$, where $L: z=0, x^2+y^2=1$.

So we obtain:

$\oint\limits_L ydx = \int\limits_0^{2\pi} \sin t d\cos t = -\int\limits_0^{2\pi} \sin^2 t dt = -\pi$. Answer: $-\pi$.

Hometask:

Calculate the circulation of the field $\vec{A}$ over the closed contour L immediately and by using the Stokes' formula:

5. $\vec{A} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, $L: x^2+y^2+z^2=25$, $y=4$. Answer: 0 .
6. $\vec{A} = z\vec{i} - x\vec{j} + y\vec{k}$, $L: x^2+y^2=10+z$, $z=-1$. Answer: -9π .
7. $\vec{A} = y\vec{i} + x\vec{j} + z\vec{k}$, $L: x^2+y^2=z^2$, $z=-2$. Answer: 8π .
8. $\vec{A} = y\vec{i} - 2z\vec{j} + x\vec{k}$, $L: 2x^2-y^2+z^2=4$, $y=x$. Answer: 12π .

3. 9. Differential operations and investigation of vector fields.

Hamilton operator: $\overline{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

Differential operations of the first order:

$$divA = \overline{\nabla}\cdot\overline{A},\ gradU = \overline{\nabla}\cdot U,\ rotA = \nabla\times\overline{A} .$$

Example 18. Find $\overline{\nabla}\varphi(r)$, where $r = \sqrt{x^2+y^2+z^2}$.

Solution. $\overline{\nabla}\varphi(r) = \varphi'_r r'_x\vec{i} + \varphi'_r r'_y\vec{j} + \varphi'_r r'_z\vec{k}$; here $r'_x = \frac{x}{r}$, $r'_y = \frac{y}{r}$, $r'_z = \frac{z}{r}$, so

$\overline{\nabla}\varphi(r) = \varphi'_r\frac{\vec{r}}{r}$.

Hometask:

1. Find $\nabla\cdot\vec{r}\varphi(r)$. Answer: $3\varphi(r) + r\varphi'_r(r)$.

Differential operations the second order:

1. $div(gradU) = \overline{\nabla}\cdot\overline{\nabla}U = \overline{\nabla}^2 U = \Delta U$;

$$\Delta = \overline{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ;$$

2. $rot(gradU) = \overline{\nabla}\times\overline{\nabla}U \equiv 0$;
3. $grad(divA) = \overline{\nabla}(\overline{\nabla}A)$;
4. $div(rot\overline{A}) = \overline{\nabla}\overline{\nabla}\times\overline{A} \equiv 0$;

5. $rot(rot\overline{A}) = \nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \overline{A}) - \nabla^2 \overline{A}$.

Hometask:

2. $\nabla(\nabla \cdot (x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k})) = ?$ Answer: $6\vec{r}$.
3. $\overline{\nabla \times}(\overline{\nabla} \times (xy^2 \vec{i} + yz^2 \overline{j} + zx^2 \overline{k})) = ?$ Answer: 0
4. $\overline{\nabla}(\overline{\nabla} \times (y\vec{i} - xz\vec{j} + y\vec{k})) = ?$

Find the potential of the field $\vec{A}$:

Example 19. $\vec{A} = (yz - xy)\vec{i} + (xz - \frac{x^2}{2} + yz^2)\vec{j} + (xy + y^2 z)\vec{k}$.

Solution. The condition $rot\vec{A} = 0$ is true:

$\nabla \times \vec{A} = \vec{i}(x + 2yz - x - 2zy) - \vec{j}(y - y) + \vec{k}(z - x - z + x) = 0$.

By using the third property we have:

$$u(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \overline{A}\overline{dr} + c = \int_{(0,0,0)}^{(x,0,0)} (yz - xy)dx + \int_{(x,0,0)}^{(x,y,0)} (xz - \frac{x^2}{2} + yz^2)dy +$$

$$+ \int_{(x,y,0)}^{(x,y,z)} (xy + y^2 z)dz + c = -\frac{x^2}{2}\int_0^y dy + \int_0^z (xy + y^2 z)dz + c = xyz - \frac{x^2}{2}y + y^2 \frac{z^2}{2} + c$$

Hometask: Find the potential of the field $\vec{A}$:

5. $A = (3x^2 y - y^3)\vec{i} + (x^3 - 3xy^2)\vec{j}$ Answer: $x^3 y - xy^3 + c$
6. $\overline{A} = 2xy\vec{i} + (x^2 - 2yz)\vec{j} - y^2\overline{k}$ Answer: $x^2 y - y^2 z + c$

Test the function u (is it harmonic?)

7. $u = \sqrt{x^2 + y^2 + z^2} - x$ Answer: No.
8. $u = ax^3 + 3bx^2 y + 3cxy^2 + dy^3$ Answer: Yes, if $a + c = 0$, $b + d = 0$.

3.10. Expressing divergence in Cartesian coordinates.

Let a Cartesian coordinate system $Oxyz$ be given in space. Then a vector field $\overline{A}$ can be represented in the form
$\overline{A} = A_x(x,y,z)\vec{i} + A_y(x,y,z)\vec{j} + A_z(x,y,z)\vec{k}$. In this case we can deduce a simple formula for computing the divergence $div\overline{A}$. To do this we take into account the fact that the particular form of an infinitesimal domain $(\Delta\Omega)$ which occurs in the definition of a divergence is inessential. Therefore we can take a small rectangular parallelepiped with faces parallel to the coordinate planes
(see Fig. 25).

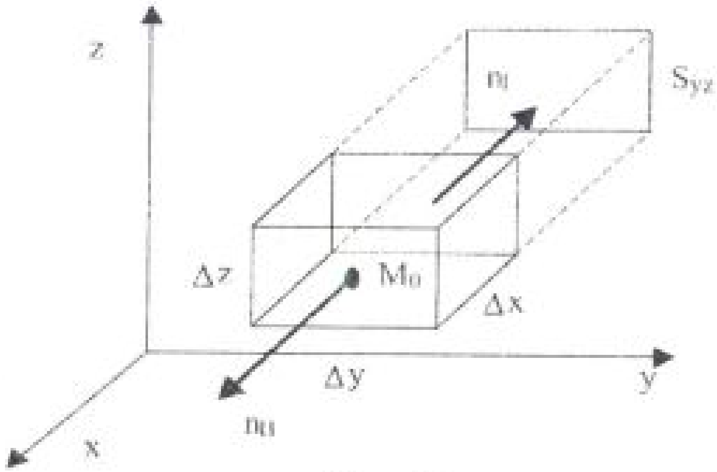

Fig. 25

Then $\oiint\limits_{(\Delta\sigma)} A_n d\sigma = \iint\limits_{I}... + \iint\limits_{II}... + \iint\limits_{III}... + \iint\limits_{IV}... + \iint\limits_{V}... + \iint\limits_{VI}...$ the flux can be represented as a sum of six summands corresponding to the six faces of the parallelepiped. We now consider the sum of two summands corresponding to the faces (designated by I and II in Fig. 25), which are parallel to the yOz-plane and whose unit outer normals are denoted as n_I and n_{II}. We have $(A_n)_I = -(A_x)_I$, and on the basis of Taylor's formula we can write $(A_n)_{II} = -(A_x)_{II} = (A_x)_I + (\partial_x A_x)_I + ...$, where the expression $\partial_x \Delta_x = \frac{\partial A_x}{\partial x}\Delta x$ is the partial differential with respect to x which appears here because the points belonging to the faces I and II differ by Δx in the values of their abscissas x. The integration over these faces reduce to the integration over their projections onto the yOz plane (i.e. over S_{yz}), so we have

$$\iint\limits_{I}... + \iint\limits_{II}... = \iint\limits_{S_{yz}}\left[-(A_x)_I + (A_x)_I + \left(\frac{\partial \Delta_x}{\partial x}\right)_I \Delta x + ...\right]dydz =$$

$$= \left(\iint\limits_{S_{yz}}\left(\frac{\partial A_x}{\partial x}\right)_I dydz\right)\Delta x + ... = \left(\frac{\partial A_x}{\partial x}\right)_M \Delta x\Delta y\Delta z + ...$$

The dots here designate the terms of higher order of smallness relatively to the terms which are written in full. The subscripts I, II and M mean that the corresponding terms are taken for the points belonging to the faces I, II or for the point M, respectively.

Performing similar calculations for the other two pairs of faces and summing up all the expressions we arrive at the formula for the flux through the whole boundary surface of the parallelepiped:

$$\oiint\limits_{(\Delta\sigma)} A_n d\sigma = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)_M \Delta x\Delta y\Delta z + ...$$

In this case we have $\Delta V = \Delta x \Delta y \Delta z$ and therefore:

$$\frac{1}{\Delta V} \oiint_{(\Delta\sigma)} A_n d\sigma = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)_M + \ldots$$

Passing to the limits we finally obtain the expression:

$$divA = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

We have not written the subscript M here because the formula holds for any point of the field.

3.11. Expression of circulation taken along an infinitesimal closed loop.

To obtain the expression we assume that the vector field $\vec{A}$ is represented in the form

$$\vec{A} = A_x(x,y,z)\vec{i} + A_y(x,y,z)\vec{j} + A_z(x,y,z)\vec{k},$$

i. e. is being resolved into components along the unit vectors of the Cartesian axes. Let the loop (ΔL) be placed near a point M_0 of space. Now we compute the integral of the first summand, which occurs into the right-hand side of the formula

$$\oint_{(\Delta L)} \vec{A}\, d\vec{r} = \oint_{(\Delta L)} A_x dx + A_y dy + A_z dz.$$

$$\oint_{(\Delta L)} A_x(x,y,z)dx = \oint_{(\Delta L)} [(A_x)_0 + (\frac{\partial A_x}{\partial x})_0 (x - x_0) + (\frac{\partial A_x}{\partial y})_0 (y - y_0) +$$

$$+ (\frac{\partial A_x}{\partial z})_0 (z - z_0) + \ldots]dx.$$

Here we have applied Taylor's formula. The subscript "zero" indicates that the corresponding quantities are taken at the point M_0, and the dots designate the terms of higher order of smallness.

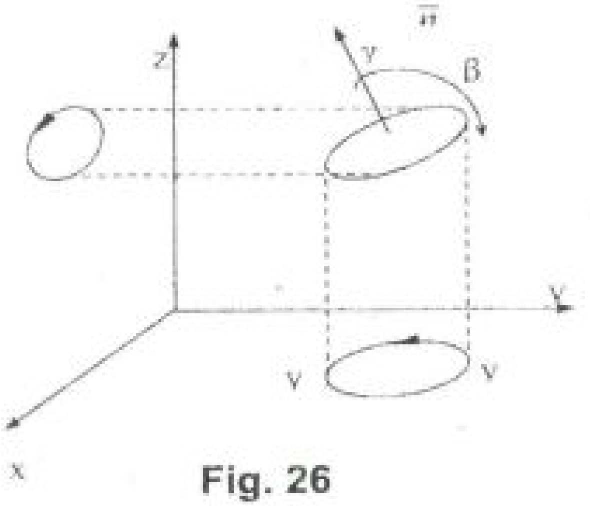

Fig. 26

Since

$$\oint\limits_{(\Delta L)} [(A_x)_0 + (\frac{\partial A_x}{\partial x})_0 (x - x_0) - (\frac{\partial A_x}{\partial y})_0 y_0 - (\frac{\partial A_x}{\partial z})_0 z_0]dx =$$

$$= \int\limits_{(\Delta L)} [c_1 + c_2 x]dx = 0,$$

$$\oint\limits_{(\Delta L)} y dx = -\Delta S \cos\gamma \quad \text{and} \quad \oint\limits_{(\Delta L)} z dx = \Delta S \cos\beta,$$

we obtain

$$\oint\limits_{(\Delta L)} A_x dx = [(\frac{\partial A_x}{\partial z})_0 \cos\beta - (\frac{\partial A_x}{\partial y})_0 \cos\gamma]\Delta S + ...$$

Evaluating the other two integrals in a similar manner and summing up the results we deduce:

$$\oint\limits_{(\Delta L)} \bar{A}\, d\bar{r} = (\overline{rot A})_0 \cdot \bar{n}\Delta S + ...$$

where ΔS is the area of the surface bounded by the loop, $\bar{n}$ is the unit outer normal.

3.12. Exercises for homework.

A) Find derivative of given scalar field u at the point M in the direction $\bar{n}$, where $\bar{n}$ is a normal to the surface S (the angle between $\bar{n}$ and positive direction of z-axis is acute.)

1. $u = 4\ln(3 + x^2) - 8xyz,$ $\quad S: x^2 - 2y^2 + 2z^2 = 1, M(1,1,1)$

2. $u = x\sqrt{y} + y\sqrt{z},$ $\quad S: 2x^2 - y^2 + 4z = 0, M(2,4,4)$

3. $u = -2\ln(5 + x^2) - 4xyz,$ $\quad S: x^2 + 2y^2 - 2z^2 = 1, M(1,1,1)$

4. $u = \frac{1}{4}x^2 y - \sqrt{x^2 + 5z^2}$ $\quad S: z^2 = x^2 + 4y^2 - 4, M(-2, \frac{1}{2}, 1)$

5. $u = xz^2 - x^3 y,$ $\quad S: 2x^2 - y^2 - 3z + 12 = 0, M(2,2,4)$

6. $u = x\sqrt{y} - yz^2,$ $\quad S: x^2 + y^2 = 4z, M(2,1,-1)$

7. $u = -7\ln(\frac{1}{13} + x^2) - 4xyz,$ $S: 7x^2 - 4y^2 + 4z^2 = 7, M(1,1,1)$

8. $u = arctg\,\frac{y}{x} + xz,$ $S: x^2 + y^2 - 2z = 10, M(2,2,-1)$

9. $u = \ln(1 + x^2) - xy\sqrt{z},$ $S: 4x^2 - y^2 + z^2 = 16, M(1,-2,4)$

10. $u = \sqrt{x^2 + y^2} - z,$ $S: x^2 + y^2 = 24z, M(3,4,1)$

11. $u = x\sqrt{y} - (z + y)\sqrt{x},$ $S: x^2 - y^2 + z^2 = 4, M(1,1,-2)$

12. $u = \sqrt{xy} - \sqrt{4 - z^2},$ $S: z = x^2 - y^2, M(1,1,0)$

13. $u = (x^2 + y^2 + z^2)^{3/2},$ $S: 2x^2 - y^2 + z^2 - 1 = 0, M(0,-3,4)$

14. $u = \ln(1 + x^2 + y^2) - \sqrt{x^2 + z^2},$ $S: x^2 - 6x + 9y^2 + z^2 = 4z + 4, M(3,0,-4)$

B) Find derivative of given scalar field u at the point M in the direction $\vec{l}$:

1. $u = (x^2 + y^2 + z^2)^{3/2},$ $\vec{l} = \vec{i} - \vec{j} + \vec{k}, M(1,1,1)$

2. $u = xy^2 - \sqrt{xy + z^2},$ $\vec{l} = 2\vec{i} - 2\vec{k}, M(1,5,-2)$

3. $u = x + n(z^2 + y^2),$ $\vec{l} = -2\vec{i} + \vec{j} - \vec{k}, M(2,1,1)$

4. $u = x(\ln y - arctgz),$ $\vec{l} = 8\vec{i} + 4\vec{j} + 8\vec{k}, M(-2,1,-1)$

5. $u = \ln(3 - x^2) + xy^2z,$ $\vec{l} = -\vec{i} + 2\vec{j} - 2\vec{k}, M(1,3,2)$

6. $u = \sin(x + 2y) + \sqrt{xyz},$ $\vec{l} = 4\vec{i} + 3\vec{j}, M(\frac{\pi}{2}, \frac{3\pi}{2}, 3)$

7. $u = x^2y^2z - \ln(z - 1),$ $\vec{l} = 5\vec{i} - 6\vec{j} + 2\sqrt{5}\vec{k}, M(1,1,2)$

8. $u = x^3 + \sqrt{y^2 + z^2},$ $\vec{l} = \vec{j} - \vec{k}, M(1,-3,4)$

9. $u = \frac{\sqrt{x}}{y} - \frac{yz}{x+\sqrt{y}}$, $\quad \bar{l} = 2\bar{i} + \bar{k},\ M(4,1,-2)$

10. $u = \sqrt{xy} + \sqrt{9 - z^2}$, $\quad \bar{l} = -2\bar{i} + 2\bar{j} - \bar{k},\ M(1,1,0)$

11. $u = 2\sqrt{x+y} + yarctgz$, $\quad \bar{l} = 4\bar{i} - 3\bar{k},\ M(3,-2,1)$

12. $u = z^2 + 2arctg(x-y)$, $\quad \bar{l} = \bar{i} + 2\bar{j} - 2\bar{k},\ M(1,2,-1)$

13. $u = \ln(x^2 + y^2) + xyz$, $\quad \bar{l} = \bar{i} - \bar{j} + 5\bar{k},\ M(1,-1,2)$

14. $u = xy - \frac{x}{z}$, $\quad \bar{l} = 5\bar{i} + \bar{j} - \bar{k},\ M(-4,3,-1)$

15. $u = \ln(x + \sqrt{y^2 + z^2})$, $\quad \bar{l} = -2\bar{i} - \bar{j} + \bar{k},\ M(1,-3,4)$

16. $u = x^2 - arctg(y+z)$, $\quad \bar{l} = 3\bar{j} - 4\bar{k},\ M(2,1,1)$

C) Find the vector lines of the vector field $\bar{a}$:

1. $\bar{a} = 4y\bar{i} - 9x\bar{j}$ 2. $\bar{a} = 2x\bar{i} + 4y\bar{j}$ 3. $\bar{a} = x\bar{i} + 4y\bar{j}$ 4. $\bar{a} = 4z\bar{i} - 9x\bar{j}$

5. $\bar{a} = 4y\bar{j} + 8z\bar{k}$ 6. $\bar{a} = 2x\bar{i} + 8z\bar{k}$ 7. $\bar{a} = 4z\bar{j} - 9z\bar{k}$ 8. $\bar{a} = 5x\bar{i} + 10y\bar{j}$

9. $\bar{a} = y\bar{j} + 4z\bar{k}$ 10. $\bar{a} = 9y\bar{i} - 4x\bar{j}$ 11. $\bar{a} = 6x\bar{i} + 12z\bar{k}$ 12. $\bar{a} = 4x\bar{i} + y\bar{j}$

13. $\bar{a} = x\bar{i} + z\bar{k}$ 14. $\bar{a} = 7y\bar{j} + 14z\bar{k}$ 15. $\bar{a} = 4x\bar{i} + y\bar{j}$ 16. $\bar{a} = 9z\bar{j} + 4y\bar{k}$

17. $\bar{a} = 2y\bar{i} + 3x\bar{j}$ 18. $\bar{a} = x\bar{i} + 3y\bar{j}$ 19. $\bar{a} = 3x\bar{i} + 6z\bar{k}$ 20. $\bar{a} = 2z\bar{i} + 3x\bar{k}$

21. $\bar{a} = y\bar{j} + 3z\bar{k}$ 22. $\bar{a} = x\bar{i} + 3z\bar{k}$ 23. $\bar{a} = 2z\bar{j} + 3y\bar{k}$ 24. $\bar{a} = 2x\bar{i} + 6y\bar{j}$

25. $\bar{a} = x\bar{i} + y\bar{j}$ 26. $\bar{a} = 5y\bar{i} + 7x\bar{j}$ 27. $\bar{a} = 2y\bar{j} + 6z\bar{k}$ 28. $\bar{a} = 9z\bar{i} - 4x\bar{k}$

29. $\bar{a} = 5z\bar{i} + 7x\bar{k}$ 30. $\bar{a} = 2x\bar{i} + 6z\bar{k}$ 31. $\bar{a} = 5z\bar{j} + 5y\bar{k}$

D) Find the flux of the vector field $\vec{a}$ through the part of the plane P which is situated in the first octant (the angle between its normal and z-axis is acute.)

1. $\vec{a} = 7x\vec{i} + (5\pi y + 2)\vec{j} + 4\pi z\vec{k}$, P: $x + y/2 + 4z = 1$

2. $\vec{a} = 9\pi x\vec{i} + y\vec{j} - 3z\vec{k}$, P: $x/3 + y + z = 1$

3. $\vec{a} = 2\pi x\vec{i} + (7y + 2)\vec{j} + 7\pi z\vec{k}$, P: $x + y/2 + z/3 = 1$

4. $\vec{a} = (2x + 1)\vec{i} - y\vec{j} + 3\pi z\vec{k}$, P: $x/3 + y + 2z = 1$

5. $\vec{a} = 7x\vec{i} + 9\pi y\vec{j} + \vec{k}$, P: $x + y/3 + z = 1$

6. $\vec{a} = \vec{i} + 5y\vec{j} + 11\pi z\vec{k}$, P: $x + y + z/3 = 1$

7. $\vec{a} = x\vec{i} + (\pi z - 1)\vec{k}$, P: $2x + y/2 + z/3 = 1$

8. $\vec{a} = 2\vec{i} - y\vec{j} + \frac{3\pi}{2} z\vec{k}$, P: $x/3 + y + z/4 = 1$

9. $\vec{a} = 5\pi x\vec{i} + (9y + 1)\vec{j} + 4\pi z\vec{k}$, P: $x/2 + y/3 + z/2 = 1$

10. $\vec{a} = 9\pi x\vec{i} + (5y + 1)\vec{j} + 2\pi z\vec{k}$, P: $3x + y + z/9 = 1$

11. $\vec{a} = 7\pi x\vec{i} + 2\pi y\vec{j} + (7z + 2)\vec{k}$, P: $x + y + z/2 = 1$

12. $\vec{a} = \pi y\vec{i} + (4 - 2z)\vec{k}$, P: $2x + y/3 + z/4 = 1$

13. $\vec{a} = (3\pi - 1)x\vec{i} + (9\pi y + 1)\vec{j} + 6\pi z\vec{k}$, P: $x/2 + y/3 + z/9 = 1$

14. $\vec{a} = \pi x\vec{i} + \frac{\pi}{2} y\vec{j} + (4 - 2z)\vec{k}$, P: $x + y/3 + z/4 = 1$

15. $\vec{a} = (5y + 3)\vec{i} + 11\pi z\vec{k}$, P: $x + y/3 + 4z = 1$

16. $\vec{a} = 9\pi y\vec{j} + (7z + 1)\vec{k}$, P: $x + y + z = 1$

17. $\vec{a} = \pi y\vec{j} + (1 - 2z)\vec{k}$, P: $x/4 + y/3 + z = 1$

18. $\bar{a} = (27\pi - 1)x\bar{i} + (34\pi + 3)y\bar{j} + 20\pi z\bar{k}$, P: $3x + y/9 + z = 1$

19. $\bar{a} = \pi x\bar{i} + 2\bar{j} + 2z\bar{k}$, P: $x/2 + y/3 + z = 1$

20. $\bar{a} = 4\pi x\bar{i} + 7\pi y\bar{j} + (2z + 1)\bar{k}$, P: $2x + y/3 + 2z = 1$

21. $\bar{a} = 3\pi x\bar{i} + 6\pi y\bar{j} + 10\bar{k}$, P: $2x + y + z/3 = 1$

22. $\bar{a} = \pi x\bar{i} - 2y\bar{j} + \bar{k}$, P: $2x + y/6 + z = 1$

23. $\bar{a} = (21\pi - 1)x\bar{i} + 62\pi y\bar{j} + (1 - 2\pi z)\bar{k}$, P: $8x + y/2 + z/3 = 1$

24. $\bar{a} = \pi x\bar{i} + 2\pi y\bar{j} + 2\bar{k}$, P: $x/2 + y/4 + z/3 = 1$

25. $\bar{a} = 9\pi x\bar{i} + 2\pi y\bar{j} + 8\bar{k}$, P: $2x + 8y + z/3 = 1$

26. $\bar{a} = 3\pi x\bar{i} + 6\pi y\bar{j} + 10\bar{k}$, P: $2x + y + z/3 = 1$

27. $\bar{a} = \pi x\bar{i} - 2y\bar{j} + \bar{k}$, P: $2x + y/6 + z = 1$

28. $\bar{a} = (21\pi - 1)x\bar{i} + 62\pi y\bar{j} + (1 - 2\pi z)\bar{k}$, P: $8x + y/2 + z/3 = 1$

29. $\bar{a} = \pi x\bar{i} + 2\pi y\bar{j} + 2\bar{k}$, P: $x/2 + y/4 + z/3 = 1$

30. $\bar{a} = 9\pi x\bar{i} + 2\pi y\bar{j} + 8\bar{k}$, P: $2x + 8y + z/3 = 1$

31. $\bar{a} = 7\pi x\bar{i} + (4y + 1)\bar{j} + 2\pi z\bar{k}$, P: $x/3 + 2y + z = 1$

32. $\bar{a} = 6\pi x\bar{i} + 3\pi y\bar{j} + 10\bar{k}$, P: $2x + y/2 + z/3 = 1$

33. $\bar{a} = (\pi - 1)x\bar{i} + 2\pi y\bar{j} + (1 - \pi z)\bar{k}$, P: $x/4 + y/2 + z/3 = 1$

34. $\bar{a} = \frac{\pi}{2}x\bar{i} + \pi y\bar{j} + (4 - 2z)\bar{k}$, P: $x + y/3 + z/4 = 1$

35. $\bar{a} = 7\pi x\bar{i} + 4\pi y\bar{j} + 2(z + 1)\bar{k}$, P: $x/3 + y/4 + z = 1$

36. $\bar{a} = 5\pi x\bar{i} + (1 - 2y)\bar{j} + 4\pi z\bar{k}$, P: $x/2 + 4y + z/3 = 1$

E) Find the flux of the vector field a through the closed surface S (choosing outer normal).

1. $\vec{a} = (x+z)\vec{i} + (z+y)\vec{k}$, $S: \begin{cases} x^2+y^2=9 \\ z=x, z=0 (z \ge 0) \end{cases}$

2. $\vec{a} = 2x\vec{i} + z\vec{k}$, $S: \begin{cases} z = 3x^2+2y^2+1 \\ x^2+y^2=4, z=0 \end{cases}$

3. $\vec{a} = 2x\vec{i} + 2y\vec{j} + z\vec{k}$, $S: \begin{cases} y=x^2, y=4x^2, y=1 (x \ge 0) \\ z=y, z=0. \end{cases}$

4. $\vec{a} = 3x\vec{i} - z\vec{k}$, $S: \begin{cases} z=6-x^2-y^2 \\ z^2=x^2+y^2, (z \ge 0). \end{cases}$

5. $\vec{a} = (z+y)\vec{i} + y\vec{j} - x\vec{k}$, $S: \begin{cases} x^2+y^2=2y \\ y=2. \end{cases}$

6. $\vec{a} = x\vec{i} - (x+2y)\vec{j} + y\vec{k}$, $S: \begin{cases} x^2+y^2=1, z=0, \\ x+2y+3z=6. \end{cases}$

7. $\vec{a} = 2(z-y)\vec{j} + (x-z)\vec{k}$, $S: \begin{cases} z=x^2+3y^2+1, \\ x^2+y^2=1. \end{cases}$

8. $\vec{a} = x\vec{i} + z\vec{j} - y\vec{k}$, $S: \begin{cases} z=4-2(x^2+y^2), \\ z=2(x^2+y^2). \end{cases}$

9. $\vec{a} = z\vec{i} - 4y\vec{j} + 2x\vec{k}$, $S: \begin{cases} z=x^2+y^2, \\ z=1. \end{cases}$

10. $\vec{a} = 4x\vec{i} - 2y\vec{j} - z\vec{k}$, $S: \begin{cases} 3x+2y=12, 3x+y=6, y=0, \\ x+y+z=6, z=0. \end{cases}$

11. $\vec{a}=8x\vec{i}-2y\vec{j}+x\vec{k}$, $S:\begin{cases} x+y=1,\ x=0,\ y=0, \\ z=x^2+y^2,\ z=0. \end{cases}$

12. $\vec{a}=z\vec{i}+x\vec{j}-z\vec{k}$, $S:\begin{cases} 4z=x^2+y^2, \\ z=4. \end{cases}$

13. $\vec{a}=6x\vec{i}-2y\vec{j}-z\vec{k}$, $S:\begin{cases} z=3-2(x^2+y^2), \\ z^2=x^2+y^2\ (z\geq 0). \end{cases}$

14. $\vec{a}=(z+y)\vec{i}+(x-z)\vec{j}+z\vec{k}$, $S:\begin{cases} x^2+4y^2=4, \\ 3x+4y+z=12,\ z=1. \end{cases}$

15. $\vec{a}=(y+2z)\vec{i}-y\vec{j}+3x\vec{k}$, $S:\begin{cases} 3z=27-2(x^2+y^2), \\ z^2=x^2+y^2,\ (z\geq 0). \end{cases}$

16. $\vec{a}=(y+6x)\vec{i}+5(x+z)\vec{j}+4y\vec{k}$, $S:\begin{cases} y=x,\ y=2x,\ y=2, \\ z=x^2+y^2,\ z=0. \end{cases}$

F) Find the work performed by the force $\vec{F}$ while moving along L from point M to point N.

1. $\vec{F}=(x^2-2y)\vec{i}+(y^2-2x)\vec{j}$, L: Segment MN, $M(-4,0)$, $N(0,2)$.

2. $\vec{F}=(x^2+2y)\vec{i}+(y^2+2x)\vec{j}$, L: Segment MN, $M(-4,0)$, $N(0,2)$.

3. $\vec{F}=(x^2+2y)\vec{i}+(y^2+2x)\vec{j}$, L: $2-\dfrac{x^2}{8}=y$, $M(-4,0)$, $N(0,2)$.

4. $\vec{F}=(x+y)\vec{i}+2x\vec{j}$, L: $x^2+y^2=4,\ (y\geq 0)$, $M(2,0)$, $N(-2,0)$.

5. $\vec{F}=x^3\vec{i}-2y^3\vec{j}$, L: $x^2+y^2=4,\ (x\geq 0,\ y\geq 0)$, $M(2,0)$, $N(0,2)$.

6. $\vec{F}=(x+y)\vec{i}+2x\vec{j}$, L: $y=x^2$, $M(-1,1)$, $N(1,1)$.

7. $\vec{F}=x^2y\vec{i}+y\vec{j}$, L: Segment MN, $M(-1,0)$, $N(0,1)$.

8. $\vec{F}=(2xy-y)\vec{i}+(x^2+x)\vec{j}$, L: $x^2+y^2=9,\ (y\geq 0)$, $M(3,0)$, $N(-3,0)$.

9. $\vec{F}=(x+y)\vec{i}+(x-y)\vec{j}$, L: $x^2+\frac{y^2}{9}=1$, $(x\geq 0,\ y\geq 0)$, $M(1,0)$, $N(0,3)$.

10. $\vec{F}=y\vec{i}-x\vec{j}$, L: $x^2+y^2=1$, $(y\geq 0)$, $M(1,0)$, $N(-1,0)$.

11. $\vec{F}=(x^2+y^2)\vec{i}+(x^2-y^2)\vec{j}$, L: $y=\begin{cases} x,\ 0\leq x\leq 1 \\ 2-x,\ 1\leq x\leq 2\end{cases}$, $M(2,0)$, $N(0,0)$.

12. $\vec{F}=xy\vec{i}+2y\vec{j}$, L: $x^2+y^2=1$, $(x\geq 0, y\geq 0)$, $M(1,0)$, $N(0,1)$.

13. $\vec{F}=y\vec{i}-x\vec{j}$, L: $2x^2+y^2=1$, $(y\geq 0)$, $M(\frac{1}{\sqrt{2}},0)$, $N(-\frac{1}{\sqrt{2}},0)$.

14. $\vec{F}=(x^2+y^2)(\vec{i}+2\vec{j})$, L: $x^2+y^2=R^2$, $(y\geq 0)$, $M(R,0)$, $N(-R,0)$.

15. $\vec{F}=(x+y\sqrt{x^2+y^2})\vec{i}+(y-x\sqrt{x^2+y^2})\vec{j}$, L: $x^2+y^2=1$, $(y\geq 0)$, $M(1,0)$, $N(-1,0)$.

16. $\vec{F}=x^2y\vec{i}+xy^2\vec{j}$, L: $x^2+y^2=4$, $(x\geq 0,\ y\geq 0)$, $M(2,0)$, $N(0,2)$.

17. $\vec{F}=(x+y\sqrt{x^2+y^2})\vec{i}+(y-x\sqrt{x^2+y^2})\vec{j}$, L: $x^2+y^2=16$, $(x\geq 0,\ y\geq 0)$, $M(4,0)$, $N(0,4)$.

18. $\vec{F}=y^2\vec{i}+x^2\vec{j}$, L: $x^2+y^2=9$, $(x\geq 0,\ y\geq 0)$, $M(3,0)$, $N(0,3)$.

19. $\vec{F}=(x+y)^2\vec{i}+(y^2+x^2)\vec{j}$, L: Segment MN, $M(1,0)$, $N(0,1)$.

20. $\vec{F}=(x^2+y^2)\vec{i}+y^2\vec{j}$, L: Segment MN, $M(2,0)$, $N(0,2)$.

21. $\vec{F}=x^2\vec{j}$, L: $x^2+y^2=9$, $(x\geq 0,\ y\geq 0)$, $M(3,0)$, $N(0,3)$.

22. $\vec{F}=(y^2-y)\vec{i}+(2xy+x)\vec{j}$, L: $x^2+y^2=9$, $(x\geq 0,\ y\geq 0)$, $M(3,0)$, $N(-3,0)$.

23. $\vec{F}=xy\vec{i}$, L: $y=\sin x$, $M(\pi,0)$, $N(0,0)$.

24. $\vec{F}=(xy-y^2)\vec{i}+x\vec{j}$, L: $y=x^2$, $M(0,0)$, $N(1,1)$.

25. $\vec{F} = x\vec{i} + y\vec{j}$, $\quad$ L: Segment MN, $\quad M(1,0)$, $N(0,3)$.

26. $\vec{F} = (xy - x)\vec{i} + \dfrac{x^2}{2}\vec{j}$, $\quad$ L: $y = 2\sqrt{x}$, $\quad M(0,0)$, $N(1,2)$.

27. $\vec{F} = -x\vec{i} + y\vec{j}$, $\quad$ L: $x^2 + \dfrac{y^2}{9} = 1$, $(x \geq 0,\ y \geq 0)$, $M(1,0)$, $N(0,3)$.

28. $\vec{F} = -y\vec{i} + x\vec{j}$, $\quad$ L: $y = x^3$, $\quad M(0,0)$, $N(2,8)$.

29. $\vec{F} = (x^2 - y^2)\vec{i} + (x^2 + y^2)\vec{j}$, $\quad$ L: $\dfrac{x^2}{9} + \dfrac{y^2}{2} = 1$, $(y \geq 0)$, $\quad M(3,0)$, $N(-3,0)$.

30. $\vec{F} = (x - y)\vec{i} + \vec{j}$, $\quad$ L: $x^2 + y^2 = 4$, $(y \geq 0)$, $M(2,0)$, $N(0,2)$.

<u>Task 6.</u>

Find the modulus of the circulation of the field $\vec{a}$ along the loop Γ.

1. $\vec{a} = (x^2 - y)\vec{i} + x\vec{j} + \vec{k}$, $\ \Gamma: \begin{cases} x^2 + y^2 = 1, \\ z = 1. \end{cases}$

2. $\vec{a} = xz\vec{i} - \vec{j} + y\vec{k}$, $\ \Gamma: \begin{cases} z = 5(x^2 + y^2) - 1, \\ z = 4. \end{cases}$

3. $\vec{a} = yz\vec{i} + 2xz\vec{j} + xy\vec{k}$, $\ \Gamma: \begin{cases} x^2 + y^2 + z^2 = 25, \\ x^2 + y^2 = 9,\ (z \geq 0). \end{cases}$

4. $\vec{a} = x\vec{i} + yz\vec{j} - x\vec{k}$, $\ \Gamma: \begin{cases} x^2 + y^2 = 1, \\ x + y + z = 1. \end{cases}$

5. $\vec{a} = (x - y)\vec{i} + x\vec{j} - z\vec{k}$, $\ \Gamma: \begin{cases} x^2 + y^2 = 1, \\ z = 5. \end{cases}$

6. $\vec{a} = y\vec{i} - x\vec{j} + z^2\vec{k}$, $\Gamma: \begin{cases} z = 3(x^2 + y^2) + 1, \\ z = 4. \end{cases}$

7. $\vec{a} = yz\vec{i} + 2xz\vec{j} + y^2\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 + z^2 = 25, \\ x^2 + y^2 = 1, (z \geq 0). \end{cases}$

8. $\vec{a} = xy\vec{i} + yz\vec{j} + xz\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 = 9, \\ x + y + z = 1. \end{cases}$

9. $\vec{a} = y\vec{i} + (1 - x)\vec{j} - z\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 + z^2 = 4, \\ x^2 + y^2 = 1, (z > 0). \end{cases}$

10. $\vec{a} = y\vec{i} - x\vec{j} + z^2\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 = 1, \\ z = 4. \end{cases}$

11. $\vec{a} = 4x\vec{i} + 2\vec{j} - xy\vec{k}$, $\Gamma: \begin{cases} z = 2(x^2 + y^2) + 1, \\ z = 7. \end{cases}$

12. $\vec{a} = 2y\vec{i} - 3x\vec{j} + z^2\vec{k}$, $\Gamma: \begin{cases} z = x^2 + y^2, \\ z = 1. \end{cases}$

13. $\vec{a} = -3z\vec{i} + y^2\vec{j} + 2y\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 = 4, \\ x - 3y - 2z = 1. \end{cases}$

14. $\vec{a} = 2y\vec{i} + 5z\vec{j} + 3x\vec{k}$, $\Gamma: \begin{cases} 2x^2 + 2y^2 = 1, \\ x + y + z = 3. \end{cases}$

15. $\vec{a} = 2y\vec{i} + \vec{j} - 2yz\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 - z^2 = 0, \\ z = 2. \end{cases}$

16. $\vec{a}=(x-y)\vec{i}+x\vec{j}+z^2\vec{k}$, $\Gamma:\begin{cases} x^2+y^2-4z^2=0, \\ z=\dfrac{1}{2}. \end{cases}$

17. $\vec{a}=xz\vec{i}-\vec{j}+y\vec{k}$, $\Gamma:\begin{cases} x^2+y^2+z^2=4, \\ z=1. \end{cases}$

18. $\vec{a}=2yz\vec{i}+xz\vec{j}-x^2\vec{k}$, $\Gamma:\begin{cases} x^2+y^2+z^2=25, \\ x^2+y^2=9, (z>0). \end{cases}$

19. $\vec{a}=-y\vec{i}+2\vec{j}+\vec{k}$, $\Gamma:\begin{cases} x^2+y^2-z^2=0, \\ z=1. \end{cases}$

20. $\vec{a}=y\vec{i}+3x\vec{j}+z^2\vec{k}$, $\Gamma:\begin{cases} z=x^2+y^2-1, \\ z=3. \end{cases}$

21. $\vec{a}=2yz\vec{i}+xz\vec{j}+y^2\vec{k}$, $\Gamma:\begin{cases} x^2+y^2+z^2=25, \\ x^2+y^2=16, (z>0). \end{cases}$

22. $\vec{a}=(2-xy)\vec{i}-yz\vec{j}-xz\vec{k}$, $\Gamma:\begin{cases} x^2+y^2=1, \\ x+y+z=1. \end{cases}$

23. $\vec{a}=-y\vec{i}+x\vec{j}+3z^2\vec{k}$, $\Gamma:\begin{cases} x^2+y^2+z^2=9, \\ x^2+y^2=1, (z>0). \end{cases}$

24. $\vec{a}=y\vec{i}-x\vec{j}+2z\vec{k}$, $\Gamma:\begin{cases} x^2+y^2-\dfrac{z^2}{4}=0, \\ z=2. \end{cases}$

25. $\vec{a}=x^2\vec{i}+yz\vec{j}+2z\vec{k}$, $\Gamma:\begin{cases} x^2+y^2+z^2=25, \\ z=4. \end{cases}$

26. $\vec{a} = y\vec{i} - 2x\vec{j} + z^2\vec{k}$, $\Gamma: \begin{cases} z = 4(x^2 + y^2) + 2, \\ z = 6. \end{cases}$

27. $\vec{a} = 3z\vec{i} - 2y\vec{j} + 2y\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 = 4, \\ 2x - 3y - 2z = 1. \end{cases}$

28. $\vec{a} = (x + y)\vec{i} - x\vec{j} + 6\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 = 1, \\ z = 2. \end{cases}$

29. $\vec{a} = 4\vec{i} + 3x\vec{j} + 3xz\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 - z^2 = 0, \\ z = 3. \end{cases}$

30. $\vec{a} = yz\vec{i} - xz\vec{j} + xy\vec{k}$, $\Gamma: \begin{cases} x^2 + y^2 + z^2 = 9, \\ x^2 + y^2 = 9. \end{cases}$

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