МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ УКРАИНЫ

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MECHANICS

Part I

Учебное пособие по физике
для студентов-иностранцев

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На английском языке изложены основные вопросы, предусмотренные учебной программой по курсу «Экспериментальная и теоретическая физика» из разделов «Кинематика» и «Динамика». Каждая глава пособия содержит теоретический раздел, в котором изложен материал по данной тематике, примеры решения наиболее общих задач, набор контрольных вопросов для самопроверки и задач для самостоятельного решения.

Для студентов-иностранцев технических вузов.

In this textbook there are textual material of lectures, solved examples and problems on classical mechanics: kinematics and dynamics of translational and rotational motion, work and energy.

This textbook is intended for the course of physics for students of the Aviation University.

Ил. 45. Табл. 1. Библиогр.: 7 назв.

Рецензенты: канд. физ.-мат. наук, доц. Ю.В. Сидоренко,
канд. физ.-мат. наук, доц. И.Н. Кудрявцев

1. Products of vector

Many physical relationships can be expressed concisely by the use of products of vectors. There are two different kinds of vector products. The first, called the scalar product, yields a result that is the scalar quantity, while the second, the vector product, yields another vector.

1. **Scalar product (dot product)** of two vectors $\vec{A}$ and $\vec{B}$. We draw the two vectors from a common point as in Fig.1 a). The angle between their directions is θ, as shown. We define the scalar product, denote by $\vec{A} \cdot \vec{B}$, as

$$\vec{A} \cdot \vec{B} = AB\cos\theta = |\vec{A}||\vec{B}|\cos\theta .$$

It is a scalar quantity, not a vector, and it may be positive or negative. When θ is between zero and 90°, the scalar product is positive; when θ is between 90° and 180°, it is negative; and when $\theta = 90°$, $\vec{A} \cdot \vec{B} = 0$. The scalar product of two perpendicular vectors is always zero.

The order of the two vectors doesn't matter. For any two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} .$$

If we the components of vectors $\vec{A}$ and $\vec{B}$, we can represent their scalar product as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z .$$

2. **Vector product (cross product)** of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{C} = \vec{A} \times \vec{B}$ or $\vec{C} = \lfloor \vec{A} \cdot \vec{B} \rfloor$. To define the vector product we again draw $\vec{A}$ and $\vec{B}$ from a common point. The two vectors $\vec{A}$ and $\vec{B}$ then lie in a plane. We define the vector product $\vec{C}$ as a vector quantity with a direction perpendicular to this plane (i.e., perpendicular to both $\vec{A}$ and $\vec{B}$) and a magnitude given by $AB\ sin\theta$. That is, if $\vec{C} = \vec{A} \times \vec{B}$, then

$$C = AB\sin\theta.$$

We measure the angle θ from $\vec{A}$ toward $\vec{B}$ and take it always positive. We note also that when $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, θ = 0 or θ = 180° then C = 0.

There are always *two* directions perpendicular to a given plane. To distinguish between these, we imagine rotating vector $\vec{A}$ about the perpendicular line until it is aligned with $\vec{B}$ (choosing the smaller of the two possible angle). We then curl the fingers of the right hand around this perpendicular line so that the fingertips point in the direction of rotation; the thumb then given the direction of the vector product. Alternatively, the direction of the vector product is the direction a right-hand-thread screw advanced if turned in the sense from $\vec{A}$ toward $\vec{B}$ (Fig. 1b).

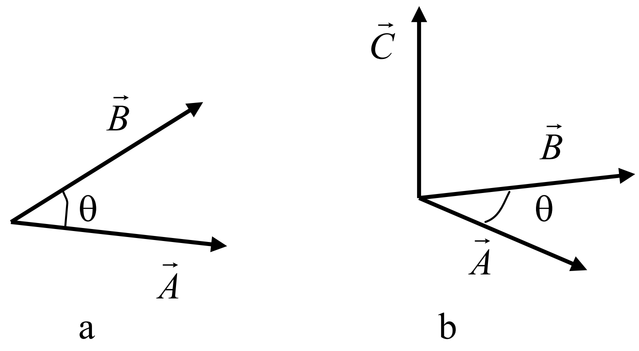


Fig. 1

2. Kinematics of translation motion

Our study of physics begins in the area of mechanics, which deals with the relations among force, matter and motion. Mechanics consists of kinematics, dynamics and static. *Kinematics studies motion of the body without the reason of this motion.*

Motion is a continuous change of position. We can think of a moving body as *a particle* when it is small and when there is no rotation or change of shape. We say that the particle is a model of a moving body; it provides a simplified, idealized description of the position and motion of the body.

2.1. Motion along a straight line

The simplest case is motion of a particle along a straight line, and we will always take that line to be a *coordinate axis*. Then we will consider more general motion in space, but these can always be represented by means of their projections onto three coordinate axis. Displacement is in general a vector quantity, but we first consider situations in which only one component is different from zero. Then the particle moves along one coordinate axis, and its position is described by a single coordinate.

2.1.1. Displacement and average velocity

Let us consider a hockey player skating the puck down the center line of the ice toward the opposition's net. The puck moves along a straight line, which we will use as the x-axis of our coordinate system, as shown in Fig. 2. The puck's distance from the origin O at center court is given by the coordinate x, which varies with time. At time t_1 the puck is at point P with coordinate x_1 and at time t_2 it is at point Q where its coordinate is x_2. The *displacement* during the time interval from t_1 to t_2 is the vector from P to Q; the x- component of this vector is $(x_2 – x_1)$ and all other components are zero.

It is convenient to represent the quantity $(x_2 – x_1)$, the change in x, using the Greek latter Δ (capital delta).

$$\Delta x = x_2 - x_1. \qquad (1)$$

Similarly, we denote the time interval from t_1 to t_2 as $\Delta t = t_2 - t_1$.

Fig. 2

The *average velocity* of the puck is defined as the ratio of the displacement Δx to the time interval Δt. We represent average velocity by latter υ with a subscript "av" to signify average value. Thus

$$\upsilon_{av} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}. \qquad (2)$$

Strictly speaking, average velocity is a vector quantity and this defines the x-component of the average velocity.

We have not specified whether the speed of the hockey puck is or is not constant during the time interval $\Delta t = t_2 - t_1$. It way has started from rest, reached maximum speed and then slowed down. But that does not matter. To calculate the average velocity we need only the total displacement $\Delta x = x_2 - x_1$ and the total time interval $\Delta t = t_2 - t_1$.

2.1.2. Instantaneous velocity

Even when the velocity of a moving particle varies, we can still define a velocity at any instant of time or at any specific point in the path. Such a velocity is called *instantaneous velocity*.

The instantaneous velocity is defined as

$$\upsilon = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \qquad (3)$$

The limit of $\frac{\Delta x}{\Delta t}$ as Δt approaches zero is written $\frac{dx}{dt}$ and is called the derivative of x respect to t. Since Δt is assumed positive, υ has the positive x-axis points to the right, a positive velocity indicates motion toward the right.

In more general motion, velocity must be treated as a vector quantity. In that case Eq. (3) becomes the x-component of the instantaneous velocity. When we use the term velocity, we always mean instantaneous rather then average velocity, unless we state otherwise.

The instantaneous velocity at any point of coordinate – time graph equals the slope of tangent to the graph at the point. If the tangent slopes upward to the right, its slope is positive, the velocity is positive, and the motion is toward the right. If the tangent slopes downwards to the right, the velocity is negative. At a point where the tangent is horizontal, its slope is zero and the velocity is zero.

If distance is expressed in meter and time in second velocity is expressed in meter per second $\left(\frac{m}{s}\right)$.

The term speed has two different meanings. It may mean the magnitude of the instantaneous velocity. For example, when two cars travel at $50\frac{km}{hour}$ one north and the other south, both have speeds of $50\frac{km}{hour}$. In a different sense, referring to an average quantity, the speed of a body is the total length, divided by the elapsed time. Thus if a car travels 90 km in 3hr, its average speed is $30\frac{km}{hour}$, even if the trip starts and ends at the same point. In the latter case the average velocity would be zero, because the total displacement is zero.

2.1.3. Average and instantaneous acceleration

When the velocity of a moving body changes with time, we say that the body has acceleration. Just as velocity is a quantitative description of the rate of change of position with time, so acceleration is a quantitative description of the rate of change of velocity with time.

Considering again the motion of particle along the x-axis, suppose that at time t_1 the particle is at point P and has velocity υ_1, and that at a latter time t_2 it is at point Q and has velocity υ_2.

The *average acceleration* a_{av} of the particle as it moves from P to Q is defined as the ratio of the change in velocity to the elapsed time:

$$a_{av} = \frac{\upsilon_2 - \upsilon_1}{t_2 - t_1} = \frac{\Delta\upsilon}{\Delta t}. \tag{4}$$

Again, strictly speaking, υ_1 and υ_2 are values of the x-component of instantaneous velocity, and Eq. (4) defines the x-component of average acceleration.

We can define *instantaneous acceleration*, following the same procedure used for instantaneous velocity.

$$a = \lim_{\Delta t \to 0} \frac{\Delta\upsilon}{\Delta t} = \frac{d\upsilon}{dt}. \tag{5}$$

Instantaneous acceleration plays an essential role in the laws of mechanics, while average acceleration is less frequently used. When we used the term acceleration, we always mean instantaneous acceleration.

The instantaneous acceleration at any point on the graph equals the slope of the line tangent to the curve at that point.

The acceleration $a = \frac{d\upsilon}{dt}$ can be expressed in various ways. Since $\upsilon = \frac{dx}{dt}$ it follows that

$$a = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}. \tag{6}$$

The a is therefore the second derivative of the coordinate with respect to time.

If we express velocity in meter per second and time in seconds, then acceleration is in meter per second, per second $\left(\frac{m \cdot 1}{s \cdot s}\right)$. This is usually written as $\frac{m}{s^2}$, and is read meters per second squared. A few remarks about the sing of acceleration may be helpful. When the acceleration and velocity of a body have the same sing, the body is speeding up. If both are positive, the body moves in the positive direction with increasing speed. If both are negative, the body moves in the negative direction with a velocity that more and more negative with time, and again the body's speed increases.

When υ and *a* have opposite sing, the body is slowing down. If υ is positive and *a* negative, the body moves in the positive direction with decreasing speed.

Example 1. The velocity of the car is given by the equation $\upsilon = m + nt^2$, when $m = 10$ m/s; $n = 2$ m/s.

a) Find the change in velocity of the car in the time interval between $t_1 = 2$ s and $t_2 = 5$ s.

Solution.

At time t_1 $\upsilon_1 = m + nt_1^2 = 10 + 2 \cdot 2^2 = 18$ m/s.

At time t_2 $\upsilon_2 = m + nt_2^2 = 10 + 2 \cdot 5^2 = 60$ m/s.

The change in velocity is therefore $\Delta\upsilon = \upsilon_2 - \upsilon_1 = 60 - 18 = 42$ m/s.

b) Find the average acceleration in this time interval.

Solution.

$$a_{av} = \frac{\Delta\upsilon}{\Delta t} = \frac{42}{3} = 14 \text{ m/s}^2.$$

c) Find the acceleration at time $t_1 = 2$ s.

Solution.

$$a = \frac{d\upsilon}{dt} = 2nt = 2 \cdot 2 \cdot 2 = 8 \text{ m/s}^2.$$

2.1.4. Motion with constant acceleration

How to find the velocity at any time

The simplest accelerated motion is straight–line motion with constant acceleration, when the velocity changes at the same rate through the motion. The graph of velocity as a function of time is then a straight line; the velocity increases by equal amounts in equal time intervals (Fig. 3).

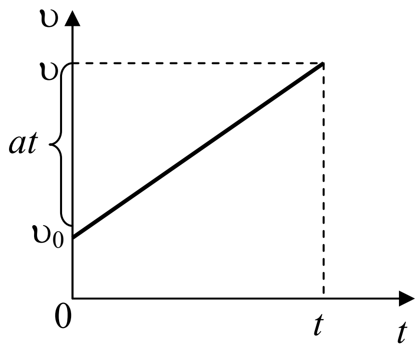


Fig. 3

Hence in Eq. (4) we have

$$a = \frac{\upsilon_2 - \upsilon_1}{t_2 - t_1}. \quad (7)$$

Now let $t_1 = 0$ and t_2 be any arbitrary later time t. Let υ_0 represent the velocity when $t = 0$ (called the initial velocity), and let υ be the velocity at the later time *t*. Then Eq. (7) becomes

$$a = \frac{\upsilon - \upsilon_0}{t - 0}$$

or

$$\boldsymbol{\upsilon = \upsilon_0 + at}. \quad (8)$$

We can interpret this equation as follows. The acceleration a is the constant rate of change of velocity, or the change per unit time. The term at is the product of the change in velocity per unit time and the time interval t. Therefore it equals the total change in velocity. The velocity υ at any time t then equals the initial velocity υ_0 (at time $t = 0$) plus the change in velocity at. Graphically, we can consider the ordinate υ at time t as the sum of two segments: one with length υ_0 equal to the initial velocity, the other with length at equal to the change in velocity during time t.

Here is an alternative route to Eq. (8). We know that υ is some function of t. If the derivative of that function with respect to t is the constant a, what must the function itself be? (Such a function might be indefinite integral). One possibility is the function at; its derivative with respect to t is the constant a. But the derivative of the function $(at+C)$, where C is any constant, is also equal to a, because the derivative of any constant is zero. Thus we conclude that the function for v must have the form

$$\upsilon = at + C. \tag{9}$$

Now this function must also satisfy the additional requirement that at time $t = 0$ it yields the value υ_0. Substituting $t = 0$ and equating the result to υ_0, we find

$$\upsilon_0 = a(0) + C \quad \text{or} \quad C = \upsilon_0 . \tag{10}$$

Putting all this together, we again obtain

$$\upsilon = \upsilon_0 + at .$$

How to find the position at any time

We may use a similar procedure to find an expression for the position x of the particle as a function of time, if the position at time, if the position at time $t = 0$ is denoted by x_0. We know that $\upsilon = \frac{dx}{dt}$; what function of t must x be, in order for its derivative with respect to t to equal $(\upsilon_0 + at)$?

The function

$$x = \upsilon_0 + \frac{at^2}{2} + D. \tag{11}$$

Where D is any constant, satisfies this requirement; this may be checked easily by taking the derivative of Eq. (11).

2.1.5. Velocity and coordinate by integration

When x varies with time we use the relation $\upsilon = \frac{dx}{dt}$ to find the velocity υ as a function of time. Similarly, we can use $a = \frac{d\upsilon}{dt}$ to find the acceleration a as a function of time if the velocity υ is given as a function of time.

We can also reverse this process. Suppose υ is known as function of time; how can we find x as a function of time? It is simply the integral of υ from t_1 to t_2.

As $$dx = \upsilon(t)dt,$$

then $$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} \upsilon(t)dt,$$

or $$x_2 - x_1 = \int_{t_1}^{t_2} \upsilon(t)dt,$$

and similarly: $$d\upsilon = a(t)dt,$$

$$\int_{\upsilon_1}^{\upsilon_2} d\upsilon = \int_{t_1}^{t_2} a(t)dt,$$

$$\upsilon_2 - \upsilon_1 = \int_{t_1}^{t_2} a(t)dt.$$

<u>Example 2.</u> An automobile travels along a straight highway. It's acceleration is given as a function of time by $a = D - bt$, where $D = 2$ m/s^2; $b = 0{,}1$ m/s^3.The position at time $t = 0$ is given by $x_0 = 0$, and the velocity at $t = 0$ is $\upsilon_0 = 10$ m/s. Derive expressions for the velocity and position as function of time.

1. As $$a(t) = \frac{d\upsilon}{dt},$$

then $$d\upsilon = adt.$$

Hence $$\upsilon = \int a(t)dt = \int (D - bt)dt = \int Ddt - \int btdt = Dt - \frac{1}{2}bt^2 + C_1.$$

Constant of integration C_1 is defined from initial condition $\upsilon = \upsilon_0$ at $t = 0$, i.e.

$$\upsilon(0) = 0 \cdot D + 0{,}5\, b \cdot 0 + C_1 = \upsilon_0, \quad C_1 = \upsilon_0;$$

and $$\upsilon = \upsilon_0 + Dt - 0{,}5bt^2.$$

2. As $\upsilon = \dfrac{dx}{dt}$, then $dx = \upsilon dt$, and

$$x = \int \upsilon dt = \int\left(\upsilon_0 + Dt - \frac{1}{2}bt^2\right)dt = \int \upsilon_0 dt + \int Dtdt - \int \frac{1}{2}bt^2 dt = \upsilon_0 t + \frac{1}{2}Dt^2 - \frac{1}{6}bt^3 + C_2.$$

Constant of integration C_2 we can obtain using initial condition $x(0) = 0$,

i.e. $$x(0) = \upsilon_0 \cdot 0 + \frac{1}{2}D \cdot 0 - \frac{1}{6}b \cdot 0 + C_2; \quad C_2 = 0.$$

Finally $$x = \upsilon_0 t + \frac{1}{2}Dt^2 - \frac{1}{6}bt^3.$$

2.2. Motion in space

Let our body moves along some curve OO′ (Fig. 6). To define the body's position we use vector $\vec{r}$ – so–called *position vector.* Position vector is the vector from origin to the point of body's location.

Let at time t_1 the body is at point A. Its position vector is $\vec{r}_1$. At time t_2 the body is at point B and its position vector is $\vec{r}_2$.

The vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$ is called the *displacement vector*. When body moves its position vector changes with time, i.e.

$$\vec{r} = \vec{r}(t).$$

Position vector (as any vector) can be represented in components:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

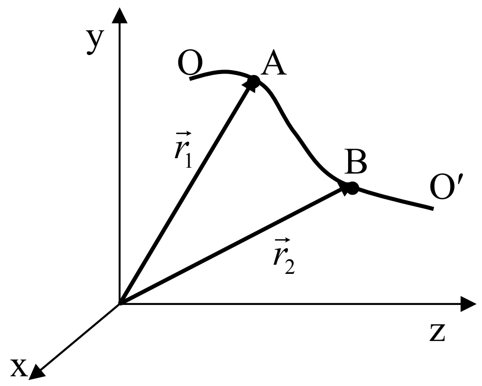


Fig. 6

To define how rapidly body's position changes we use velocity. Velocity is defined as

$$\vec{\upsilon} = \lim_{\Delta t \to o} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt},$$

and is directed along the tangential to the trajectory.

When the motion is not uniform the velocity is not constant. The change of velocity is characterized by vector of acceleration:

$$\vec{a} = \lim_{\Delta t \to o} \frac{\Delta\vec{\upsilon}}{\Delta t} = \frac{d\vec{\upsilon}}{dt}.$$

2.2.1. Normal and Tangential Components of Acceleration

When particle moves in the xy-plane the acceleration of a particle moving in curved path can also be represented in terms of rectangular component $a_{\perp}$ and $a_{\parallel}$ in direction *normal* (perpendicular) and *tangential* (parallel) to the path as shown in Fig. 4. Unlike the rectangular components referred to a set of fixed axes, the normal and tangential components do not have fixed direction in space. They do, however, have a direct physical significance. The parallel component $a_{\parallel}$ corresponds to a change in the *magnitude* of the velocity vector $\vec{\upsilon}$. While the normal component $a_{\perp}$ is associated with a change in the *direction* of the velocity.

Let's consider the case, when a particle moves in a circle with constant speed. Fig. 5 shows a particle moving in a circular path of radius R with center at O. The vector change in velocity, $\Delta\vec{\upsilon}$, is shown in Fig. 5(b). The particle moves from P to Q in a time Δt. The triangles OPQ and *opq* in Fig.5(a) and Fig. 5(b) are similar, since both are isosceles triangles and the angles labeled $\Delta\theta$ are the same.

Hence $\dfrac{\Delta \upsilon}{\upsilon_1} = \dfrac{\Delta S}{R}$ or $\Delta \upsilon = \dfrac{\upsilon_1}{R} \Delta S$.

The magnitude of the average normal acceleration $(a_\perp)_{av}$ during Δt is therefore

$$\left(a_\perp\right)_{av} = \frac{\Delta \upsilon}{\Delta t} = \frac{\upsilon_1}{R} \cdot \frac{\Delta S}{\Delta t}.$$

The instantaneous acceleration $a_\perp$ at point is the limit of this expression, as point Q is taken closer and closer point P:

$$a_\perp = \lim_{\Delta t \to 0}\left(\frac{\upsilon_1}{R} \cdot \frac{\Delta S}{\Delta t}\right) = \frac{\upsilon_1}{R} \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}.$$

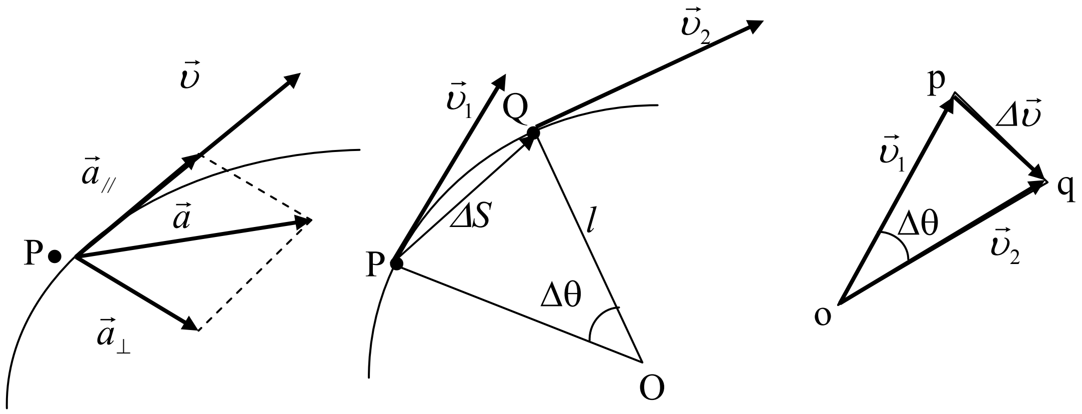


Fig. 5

However, the limit of $\dfrac{\Delta S}{\Delta t}$ is the speed υ_1 at point P, and since P can be any point of the path, we can drop the subscript from υ_1 and let υ represent the speed at any point. Then

$$a_n = a_\perp = \frac{\upsilon^2}{R}. \qquad (12)$$

The magnitude of the normal acceleration is equal to the square of the speed divided by the radius of the circle. The direction is perpendicular to υ and inward along the radius.

We have assumed that the particle's speed is constant. If the speed varies, Eq. 12 still gives the normal component of acceleration the tangential component of acceleration is equal to the rate of change of speed:

$$a_\tau = a_{//} = \lim_{\Delta t \to 0} \frac{\Delta \upsilon}{\Delta t} = \frac{d\upsilon}{dt}. \qquad (13)$$

Velocity can change its magnitude and its direction. The tangential acceleration $a_\tau = \frac{d\upsilon}{dt}$ defines the change in the magnitude of velocity and is directed along the tangential to the trajectory (as velocity vector).

The normal acceleration $a_n = \frac{\upsilon^2}{R}$ characterizes the change of velocity direction and is directed along the radius of curvature to the center of the curve and normally to the velocity and tangential acceleration.

The vector sum of normal and tangential acceleration gives the acceleration of the body (Fig. 4):

$$\vec{a} = \vec{a}_\tau + \vec{a}_n , \tag{14}$$

$$a^2 = a_\tau^2 + a_n . \tag{15}$$

2.2.2. Principle Problems of Kinematics

There are two principle problem of kinematics.

First: from known position vector $\vec{r}$ calculate all the characteristics of motion (i.e. distance from the origin $|\vec{r}|$, velocity vector $\vec{v}$ and its magnitude $|\vec{v}|$, vector of acceleration $\vec{a}$ and its magnitude $|\vec{a}|$, normal a_n and tangential a_τ components of acceleration and radius of curvature R. To obtain the solution of this type of problem we have to take derivatives of position vector and velocity with respect to time.

Second: from known acceleration $|\vec{a}|$ calculate the other characteristics of motion: velocity vector $\vec{v}$ and its magnitude $|\vec{v}|$, position vector $\vec{r}$, distance from the origin $|\vec{r}|$, normal a_n and tangential a_τ components of acceleration and radius of curvature R. To solve such kind of problems we have to take integrals.

Lets consider the first class of problems. When we are given the position vector as function of time we can find all characteristics of the motion as follows:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} .$$

The distance of the body from the origin is the module of the position vector:

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} .$$

a) The displacement is $\Delta\vec{r} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

and $$\Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} .$$

b) Velocity is calculated as $\vec{\upsilon} = \frac{d\vec{r}}{dt}$,

or $$\vec{\upsilon} = \frac{d}{dt}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \upsilon_x\vec{i} + \upsilon_y\vec{j} + \upsilon_z\vec{k} ,$$

where $\upsilon_x = \frac{dx}{dt}; \upsilon_y = \frac{dy}{dt}; \upsilon_z = \frac{dz}{dt}$ - are the components of velocity along x -, y - and z -axis, correspondingly.

Magnitude of the velocity is equal to the module of the velocity vector.

c) Acceleration of the body is

$$\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d}{dt}\left(\upsilon_x\vec{i} + \upsilon_y\vec{j} + \upsilon_z\vec{k}\right) = \frac{d\upsilon_x}{dt}\vec{i} + \frac{d\upsilon_y}{dt}\vec{j} + \frac{d\upsilon_z}{dt}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k},$$

where $a_x = \frac{d\upsilon_x}{dt} = \frac{d^2x}{dt^2}; a_y = \frac{d\upsilon_y}{dt} = \frac{d^2y}{dt^2}; a_z = \frac{d\upsilon_z}{dt} = \frac{d^2z}{dt^2}$ are the components of the acceleration vector along x -, y - and z - axis, correspondingly. Magnitude, that is module of acceleration vector we can obtain as

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

Tangential acceleration is $a_\tau = \frac{d\upsilon}{dt} = \frac{d}{dt}\left(\sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2}\right)$.

To calculate the normal acceleration we need the radius of curvature R. If we are not given R we can use the formula $a^2 = a_n^2 + a_\tau^2$, then $a_n = \sqrt{a^2 - a_\tau^2}$. And, finally, radius of curvature R we get as $R = \frac{\upsilon^2}{a_n}$.

For the second type of problems we have to apply the following strategy: as acceleration $\vec{a} = \frac{d\vec{\upsilon}}{dt}$, then $\int d\vec{\upsilon} = \int \vec{a}(t)dt$, and velocity $\vec{\upsilon} = \int \vec{a}(t)dt + \vec{C}$.

When acceleration is constant, then $\vec{\upsilon} = \vec{a}t + \vec{C}$.

As $\vec{\upsilon}(y) = \frac{d\vec{r}(t)}{dt}$, then $d\vec{r}(t) = \vec{\upsilon}(t)dt$ and $\vec{r}(t) = \int \vec{\upsilon}(t)dt$.

Now it is time to consider several examples.

Example 1. The position vector of body changes with time as $\vec{r} = \left(A + Bt^2\right)\vec{i} + Ct\vec{j}$, where A = 10 m, B = –5 m/s^2, C = 10 m/s. Calculate velocity, acceleration, tangential and normal components of acceleration and radius of trajectory curvature.

Solution.

1. Module of position vector is $r = \sqrt{x^2 + y^2}$. In our case $x = A + Bt^2;\ y = Ct$.

   That is $$r = \sqrt{\left(A + Bt^2\right)^2 + C^2t^2}.$$

2. Velocity we can find as follows

   $$\upsilon_x = \frac{dx}{dt} = \frac{d}{dt}\left(A + Bt^2\right) = 2Bt; \upsilon_y = \frac{dy}{dt} = \frac{d}{dt}(Ct); \upsilon_z = 0$$

   and velocity vector $\vec{\upsilon} = \upsilon_x\vec{i} + \upsilon_y\vec{j} = 2Bt\vec{i} + C\vec{j}; \vec{\upsilon} = \left(-10t\vec{i} + 10\vec{j}\right)$ m/s.

And then the magnitude of velocity, i,e, module of vector $\vec{\upsilon}$ is

$\upsilon = \sqrt{(2Bt)^2 + C^2}; \upsilon = 10\sqrt{1+t^2}$ m/s.

3. Acceleration $\vec{a}$ is $\vec{a} = a_x\vec{i} + a_y\vec{j}$.

   In this case $a_x = \frac{d\upsilon_x}{dt} = \frac{d}{dt}(2Bt) = 2B; a_y = \frac{d\upsilon_y}{dt} = \frac{dC}{dt} = 0$

   and $\vec{a} = 2B\vec{i}$ then $a = \sqrt{a_x^2 + a_y^2} = \sqrt{(2B)^2} = 2B = 10$ m/s².
4. Tangential acceleration is

$$a_\tau = \frac{d\upsilon}{dt} = \frac{d}{dt}\sqrt{(2Bt)^2 + C^2} = \frac{10t}{\sqrt{1+t^2}} \text{ m/s}^2.$$

5. Normal acceleration is $a_n = \frac{\upsilon^2}{R}$.

As we are not given R, we have to use connection $a^2 = a_n^2 + a_\tau^2$, hence

$$a_n = \sqrt{a^2 - a_\tau^2} = \sqrt{4B^2 - \left(\frac{10t}{\sqrt{1+t^2}}\right)} = \frac{10}{\sqrt{1+t^2}}.$$

6. And radius R will be $R = \frac{\upsilon^2}{a_n} = \left(10\sqrt{1+t^2}\right)^2 \frac{\sqrt{1+t^2}}{10} = 10\left(1+t^2\right)^{\frac{3}{2}}$.

Short answer questions

1. Is it possible that displacement is zero but not the distance?
2. Can a body have a constant velocity but a varying speed?
3. Are the magnitude of average speed and velocity equal?
4. The distance traveled by a body is found to be directly proportional to the square of time. Is the body moving with uniform velocity or with the uniform acceleration?

True – false type questions

1. A body can have eastward velocity while experiencing a westward acceleration.
2. A lorry and a car moving with the same kinetic energy are brought to rest by the application of brakes, which provide equal retarding forces. Both come to rest in equal distance.
3. A body is moving with a uniform velocity in one frame A, then there is another frame B in which it is accelerating.
4. A car covers the first half of its distance between two places at a speed of 40 km/hour and the second half at 60 km/hour. The average speed of the car is then 50 km/hour.

5. A car covers the first half of its time between two places at a speed of 40 km/hour and the second half at 60 km/hour. The average speed of the car is then 50 km/hour
6. A train is moving with a speed of 60 km/hour and a car is moving by its side in the same direction with a speed of 20 km/hour. The speed of the car relative to train is 80 km/hour.
7. Two cars A and B are moving in the same direction with equal speeds. A passenger in the car A finds that the car B is at rest.

Examples

1. A body travels 200 sm in the first two seconds and 220 sm in the next four seconds. What will be the velocity at the end of the seventh from the start (υ_0 = 115 sm/s, a = – 15 sm/s^2)?
2. A point moving with constant acceleration from A to B in the straight line AB has velocity υ_0 and υ at A and B respectively. Find its velocity at C, the mid – point of AB. Also show that if the time from A to C is twice that from C to B, then $\upsilon = 7\upsilon_0$.
3. An α-particle travels along the inside of straight hollow tube, 2,0 meter long, of a particle accelerator. Under uniform acceleration, how long is the particle in the tube if it enters at a speed of 1000 m/s and leaves at 9000 m/s. What is its acceleration during this interval ($a = 2{,}0\cdot10^7$ m/s^2)?
4. A truck starts from rest with an acceleration of 1.5 m/s^2 while a car 150 meter behind stars from rest with an acceleration of 2 m/s^2. How long will it take before both the truck and side, and how much distance is traveled by each (S_1 = 450 m (truck) and S_2 = 600 m (car), t = 24.5 s)?
5. A particle is moving in a plane with velocity given by $\vec{\upsilon} = \upsilon_0\vec{i} + (a\omega cos\,\omega t)\vec{j}$, where $\vec{i}$ and $\vec{j}$ are unit vectors along x and y axes respectively. If particle is at the origin at t =0, calculate the trajectory of the particle and find the distance from the origin at time $t = \dfrac{3\pi}{2\omega}$. $\left(S = \sqrt{\left(\dfrac{3\pi\upsilon_0}{2\omega}\right)^2 + a^2}\,;\, y = a\,sin\dfrac{\omega\cdot x}{\upsilon_0}\right)$.
6. The body is moving with velocity given by $\vec{\upsilon} = \alpha\cdot\vec{i} + \beta x\cdot\vec{j}$, where α and β are constant. If body is at origin at t =0, find its radius-vector, velocity acceleration as functions of time.
   ($\vec{r} = \alpha\cdot t\vec{i} + \dfrac{\alpha\beta}{2}t^2\vec{j};\vec{\upsilon} = \alpha\cdot\vec{i} + \alpha\beta\cdot t\vec{j};\vec{a} = \alpha\beta\cdot\vec{j}$).
7. The radius-vector of material point is $\vec{r} = (A + Bt^2)\vec{i} + Ct\vec{j}$, where A = 10m, B = – 5 m/s^2, C = 10 m/s. Calculate velocity $\vec{\upsilon}$, acceleration $\vec{a}$, tangential and

normal accelerations. ($\vec{\upsilon} = -10t\vec{i} + 10\vec{j}$ m/s; $\vec{a} = -10\vec{i}$ m/s^2; $\upsilon = 10\sqrt{1+t^2}$ m/s; $a = 10$ m/s^2; $a_\tau = \dfrac{10t}{\sqrt{1+t^2}}$ m/s^2; $a_n = \dfrac{10}{\sqrt{1+t^2}}$ m/s^2).

8. Radius-vector of the body is $\vec{r} = (2\cos\omega t)\vec{i} + (2\sin\omega t)\vec{j}$, where ω = const. Find the trajectory, velocity, acceleration, tangential and normal accelerations and radius of curve as function of time.
(Trajectory is the circle $x^2+y^2=4$. $\vec{\upsilon} = 2\omega(-\vec{i}\sin\omega t + \vec{j}\cos\omega t)$; $\upsilon = 2\omega$; $\vec{a} = -2\omega^2(\vec{i}\cos\omega t + \vec{j}\sin\omega t)$; $a = 2\omega^2$; $a_\tau = 0$; $a_n = 2\omega^2$; $R = 2$).

9. The acceleration of the point is $a = -r\upsilon$, where r is constant. Calculate the speed and distance as function of time. At time $t = 0$, $S(0) = 0$, $\upsilon(0) = \upsilon_0 = 0$.
($\upsilon = \upsilon_0 e^{-rt}$; $S = \dfrac{\upsilon_0}{2}\left(1 - e^{-rt}\right)$).

9. The point is moving along straight line with the speed $\upsilon = k\sqrt{S}$, where k is constant. Calculate speed and acceleration as function of time. At time $t = 0$ particle is at origin.

2.3. Projectiles motion

When a body is projected in air in any direction, then the body is called a *projectile*. The angle, which the direction of projection makes with the horizontal, is known as *angle of projection*. Fig. 7 shows a body projected with an initial velocity υ_0 at an angle θ with the horizontal. The path traced out by the body in its journey is called as *trajectory*.

The distance between the point of projection and point where the trajectory meets any plane drawn through the point of projection is called the *range*. In Fig. 7 AB is the range. The time that elapses before the body again meets the horizontal plane through the point of projection is known as time of flight. At a certain point the angle θ, which the velocity of the projectile makes with the horizontal, is called the direction of motion of the body at that point.

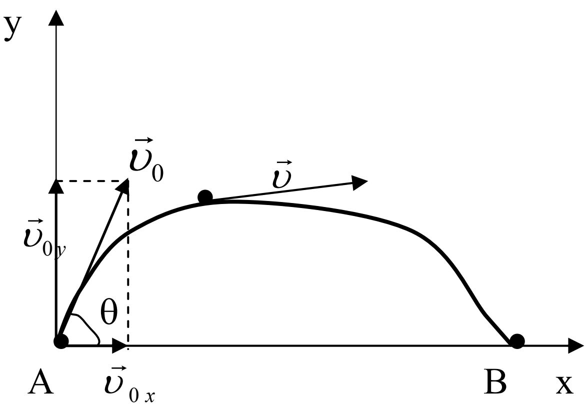


Fig. 7

The velocity υ of the projectile can be resolved along x and y axes as follows:

–the vertical component $\upsilon_y = \upsilon\sin\theta$;

–the horizontal component $\upsilon_x = \upsilon\cos\theta$.

Neglecting air resistance the horizontal component remains unaffected by gravity, i.e. constant, while the vertical component changes due to acceleration of gravity.

The vertical displacement y at any time t is given by

$$y = \upsilon_{0y} t - \frac{1}{2} g t^2 \qquad (16)$$

or

$$y = (\upsilon_0 \sin\theta) t - \frac{1}{2} g t^2 . \qquad (17)$$

The vertical component of velocity at any time t is

$$\upsilon_y(t) = \upsilon_0 \sin\theta - gt. \qquad (18)$$

1. <u>Time taken to reach the maximum height.</u>

At highest point the vertical component of velocity of projectile is zero. From Eq. 18, we get

$$0 = \upsilon_0 \sin\theta - gt ,$$

or

$$t = \frac{\upsilon_0 \sin\theta}{g} .$$

2. <u>Greatest height attained.</u>

At greatest height h, we have $\upsilon_y = 0$, $\upsilon_{0y} = \upsilon_0 \sin\theta$, acceleration $= -g$ now using $0^2 = \upsilon_{0y}^2 + 2gh$,

we have $$0 = (\upsilon_0 \sin\theta)^2 - 2gh$$

or $$h = \frac{(\upsilon_0 \sin\theta)^2}{2g} .$$

3. <u>Total time of flight.</u>

When the body returns to the same horizontal level, the displacement y in the vertical direction is zero. Using Eq. 17 we obtain:

$$0 = (\upsilon_0 \sin\theta) t - \frac{1}{2} g t^2$$

or $$t = \frac{2\upsilon_0 \sin\theta}{g} .$$

4. <u>Horizontal range.</u>

During time t, the body has moved horizontally with a constant velocity $\upsilon_{0x} = \upsilon \cos\theta$.

Then horizontal range

$$x = (\upsilon_0 \cos\theta) t = \upsilon_0 \cos\theta \frac{2\upsilon_0 \sin\theta}{g} = \frac{\upsilon_0^2 \sin 2\theta}{g} .$$

The range will be maximum when θ = 45°, i.e.

$$x_{max} = \frac{\upsilon_0^2}{g}.$$

5. <u>Equation of trajectory.</u>

In a projectile motion

$$x = (\upsilon_0 \cos\theta)t, \tag{19}$$

$$y = (\upsilon_0 \sin\theta)t - \frac{1}{2}gt^2. \tag{20}$$

From Eq. (19)

$$t = \frac{x}{\upsilon_0 \cos\theta}.$$

Putting this value in Eq. (20), we get

$$y = (\upsilon_0 \sin\theta)\frac{x}{\upsilon_0 \cos\theta} - \frac{1}{2}g\left(\frac{x}{\upsilon_0 \cos\theta}\right)^2$$

or

$$y = x\tan\theta - \frac{g^2}{2\upsilon_0^2 \cos^2\theta}.$$

This is the equation of a parabola. Thus the equation of the trajectory is a parabola.

Range and time of flight on an inclined plane

Let us consider an inclined plane, which makes an angle β with the horizontal as shown in Fig. 8. Let a particle is projected at an angle α with the horizon with a velocity υ_0. This particle strikes the inclined plane at a point A. Our aim is to find out the time of flight and range OA. The initial velocity υ_0 can be resolved in two components:

a) $\upsilon_0 \cos(\alpha-\beta)$ along the plane and
b) $\upsilon_0 \sin(\alpha-\beta)$ perpendicular to the plane

Similarly, g can be resolved in two components:

a) $g_{//} = g \sin\beta$ - parallel to the plane and
b) $g_{\perp} = g \cos\beta$ - perpendicular to the plane.

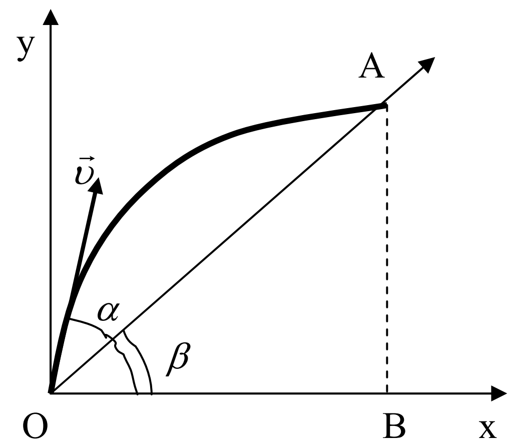


Fig. 8

If t be the time taken by the particle to go from O to A, then in this time the distance described perpendicular to OA is zero. Applying equation

$$S = \upsilon_0 t + \frac{1}{2}gt^2$$

we get $$0 = (\upsilon_0 \sin(\alpha - \beta))t - \left(\frac{1}{2} g \cos\beta\right) t^2$$

or $$t = \frac{2\upsilon_0 \sin(\alpha - \beta)}{g \cos\beta}. \tag{21}$$

Equation (21) represents the time of flight. During this time t, the horizontal velocity $\upsilon_0 \cos\alpha$ along OB is given by

$$OB = (\upsilon_0 \cos\alpha)\frac{2\upsilon_0 \sin(\alpha - \beta)}{g\cos\beta} = \frac{2\upsilon_0^2 \sin(\alpha - \beta)\cos\alpha}{g\cos\beta}.$$

From ΔOAB:

$$OA = \frac{OB}{\cos\beta} = \frac{2\upsilon_0^2 \sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}.$$

Questions

1. A ball is dropped from the window of a moving train on horizontal rails. What is the path followed by the ball on reaching the ground? (Ans.: A parabolic path).
2. A passenger sitting in a train moving with constant horizontal velocity drops a ball vertically downward. What is the path observed by (a) a man in the train, (b) a man standing on the ground near the train and (c) a man in a second train moving in the opposite direction to the first train on a parallel track? (Ans.: (a) A straight line (vertically downward); (b) parabola and (c) parabola).
3. Why does the direction of motion of a projectile become horizontal at the highest point of its trajectory? (Ans.: At the highest point the vertical velocity becomes zero).
4. At what point of the projectile path the speed is minimum? At which point maximum? (Ans.: At the highest point, at the projection point).
5. At what angle with the horizontal a player should throw a ball so that it may go to (a) a maximum distance? (b) For maximum height? (Ans.: (a) 45°, (b) 90°.)
6. Two bombs of 10 and 15 kg are thrown from cannon with the same velocity in the same direction. Which bomb will reach the ground first? (Ans.: Both reach simultaneously, because the time of flight does not depend upon the mass (if air resistance is negligible)).
7. A ball is dropped gently from the top of a tower and another ball is thrown horizontally at the same time. Which ball will hit the ground earlier?

(Ans.: Vertical component of velocity of both balls is zero, therefore, time of flight of each ball $t = \frac{\sqrt{2h}}{g}$).

Solved examples

Example 1. A stone is projected from the ground with a velocity of 25 m/s. Two seconds letter it just meets a wall 5 meters high. Find (a) the angle of projection of the stone, (b) the greatest height reached, (c) how far beyond the wall the stone again hits the ground. Neglect air resistance.

Solution. Let the stone be projected at angle θ above the horizontal. Horizontal component of initial velocity $\upsilon_{0x} = 25\cos\theta$, vertical component of initial velocity $\upsilon_{0y} = 25\sin\theta$.

a) We consider the vertical motion of the stone. The upward direction is taken as positive. Here $\upsilon_{0y} = 25\sin\theta$, $g = -10$ m/s^2, h = 5 m, t = 2 s.

Using $$h = \upsilon_0 t + \frac{1}{2}gt^2,$$

we have $$5 = (25\sin\theta)\cdot 2 - \frac{1}{2}\cdot 10\cdot 4$$

or $$25 = 50\sin\theta;\quad \sin\theta = \frac{1}{2};\quad \theta = 30^0.$$

b) To calculate the greatest height reached, we use the formula
$$\upsilon_y^2 = \upsilon_{0y}^2 + 2gh;\ \upsilon_y = 0;\ \upsilon_{0y} = 25\sin 30^0 = 12{,}5,$$
i.e. $$0 = 12{,}5^2 - 2\cdot 10h;\ h = 7{,}8 \text{ m}.$$

c) Total time of flight $t = \dfrac{2\upsilon_0\sin\theta}{g} = \dfrac{2\cdot 25}{2\cdot 10} = 2{,}5$ s.

According to given problem, the time taken to clear the wall is two seconds, hence time in air after clearing the wall $t_1 = (2{,}5\text{-}2) = 0{,}5$ s. Horizontal distance traveled during the interval $S = \upsilon_0 t\cos\theta = 25\cdot 0{,}5\cos 30° = 25\dfrac{\sqrt{3}}{2}\cdot 0{,}5 = 10{,}8$ m.

Example 2. A stone is thrown from the top of a tower of height 50 m with a velocity of 30 m/s at an angle of 30° above the horizontal (Fig. 9).

Find:

(a) The time during which the stone will be in air.
(b) The distance from the tower base to where the stone will hit the ground.
(c) The speed with which the stone will hit the ground.
(d) The angle formed by trajectory of the stone with the horizontal at the point of hit.

Solution. The situation is shown (a) horizontal component of velocity $\upsilon_{0x} = 30\cos 30° = 25{,}98$ m/s.

Let t be the time taken by stone to reach the ground, i.e. the time during which the stone will be in air. Taking the upward direction as positive, we have
$$h = \upsilon_{0y}t + \frac{1}{2}gt^2,$$
as $$-50 = 15t - 0{,}5\cdot 10t^2.$$
Solving for t, we get $t = 5$ s.

(b) The distance S where the stone will hit the ground

$S = \upsilon_{0x}t = 25{,}98 \cdot 5 = 104{,}98$ m.

(c) From Fig. 9 $\upsilon_x = \upsilon_{0x} = 25{,}98$ m/s and $\upsilon_y = -15 + (10 \cdot 5) = 35$ m/s. Now

$$\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2} = \sqrt{25{,}98^2 + 35^2} = 43{,}6 \text{ m/s}$$

(d)
$$tan\,\theta = \frac{\upsilon_y}{\upsilon_x}$$

or
$$\theta = arctan\frac{35}{25{,}98}.$$

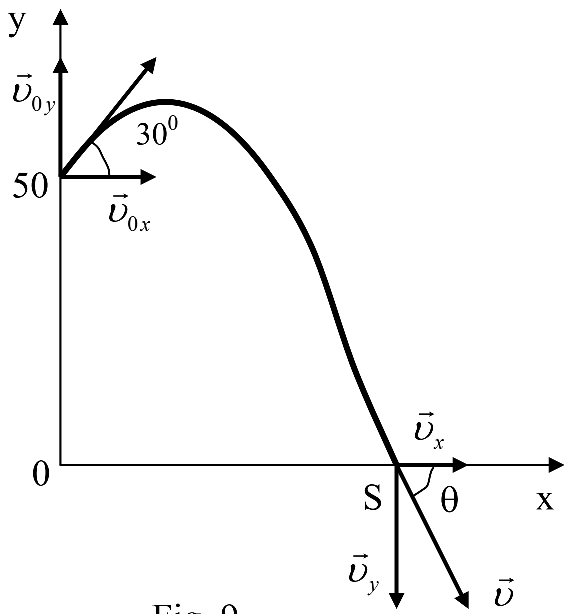


Fig. 9

Problems

1. An airplane is flying in a horizontal direction with a velocity 600 km/hour and at height of 1960 m. When it is vertically below the point A on the ground, a body is dropped from its. The body strikes the ground at point B. Calculate the distance AB. (Ans.: 3,333 km).
2. A stone is thrown from the ground towards a wall 6 m high at a distance of 4 m such that it just clears the top of the wall. Find the speed of projection of the stone. (Ans.: 11,43 m/s).
3. From the top of a tower of height 40 m, a ball is projected upwards with a speed of 20 m/s at an angle of 30° to the horizontal. When and at what distance from the foot of the tower does the ball hit the ground? What is the velocity of the ball at this instant? (Ans.: 4 s; $40\sqrt{3}$ m; 34,64 m/s).

3. Dynamics

In this chapter we begin to study more general problems involving the relation of motion to its causes. These problems form the area called *dynamics*. All of dynamics is based on three principles called *Newton laws of motion.* The first law states that when the vector sum of forces on a body is zero, the acceleration of the body is also zero. The second law relates force to acceleration when the vector sum of forces is not zero. And the third law relates the pair of forces that interacting body exert on each other.

3.1. Mass and Second Newton Law

We know from experience that a body at rest never starts to move by itself; some other body has to apply a push or pull on it. Similarly, when a body is already in motion, a force is required to slow it down or stop it. To make a moving body deviate from straight-line motion, we must apply a sideways force. All these processes involve a change in either the magnitude or the direction of the velocity. In each case the body has acceleration and the force must act on it to cause this acceleration.

Before the time of Galileo and Newton, it was generally believed that a force was necessary just to keep a body moving, even on a level, frictionless surface or in outer space. Galileo and Newton realized that *no* net force is necessary to keep a body moving, once it has been set in motion, and that the effect of a force is not to *maintain* the velocity of a body, but to *change* its velocity. The *rate of change* of velocity for a given body is directly proportional to the force acting on it.

To say that the acceleration of a body is directly proportional to the force exerted on it is to say that the *ratio* of the force to the acceleration is a constant, regardless of the magnitude of the force. This ratio is called the **mass** *m* of the body. Thus

$$m = \frac{F}{a}$$

or

$$F = ma. \qquad (22)$$

We can think of the mass of a body as *the force per unit of acceleration.* For example, if the acceleration of a certain body is 5 $m \cdot s^{-2}$ when the force is 20 N, the mass of the body is

$$m = \frac{20\ N}{5m \cdot s^{-2}} = 4\ N \cdot m^{-1} s^{2}$$

and a force of 4 N must be exerted on the body for each $m \cdot s^{-2}$ of acceleration This relationship can also be used to compare masses quantitatively. Suppose we apply a certain force F to a body having mass m_1 and observe an acceleration of a_1.. We then apply the *same* force to another body having mass

m_2, observing an acceleration a_2. Then, according to Eq. (22), $m_1 a_1 = m_2 a_2$; or

$$\frac{m_2}{m_1} = \frac{a_1}{a_2}.$$

We can use this relation to compare any mass with a standard mass. If m_1 is a standard mass and m_2 an unknown mass, we can apply the same force to each and measure the accelerations; the ratio of the masses is the inverse of the ratio of the accelerations. When a large force is needed to give a body a certain acceleration (i.e., speed it up, slow it down, or deviate it if it is in motion), the mass of the body is large; if only a small force is needed for the same acceleration, the mass is small. Thus the mass of a body is a quantitative measure of the property described in everyday language as *inertia.*

To identify another important property of mass, we measure the masses of two bodies, using the procedure just described, and then fasten them together and measure the mass of the composite body. If m_1 and m_2 are the individual masses, the mass of the composite body is always found to be $(m_1 + m_2)$. This very important result shows that mass is an *additive* quantity, and that it is directly correlated with quantity of matter. Indeed, the concept of mass is one way to give the term *quantity of matter* a precise meaning.

In the discussion above, the particle moves along a straight line (the *x*-axis), and the force also lies along this direction. This is of course a special case. More generally, the force may also have a component in the y-direction, and the particle's motion need not be confined to a straight line. Furthermore, more than one force may act on the particle. Thus this formulation needs to be generalized to include motion in a plane or in space and the possibility of several forces acting simultaneously.

Experiments show that *when several forces act on a particle at the same time, the acceleration is the same as would be produced by a single force equal to the vector sum of these forces.* This sum is usually most conveniently handled by using the method of components. When several forces act on a particle moving along the x-axis,

$$\sum F_x = m a_x .$$

When a particle moves in a plane, with position described by coordinates *(x,y),* the velocity is a vector quantity with components υ_x and υ_y equal to the time rates of change of *x* and *y,* respectively, and the acceleration is a vector quantity with components a_x and a_y equal to the rates of change of υ_x and υ_y, respectively. Then a more general formulation of the relation of force to acceleration is

$$\sum F_x = ma_x\,,\ \ \sum F_y = ma_y\,.$$

This pair of equations is equivalent to the single vector equation

$$\sum \vec{F} = m\vec{a}\,, \tag{23}$$

where we write the left-hand side explicitly as $\sum \vec{F}$ to emphasize that the acceleration is determined by the resultant of all the forces acting on the particle. If the particle moves in three dimensions, then of course Eq. (23) include a third equation for the z-components $\sum F_z = ma_z$.

Equation (23), is the mathematical statement of Newton's *second law of motion.* The acceleration of a body (the rate of change of its velocity) is equal to the resultant (vector sum) of all forces acting on the particle, divided by its mass, and has the same direction as the resultant force.

To introduce the concept of impulse and momentum let us consider a particle of mass *m* moving in space and acted on by a varying resultant force *F*. Newton's second law states that $\vec{F} = m\vec{a} = m\dfrac{d\vec{\upsilon}}{dt}$

or
$$\vec{F}dt = md\vec{\upsilon}\,.$$

Then
$$\int_{t_1}^{t_2} \vec{F}dt = \int_{\upsilon_1}^{\upsilon_2} md\vec{\upsilon}\,.$$

The integral $\int_{t_1}^{t_2} \vec{F}\,dt$ is the impulse of force $\vec{F}$ in the time interval $(t_2 - t_1)$ and is the vector quantity. The integral $\int_{\upsilon_1}^{\upsilon_2} md\vec{\upsilon}$ is momentum. So, Newton's second law can be written

as
$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{\boldsymbol{dt}}\,. \tag{24}$$

<u>*Important points regarding Newton's law*</u>

1. If a body is in equilibrium, then it does not mean that no force acts on the body but it simply means that the net force (resultant of a number of force acting on the body) on the body is zero.
2. Action and reaction are always equal and opposite and they act on different bodies.
3. Whenever a force acts on a body, the reaction R always acts normal to surface of the body. Consider the case of a book placed on the table. The book applies action force on the table equal to its weight mg downward. Now the table exerts an equal reactionary force mg on the book in the upward direction.

4. Consider the case of a mass *m* attached to a linear spring. Let, by application of a force F_1 the extension or compression in the spring be *x* then $F = kx$ where *k* is known as force constant of the spring. The unit of *k* is Newton/meter.

3.2. Weight of a body in a lift

Earth attracts every body towards its center. The force of attraction exerted by the earth on the body is called *gravity force*. If m be the mass of the body then the gravity force on it will be mg. Generally, the weight of a body is equal to the gravity force $P = mg$. But when the body is on an accelerated platform, the weight of a body appears. The new weight is called as *apparent weight*. Here we shall consider the apparent weight of a man standing in a lift which is in motion. We consider the following cases:

1. The lift is not accelerated (i.e. $\upsilon = 0$ or constant). The situation is shown in Fig. 10 (a). In this case $F_R = ma = 0$. Hence apparent weight

$$\boldsymbol{P' = \text{actual weight} = mg.}$$

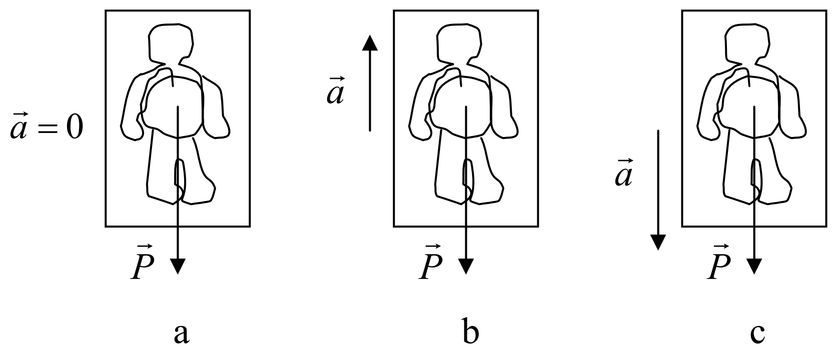


Fig. 10

2. When the lift is accelerated upward. In this case, there will be two forces acting on the man, i.e. weight mg and reaction $F_R = ma$ both acting in the downward direction as shown in Fig. 10 (b).
   Apparent weight $P' = mg + F_R = mg + ma = \boldsymbol{m(g + a)}$ or apparent weight > > actual weight.
3. When the lift is accelerated downward. This situation is shown in Fig. 10 (c). Here the weight *mg* acts downward while the reaction $F_R = ma$ acts upward. We assume that $a < g$. Hence apparent weight $P' = mg - F_R = mg - ma = \boldsymbol{m(g - a)}$ i.e. apparent weight $P' <$ actual weight *P*. When the lift is accelerated downward such that $a > g$, then $F_R = ma$ is greater than weight *mg*. Apparent weight $\boldsymbol{P' = m(g - a)} < 0$ – negative so the man will be accelerated upward and will stay at the ceiling of the lift.

   Now we consider the special case when $g = a$. In this case apparent weight $\boldsymbol{P' = 0}$. Thus in a freely falling lift, the man will experience a state of weightlessness.

3.3. Motion of connected bodies

1) Two bodies. Let us consider the case of two bodies of mass m_1 and m_2 connected by a thread and placed on a smooth horizontal surface as shown in Fig. 11. A force $\vec{F}$ is applied on the body of mass m_2 in forward direction as shown. Our aim is to consider acceleration of the system and the tension T in the thread. The forces acting separately on two bodies are also shown in the figure.

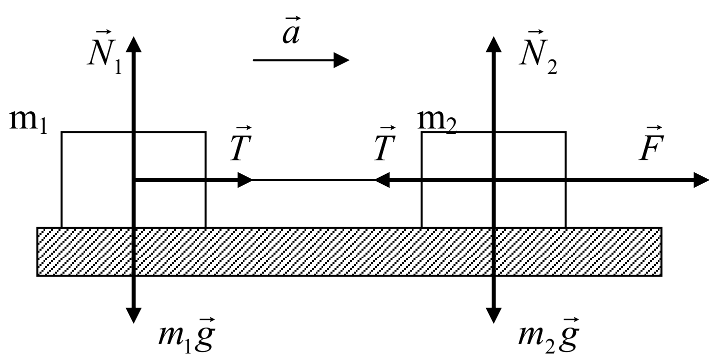


Fig. 11

From Fig. 11
$$\begin{cases} T = m_1 a, \\ F - T = m_2 a. \end{cases}$$

Adding first and second equations $a = \dfrac{F}{m_1 + m_2}$

and $F = (m_1 + m_2)\, a,$

tension $T = m_1 a = \dfrac{m_1 F}{m_1 + m_2}.$

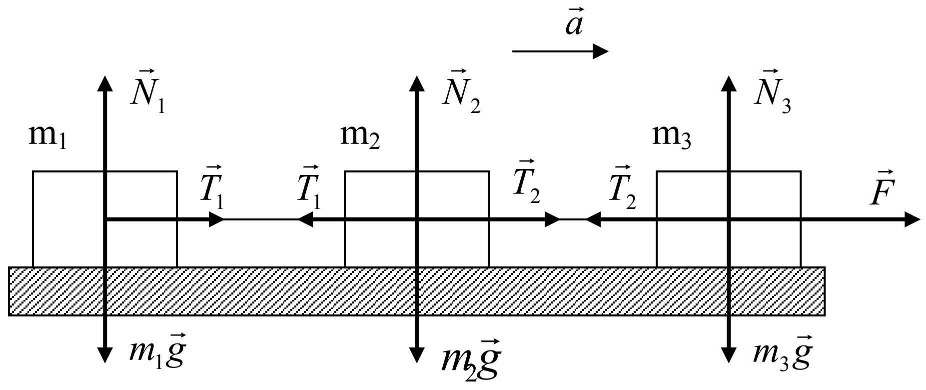


Fig. 12

2) Three bodies. In case of three bodies, the situation is shown in Fig. 12.

Acceleration $a = \dfrac{F}{m_1 + m_2 + m_3},$

$$T_1 = m_1 a = \frac{m_1 F}{m_1 + m_2 + m_3},$$

$$F - T_2 = m_3 a$$

and
$$T_2 = F - \frac{m_3 F}{m_1 + m_2 + m_3}.$$

3.4. Motion of a body on a smooth inclined plane

Let us consider the case of a body of mass m placed over a smooth fixed plane AB making an angle θ with the horizontal as shown in Fig. 13. *N* is normal reaction of the smooth surface on the body, *mg* is the weight of the body acting downward.

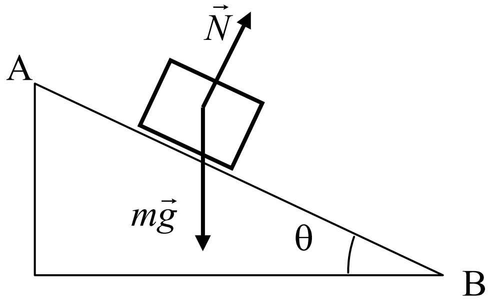


Fig. 13

Resolving these two forces along AB and perpendicular to AB, we have

$$\begin{cases} ma = mg\sin\theta, \\ N = mg\cos\theta. \end{cases}$$

So we have $a = g\sin\theta$ and $N = mg\cos\theta$.

The same results can also be obtained by resolving the forces horizontally and vertically.

3.5. Motion of two bodies connected by a string

Case 1. Let us consider the case of two bodies of mass m_1 and m_2, which are connected by light inextensible string passing over a light smooth pulley (Fig. 14).

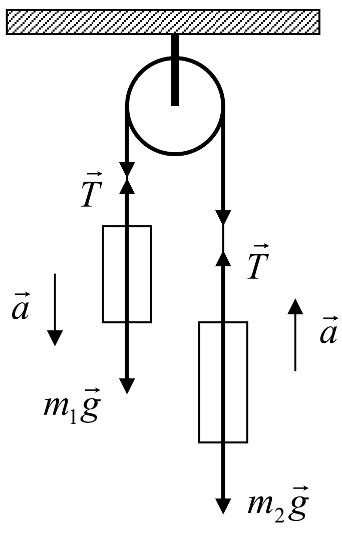


Fig. 14

Here it is assumed that $m_1 > m_2$. Our aim is to find out the acceleration of the system and tension in the string.

The mass m_1 has downward acceleration hence

$$m_1 g - T = m_1 a. \quad (25)$$

The mass m_2 has upward acceleration hence

$$T - m_2 g = m_2 a. \quad (26)$$

Solving Eqs. (25) and (26), we get

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

and
$$T = \frac{2m_1 m_2}{m_1 + m_2} g.$$

Case 2. Let us consider the case of a body of mass m_1, to which a light and inextensible string is attached, rests on a smooth horizontal table. The string passes over a frictionless pulley fixed at the end of table. Another end of the string carries a mass m_2 as shown in Fig. 15. Our aim is to calculate the acceleration of the system and tension in the string. Here we have

$$m_2g - T = m_2a \tag{27}$$

and

$$T = m_1a. \tag{28}$$

Solving Eqs. (27) and (28), we have $a = \dfrac{m_2}{m_1 + m_2}g$

and

$$T = m_1a = \frac{m_1m_2}{m_1 + m_2}g .$$

Case 3. Here we shell consider the above case with a difference that m_1 is placed on smooth inclined plane making an angle θ with horizontal as shown in Fig. 16. In this case

$$\begin{cases} T - m_1g\sin\theta = m_1a, \\ m_2g - T = m_2a \end{cases}$$

and

$$T = m_2g\left[1 - \frac{m_2 - m_1\,sin\,\theta}{m_1 + m_2}\right].$$

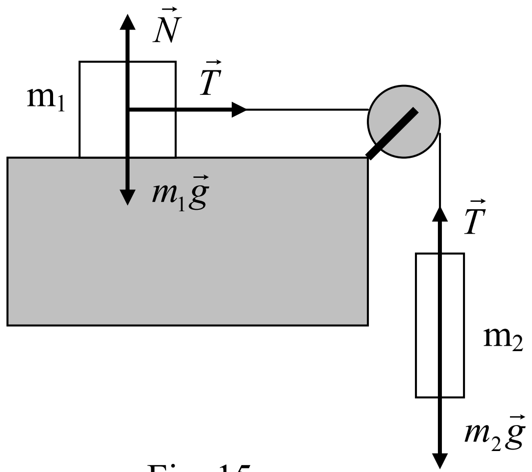


Fig. 15

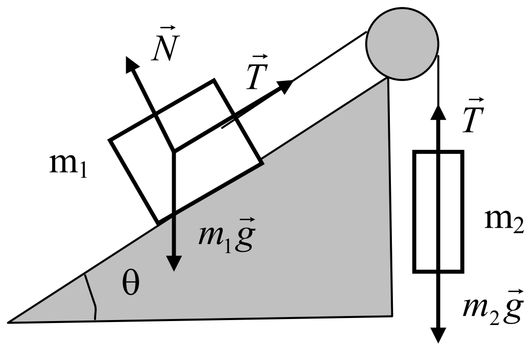


Fig. 16

Case 4. Let us consider the case when masses m_1 and m_2 are on inclined plane making angle α and β with horizontal respectively as shown in Fig.17. Here we have

$$\begin{cases} m_1g\sin\alpha - T = m_1a, \\ T - m_2g\sin\beta = m_2a. \end{cases}$$

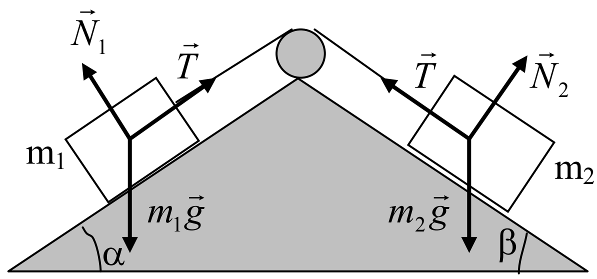


Fig.17

Solving system we get

$$a = \frac{g(m_1 \sin\alpha - m_2 \sin\beta)}{m_1 + m_2}$$

and $T = m_2 g\left(\frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2} + \sin\beta\right)$

or
$$T = \frac{m_1 m_2}{m_1 + m_2} g(\sin\alpha + \sin\beta).$$

Case 5. The body of mass *m* is moving in plane *xy* according the law $x = A \sin\omega t,\ y = B \cos\omega t$ and vector of force acting on the body.

Solution. According the second Newton's law:

$$\vec{F} = m\vec{a}.$$

As we are given mass *m*, acceleration $\vec{a}$ we get as follows:
as $x = A \sin\omega t$ then

$$\upsilon_x = \frac{dx}{dt} = A\omega \cos\omega t$$

and
$$a_x = \frac{d\upsilon_x}{dt} = -A\omega^2 \sin\omega t,$$

as $y = B \cos\omega t$ then
$$\upsilon_y = \frac{dy}{dt} = -B\omega \sin\omega t$$

and
$$a_y = \frac{d\upsilon_y}{dt} = -B\omega^2 \cos\omega t.$$

Hence acceleration vector is

$$\vec{a} = a_x\vec{i} + a_y\vec{j} = -\omega^2\left(A\sin\omega t \cdot \vec{i} + B\cos\omega t \cdot \vec{j}\right) = -\omega^2\vec{r},$$

where
$$\vec{r} = x\vec{i} + y\vec{j} = A\sin\omega t \cdot \vec{i} + B\cos\omega t \cdot \vec{j}.$$

Force $\vec{F}$ is
$$\vec{F} = m\vec{a} = -m\omega^2\vec{r}$$

and module of force vector $F = \sqrt{F_x^2 + F_y^2} = m\omega^2\sqrt{x^2 + y^2}$.

Case 6. Body with mass $m = 1$ kg is at rest at the origin. At time $t = 0$ the force $\vec{F} = 2\vec{i} + 6t\vec{j}$ begins to act on the body. Define the body's trajectory.

The Newton's equation is $m\frac{d\vec{\upsilon}}{dt} = 2\vec{i} + 6t\vec{j}$.

This equation in components is written as

$$m\frac{d\upsilon_x}{dt} = 2;\ m\frac{d\upsilon_y}{dt} = 6t.$$

Hence $\int d\upsilon_x = 2\int dt + C_1;\ \int d\upsilon_y = 6\int t dt + C_2$, (as $m = 1$)

and $\upsilon_x = 2t + C_1$; $\upsilon_y = 3t^2 + C_2$.

We define constants of integration $C_1$and C_2 by using initial conditions $\upsilon_x(0) = 0$, $\upsilon_y(0) = 0$: $C_1 = 0$ and $C_2 = 0$.

Then
$$\begin{cases} \upsilon_x = \dfrac{dx}{dt} = 2t, \\ \upsilon_y = \dfrac{dy}{dt} = 3t^2 \end{cases}$$

and $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$,

$\int dx = \int 2t dt$, $\int dy = 3\int t^2 dt$,

$x = 2\dfrac{t^2}{2} + C_3 = t^2 + C_3$, $y = t^3 + C_4$,

$x(0) = t^2 + C_3 = 0 \Rightarrow C_3 = 0$, $y(0) = t^3 + C_4 = 0 \Rightarrow C_4 = 0$,

$x(t) = t^2$; $y(t) = t^3$.

To define trajectory we have to obtain to connection between x and y. As $x = t^2$ hence $t = \sqrt{x}$ and $y = t^3 = \left(\sqrt{x}\right)^3 = x^{\frac{3}{2}}$, i.e.
$$y = x^{\frac{3}{2}}.$$

Questions

1. A person sitting in a train moving with constant velocity throws a ball vertical upward. Will the ball return to thrower’s hand?
2. According to Newton’s third law every force is accompanied by on equal and opposite force. How can a movement ever take place?
3. A cord from a ceiling of a motorcar suspends a ball. What will be the effect on the position of the ball if:
   a) the car is moving with constant velocity;
   b) the car is moving with acceleration motion;
   c) the car is turning towards right?
4. Air is thrown on a sail attested to boat from an electric fan placed on the boat. Will the boat start moving?
5. Two bodies of mass M and m are allowed to fall from the same height. If air resistance for each be the same, then will both the bodies reach the earth simultaneously?
6. A man stands in a lift going downward with uniform velocity. He experiences a loss of weight at the start but not when lift is in uniform motion. Explain why?

Problems

1. A body of 0,02 kg falls from a height of 5 meter into a pile of sand. The body penetrates the sand a distance of 5 cm before stopping. What force has the sand exerted on the body? ($F = -$ 19,6 N).

2. A block of ice slides down from the top of an inclined roof of a house (angle of inclination of roof = 30° to the horizontal). The highest and lowest point of the roof are at height of 8,1 m and 5,6 m respectively from the ground. At what horizontal distance from the starting point will the block hit the ground (neglect friction)? (S = 8,93 m).

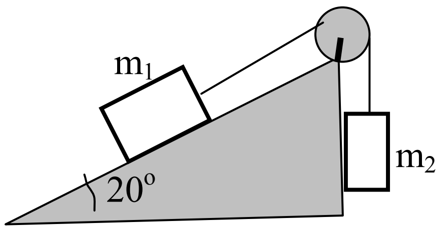


Fig. 18

3. A block mass m_1 = 4 kg on a smooth inclined plane of 30° is connected by a cord over a small, frictionless pulley to a second block of mass m_2 = 5 kg, hanging vertically (Fig. 18). Calculate the acceleration with which the block moves and also the tensions in the cord.
(T = 33,3 nt ; a = 3,33 m/s^2).

4. The force applied on the body is $\vec{F} = 4t\vec{i} + 2t\vec{j}$. Determine the change of impulse of the body in the time interval $0 \le t \le \tau$. ($\Delta\vec{p} = \tau^4\vec{i} + \tau^2\vec{j}$).

5. The body of mass m is in uniform motion along x-axis with velocity $\upsilon_{ox} = \upsilon_o$. At time $t = 0$ the force $\vec{F} = bt\vec{j}$ is applied on the body (b = const > 0). Calculate the trajectory of body's motion if at time $t = 0$ body was at the origin.

Answer: $\left(y = \dfrac{bx^3}{6m\upsilon_0^3} \right)$.

<u>3.6. Center of mass of a system of particles and rigid bodies</u>

The point at which the whole mass of the body may be supposed to be concentrated is called the *center of mass*.

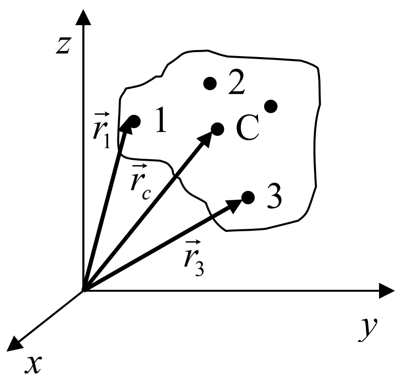


Fig. 19

Consider the case of a body of arbitrary shape as shown in Fig.19. Let the body consists of a number of particles P$_1$, P$_2$, P$_3$ … of masses m_1, m_2, m_3 … and coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) … . If (x_c, y_c, z_c) be the coordinates of center of mass, then

$$x_c = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3 + ...}{m_1 + m_2 + m_3 + ...};$$

$$y_c = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3 + ...}{m_1 + m_2 + m_3 + ...};$$

$$z_c = \frac{z_1 m_1 + z_2 m_2 + z_3 m_3 + ...}{m_1 + m_2 + m_3 + ...}.$$

When there is a continuous distribution of mass instead of being discrete, we treat an infinitesimal element of the body of mass dm_1 whose is (x, y, z). Then we have

$$x_c = \frac{\int xdm}{\int dm} = \frac{\int xdm}{M}; y_c = \frac{\int ydm}{\int dm} = \frac{\int ydm}{M}; z_c = \frac{\int zdm}{\int dm} = \frac{\int zdm}{M},$$

where M is total mass.

Following points should be remembered in case of center of mass:

a) The position of center of mass is independent of the coordinate system chosen;
b) The position of center of mass depends upon the shape of the body and distribution of mass. The center of mass of a circular disk is within in material of the body while that of a circular ring is outside the material of the body.
c) In symmetrical bodies in which the distribution of mass is homogeneous, the center of mass is coincides with the symmetry i.e. geometrical center.

Solved examples

Ex.1. A circular plate of uniform thickness has a diameter D = 56 cm. A circular portion of diameter d = 42 cm is removed from one edge of the plate as shown in Fig. 20. Find the position of center of mass of the remaining portion.

Solution. Suppose the plate is uniform. If O be the center of mass of the whole plate and C_1, the center of mass of the cut out circular portion, then the center of mass of the remaining portion will lie on the C_1O. Let C_2 be the center of mass of the remaining portion.

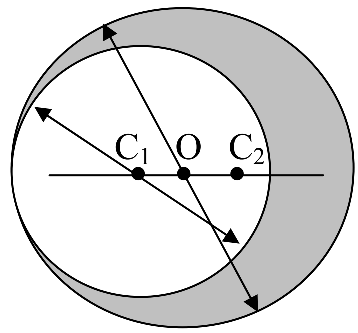


Fig. 20

Area of the whole plate $S = \frac{\pi D^2}{4}$.

Area of the cut out portion $S_1 = \frac{\pi d^2}{4}$.

Area of remaining portion $S_2 = S - S_1$.

Since the weights are proportional to areas then

$$\frac{weigt \quad of \quad cut \quad out \quad portion}{weigt \quad of \quad remain \quad portion} = \frac{S_1}{S_2} = \frac{9}{7} = \frac{m_1 g}{m_2 g}.$$

Taking moment about C_1 we get $OC_1 = \frac{D}{2} - \frac{d}{2} = 7$ cm,

or $m_1 g \cdot OC_1 = m_2 g \cdot OC_2$,

as $OC_2 = \frac{m_1 g}{m_2 g} OC_1 = \frac{9}{7} \cdot 7 = 9$ cm.

Ex.2. Three ball with masses m, $2m$, and $3m$ are located along x-axis in such a way that distance between their center is l (Fig. 21). Find the position of center of mass of the system.

Solution. Suppose the origin is in the center of the first ball. Then its coordinate is $x_1 = 0$. The coordinate of the second ball is $x_2 = l$ and of the third ball is $x_3 = 2l$.

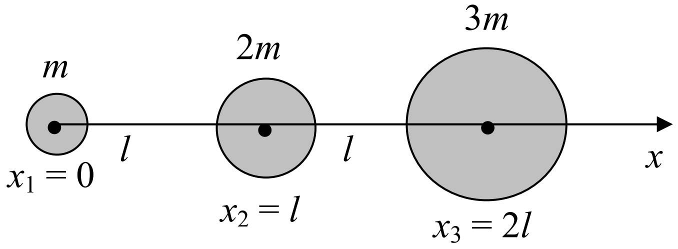


Fig. 21

Then position of the center of mass of system can be obtained as follows

$$x_c = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{0 + 2ml + 3m2l}{m + 2m + 3m} = \frac{4}{3} l .$$

Problems

1. Cylindrical rod with length l is located along x-axis. It's density changes with x according the law: $\rho = \rho_0 \left(1 - \frac{x}{l}\right)$. Find the position of the center of mass of this rod. ($x_c = \ell/3$).

2. Locate the center of a system of particles of masses $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg, situated at the corners of an equilateral triangle of side 1,0 m. ($x_c = \frac{3,5}{6} m; y_c = \frac{\sqrt{3}}{4} m$).

3. Calculate the position of center of mass of the system Earth-Moon. Distance between Earth and Moon $r = 3,84 \cdot 10^8$ m, mass of the Earth $m_E = 5,96 \cdot 10^{24}$ kg, mass of the Moon $m_M = 7,3 \cdot 10^{22}$ kg. ($r_c = 4600$ km from the center of Earth).

4. Work and energy

Energy is one of the most important concepts in all physics science. Its importance stems from the principle of **conservation of energy**, which state that it any isolated system the total energy of all forms is constant.

4.1. Work

In every day life **work** is any activity that requires muscular or mental exertion. Physicist use the term work in a much more specific sense, involving a force acting on a body while the body undergoes a displacement. When a body moves a distance S along straight line while a constant force of magnitude F,

$$A = FS$$

directed along the line acts on it, the work A done by the force is defined as.

The force need not have the same direction as the displacement. In Fig. 22, the force $\vec{F}$, assumed constant makes an angle θ with the displacement. The work A done by this force when its point of application undergoes a displacement $\vec{S}$ is defined as the product of the magnitude of the displacement and the component of force in the direction of the displacement.

The component of $\vec{F}$ in the direction of $\vec{S}$ is $F\cos\theta$. Then

$$A = (F\cos\alpha)S. \qquad (29)$$

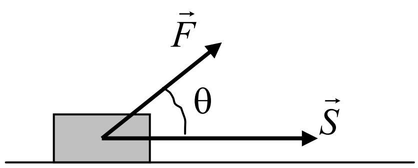


Fig. 22

An alternative interpretation of Eq. (29) is that $S\cos\theta$ is the component of displacement in the direction of $\vec{F}$. Thus the work is also the component of displacement in the direction of $\vec{F}$ multiplied by the magnitude of F.

In other words, work can be expressed as scalar product of two vectors:

$$\boldsymbol{A} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{S}}.$$

Work itself is a scalar quantity. Work is an algebraic quantity: it can be positive or negative. When the component the force is the same direction, as the displacement, the work A is positive. When it is opposite to the displacement, the work is negative. If the force is at right angles to the displacement, it has no component in the direction of the displacement, and the work is zero.

When several external forces act an a body, it is useful to consider the work done by each separate force. Each of these may be computed from the definition of work in Eq. (29). Then since work is scalar quantity, the total work is the algebraic sum of the individual works. When several forces act on a body, there are always two equivalent ways to calculate the total work. We may calculate the work done by each force separately and take the algebraic sum of these works, or we may compute the vector sum or resultant of the forces and compute the work done by the resultant.

4.2. Work done by a varying force

After work is done by a force that varies in magnitude or direction. Suppose a particle moves along a line under the action of a force directed along the line but varying with particle's position. In Fig. 23 the force magnitude is shown as a function of the particle's coordinate x.

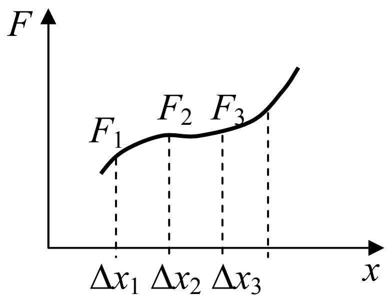


Fig. 23

To find the work, done by this force, we divide the displacement into short segments Δx. We approximate the varying force by one that is constant with in each segment. The force then has approximately the value F_1 in segment Δx_1, F_2 in segment Δx_2, and so on. The work done in the first segment is then $F_1\Delta x_1$ that in second is $F_2\Delta x_2$, and so on. The total work is

$$A = F_1\Delta x_1 + F_2\Delta x_2 + F_3\Delta x_3 + \ldots$$

As the number of segments becomes very large and the size of each very small, this sum becomes (in the limit) the integral of F from x_1 to x_2:

$$A = \int_{x_1}^{x_2} F dx. \qquad (30)$$

Note, that integral represents the area under the curve in Fig. 23.

If the force also varies in direction during the displacement, then F in Eq. (30) must be replaced by the component of force in the direction of displacement. Then we have

$$A = \int_{x_1}^{x_2} F \cos\theta dx.$$

The definition of work can be generalized further to include motion along a curved path (Fig. 24). We imagine dividing the portion of the curve between points P_1 and P_2 into many infinitesimal vector displacements, and we call a typical one of them $d\vec{l}$. Each $d\vec{l}$ is tangent to the path at its position. Let $\vec{F}$ be the force along the path, and θ the angle between $\vec{F}$ and $d\vec{l}$. Then the small element dA may written as

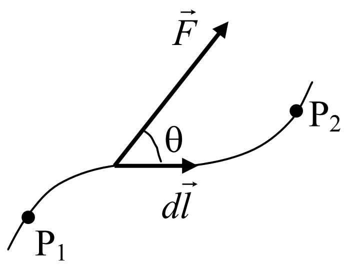


Fig. 24

$$\boldsymbol{dA = F \cos\theta dl = F_{//} dl = \vec{F} d\vec{l}}. \qquad (31)$$

The total work then is

$$\boldsymbol{A = \int_1^2 F \cos\theta dl = \int_1^2 F_{//} dl = \int_1^2 \vec{F} d\vec{l}}.$$

This integral is called a line integral.

4.3. Kinetic energy

Isolated system. The bodies system is called isolated one when no external forces act on it or when all external forces compensate each other.

Suppose for simplicity that system consists only from one particle. Then the equation of motion, i.e. Newton's second law is $m\frac{d\vec{\upsilon}}{dt} = \vec{F}$.

Multiplying this equation on displacement of the particle $d\vec{r} = \vec{\upsilon}dt$ we obtain

$$m\frac{d\vec{\upsilon}}{dt}\vec{\upsilon}dt = \vec{F}d\vec{r}\,;m\vec{\upsilon}d\vec{\upsilon} = \vec{F}d\vec{r}$$

and
$$d\left(\frac{m\upsilon^2}{2}\right) = \vec{F}d\vec{r}\,.$$

We denote $\boldsymbol{K} = \frac{\boldsymbol{m\upsilon^2}}{\boldsymbol{2}}$ and call it kinetic energy.

Kinetic energy can be expressed by momentum $\vec{p} = m\vec{\upsilon}$

as
$$\boldsymbol{K} = \frac{\boldsymbol{p^2}}{\boldsymbol{2m}}\,. \tag{32}$$

When system is isolated then force $\vec{F}$ is zero and then $d\left(\frac{m\upsilon^2}{2}\right) = 0$. Hence

$$K = \frac{m\upsilon^2}{2} = const\,, \tag{33}$$

i.e. in isolated system the kinetic energy is conserved.

4.4. Work and kinetic energy

The work done on a body is related to the resulting change in the body's motion. Let's consider a body of mass m moving along a straight line under the action of a constant resultant force of magnitude *F* directed along the line. The body's acceleration is given by Newton's second law, $F = ma$. Suppose the speed increases from υ_1 to υ_2 while the body undergoes a displacement $S = x_2 - x_1$.

Then, we have
$$a = \frac{\upsilon_2^2 - \upsilon_1^2}{2S}\,.$$

Hence
$$F = ma = m\frac{\upsilon_2^2 - \upsilon_1^2}{2S}$$

as
$$FS = \frac{1}{2}m\upsilon_2^2 - \frac{1}{2}m\upsilon_1^2 \tag{34}$$

and
$$K = \frac{m\upsilon^2}{2}\,.$$

The product *FS* is the work *A* done by force *F*. The quantity $K = \frac{1}{2} m\upsilon^2$, one – half of the product of the mass of the body and the square of speed, is called its kinetic energy *K*.

The first term of the right side of Eq. (34) is the final kinetic energy of the body, $K_2 = \frac{1}{2} m\upsilon_2^2$, and the second term is the initial kinetic energy, $K_1 = \frac{1}{2} m\upsilon_1^2$. The difference between these terms is the change in kinetic energy and we have the important result that the work done by the resultant external force on a body is equal to the change in kinetic energy of the body

$$\boldsymbol{A = K_2 - K_1 = \Delta K.} \qquad (35)$$

Kinetic energy is a scalar quantity, even thought the particle's velocity is a vector quantity. *K* depends only on speed (the magnitude of velocity) but not on the direction in which particle is moving. The change in kinetic energy depends only on the work $A = FS$, and not on the individual value of *F* and *S*.

If the work *A* is positive, the final kinetic energy is greater that the initial kinetic energy and the kinetic energy increase. If the work is negative, the *K* decreases. In case in which the work is zero, the *K* remains constant.

In Eq. (35) A is the work done by the resultant force. Alternatively, we may calculate the work done by each separate force. *A* is then the algebraic sum of all these quantities of work.

Although we derived Eq. (35) for case of constant resultant force, it is true even when the force varies in an arbitrary way. We divide the total displacement *x* into a large number of small segments Δx. The change of kinetic energy in segment Δx_1 is equal to the work $F_1 \Delta x_1$ and so on. The total kinetic energy change is the sum of the changes in the individual segments and is thus equal to the total work done

$$A = \sum_i F_i \Delta x_i ,$$

or, in general

$$A = \int \vec{F} d\vec{S} . \qquad (36)$$

4.5. Gravitational potential energy

When a gravitational force acts on a body while the body undergoes a velocity displacement, the force does work on the body. This work can be expressed conveniently in terms of the initial and final position of the body. In Fig. 25 a body, having mass m and weigh $P = mg$ moves vertically, from a height above some reference level to a height h_2. The positive direction for *h* is upward. In this Fig. 25 $\vec{F}$ represents the result of all forces on the body. The direction of $\vec{g}$ is opposite to the upward displacement and work done by this force is

$$A_{grav} = FS = -mg(h_1 - h_2).$$

Thus we can express A_{grav} in terms of values of the quantity *mgh* at the beginning and the end of the displacement. This quantity, the product of the weight *mg* and the height *h* above the reference level (the origin of coordinates), is called the gravitational potential energy, *U*:

$$\boldsymbol{U = mgh}. \tag{37}$$

We can express the work A_{grav} done by gravitational force during the displacement from h_1 to h_2

$$A_{grav} = U_1 - U_2 = \Delta U. \tag{38}$$

Thus, when the body moves downward, *h* decreases, the gravitational force does positive work and the potential energy decreases. When the body moves upward, the work done by gravitational force is negative and the potential energy increases.

Note, that if we shift the origin for *h*, then h_1 and h_2 change, but the difference $(h_1 - h_2)$ does note. Similarly U_1 and U_2 change, but the difference $(U_1 - U_2)$ is the same as before. The choice of origin is arbitrary; the physically significant quantity is not the value of *U* at a particular point, but only the difference in *U* between two points.

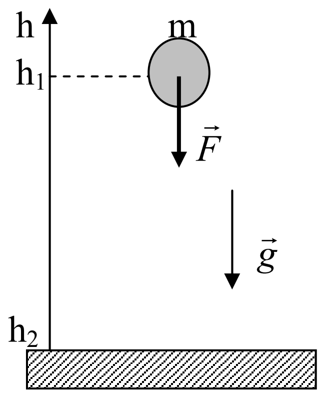


Fig. 25

Now let A_{other} represents the total done by $\vec{F}$ that is, by all forces other then gravitational force. The total work done by all forces is then

$$A = A_{grav} + A_{other}.$$

Since the total work equals to the change in kinetic energy

$$A_{grav} + A_{other} = K_2 - K_1 = \Delta K,$$

$$A_{other} - (mgh_2 - mgh_1) = \frac{1}{2}m\upsilon_2^2 - \frac{1}{2}m\upsilon_1^2. \tag{39}$$

The quantities $\frac{1}{2}m\upsilon_2^2$ and $\frac{1}{2}m\upsilon_1^2$ depends only on the final and initial speeds; the quantities mgh_2 and mgh_1 depend only on the initial and final elevations. Then, Eq. (39) may be written as :

$$A_{other} = \left(\frac{1}{2}m\upsilon_2^2 - \frac{1}{2}m\upsilon_1^2\right) + \left(mgh_2 - mgh_1\right) = \Delta K + \Delta U. \tag{40}$$

The sum of kinetic and potential energies is called the total mechanical energy,

$$\boldsymbol{E = K + U.}$$

Eq.(40) can also be written

$$A_{other} = \left(\frac{1}{2}m\upsilon_2^2 + mgh_2\right) - \left(\frac{1}{2}m\upsilon_1^2 + mgh_1\right) = \\ = \left(K_2 + U_2\right) - \left(K_1 + U_1\right) = E_2 - E_1 = \Delta E. \tag{41}$$

Hence the work done by all forces acting on the body, with the exception of the gravitational force, equals the change in the total mechanical energy of the body. If A_{other} is positive, the mechanical energy increases. If A_{other} is negative, the mechanical energy decreases.

In a case where the only force acted on the body is the gravitational force, the work A_{other} is zero Eq.(41) can be written as $E_1 = E_2$, or

$$\boldsymbol{K_2 + U_2 = K_1 + U_1.}$$

That is total mechanic energy is constant, that is, conserved. This is a particular case of the *principle of conservation mechanical energy*.

Note, that in the case when body travels from initial elevation h_1 to a final elevation h_2 along a slanted or curved path the work done by the gravitational force is the same as when the body travels straight up. To prove this divide, the path into a large number of small segments ΔS; the work done during this displacement is the component of displacement in the direction of the force, multiplied by the magnitude of the force (Fig. 26). The vertical component of displacement is

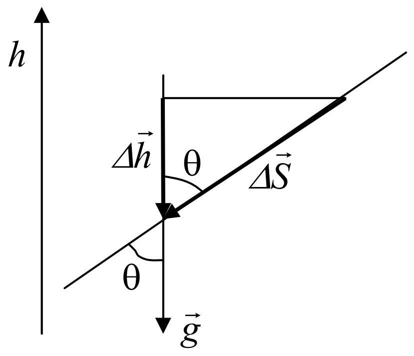


Fig. 26

$$\Delta S\cos\theta = \Delta h.$$

And therefore $A_{grav} = mg(h_1 - h_2) = mg\Delta h$.

Similarly, the work done by a stretched or compressed spring that exert a force $F = -kx$ on a particle where x is the amount of stretch or compression, can be represented in term of a potential energy function

$$U = \frac{1}{2}kx^2, \qquad (42)$$

$$A_{el} = U_1 - U_2 = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2. \qquad (43)$$

4.6. Conservative and dissipative force

We have seen that when a body acted on by a gravitational force moves from one position to another; the work done by the gravitational force is independent of the body's path. A similar situation occurs when a body is attached to a spring and moves from one position to another, changing the extension or compression of the spring. In both case the total mechanical energy is constant or conserved. For this reason the gravitational and elastic force are called a conservative force and work done by these force can be represented as

$$A = U_1 - U_2.$$

The work reversible on the return trip is always exactly the negative of that on the first of the trip. Thus the work done by a conservative force always has these properties:

1. It is independent of the path of the body and depends only on the starting point and end point.
2. It is equal to the difference between initial and final values of the potential energy function.
3. It is completely reversible.
4. When the starting point and end point are the same – that, the path forms a closed loop the total work is zero:

$$A = \oint \vec{F} d\vec{l} = 0. \tag{44}$$

For comparison a function force is a dissipative force, and the total mechanical energy, is not conserved and we have to describe the energy relation in terms of additional kinds of energy.

4.7. Conservation of Momentum

The concept of momentum is most useful in situations involving several interacting bodies. Let's consider first a system consisting of two bodies that interact with each other but not with anything else each body exert a force on the other, so the momentum of each body changes. According to Newton's third law, the forces the bodies exert on each other are always equal in magnitude and opposite in direction. Thus the impulses given to the two bodies in any time interval are also equal and apposite and therefore the momentum changes of the two bodies are equal and apposite.

We define the <u>total momentum</u> of the system $\vec{P}$ as the vector sum of momenta of the bodies in the system:

$$\vec{P} = m\vec{\upsilon}_1 + m\vec{\upsilon}_2 + m\vec{\upsilon}_3 + ... = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... \ . \tag{45}$$

If the change in the momentum of one body is exactly the negative of that of the other, then the change in the total momentum must be zero. Thus *when bodies interact only with each other, their total momentum is constant.*

A force that one part of system exerts on another is called an interval force and a force exerted on a part of the system by some agency outside the system is an <u>external forces</u> act on a system, we call it an isolated system. Thus we may state the principle of <u>conservation of momentum</u> as follows:

The total momentum of an isolated system is constant, or conserved.

4.8. Elastic collision

If the total kinetic energy of two bodies remains to be same both after and before the impact the collision is said to be perfectly elastic. Collisions between atomic, nuclear and fundamental particles are examples of elastic collision.

Consider two smooth spheres of mass m_1 and m_2 moving along the line joining their centers with velocities υ_1 and υ_2 respectively. Let after collision, their velocities become u_1 and u_2 respectively. As momentum is conserved:

momentum before collision = momentum after collision

$$m_1\upsilon_1 - m_2\upsilon_2 = m_1u_1 - m_2u_2$$

or $$m_1(u_1 - \upsilon_1) = m_2(u_2 - \upsilon_2). \qquad (46)$$

In elastic collision the total energy remains conserved, i.e.

energy before collision = energy after collision

or $$\frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2,$$

or $$m_1\left(u_1^2 - \upsilon_1^2\right) = m_2\left(u_2^2 - \upsilon_2^2\right). \qquad (47)$$

Dividing Eq. (47) by Eq. (46), we have $\upsilon_1 + u_1 = \upsilon_2 + u_2$.

The velocity υ_1 and υ_2 may be obtain in the following way: Eq. (3) we have

$$u_1 = u_2 - \upsilon_1 + \upsilon_2 . \qquad (48)$$

Substituting this value of υ_1 in Eq. (46), we get

$$m_1(\upsilon_1 + \upsilon_1 - \upsilon_2 - u_2) = m_2(u_2 - \upsilon_2) ,$$

or $$2m_1\upsilon_1 - m_1u_2 - m_1\upsilon_2 = m_2u_2 - m_2\upsilon_2 ,$$

or $$2m_1\upsilon_1 - m_1\upsilon_2 + m_2\upsilon_2 = m_1u_2 + m_2u_2 ,$$

or $$2m_1\upsilon_1 + (m_2 - m_1)\upsilon_2 = (m_1 + m_2)u_2 .$$

Hence

$$u_2 = \frac{m_2 - m_1}{m_1 + m_2}\upsilon_2 + \frac{2m_1}{m_1 + m_2}\upsilon_1 . \qquad (49)$$

Similarly, we can obtain the value of υ_1:

$$u_1 = \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1 + \frac{2m_2}{m_1 + m_2}\upsilon_2 . \qquad (50)$$

Special cases:

1) When $m_1 = m_2$, then from Eq. (46): $u_1 - \upsilon_1 = u_2 - \upsilon_2$ on comparing with Eq. (48), we get $u_1 = \upsilon_2$, and $u_2 = \upsilon_1$; i.e. in one dimensional elastic collision of two bodies of equal masses the bodies simply exchange velocities as a result of collision.

2) When $\upsilon_2 = 0$ (the second mass is at rest), then from Eqs. (49) and (50), we have

$$u_2 = \frac{2m_1}{m_1 + m_2}\upsilon_1$$

and $$u_1 = \frac{m_1 - m_2}{m_1 + m_2}\upsilon_1 . \qquad (51)$$

Now we consider the following three case:

1. If $m_1 = m_2$, then $\upsilon_1 = \upsilon$ and $\upsilon_2 = 0$.

Thus, both the momentum and kinetic energy of the first body are completely transferred to the second as $u_1 = 0$, i.e. the first body is stopped.

2. If $m_2 >> m_1$, then $u_1 \cong -\upsilon_1$ and $u_2 \cong \upsilon_2 = 0$.

Thus, when a light body collides with a much heavier body at rest, the velocity of light body is approximately reversed and heavier body remains approximately at rest.

3. If $m_2 << m_1$, then $u_1 \cong \upsilon_1$ and $u_2 = 2\upsilon_1$.
   Thus when a heavy body collides a much lighter body at rest, the velocity of the heavy body remains practically unchanged while the light body rebounds with approximately twice the velocity of heavy body.

4.9. Maximum energy transfer in a head on elastic collision

Consider a ball of mass m_1 moving with velocity υ_1 collides with a ball of mass m_2 at rest. Let the velocity of the first ball after collision be u_1. Now the initial kinetic energy of first ball $K_i = \frac{1}{2} m_1 \upsilon_1^2$, the final kinetic energy of first ball $K_f = \frac{1}{2} m_1 u_1^2$.

The fractional decrease in kinetic energy is

$$\frac{K_i - K_f}{K_i} = \frac{\upsilon_1^2 - u_1^2}{\upsilon_1^2} = 1 - \frac{u_1^2}{\upsilon_1^2}. \tag{52}$$

According to Eq. (51) $\quad u_1 = \frac{m_1 - m_2}{m_1 + m_2} \upsilon_1$

or

$$\frac{u_1^2}{\upsilon_1^2} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2. \tag{53}$$

Substituting the value of $\frac{u_1^2}{\upsilon_1^2}$ from Eq. (53) into Eq. (52), we have

$$\frac{K_i - K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 = \frac{4 m_1 m_2}{(m_1 + m_2)^2}.$$

If $m_1 = m$ and $m_2 = nm$, then $\frac{K_i - K_f}{K_i} = \frac{4n}{(1+n)^2}$.

The transfer of energy will be maximum when $K_f = 0$. For $n = 1$

$$\frac{4n}{(1+n)^2} = 1.$$

Thus when the mass ratio is unity, the whole of the kinetic energy of the moving ball is transferred to the ball initially at rest.

4.10. Perfectly inelastic collision

The collision is known as perfectly inelastic when there is a loss of kinetic energy during collision and colliding bodies stick together and move as a single unit. For example the collision between a bullet and a target is perfectly inelastic when the bullet remains embedded in the target. In this case kinetic energy is not conserved. Between the two limits of perfectly elastic and perfectly inelastic collisions, all other collisions are imperfectly elastic.

Now we shall calculate the change of kinetic energy in an imperfectly elastic collision. Let us consider the case of two bodies of masses m_1 and m_2 moving along the joining centers with velocities υ_1 and υ_2 respectively. Let after collision they velocity u_1 and u_2 ($u_1 = u_2 = u$). As total momentum remains constant

$$m_1\upsilon_1 + m_2\upsilon_2 = m_1u_1 + m_2u_2 = (m_1 + m_2)u \qquad (54)$$

and

$$u = \frac{m_1\upsilon_1 + m_2\upsilon_2}{m_1 + m_2}. \qquad (55)$$

Decrease in kinetic energy

$$\Delta E = (\frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2) - (\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2).$$

Short answer questions

1. A body is kept moving with uniform speed in a circle by centripetal force acting on it. However the work done by this force is zero. Is it true? Explain.
2. The Earth moving round the sun in a circular orbit is acted upon by a force and hence work must be done on the earth by this force. Do you agree with this statement?
3. Is it possible that a body be in accelerated motion under force acting on the body, yet no work is being done by the force?
4. Springs A and B are identical except that A is stiffer than B, i.e. force constant $k_A > k_B$. In which spring is more work expended if:

a) they are stretched by the same amount?
b) they are stretched by the same force?

5. A lorry and a car moving with the same kinetic energy are brought to rest by the application of brakes, which provide equal retarding forces. Which of them will came to rest in a shorter distance?
6. When a constant force is applied to a body moving with constant acceleration, is power of the force constant? If not, how would force have to vary with speed for the power to be constant?

Solved examples

1. An object of mass 5 kg falls from the rest through a vertical distance of 20 m and reaches a velocity of 10 m/s. How much work is done by the push of the air on the object?

Solution. The motion of the body is shown in Fig. 27. The following two forces are acting on the body:
a)weight mg is acting vertically downward;
b) the push of the air is acting upward.
As the body is accelerating downward, the resultant force is

$$(mg - F).$$

Work done by the resultant force to ball through a vertical distance h = 20 m is $A = (mg - F)\,h$.

Gain in the kinetic energy $\Delta K = K_f - K_i = \frac{1}{2}m\upsilon^2$.

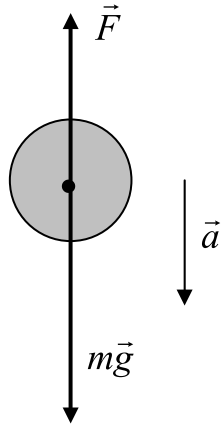


Fig. 27

Now the work down by the resultant force is equal to the change in kinetic in energy, i.e. $(mg - F)h = \frac{1}{2}m\upsilon^2$,

or $$Fh = mgh - \frac{1}{2}m\upsilon^2.$$

Work done by the force F is A = - 750 Joule. The negative sign is used because the push of the air is upward while the displacement is downwards.

2. AB is a quarter of a smooth circular track of radius R = 4 m as shown in Fig. 28 (a). A particle P of mass m = 5 kg moves along the track from A to B under the action of the following forces:
   a) A force F_a is directed always toward to B, its magnitude is constant and equals F_a = 4 N.
   b) A force F_b that is directed along the instantaneous tangent to the circular track; its magnitude is $F_b = (20 - S)$ N, where S is the distance traveled in meter;
   c) A horizontal force F_c = 25 N.

If the particle starts with a speed υ = 10 m/s, what is its speed at point B.

Solution.

a) Work done by force F_a. As shown in Fig. 28 (b), let the particle be at point P at some instant of time t. The particle moves from position P to position Q in small interval of time dt. The direction of force on particle P will be in the direction PB. The small amount of work done in time dt is

$$dA_a = F_a dS\cos\theta.$$

As $$dS = Rd\theta$$

then $$dA_a = F_a\cos\theta Rd\theta.$$

Now the total work done as the particle moves from A to B is given by

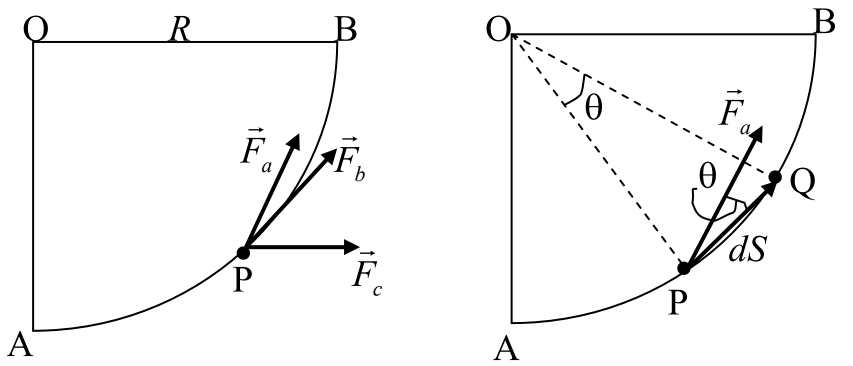
$$A_a = \int_0^{\pi/4} F_a \cos\theta R d\theta = F_a R \int_0^{\pi/4} \cos\theta d\theta = F_a R \sin\theta \Big|_0^{\pi/4},$$

$$A_a = 11,32 \text{ (joule)}.$$

b) Work done by force F_b: $dA_b = F_b dS$.

In this case $dA_b = (20 - S)\cdot dS$

but $S = R\theta$, and $dS = Rd\theta$.



a b

Fig. 28

So $dA_b = (20 - R\theta)\cdot Rd\theta$.

Or $$A_b = \int_0^{\pi/4} (20 - R\theta)\cdot Rd\theta = \int_0^{\pi/4} 20Rd\theta - \int_0^{\pi/4} R\theta \cdot Rd\theta = 20R\theta \Big|_0^{\pi/4} - R^2 \frac{\theta^2}{2}\Big|_0^{\pi/4}$$

and $$A_b = \frac{80\pi}{4} - \frac{16}{2}\left(\frac{\pi}{4}\right)^2 = 248,67 \text{ (Joule)}.$$

c) Work done by force F_c:
The magnitude of F_c is 25 N, which is always horizontal. The net displacement of the particle is OB. Hence the work done

$$A_c = F_c R,$$
$$A = 100 \text{ (Joule)}.$$

d) There would be some work done against weight. The net vertical displacement would be equal to the radius of the track R:

$$A_n = -mgR = -196 \text{ (Joule)}.$$

Negative sign is used because the force of weight and displacement are in opposite directions. Total work done $A = A_a + A_b + A_c + A_n$,

$$A = 11,32 + 248,67 + 100 - 196 = 163,99 \text{ (Joule)}.$$

Let υ_a and υ_b be the velocity of the particle at A and B respectively,

$$\frac{1}{2}m\upsilon_b^2 - \frac{1}{2}m\upsilon_a^2 = A.$$

Hence $$\frac{1}{2}m\upsilon_b^2 = A + \frac{1}{2}m\upsilon_a^2.$$

Speed υ_b can be obtained as follows: $\upsilon_b = \sqrt{\frac{2}{m}\left(A + \frac{1}{2}m\upsilon_a^2\right)}$

or $$\upsilon_b = 12{,}85 \text{ m/s}.$$

3. A projectile of mass m = 50 kg is shot vertically upward with an initial velocity of υ_0 = 100 m/s. After t = 5 s it explodes into two fragments, one of which having mass m_1 = 20 kg travels vertically up with a velocity υ_1 = 150 m/s.

What is velocity of the other fragment at that instant?

<u>Solution.</u>

After explosion, one fragment of mass m_1 = 20 kg goes upward with velocity υ_1 = 150 m/s. It is quite obvious that the second fragment of mass m_2 = 30 kg will go downward with a velocity, say υ_2.

The velocity of the projectile after t = 5 s is
$$\upsilon = \upsilon_o - gt = 100 - 9{,}8\cdot 5 = 51 \text{ m/s}.$$

The momentum of projectile before explosion
$$p = m\upsilon = 50\cdot 51 = 2550 \text{ kg}\cdot\text{m/s}.$$

Momentum of projectile after explosion $\vec{p} = \vec{p}_1 + \vec{p}_2$.

Where $p_1 = m_1\upsilon_1$ - momentum of the first fragment,
$p_2 = m_2\upsilon_2$ - momentum of the second fragment,

i.e. $$m\upsilon = m_1\upsilon_1 - m_2\upsilon_2$$

and $$\upsilon_2 = \frac{m_1\upsilon_1 - m\upsilon}{m_2} = 15 \text{ m/s}.$$

4. A rod of length l = 1 meter and mass m = 0,5 kg is fixed at one end is initially hanging vertical. The other end is now raised until it makes an angle 60° with the vertical (Fig. 29). How much work is required?

<u>Solution.</u>

The weight mg of the rod acts as the center of gravity. When the rod is rotated through 60°, Y moves Y´, i.e. it is raised through height
$$h = AY = OY - OA = OY - OY'\cos 60^\circ = $$
$$= OY(1 - \cos 60^\circ),$$
as $$OY = OY' = \frac{l}{2}.$$

Gain in potential energy $U = mgh$. The work done A has been stored as potential energy, i.e.
$$A = U = mgh = 1{,}225 \text{ (Joule)}.$$

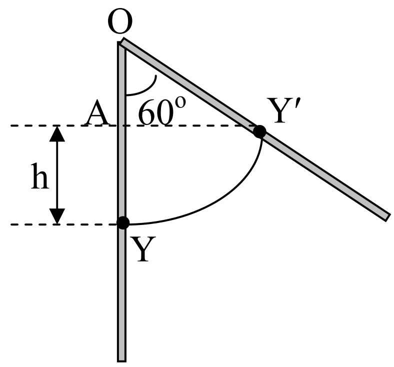


Fig. 29

4. A uniform chain is held on a frictionless table with one-fifth of its hanging over the edge (Fig. 30). If the chain has a length *l* and a mass *m*, how much work is required to pull the hanging part back on the table?

<u>Solution.</u> Mass of the hanging part of the chain is $m_1 = \frac{1}{5}m$. The weight $\frac{1}{5}mg$ acts at the center of gravity of the hanging chain, i.e. at a distance $l_1 = \frac{l}{10}$ below the surface of a table.

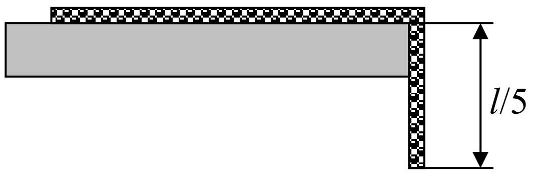


Fig. 30

The gain in potential energy in pulling the hanging part on the table

$$U = m_1 g l_1 = \frac{m}{5} g \frac{l}{10} = \frac{mgl}{50}.$$

Hence work done will be

$$A = U = 0{,}02mgl.$$

Problems and exercises

Ex.1. A small ball A slides down the quadrant of a circle as shown in Fig. (31), and hits the ball B of equal mass which is initially at rest. Find the velocities of both balls after collision. Neglect the effect of friction and assume the collision to be elastic.

<u>Answer</u>: $\upsilon_1 = 0$ and $\upsilon_2 = 1{,}4$ m/s.

Ex.2. A bullet of mass *m* moving with a horizontal velocity υ, strikes a stationary block of mass *M* suspended by a string of length *L* (Fig. 32). The bullet gets embedded in the block. What is the maximum angle made by the string after impact?

<u>Answer</u>: $cos\,\theta = \left[1 - \frac{m^2\upsilon^2}{2gl(m+M)^2}\right]$.

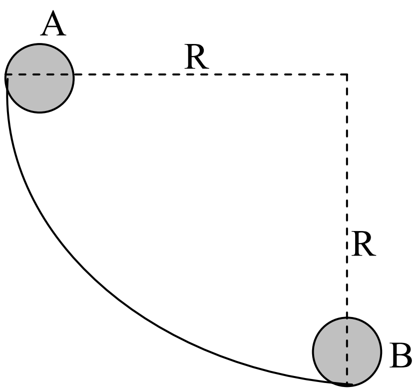


Fig. 31

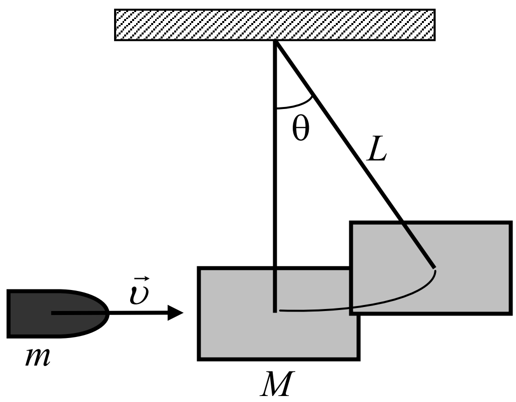


Fig. 32

Ex.3. A running man has half the kinetic energy that a boy of half his mass has. The man speeds up by 1,0 m/s and then has the same kinetic energy as the boy. What were the original speeds of man and boy?
Answer: υ_{man} = 2,4 m/s; υ_{boy} = 4,8 m/s.

Ex.4. A proton of mass $m_p = 1{,}6\cdot10^{-27}$ kg undergoes a head on collision with an α-particle initially at rest. After the collision, the α-particle moves with a speed of $8\cdot10^5$ m/s. Calculate the velocity of the proton before and after the collision. Mass of α-particle $m_\alpha = 6{,}58\cdot10^{-27}$ kg.
Answer: $\upsilon_{before} = 2\cdot10^6$ m/s; $\upsilon_{after} = 1{,}2\cdot10^6$ m/s.

Ex.5. What is the minimum stopping distance for a car of mass *m*, moving with speed υ along a level road, if the coefficient of static friction between the tubes and road is μ?
Answer: $S = \dfrac{\upsilon^2}{2\mu\cdot gl}$.

5. Rotational Motion

5.1. Kinematics of Rotational Motion

Let us consider a rigid body that rotates about a stationary axis (in Fig. 33), A rigid body rotates about a stationary line passing through point O perpendicular to the plane of the diagram. Line OP is fixed in the body and rotates with it. The angle between this line and the horizontal line is θ. The angle θ describes the position of the body completely. Thus θ serves as a coordinate to describe the rotational of the body.

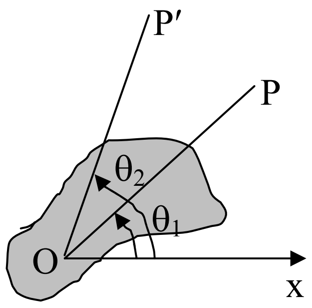


Fig. 33

Rotational motion of a body can be described in terms of the rate of change of angle θ. In Fig. 33 a reference line OP makes an angle θ_1 with the reference line OX, at a time t_1, at a later time t_2 the angle has changed to θ_2. We defined the angular velocity as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \tag{56}$$

Angular acceleration is the limit of ratio $\Delta\omega/\Delta t$ as $\Delta t \to 0$:

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}. \tag{57}$$

Because $\omega = d\theta/dt$, the angular acceleration can be written as

$$\varepsilon = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}. \tag{58}$$

It is convenient to represent qualities $d\theta$, ω and ε as vectors, strictly speaking, as pseudo-vectors. These vectors are directed along the axis of rotation according to the rule of right-handed screw. In Fig. 34 two cases are shown:

a) Particle is moving in clockwise direction vectors $d\vec{\theta}$ and $\vec{\omega}$ are directed downward vector of angular acceleration $\vec{\varepsilon}$ has the same direction when

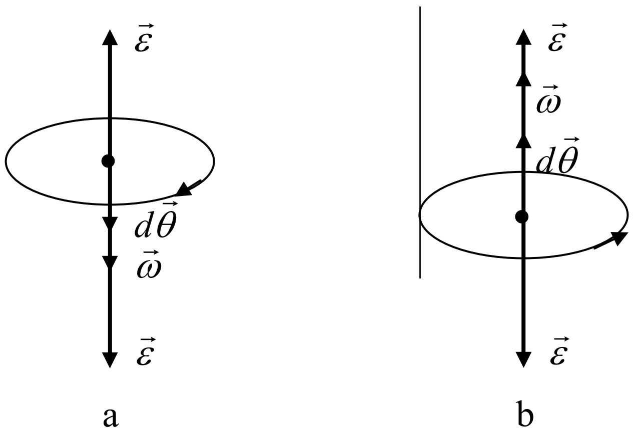


Fig. 34

angular velocity increases and opposite direction when angular velocity decreases.

b) Particle is moving in anticlockwise direction. In this case $d\vec{\theta}$ and $\vec{\omega}$ are directed upward, angular acceleration $\vec{\varepsilon}$ – upward or downward.

5.1.1. Rotation with constant angular acceleration

When the angular acceleration is constant, it is easy to derive equations for angular velocity and angular position as function of time by integration.

$$\frac{d\omega}{dt} = \varepsilon = const.$$

Then $$\int d\omega = \int \varepsilon \cdot dt, \qquad \omega = \varepsilon t + C_1,$$

where C_1 is an integration constant. If ω_0 is the angular velocity when $t = 0$, C_1 is equal to ω_0 and

$$\boldsymbol{\omega = \omega_0 + \varepsilon t.} \tag{59}$$

Also, $\omega = \dfrac{d\theta}{dt}$; integrating again, we find

$$\int d\theta = \int \omega_0 dt + \int \varepsilon \cdot t dt + C_2 .$$

The integration constant C_2 is the value of θ when $t = 0$ (the initial position), which we denote θ_0. Thus

$$\boldsymbol{\theta = \theta_0 + \omega_0 t + \frac{\varepsilon \cdot t^2}{2}}. \tag{60}$$

We can also derive an equation-relating ω and θ. The final result is:

$$\omega^2 = \omega_0^2 + 2\varepsilon(\theta - \theta_0). \tag{61}$$

5.1.2. Relations between angular and linear velocity and acceleration

When a rigid body rotates about a stationary axis, every particle of the body moves in a circle lying in a plane perpendicular to this axis, with the center of the circle on the axis. Earlier we get a relation for the acceleration of a particle moving in a circular path, in terms of its speed and the radius; this relation is steel valid when the particle is part of the rotating rigid body.

The speed of a particle in rigid body is directly proportional to the body's angular velocity. In Fig. 35 point P is at a distance R away from the axis of rotation, and it moves in a circle of radius R. When the angle θ increases by a small amount $\Delta\theta$ in a time interval Δt, the particle moves through an arc length $\Delta S = R\,\Delta\theta$. If $\Delta\theta$ is very small, this arc is nearly a straight line, and the average speed of the particle is given by

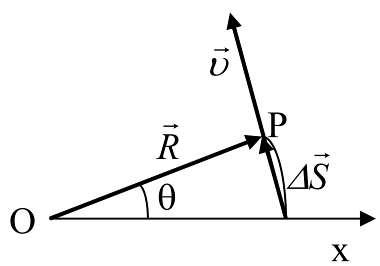


Fig. 35

$$\upsilon_{av} = \frac{\Delta S}{\Delta t} = R \cdot \frac{\Delta \theta}{\Delta t}. \qquad (62)$$

In the limit, as $\Delta t \rightarrow 0$, this becomes

$$\upsilon = R\frac{d\theta}{dt} = R\omega. \qquad (63)$$

The direction of the particle's velocity is tangent to its circular path at each point. Eq. (63) can be written in vector form:

$$\vec{\upsilon} = \vec{\omega} \times \vec{\boldsymbol{R}}.$$

If the angular velocity changes by $\Delta\omega$, the particle's speed changes by an amount $\Delta\upsilon$ given by

$$\Delta\upsilon = R\Delta\omega.$$

This corresponds to a component of acceleration a_τ tangent to the circle. If these changes take place in a small time interval Δt, then

$$\frac{\Delta\upsilon}{\Delta t} = R\frac{\Delta\omega}{\Delta t}.$$

And in limit $\Delta t \rightarrow 0$,

$$a_\tau = R\frac{d\omega}{dt} = R\varepsilon, \qquad (64)$$

where a_τ is a tangential component of acceleration of a point at a distance R from the axis.

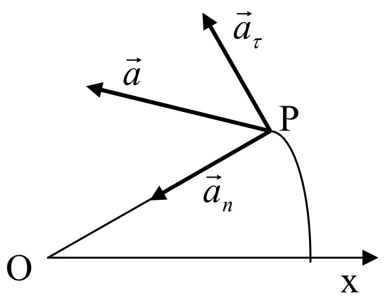


Fig. 36

The normal component

$$a_n = \frac{\upsilon^2}{R} = \omega^2 R.$$

The tangential and normal components of acceleration are shown in Fig. 36. Their sum is the acceleration a.

$$\vec{a} = \vec{a}_n + \vec{a}_\tau; \qquad (65)$$

$$|\vec{a}| = \sqrt{a_n^2 + a_\tau^2}. \qquad (66)$$

5.2. Kinetic energy of rotation

Kinetic energy of a particle with mass m is

$$\frac{1}{2}m\upsilon^2 = \frac{1}{2}mR^2\omega^2. \qquad (67)$$

The total kinetic energy of the body is the sum of the kinetic energies of all particle of the body

$$K = \frac{1}{2}m_1R_1^2\omega^2 + \frac{1}{2}m_2R_2^2\omega^2 + \ldots = \sum_i \frac{1}{2}m_iR_i^2\omega^2.$$

As the angular velocity ω is the same for all particle in the rigid body, we can rewrite this as $$K = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 .$$

To obtain the sum $\sum_i m_i r_i^2$, we subdivide the body (in our imagination) into a large number of particles, multiply the mass m; of each particle by the square of its distance from the axis, and add these products for all particles. The result is called the <u>*momentum of inertia I*</u> of the body, about the axis of rotation:

$$\boldsymbol{I = \sum_i m_i r_i^2} . \qquad (68)$$

In «**SI**» units of I is 1 kilogram-meter^2 (kg·m^2).

We can express the rotational kinetic energy of a rigid body as

$$\boldsymbol{K = \frac{I\omega^2}{2}} . \qquad (69)$$

When body is rolling it takes part in two kinds of motion: translational and rotational ones, so its kinetic energy is the sum:

$$K = K_{trans} + K_{rot} = \frac{m\upsilon^2}{2} + \frac{I\omega^2}{2} .$$

<u>5.3. Moment-of-Inertia Calculations</u>

When the body consists of a continuous distribution of matter, we can express the sum in terms of an integral.

Imagine dividing the entire volume of the body into small volume elements dV so that all points in a particular element are very nearly the same distance from the axis of rotation; we call this distance r, as before. Let dm be the mass in a volume element dV. The moment of inertia then can be expressed as

$$\boldsymbol{I = \int r^2 dm} . \qquad (70)$$

Density ρ is mass per unit volume, $\rho = \frac{dm}{dV}$, so we may also write

$$I = \int r^2 \rho dV .$$

If the body is homogeneous (uniform in density), then ρ may be taken outside the integral

$$\boldsymbol{I = \rho \int r^2 dV} . \qquad (71)$$

In using this equation, we express the volume element dV in terms of the differentials of the integration variables, usually the coordinates of the volume elements. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. For regularly shaped bodies this integration can be after be carried out quite easily. Discuss several examples.

5.3.1. Uniform slender rod; axis perpendicular to length rod

Rod has mass M and length l. We wish to compute its moment of inertia about an axis through O, at an arbitrary distance from one end. Using Eq. (70), we choose as an element of mass a short section having length dx at a distance x from point O. The ratio of the mass dm of this element to the total mass M is equal to the ratio of its length dx to the total length l.

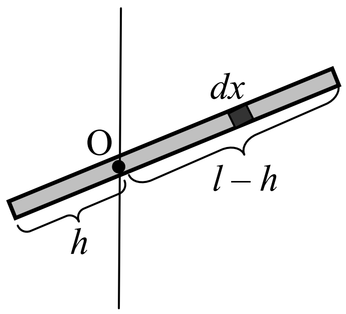


Fig. 37

Thus $$\frac{dm}{M} = \frac{dx}{l},$$

where $$dm = M\frac{dx}{l}.$$

Using (70) we obtain:

$$I_0 = \int x^2 dm = \frac{M}{l}\int_{-h}^{l-h} x^2 dm = \frac{M}{l}\cdot\frac{x^3}{3}\Bigg|_{-h}^{l-h} = \frac{1}{3}M\left(l^2 - 3hl + 3h^2\right).$$

From this general expressing we can find the momentum of inertia about an axis through any point on the rod. For example, if the axis is at the left end, $h = 0$ and

$$\boldsymbol{I = \frac{1}{3}Ml^2}. \qquad (72)$$

If the axis is the right end, $h = l$ and

$$I = \frac{1}{3}Ml^2.$$

As would be expected. If the axis passes though the center, $h = \frac{l}{2}$ and

$$\boldsymbol{I = \frac{1}{12}Ml^2}. \qquad (73)$$

5.3.2. Hollow or solid cylinder; axis of symmetry

Fig. 38 shows a hollow cylinder of length l and inner and outer radii R_1 and R_2. We choose as the most convenient volume element a thin cylindrical sheet of radius r, thickness dr and length l. The volume of this shell is very nearly equal to that of flat sheet of thickness dr, length l, and width $2\pi r$. Then

$$dm = \rho dV = 2\pi\rho l r dr.$$

The moment of inertia is given by

$$I = \rho\int r^2 dV = 2\pi\rho l\int_{R_1}^{R_2} r^3 dr = \frac{\pi\rho l}{2}\left(R_2^4 - R_1^4\right) = \frac{\pi\rho l}{2}\left[\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)\right].$$

Volume $V = \pi l\ (R_2^2 - R_1^2)$.
Hence $M = \pi\rho l\ (R_2^2 - R_1^2)$
and the moment of inertia is

$$\boldsymbol{I = \frac{1}{2} M\left(R_1^2 + R_2^2\right)} \quad (74)$$

If the cylinder is solid, $R_1 = 0$; letting outer radius be R_1 we find that the moment of inertia of a solid cylinder of radius R is

$$\boldsymbol{I = \frac{1}{2} MR^2}. \quad (75)$$

If the cylinder is very thin, R_1 and R_2, are nearly equal, if R represents this common radius

$$I = MR^2. \quad (76)$$

Note, that the moment of inertia of cylinder does not depend on the length l. It depends only on the radial distribution of mass, not on distribution along the axis.

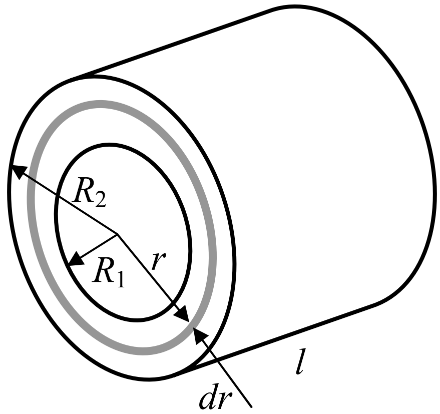


Fig. 38

5.3.3. Uniform sphere of radius *R*, axis through center

Divide the sphere into thin disks. The radius r of the disk shown in Fig. 7 is

$$r = \sqrt{R^2 - x^2}.$$

Its volume is $dV = \pi r^2 dx = \pi(R^2 - x^2)dx$
and its mass is $dm = \rho dV$.
Hence from Eq. (75) $dI = \frac{\pi\rho}{2}\left(R^2 - x^2\right)^2 dx$.

Integrating this expression from 0 to R gives the momentum of inertia of the right hemisphere: from symmetry, the total I for the entire sphere is just twice this:

$$I = \frac{2\pi\rho}{2}\int_0^R \left(R^2 - x^2\right)^2 dx.$$

Carrying out the integration, we obtain

$$I = \frac{8\pi\rho}{15} R^5.$$

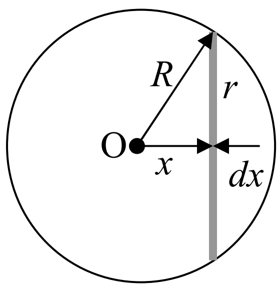


Fig. 39

The mass M of the sphere is $M = \rho V = \frac{4}{3}\pi\rho R^3$.

Hence $\boldsymbol{I = \frac{2}{5} MR^2}$.

5.4. Parallel – axis theorem

There is a theorem that is often useful in finding moments of inertia with respect to various axes. If the moment of inertia I_0 of a body about an axis through its center of mass is known, then the moment of inertia I_p about any other axis parallel to the original one but displaced from it by a distance d (Fig. 40) easily obtained my means of relation called the *parallel – axis theorem*, which states that

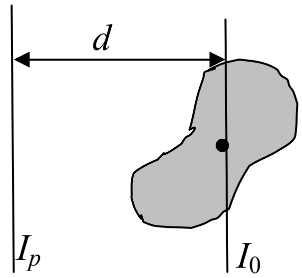


Fig. 40

$$\boldsymbol{I_p = I_0 + Md^2} \ . \tag{77}$$

To prove this theorem we consider the body shown in Fig. 41.

The origin of coordinates has been chosen to coincide with the center of mass. We wish to compute the moment of inertia about an axis through point P_1 perpendicular to the plane of the figure. Point P has coordinates (a,b), and its distance from origin is d. We note that $d^2 = a^2 + b^2$.

Let m_i be a typical mass element, with coordinates (x_i, y_i). Then the moment of inertia about an axis through O is

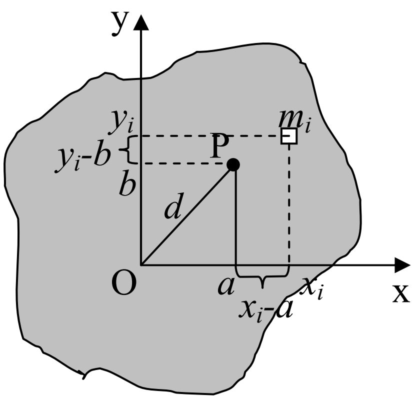


Fig. 41

$$I_0 = \sum_i m_i \left(x_i^2 + y_i^2\right),$$

and the moment of inertia about axis through is

$$I_p = \sum_i m_i \left[\left(x_i - a\right)^2 + \left(y_i - b\right)^2\right].$$

We expend the squared terms and regroup, obtaining

$$I_p = \sum_i m_i \left(x_i^2 + y_i^2\right) - 2a\sum_i m_i y_i + \left(a^2 + b^2\right)\sum_i m_i .$$

The first sum is I_o. The second and third sums are zero because they represent the x and y coordinates of the center of mass, which are zero because we have taken the origin to be center of mass. The final term is d^2 multiplied by the total mass, so the theorem is proved.

5.5. Torque

In studying dynamics of a particle, we made extensive use of Newton's second law, which relates the acceleration of a particle to the force acting on it. Now we need to develop an analogous relation between the angular acceleration of a rotating rigid body and forces acting on it. This relation includes a new concept, *torque*.

A torque is always associated with a force. Qualitatively speaking, torque is the tendency of a force to cause a rotation of the body on which it acts. This tendency depends on the magnitude and direction of the force, and also on the location of the point where it acts. For example, it is easier to push a door open by pushing near the doorknob side than near the hinge side.

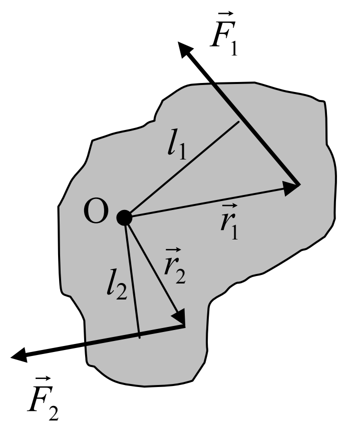


Fig. 42

Torque is always defines with reference to a special axis of rotation. To define torque quantitatively, we consider a body that can rotate about an axis perpendicular to the plane of Fig. 42 through point O. Forces F_1 and F_2 act on a body; both forces act lines that lie in a plane perpendicular to the axis. The tendency of force F_1 to cause a rotation about the axis through O depends on both the magnitude F_1 of the force and the perpendicular distance l_1 between the line of action of the force and the axis. If $l_1 = 0$, there is no tendency to cause rotation. Torque $\vec{M}$ is a vector quantity

$$\vec{\boldsymbol{M}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}\,, \qquad (78)$$

where $\vec{r}$ is the position vector of the point at which the force acts. $\vec{M}$ is a vector quantity. Its magnitude

$$\boldsymbol{M = F \sin\theta}. \qquad (79)$$

Where θ is angle between $\vec{r}$ and $\vec{F}$. The direction of $\vec{M}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$ according to the rule of right – handed screw.

5.6. Angular momentum

In translatory motion the linear momentum of a single particle is expressed as

$$\vec{p} = m\vec{\upsilon}\,.$$

In rotation motion the analogue of linear momentum is angular momentum. Consider the case of a particle, having linear momentum $\vec{p}$. The angular momentum $\vec{L}$ of the particle with respect to a fixed point O as origin is defined as cross product:

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}\,, \qquad (80)$$

where $\vec{r}$ is a vector distance of the particle from origin O. The direction of $\vec{L}$ is perpendicular to the plane of $\vec{r}$ and $\vec{p}$, and magnitude is

$$\boldsymbol{L = rp \sin\theta}, \qquad (81)$$

where θ is the angle between $\vec{r}$ and $\vec{p}$.

When a particle moves with angular velocity $\vec{\omega}$ in a circle, then its angular momentum is given by

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\upsilon} = m \cdot \vec{r} \times \vec{\upsilon}\,.$$

As $\upsilon = \omega r$ then $$L = mr\omega r \sin(\vec{r}, \vec{\upsilon}).$$
In rotation motion $\vec{\upsilon}$ is tangential hence perpendicular to the $\vec{r}$ and so θ = 90°, i.e.

$$L = mr^2 \omega,$$

or $$L = I\omega, \qquad (82)$$

or $$\vec{\boldsymbol{L}} = \boldsymbol{I}\vec{\boldsymbol{\omega}}. \qquad (83)$$

5.7. Main Law of Rotational Motion

When a force $\vec{F}$ acts a particle, then moment of the force or torque $\vec{M}$ is defined as $$\vec{M} = \vec{r} \times \vec{F}.$$

We now that $$\vec{F} = \frac{d}{dt}(m\vec{\upsilon})$$

and as $$\vec{M} = \vec{r} \times \frac{d}{dt}(m\vec{\upsilon}),$$

then $$\vec{M} = \frac{d}{dt}(\vec{r} \times m\vec{\upsilon})$$

or $$\vec{\boldsymbol{M}} = \frac{\boldsymbol{d}\vec{\boldsymbol{L}}}{\boldsymbol{dt}}. \qquad (84)$$

Thus torque equals to rate of change of angular momentum.

In case when moment of inertia *I* is constant the main law of rotational motion can be represented as follows

$$\vec{M} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt} = I\vec{\varepsilon},$$

i.e. $$\vec{\boldsymbol{M}} = \boldsymbol{I}\vec{\boldsymbol{\varepsilon}}. \qquad (85)$$

5.8. Law of Conservation of Angular Momentum

When $\vec{M}$ = 0, then $\frac{d\vec{L}}{dt} = 0$ or

$$\vec{\boldsymbol{L}} = \boldsymbol{const}, \qquad (86)$$

i.e. *in isolated systems the angular momentum is conserved*.

5.9. Rolling down an inclined plane.

Let us consider the case of rigid body of radius *R* and mass *m* rolling down without slipping a smooth inclined plane having an angle of inclination θ as shown in Fig. 43. As the body rolls down it suffer vertical descent and therefore loses its potential energy. At the same time, it acquires linear and angular speed and hence, gains kinetic energy of translation and that of rotation. If there is no loss in potential energy, then the loss of potential energy is equal to the gain in kinetic energy. Let initially the body be at A and rest and after sometime it readies B, i.e., traverses a distance *S*. Suppose υ and ω be the velocity and angular velocity respectively acquired by the center of the body.

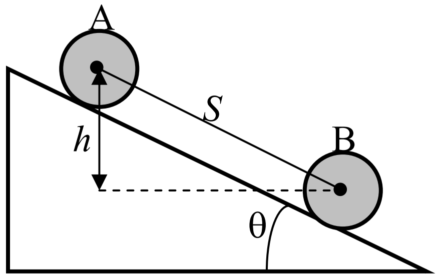


Fig. 43

The vertical distance *h* traveled by the body, $h = S\sin\theta$. Then loss of potential energy $U = mg\,S\sin\theta$ and gain in the kinetic energy of translation $K_t = \frac{1}{2}m\upsilon^2$. Gain in the kinetic energy of rotation $K_r = \frac{1}{2}I\omega^2$, where *I* is the moment of inertia of the body.

Now $$mgS\sin\theta = \frac{1}{2}m\upsilon^2 + \frac{1}{2}I\omega^2,$$

as $\omega = \frac{\upsilon}{R}$, then

$$mgS\sin\theta = \frac{1}{2}m\upsilon^2 + \frac{1}{2}I\frac{\upsilon^2}{R^2}$$

and $$\upsilon^2 = \frac{2mgSR^2\sin\theta}{mR^2 + I}.$$

It is useful to compare main characteristics and laws of translational and rotational motions.

TRANSLATIONAL MOTION	**ROTATIONAL MOTION**
Displacement $d\vec{r}$	Angular displacement $d\vec{\varphi}$
Velocity $\vec{\upsilon} = \frac{d\vec{r}}{dt}$	Angular velocity $\vec{\omega} = \frac{d\vec{\varphi}}{dt}$
Acceleration $\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d^2\vec{r}}{dt^2}$	Angular acceleration $\vec{\varepsilon} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\varphi}}{dt^2}$
Mass *m*	Moment of inertia *I*
Force $\vec{F}$	Torque $\vec{M} = \vec{r} \times \vec{F}$
Momentum $\vec{p} = m\vec{\upsilon}$	Angular momentum $\vec{L} = \vec{r} \times \vec{p}$
Kinetic energy $K_t = \frac{1}{2}m\upsilon^2$	Kinetic energy of rotation $K_r = \frac{1}{2}I\omega^2$
Work $A = \int \vec{F}d\vec{r}$	Work $A = \int \vec{M}d\vec{\varphi}$
Power $P = \vec{F}\vec{\upsilon}$	Rotational power $P = \vec{M}\vec{\omega}$

Questions

1. A body is rotating. It is necessary being acted upon by an external torque?
2. A person sits near the edge of a circular platform revolving with a uniform angular speed. What will be the change in the motion of the platform? What will happen when the persons starts moving from the edge towards the center of the platform?

(Ans.: The system tends to keep its angular momentum constant. When the person sits near the edge of the platform, the moment of inertia of platform increases and hence its angular velocity decreases. When the persons starts moving towards the center of platform, the moment of inertia decreases and hence the angular velocity increases.)

3. How a swimmer jumping from a height is able to increase the number of loops made in the air?

(Ans.: The swimmer can increase the number of loops by pooling his legs and arms inward, i.e. by decreasing the moment of inertia. By doing so the angular velocity ω increases because $I\omega$ remains constant.)

4. Why there are two propellers in a helicopter?

(Ans.: If there were only one propeller in the helicopter the helicopter itself would have turned in opposite direction due to conservation of angular momentum.)

5. A disk of metal is melted and recast in the form rolled sphere. What will be happen to the moment of inertia about a vertical axis passing through the center?

Problems

1. A particle of 10 kg mass is moving in a circle of R = 4 m radius with a constant speed of υ = 5 m/s. What is angular momentum about (a) the center of circle and (b) a point on the axis of the circle and l = 3 m distant from its center? Which of these will always be in same direction?

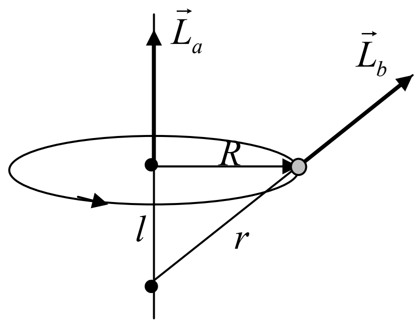


Fig. 44

Solution: The situation is shown in Fig. 44.

a) We know that

$$\vec{L} = \vec{R} \times m\vec{\upsilon},$$

$$L = Rm\upsilon \sin\theta,$$

here θ = 90°, as $\vec{\upsilon} \perp \vec{R}$.

Then $L = Rm\upsilon = 4 \cdot 10 \cdot 5 = 200$ (kg·m^2)/s.

b) In this case angular momentum is

$$L = rm\upsilon,$$

$$L = 5 \cdot 10 \cdot 5 = 250 \text{ (kg·m}^2\text{)/s}.$$

From figure it is obvious that angular momentum in first case always has same direction but in second case the direction changes.

2. A symmetrical body is rotating about its axis of symmetry. Its moment of inertia about the axis of rotating being $I = 1$ kg·m^2 and its rate of rotation $\nu_1 = 2$ rev/s.

a) What is the angular momentum?

b) What additional work will have to be done to double its rate of rotation?

Solution: a) As the body is rotating about its axis of symmetry, the angular momentum vector coincides with the axis of rotation.

Angular momentum $L = I\omega = 2I\pi\nu$;

$$L = 2\cdot 1\cdot 3{,}14\cdot 2 = 12{,}57\ (\text{kg}\cdot\text{m}^2)/\text{s}.$$

Kinetic energy of rotation $K_1 = \frac{1}{2}I\omega_1^2$, as $\omega_1 = 2\pi\nu_1$.

When the rate of rotation is doubled, i.e. $\omega_2 = 2\omega_1$

the kinetic energy is $K_2 = \frac{1}{2}I\omega_2^2 = \frac{4}{2}I\omega_1^2 = 2I\omega_1^2$.

Additional work required: $A = K_2 - K_1 = 2I\omega_1^2 - \frac{1}{2}I\omega_1^2 = 1{,}57\omega_1^2$,

$$A = 236{,}8 \text{ Joule.}$$

3. A uniform disc of radius R and mass m_1 is mounted on an axis supposed in fixed frictionless bearing. A light card is wrapped around the rim of the wheel and supposes that we hang a body of mass m_2 from the cord (Fig. 45). Find the angular acceleration of the disc and tangential acceleration of point on the rim.

Solution: Let T be the tension in the cord. Now,

$$m_2 g - T = m_2 a, \qquad (87)$$

where a is the tangential acceleration of a point on the rim of the disk.

We know that $M = I\varepsilon$.

But the resultant torque on the disk, from the other hand $M = TR\sin\alpha = TR$, as $\alpha = 90°$, because $\vec{T} \perp \vec{R}$.

Moment of inertia of the disk is

$$I = \frac{1}{2}m_1 R^2.$$

Then angular acceleration ε is connected with tangential acceleration as follows: $\varepsilon = \frac{a}{R}$.

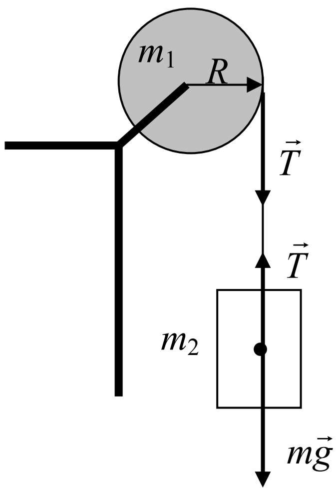


Fig. 45

Hence $TR = \frac{1}{2}m_1 R^2 \frac{a}{R}$

or $2T = m_1 a$

and $$T = \frac{m_1 a}{2}. \qquad (88)$$

From Eqs. (87) and (88) we get $m_2 g - \frac{m_1 a}{2} = m_2 a$,

or $$a = \frac{2m_2 g}{m_1 + 2m_2},$$

and $$T = m_2(g - a) = \frac{m_1 m_2 g}{m_1 + 2m_2}.$$

4. Disk with radius R = 0,2 m rotates according the law: $\varphi = A + Bt + Ct^3$, where A = 3 rad; B = – 1 rad/s, C = 0,1 rad/s^3. Find tangential a_τ, normal a_n and instantaneous acceleration of point on the rim at time t = 10 s.

Solution: We know, that angular velocity ω is

$$\omega = \frac{d\varphi}{dt} = \frac{d}{dt}\left(A + Bt + Ct^3\right) = B + 3Ct^2.$$

Then angular acceleration ε is

$$\varepsilon = \frac{d\omega}{dt} = \frac{d}{dt}\left(B + 3Ct^2\right) = 6Ct.$$

Tangential acceleration a_τ is connected with angular acceleration as follows:

$a_\tau = \varepsilon R$, then $a_\tau = 6CtR$ or $a_\tau = 6 \cdot 0,1 \cdot 10 \cdot 0,2 = 1,2$ m/s^2.

Normal acceleration is $a_n = \omega^2 R$ or $a_n = (B + 3Ct^2)^2 R$; $a_n = 168,2$ m/s^2.

And as we have got a_n and a_τ the acceleration a can be obtained as

$$a = \sqrt{a_n^2 + a_\tau^2},$$

$$a = 168,204 \text{ m/s}^2.$$

5. A sphere of mass m = 10 kg and radius R = 0,2 m rotates about axis passing through its center. Angle changes with time as $\varphi = A + Bt^2 + Ct^3$, where B = 4 rad/s^2, C = – 1 rad/s^3.

Find the torque action on the sphere as function of time.

Solution: The torque is

$$M = I\varepsilon.$$

Angular velocity is $$\omega = \frac{d\varphi}{dt} = \frac{d}{dt}\left(A + Bt^2 + Ct^3\right) = 2Bt + 3Ct^2.$$

Angular acceleration is $$\varepsilon = \frac{d\omega}{dt} = \frac{d}{dt}\left(2Bt + 3Ct^2\right) = 2B + 6Ct.$$

Then torque can be written as $M = I\,(2B + 6Ct) = 0,4\, mR^2(2B + 6Ct)$, as moment of inertia of the sphere is $I = 0,4\, mR^2$.

$$M = 0,4 \cdot 10 \cdot 0,04(2 \cdot 4 + 6 \cdot (-1) \cdot t) = 1,28 - 0,96 \cdot t.$$

Problems and Exercises

1. Calculate the angular momentum and rotational kinetic energy of Earth about its own axis.
2. A wheel of radius 6 sm is mounted so as to rotate about a horizontal axis through its center. A string of neglectible mass wrapped round its circumference carries a mass of 0,2 kg attached to its free end. When let fall the mass descends through one meter in 5 seconds. Calculate the angular acceleration of the wheel, its moment of inertia and tension in the card.
   (Ans.: $I = 8{,}75\cdot10^{-2}$ (kg·m^2)/s; $T = 1{,}94$ N).
3. A sphere, a disk and a ring of the same mass and radius are allowed to roll down an inclined plane simultaneously from the same height without slipping. Prove that the sphere reaches down first, the disc next and the ring the last.
   (Ans.: $a_1 > a_2 > a_3$).
4. A sphere of mass 1 kg and diameter 1 m rolls without sliding with a constant velocity of 10 m/s. Calculate what fraction of the total kinetic energy of the sphere is rotational? How much work has to be done to stop it?
   (Ans.: $K_{rot}/K_{total} = 2/7$; $A = 70$ Joules).
5. A wheel rotates about an axis passing through its center. Speed of the points of the rim changes with time according to law $\upsilon = 3t + t^2$. Find the normal a_n and tangential a_τ components of acceleration and angle as function of time.
   (Ans.: $a_n = \dfrac{(3t + t^2)^2}{R}$; $a_\tau = 3 + 2t$; $\varphi = \dfrac{t^2}{2}\left(\dfrac{3}{2} + \dfrac{t}{3}\right)$).

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