

МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ УКРАИНЫ

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Systems Department*

V. F. Symonov

LINEAR CONTINUOUS AUTOMATIC CONTROL SYSTEMS

Lectures

В.Ф. Симонов

ЛИНЕЙНЫЕ НЕПРЕРЫВНЫЕ СИСТЕМЫ АВТОМАТИЧЕСКОГО
УПРАВЛЕНИЯ

Конспект лекций

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Симонов В.Ф. Линейные непрерывные системы автоматического управления: консп. лекций / В.Ф. Симонов. – Х.: Нац. аэрокосм. ун-т «Харьк. авиац. ин-т», 2008. – 214 с.

Изложена современная теория автоматического управления для непрерывных линейных стационарных систем. Приведены фундаментальные принципы управления, основные законы управления, математическое описание систем автоматического управления (САУ), их классификация. Описаны методы анализа устойчивости, точности, быстродействия, управляемости, наблюдаемости, чувствительности и других показателей качества САУ.

Рассмотрены методы коррекции САУ, методы синтеза корректирующих устройств и законов управления.

Для студентов специальностей «Системы управления летательными аппаратами и комплексами», «Системы управления и автоматики» и «Технология самолёто- и вертолётостроения», изучающих курсы «Теория автоматического управления» и «Теоретические основы управления» на английском языке.

Modern automatic control theory is presented for continuous linear stationary systems. Fundamental control principles and laws, mathematical description and classification of systems are produced. The methods of stability, accuracy, swiftness, controllability, observeability and other quality indexes analysis are described.

The correction and the synthesis of the automatic control systems (ACS) methods are considered.

The textbook is meant for students specializing in “Flying Vehicle Control System and Complexes”, “System of Control and Automatics” and “Aircraft and Helicopter Building Technology” and studying the “Theoretical Basics of Control” in English.

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Рецензенты: д-р техн. наук, проф. В.М. Свищ,
канд. техн. наук С.Н. Ларьков

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Lecture №1

INTRODUCTION TO THE AUTOMATIC CONTROL THEORY

To operate – means to expect, and to expect means – to know much.

Marcus Tullius Cicero

Lecture outline:

1. Basic concepts of the automatic control theory (ACT).
2. Examples of the automatic control systems (ACS).
3. Concise history of the automatic control theory.

1.1. Basic concepts of the automatic control theory

Automatic control theory studies automatic control systems, their properties and ACS designing methods.

The mechanization and automation of human productive and cognitive activities form the core of scientific and technical progress. Mechanization is the process of replacing a person by the machinery in working process. Automation is the process of replacing a person by the machinery in both the operating and control processes.

To date most of the processes are automatized inclusive of technological, power-generating, transporting, designing, planning, calculating, and exploratory operations, information processing, troops and armoury supervision, etc. The automation of these processes results in productivity and work efficiency increase; product quality improvement; scheduling and control optimization and releases the person from monotonous work, labour under harmful and hazardous (to health) conditions and excessive toil.

Theoretically the process of automatization is based on the cybernetics, a complex science studying general principles of data acquisition, storage, transmission, processing and application to the systems of any kind. The cybernetics ideas and techniques are applied to engineering, medicine, soldiery, agriculture, criminalistics, etc. The technical cybernetics is engaged in the principles of technical control systems development.

Cybernetics has been first defined by the American mathematician Norbert Wiener (1894-1964) in his book named "Cybernetics or the control and communication in an animal and a machine". It included the automatic control theory, information theory, finite automata theory, theory of algorithms, operation study, the theory of mass service, the theory of the big systems, etc.

Technically the automatization is based on the possibility to integrate various electromechanical, electronic, hydraulic, pneumatic and other devices into the control systems (CS).

Control is a set of activities providing arranged systems structure preservation or purposeful exchange, operation modes maintenance, programme and purport organization. In other words, control is the process of transforming 'what it is' into 'what it should be'. The main **stages of control** are:

1. To define control purpose. The purposes of control may include certain process parameter stabilization, input signal monitoring, a control object state-to-state, etc. The purpose should be positive, topical and real.

2. To evaluate the possibility, methods, means, time and cost of control purpose achievement.

3. To analyze the controlled object properties (to identify the object structure and parameters, to estimate object linearity and stationarity, to evaluate control efficiency, limitation, controllability, observability, etc) and environmental interaction (the analysis of reference action and disturbance, system-with-environment interference).

4. To select the control process quality criteria (e.g. accuracy, operation speed, power and fuel consumption, cost, etc.

5. To organize control procedure, that is to select control system structural pattern, equipment, energy source, necessary data, action programme, control principle, control systems production, etc.

6. To provide control system survivability (operability monitoring, failure diagnosis and counteraction, operation modes maintenance, etc).

7. To evaluate current and future conditions: to rate and calculate state variables, disturbances, to estimate quality criteria, to forecast course of events, to compare the obtained values with expected ones.

8. To make a decision and to provide its performance that is to regard all the factors, actions and effort and controlled object influences.

9. To estimate control outcome: to compare the obtained results with those expected, to ascertain mistakes, to elaborate the control process improvements.

10. To develop the control system applying new methods of CS, facilities and components construction.

In the control process we should revert to separate stages, some of them carried out simultaneously. At all stages the system approach should be applied, i.e. the system should be considered bodily in the whole spectrum of its interactions with the environment.

The automatic control system is a set of controlled objects, the measuring, computing, amplifying, executive and control equipment,

incorporated in order to execute the task assigned without direct human participation.

Controlled object is a controlled and observable part of the environment, able to carry out the intended goal. An aircraft, a ship, a machine tool, a process installation, a shell of any kind, etc. including their combination can be regarded as a controlled object.

Semi-automatic systems and automated control systems (ACS) do not excluded a person completely from the control process. Such systems are still named ergatoid. For example, the air traffic control system includes the person - operator.

Regulation is the specific case of control.

Regulation means the maintenance of a constant physical value or its transformation according to the required principle. The altitude, velocity or flight course stabilization, signal level change monitoring, missile guidance, aircraft group maneuvering synchronization, plane glide path conduction, generator voltage and frequency stabilization, etc., can serve as the examples of regulation.

The elementary automatic control systems are automatic regulation systems (ARS), the principles of ARS construction studied by the ACT component, the automatic regulation theory (ART).

Generally the ACS could be resented as combination of automatic controlled object (ACO) and device of automatic control (DAC) (that is a regulator). Such scheme is presented on Fig. 1.1.

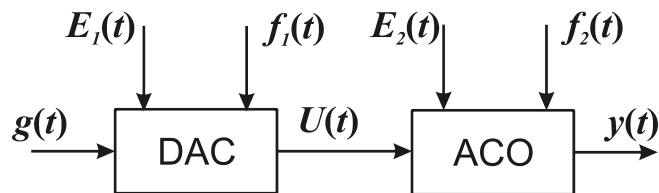


Figure 1.1 – General view of ACS

E is the ACS supply energy. E usually is not specified on the block diagrams because of the power supply energy content being essentially higher than the amount of energy consumed by the system. Signal E is designated only in the systems of energy regulation

$$g = \begin{bmatrix} g_1 \\ \dots \\ g_r \end{bmatrix} \begin{array}{l} \text{reference} \\ \text{actions} \\ \text{vector} \end{array} ; \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_l \end{bmatrix} \begin{array}{l} \text{control} \\ \text{actions} \\ \text{vector} \end{array} ; \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix} \begin{array}{l} \text{disturbance} \\ \text{actions} \\ \text{vector} \end{array} ;$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \text{ — controlled values vector ; } \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ — state vector .}$$

The vector of reference action $g(t)$ (the input signal) initializes the information of required, desirable output value, i.e. the purpose of control (e.g. the required value of the flight altitude and speed, the course of the aircraft, its roll and pitch); r — a number of reference actions. The reference action is introduced to the system by a person or higher level system, it is equal or proportional to the required output value: $g(t) = y_{req}(t)$ or $g(t) = ay_{req}(t)$.

The vector of control action $u(t)$ (control action, control) is produced by the device of automatic control (DAC) and renders purposeful influence on the automatic controlled object (ACO). The process of moving the elevator, rudder or ailerons can be an example of the vector of control action. l is a number of control actions.

The vector of disturbance action $f(t)$ (disturbance) is an influence the environment exerts to the controlled object and the control device. It shows, e.g. how the flaws influence the plane, waves effect the ship, the load biases on the engine and power supply, or the impact the measurement mistake, ACS parameter change, noise, temperature change can have. p is a number of disturbance actions.

Vector of controlled variables $y(t)$ (the controlled variables, output) is a variable linked to the state of controlled object (e.g., speed and altitude of the aircraft), m — a number of controlled variables.

Besides that the mathematical description includes a state vector $x(t)$ (state), n being the order of system. The signals describing energy in the system elements could be selected in the capacity of state variables (linear and angular speeds, capacitor voltage, inductance current, etc.).

1.2. Examples of ARS

The plane autopilot course channel (Fig. 1.2).

1. Controlled object, a plane.
2. Sensitive element, a gyroscope.
3. Amplification-transformation device.
4. Executive mechanism, a steering machine.
5. Control device, the rudder.

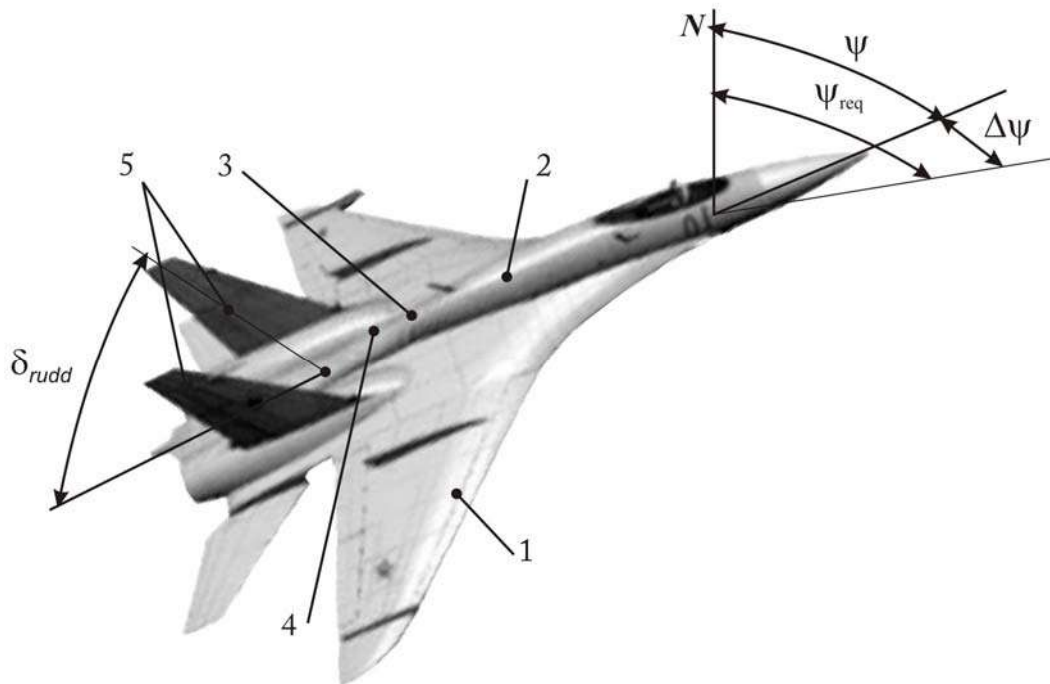


Figure 1.2 – The course channel of autopilot

The gyroscope (2) keeping constant the space direction of the axis of rotation while the plane (1) deviates from the required flight direction ψ_{req} , the sensor output voltage u_s is proportional to a deviation $\Delta\psi$.

This signal $u_s = k_s \Delta\psi$ is intensified in (3) and transmitted to the steering machine (4) which shifts a rudder (5). The plane keeps turning around until the deviation $\Delta\psi$ is eliminated.

The rotation velocity stabilization system of the lathe motor spindle (Fig. 1.3)

Any working mechanism (WM) (a lathe spindle, the radar aerial, etc) can be regarded as a controlled object.

The difference between setting potentiometer (SP) voltage and feedback potentiometer (FBP) voltage is delivered to the input of the amplification-transformation device (ATD): $\Delta u = u_s - u_{FB}$.

The tachogenerator transducer (TG) generates voltage U_{TG} which is proportional to the velocity ω of electrical motor (EM) rotation.

We specify setting voltage u_s by shifting a slider of the setting potentiometer. The increase of the load torque T causes the engine

velocity drop, the feedback voltage decrease and the voltage $\Delta u = u_S - u_{FB}$ growth.

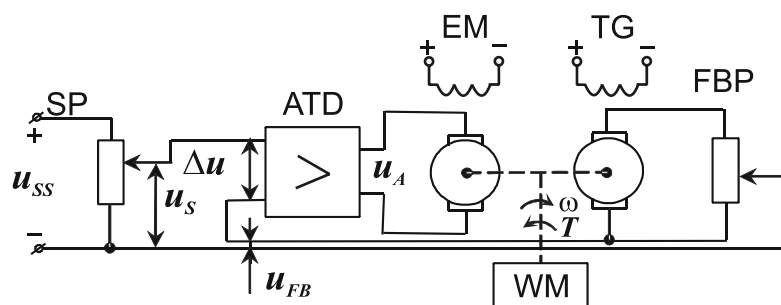


Figure 1.3 – The rotation velocity stabilization system

This results in the gain of the engine shaft rotation velocity due to the greater voltage coming from the amplifier. Thus we obtain the compensation of disturbance influence on the controlled object which remains incomplete in the described system. To obtain the full compensation we need an astatic regulator.

1.3. Concise history of automatic control

Certain automatic regulators have appeared long time ago.

At the urn of our era the Arabs provided a water clock with a floating level regulator.

In 1657 a Dutch scientist Christians Guygens (1629 - 1695) has built a pendulum rate governor into watch.

The plenty of regulators has been developed at the edge of the XVIII-XIX centuries during the industrial revolution in Europe. In 1765 a Russian heating engineer I.I. Polzunov (1728 - 1766) put forward a steam engine feed-water regulator. In 1784 an Englishman James Watt (1736 – 1819) developed a steam engine speed centrifugal governor. The first loom control timing unit based on punched cards was constructed by Frenchman Joseph Jacquard (1752 – 1834) in 1804. These regulators had opened the gates to the gush of new inventions.

But the development of complex machines with regulators required deep theoretical research.

A Russian scientist I.A. Vyshnegradsky (1831 – 1895) has become the founder of the automatic control theory, having published the work «About the general theory of regulators» in 1876. This study treated a regulator and a steam engine as an integrated system, and it introduced the methods of stability and regulation quality analysis. Unfortunately the

paper "About regulators" (1866) written by an English physicist James Maxwell (1831 – 1879) remained unnoticed by his contemporaries as the author dealt with the highly specific telescope driving mechanism.

A Russian mathematician A.M. Lyapunov (1857 – 1918) in 1892 developed the strict modern theory of mechanical systems balance and movement stability. His methods gave birth to some ideas of the modern automatic control theory. C.Ye.Tsiolkovsky (1857 – 1935) suggested applying plane rudder electric control (i.e. the first autopilot) in 1898. The first Russian textbook on the automatic control theory was written in 1909 by N.Ye. Zhukovsky (1847 – 1921).

The considerable contribution to the modern ACT was made by E. Routh, A. Gurwitz, A. V. Mikhaylov, H. Nyquist (stability criteria), V. V. Solodovnikov, Y. P. Popov (frequency methods of system analysis and synthesis), L. S. Pontryagin, A. A. Feldbaum, N. N. Krasovsky (1924) (optimal control), A. A. Krasovsky (1921), A. S. Kuhtenko (adaptive systems), N. Wiener (probabilistic approach), J. Tue, E. Jury (discrete systems) and a number of other scientists.

James Watt

(1736 – 1819)

An English inventor, the founder of the first universal thermal engine. He invented (1774 - 84) the steam engine with the double-acting cylinder which was equipped with the centrifugal governor in order to stabilize rotation velocity.



Ivan Alekseevich Vyshnegradsky

(1831 – 1895)

A Russian mathematician. Worked in the field of mechanical engineering. He investigated the dynamics of automatic control systems in terms of linearized equations. He exposed the impact the inertia moment of a flywheel working shaft, governor ball weight and force of viscous friction in the coupling have on the system stability. Vyshnegradsky was the first scientist who had solved the important problem of the industrial automatic control engineering on the basis of the linear theory





Nikolay Yegorovich Zhukovsky

(1847 – 1921)

A Russian mathematician who worked in the field of the gas and fluid mechanics, mechanics of rigid body, the aircraft theory. He was the author of classical works on aerodynamics. His lectures "Theory of machine running adjustment" were published by students in 1909 and became the first Russian textbook on automatic control where the description of regulator design was combined with mathematical research of system "machine - regulator" dynamics.



Alexander Mikhaylovich Lyapunov

(1857 – 1918)

A Russian mathematician. In the 1890ies developed the theory of ordinary differential equations solving stability. The ideas and methods of his classical works have been widely used in the control theory since the end of 1940ies. He developed the first and the second methods of nonlinear systems stability analysis. Special functions, which he had proposed, have his name.



Aurel Stodola

(1859 – 1942)

Austria-Hungary, Switzerland. An industrial engineer. He developed the scientific basis of steam and gas turbine design. In middle 1890ies he summarized the ideas suggested by A. Vyshnegradsky and investigated the complicated turbine rotational velocity automatic control system in terms of linearized equations the theoretical calculations being followed by experimental check. The linearized equations became widely applied for industrial development of automatic control systems.

Norbert Wiener

(1894 – 1964)

An American mathematician. Obtained fundamental results in the theory of random processes and other mathematical sciences. One of the founders of stochastic communication and control theory. In middle 1940ies developed the method of stationary casual processes filtration (Wiener filter). The author of the book "The Cybernetics" (1948) where he had suggested the idea of universal control science based on the principle of negative feedback and the theory of random processes.



Harry Nyquist

(1889 – 1976)

An American telecommunication engineer. Obtained fundamental achievements in the wire communication theory and technology and the theory of electronic amplifiers. In 1932 issued a paper where had suggested the analysis of linear feedback system stability based on the amplitude-phase characteristic of open-loop system. This research initiated the development of frequency methods in the automatic control theory.



Alexander Aleksandrovich Andronov

(1901 – 1952)

A Soviet physicist, worked on the fluctuation theory and the automatic control theory. The founder of nonlinear automatic control theory. The author of the phase space nonlinear system analysis. In 1940ies together with some of his students and collaborators developed the method of "point transformations". Was a remarkable teacher, so that a lot of his students became outstanding scientists and made a considerable contribution to the development of the automatic control theory.





Anatoly Isakovich Lurie

(1901 – 1980)

The USSR. Graduated from the department of mathematics the Leningrad polytechnic institute. A founder of nonlinear automatic control theory. Was the first to apply the A.M. Lyapunov's stability theory to the nonlinear system analysis. Developed Lyapunov functions building technique, the functions enabling the efficient solution of the nonlinear automatic control systems stability problem. Formulated the problem of absolute stability and found the solution of some ACS classes.



Lev Semyonovich Pontryagin

(1908 – 1988)

A Soviet mathematician. In 1930ies obtained fundamental results in topology. From the beginning of 1950ies worked in the field applied mathematics and the control theory. A founder of the mathematical theory of optimal processes. Together with V.G. Boltyansky and R.V. Gamkrelidze developed essentially new technique of solving the control theory variational problems.



Lev Semyonovich Goldfarb

(1910 – 1960)

A Soviet electrical engineer. A creator of the first electronic controllers for high-powered alternating current generators. In 1940ies suggested using frequency analysis based on harmonic linearization of nonlinear elements to study self-oscillations of ACS which proved to provide accurate results for typical cases.

**Alexander Mikhaylovich
Lyotov**

(1911 – 1974)

A Soviet mathematician. Worked on the nonlinear automatic control theory and optimal control theory. In 1960-1962 developed the method of ACS synthesis for object with one control influence with the square-law criterion (analytical regulator designing). This method took into account the control signal modulo limitations and servomotor speed limits.



**Alexander Aronovich
Feldbaum**

(1913 - 1969)

A Soviet mathematician and electrical engineer, the developer of the control theory. In 1948 formulated the problem of optimal control as variation problem. In 1949-1955 issued a series of articles where he had solved several problems on the synthesis of optimally performed systems having applied the phase space technique. Proved the n-range theorem.



**Mikhail Vladimirovich
Meyerov**

(born in 1914)

A Soviet mathematician and electrical engineer. In 1945-1960ies developed the linear ACS synthesis technique, systems able to resist regulator amplification unlimited increase for objects with one and several controlled variables.





Jacov Zalmanovich Tsytkin

(1919 – 1997)

A Soviet mathematician and electrical engineer. A developer of many branches of control theory. In 1945-1960ies conducted research on the theory of the pulse and relay systems which became fundamental to the corresponding aspects of ACT.



Richard Ernest Bellman

(1920 – 1984)

An American mathematician. Worked in the field of calculus mathematics and the control theory. A founder of the mathematical theory of optimal processes. In 1953-1965 suggested the Bellman principle of optimality and developed a new approach to solve optimization problems of technology and economics (dynamic programming method).



Stanislav Vasilyevich Yemelyanov

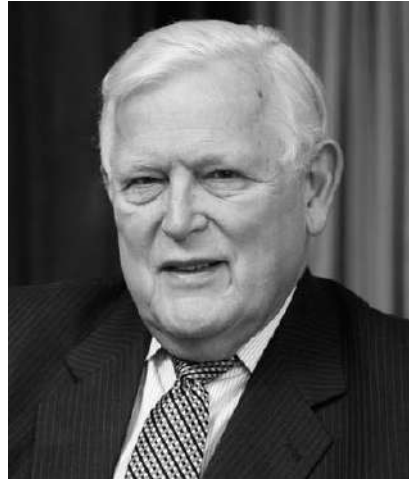
(born in 1929)

The USSR. One of the most prominent representatives of system theory and the ACT. An academician of the Russian Academy of Sciences. In 1970ies managed the development of the essentially new synthesis procedures, the synthesis of systems with variable structure. In 1980ies worked over the feedback theory, the results of this research being published in the monograph "New Feedback Types" (1997).

Rudolf Emil Kalman

(born in 1930)

The USA. A representative of the control theory and the mathematical automatic control system theory. In early 1960ies offered a new method of the solving the casual processes filtration problem (Kalman's filter) which found rapidly a wide application to various areas of engineering. In the same period developed the linear system synthesis method which was optimal from the view of square-law criterion.



The automatic control theory is still far from completion and new works and problems are developed all the time.

Lecture № 2

THEORY OF AUTOMATIC CONTROL SYSTEM CONSTRUCTION

Light of control principles, as well as light of beacons, is guiding only for those, who knows an input in harbour. The naked principle without means of its realization means nothing.

Anry Fayol

Lecture outline:

1. Principles of control.
2. Basic laws of control.

2.1. Principles of control

Principle of control is a method of control action formation. There are three fundamental principles of control: control by reference input, control by disturbance, and control by deflection.

2.1.1. Control by the reference input

Generally we can present ACS as a combination of two elements (Fig. 2.1).

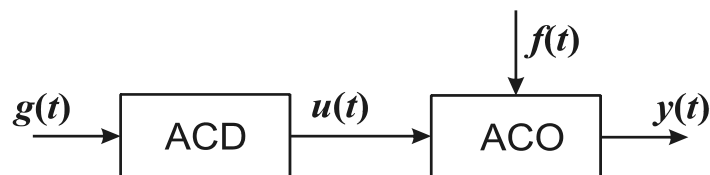


Figure 2.1 – ACS

Where

ACD is an automatic control device,

ACO is the automatic controlled object,

g is the reference input carrying the information about the required output value y ,

u is the control action altering the ACO condition purposefully,

f is disturbance action, an impact the environment has on the ACO.

For this case the control action $u(t) = u(g(t))$.

In the simplest case $y(t) = k_{ACO}u(t) - k'_{ACO}f(t)$.

The control action is generated on the basis of the information about the reference input and does not depend on disturbance, on output and on the state of the controlled object. System is open-loop.

Such principle can be apply to construct hierarchical systems (Fig. 2.2)

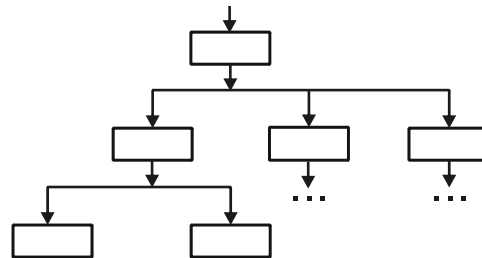


Figure 2.2 – Hierarchical system

The main advantage of this type of systems is that they are easily realized.

But such systems do not take into account the state of the object and the influence of disturbances is not compensated. The control action generation does not depend on the results of control.

Example 1

Engine speed automatic control system (Fig. 2.3):

SP is a setting potentiometer,

PA is a power amplifier,

EM is an electrical motor,

TG is a tachogenerator,

WM is a working mechanism (e.g. antenna),

V is a voltmeter.

Potentia is force,

Tachos is speed (gr.).

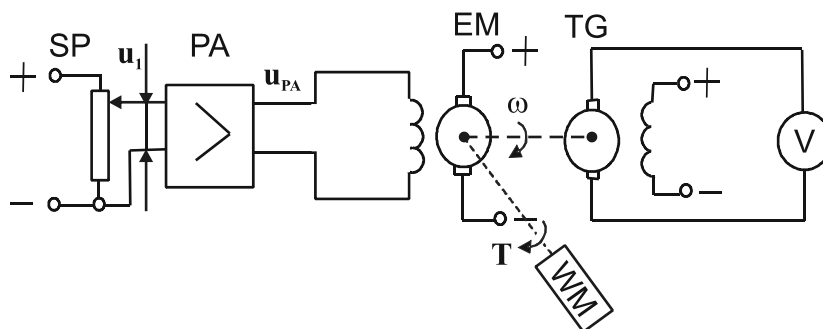


Figure 2.3 – Automatic control system of motor rotation velocity

The engine speed and voltage U_1 , U_{PA} grows up with the SP slide moved forward. Voltmeter meterage corresponds to the speed ω . The

change of the resistance torque T which is generated by the working mechanism causes the corresponding change of the engine speed which is an extremely undesirable factor (Fig. 2.4.).

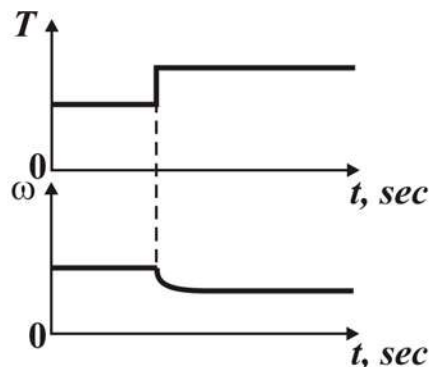


Figure 2.4 – The velocity rotation response

This principle is widely used for providing control of metal-working machinery with PNC and in automatic devices.

2.1.2. Control by disturbance (Poncelet principle, compensation control (*compensatio* (lat.) - an equilibration)

We enter the compensation block (CB) which measures disturbances and generates the signal proportional to the disturbance action. It is introduced to the system in order to compensate the harmful impact the disturbance produced on the system (Fig. 2.5.).

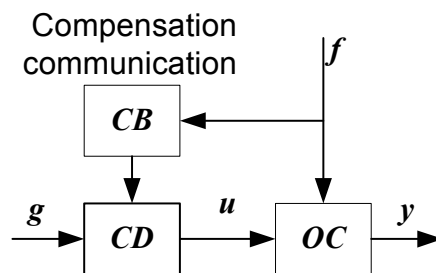


Figure 2.5 - The compensation system

The compensation block CB consists of the measuring instrument and the converter. In this case:

$$u(t) = u(g(t), f(t)). \tag{2.1}$$

The control action depends not only on the reference input, but also on the disturbance.

The system is open-loop as well as in the previous case.

Example 1

In this system the output depends not only on the reference input, but also on the disturbance action (Fig. 2.6).

Without the compensation block:

$$y(t) = k_1 k_2 g(t) - k_3 f(t). \quad (2.2)$$

With the compensation block:

$$y(t) = k_1 k_2 g(t) - k_3 f(t) + k_{CB} k_2 f(t). \quad (2.3)$$

If $k_{CB} = k_3 / k_2$ we get $y(t) = k_1 k_2 g(t).$ (2.4)

The disturbance impact is compensated in this case.

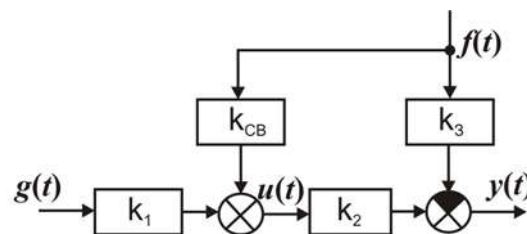


Figure 2.6 – Compensation system

Example 2

Let us introduce the measurer of torque MT to the engine speed control system. The MT voltage is added to the setting potentiometer voltage:

$$u_1 = u_{SP} + u_{MT} \quad (\text{Fig. 2.7}).$$

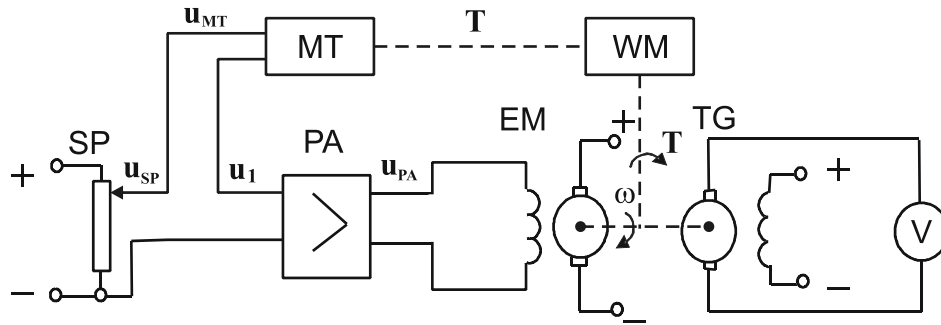


Figure 2.7 – Compensation system of motor speed stabilization

The systems with ideal compensation of disturbance are called the systems invariant by disturbance action (Fig. 2.8).

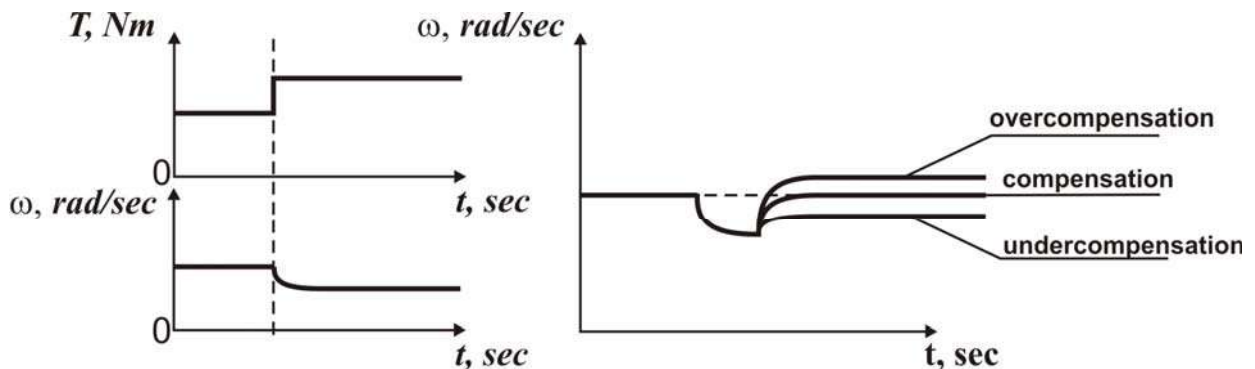


Figure 2.8 – Disturbance response Figure 2.9 – Kinds of compensations

Applying this type of ACS we can compensate the disturbances though only those which could be measured.

The sign and the transfer coefficient of *MT* should be chosen correctly, the wrong choice causing have under- or overcompensation (Fig. 2.9).

2.1.3. Control by deviation (the closed-loop control principle, the feedback principle, Poldunov-Watt principle).

Let us introduce the negative feedback. In this case we get a closed-loop system (Fig. 2.10).

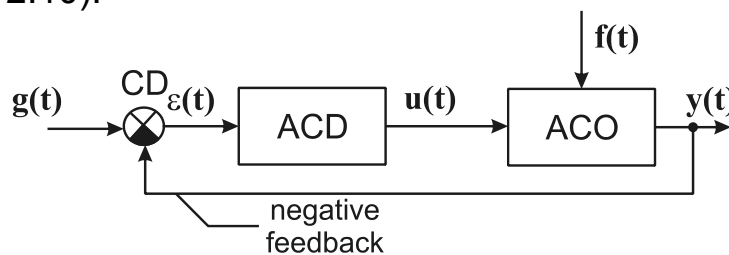


Figure 2.10 – Closed-loop system

CD is a comparing device.

The control action is an error function $u(t) = u(\varepsilon(t))$.

Let $g = y_{req}$, then $\varepsilon = g - y = y_{req} - y$ is a deflection or error.

The deflection is caused by the reference input or disturbance (load, change of parameters) actions. The control action depends on the output signal deflection from the required value and eliminates an error irrespective of its cause.

All disturbances are highly compensated but the system can become unstable.

Example

In the closed-loop system (Fig. 2.11) $u_1(t) = u_{RP}(t) - u_{TG}(t)$.

While assembling the system we should provide a correct sign of the feedback otherwise the engine can break into the saturation mode or reach the point of breakage.

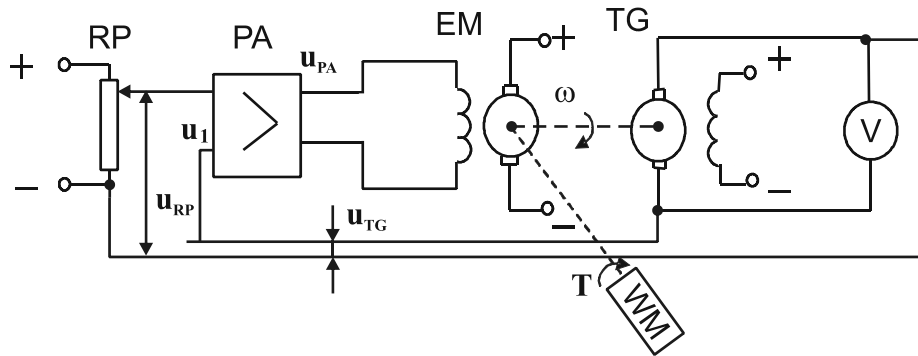


Figure 2.11 – The closed-loop system

Negative feedback (NFB) is used to get stability. Positive feedback (PFB) is used to change the state quickly or to generate oscillations.

The given system does not compensate disturbance completely (Fig. 2.12).

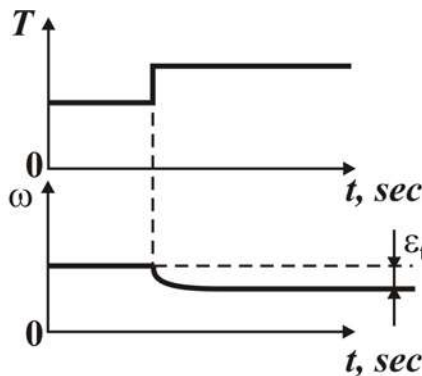


Figure 2.12 – Compensation of disturbance

The static error ε_f can be emended by the astatic regulator (e.g. integral or isodrome (isos (gr.) - equal, dromos (gr.) - run, a place for running).

The best results can be obtained by the combination of the three principles of control:

$$u(t) = u(g(t), \varepsilon(t), f(t)). \quad (2.5)$$

2.2. Basic control laws

A control law or an algorithm is the kind of subjection of the control action of the reference input or disturbance action, of the controlled value deflection or the controlled object parameters.

A control law is a rule according to which the control action is formed:

$$u = u(g, \varepsilon, f, p).$$

Let us use different feedback systems to consider some control laws (Fig. 2.13): $u = u(\varepsilon)$.

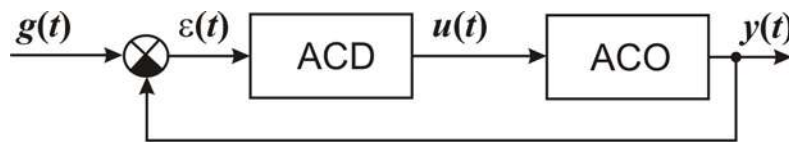


Figure 2.13 – Closed-loop system

There are linear and nonlinear laws. Further we shall consider some of them.

2.2.1. The relay control law

The simplest nonlinear law is $u = C \text{sign}(\varepsilon)$, (*signum* (lat.) - a sign)

It is used in the spacecraft stabilization systems. The engines are usually established on the opposite sides of the spacecraft. The spacecraft deviated, one of these engines is started and operates till the sign of deflection is changed, then the opposite engine is turned on (Fig. 2.14, a).

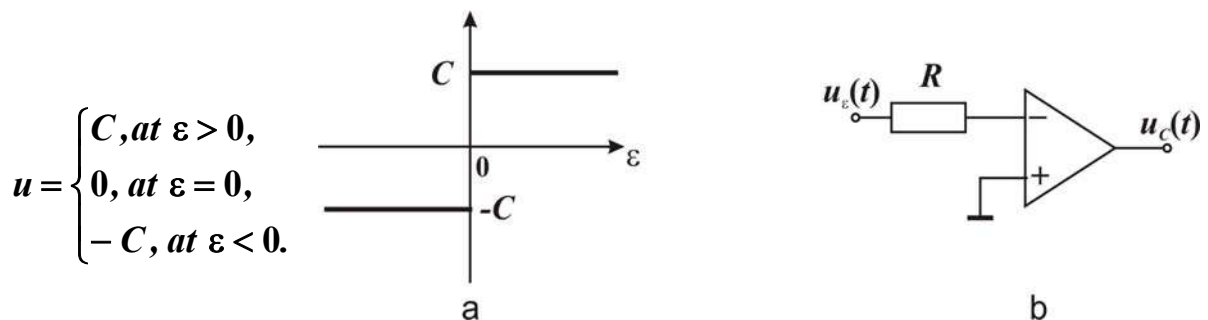


Figure 2.14 – Comparator

The system displays the simplicity of realization but it is of small efficiency. There exists also the threat of the self-oscillations to occur in the system.

2.2.2. The proportional control law

It is the simplest linear control law

$$u(t) = k \varepsilon(t).$$

The realization of the proportional regulator with the operational amplifier OA having $k = \frac{R_2}{R_1}$ is shown on Fig. 2.15.

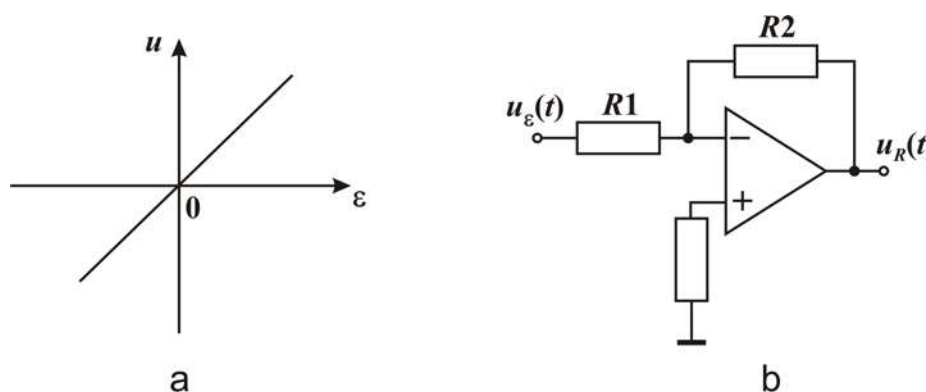


Figure 2.15 – P-regulator

The system gains greater efficiency in as compared to the systems with the relay control law applied and is simple in realization though the small deflections get not enough control.

The control evidence requires the existence of nonzero error. The operation of static object cannot occur without an error. The decrease of the error value and the acceleration of the control process need to increase the amplification coefficient k . But meanwhile the system loses stability.

2.2.3. The integral control law

The integral control law has form

$$u(t) = \frac{1}{T_I} \int_0^t \varepsilon(\tau) d\tau. \quad (2.7)$$

The realization of the I-regulator with the OA (Fig. 2.16) is widely used. Transfer function of I -regulator looks like

$$W(s) = \frac{U_R(s)}{U_\varepsilon(s)} = 1/RCs = 1/T_I s, T_I = RC, \quad (2.8)$$

where T_I is integration constant.

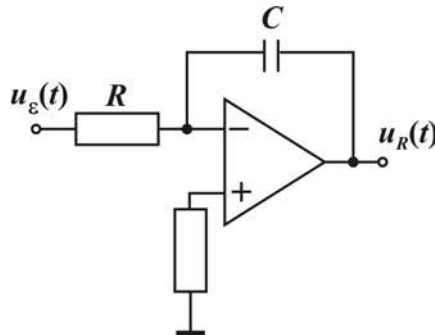


Figure 2.16 – I-regulator

We introduce an integral to eliminate the steady-state error under the constant input actions. The control is changed until the error becomes equal to zero. The control is stabilized at $\varepsilon = 0$.

The integration causes the phase delay, the process of regulation being thus slowed down and the phase stability margin decreased. The parameters chosen incorrectly, the system can become unstable.

2.2.4. The proportional - integral control law

The proportional-integral control law has form

$$u(t) = k_P \varepsilon(t) + \frac{1}{T_I} \int_0^t \varepsilon(\tau) d\tau. \quad (2.9)$$

The realization of the *PI*-regulator (isodrome) with OA is shown on Fig. 2.17.

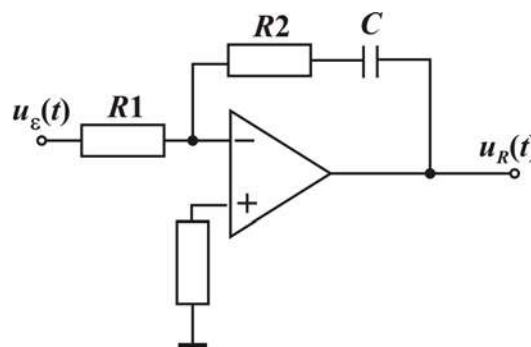


Figure 2.17 – PI-regulator

Its transfer function looks like

$$W(s) = \frac{U_R(s)}{U_\varepsilon(s)} = \frac{R_2 + 1/Cs}{R_1} = \frac{R_2}{R_1} + \frac{1}{R_1Cs} = k + \frac{1}{T_I s}, \quad (2.10)$$

where

$$k = R_2/R_1, \quad T_I = R_1C.$$

The proportional component is introduced to amplify the speed and stability of the regulation process, the integral component is imposed to increase the accuracy of the system.

2.2.5. The proportional – differential control law

The derivative is introduced into the control law in order to accelerate the regulation process. In the steady-state mode the derivative equals to zero. This component does not influence on steady-state error.

The realization of this law is shown on Fig. 2.18.

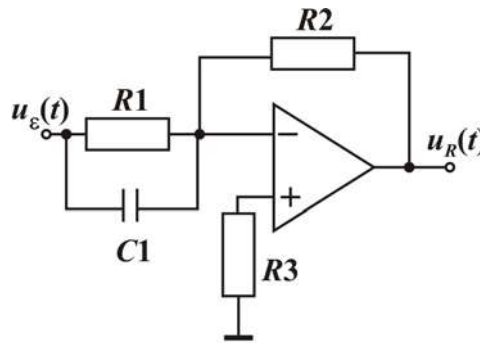


Figure 2.18 – PD-regulator

Equation of the *PD*-regulator looks like

$$u_R(t) = k\varepsilon(t) + T_D \frac{d\varepsilon}{dt}. \quad (2.11)$$

2.2.6. The proportional-integral-differential control law

This control law looks like

$$u(t) = k_P \varepsilon(t) + k_I \int_0^t \varepsilon(\tau) d\tau + T_D \frac{d\varepsilon}{dt}. \quad (2.12)$$

The realization of the PID- regulator with OA is shown on Fig. 2.19, where

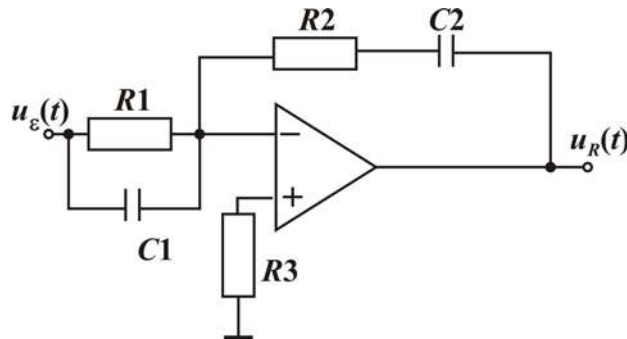
$$k_P = \frac{R_2}{R_1} + \frac{C_1}{C_2}; \quad T_I = R_1 C_2; \quad T_D = R_2 C_1. \quad (2.13)$$


Figure 2.19 – PID-regulator

Lecture №3

THEORY OF ACS CONSTRUCTION

The education of a man should start with proverbs and end with thoughts.
Lucius Annaeus Seneca

Lecture outline:

1. Functional elements of control systems.
2. Classification of ACS.

3.1. Functional elements of control systems

Let us consider the most typical ACS which is constructed on the basis of the three principles combination.

The functional diagram of this system is shown on Fig. 3.1.

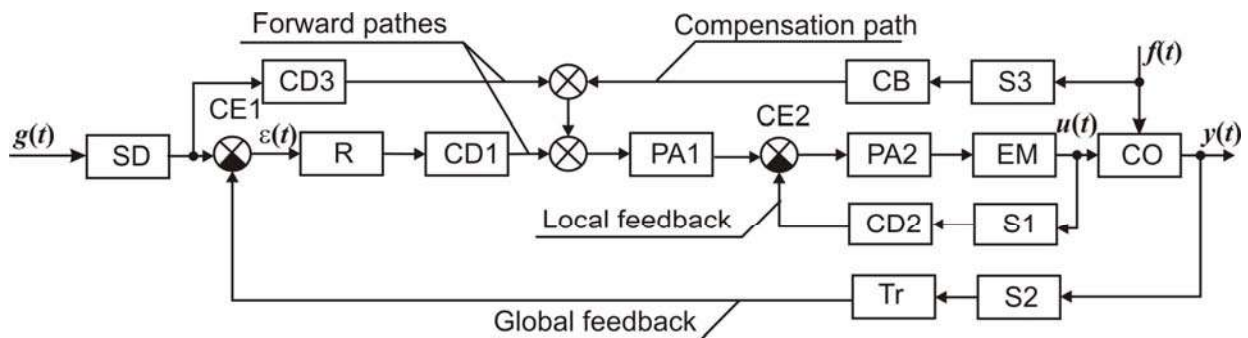


Figure 3.1– The functional diagram of the ACS, realizing the three principles control combination

SD is a setting device which transforms the setting action produced by a man or other system into the form convenient for comparison. It is used to set the required value of controlled variable.

CE is a comparison element. It compares an output signal with the reference action.

PA is a power amplifier.

R is a regulator (analogous or digital). It realizes the control law.

CD is a correcting device. It provides the stability of system and the required quality parameters (overshoot, swiftness and oscillation).

PA is a power amplifier.

EM is an executive mechanism, actuator (e.g. the aircraft servo unit). It exerts a desirable influence onto the controlled object.

CO is a controlled object.

S is a sensor (a measuring transformer). It produces the electrical signal proportional to the measured process value (shift, pressure, temperature, electric voltage, etc.).

Tr is a transformer. It transforms the feedback signal into the type convenient for comparison (a voltage divider, a phase-sensitive rectifier, an analog-to-digital converter, etc.).

CB is a compensation block.

The global feedback transfers a signal from the system output to the input.

The local feedback transfers the signal from the output of system or one of the elements to an input of certain element.

The feedback can be positive or negative, rigid (working both in the established and in the transitive modes) and flexible (working only in the transitive mode). The elements of the functional diagram are shown on Fig. 3.2.

A directional effect link transforms the information without affecting the work of the previous link (Fig. 3.2, a).

A node is an information branching point (Fig. 3.2, b)

$$y_1 = y_2 = g .$$

A summer (fig. 3.2, c) can add and subtract signals $y = x_1 \pm x_2$.

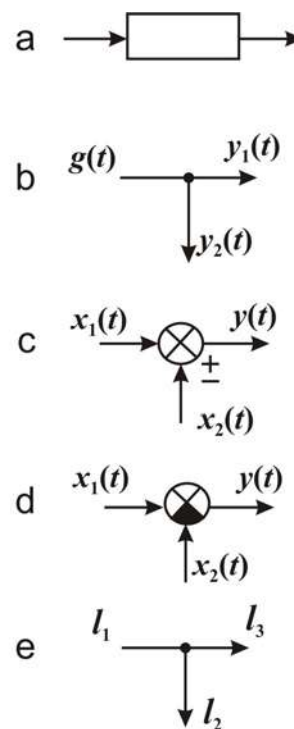


Figure 3.2- Elements of ACS diagrams

The sector is filled for a case of subtracting corresponding signal (Fig. 3.2, d).

$I_1 = I_2 + I_3$ is a node of the key chart (Fig. 3.2, e).

The various kinds of the diagrams such as a block-diagram, functional diagram, key diagrams, etc. are used to represent the control systems.

A block-diagram as an element of the design documentation is a graphical representation of the system which indicates the main blocks and their interconnections, signals and actions.

A functional diagram is the graphical representation of the system which shows all the blocks, their interconnections, signals and actions.

A key diagram is the graphical representation of system which indicates all the elements, their marks, nominal values and other characteristics, the connections between elements and blocks, signals and actions. The sockets and the wire soldering points are plotted on the circuit diagrams.

The structural diagram in terms of the control theory is the graphical representation of the system mathematical model, including links, summers, communication channels and indication of signals and actions.

3.2. Classification of the ACS

There is wide variety of automatic control systems. Therefore there is no singular indication which will become a base of division all systems into the classes totally reflecting the whole mass of properties. Let us consider ACS classifications based on certain attributes.

The ACS can be divided on the basis of:

1. Their purpose:
 - power (energy);
 - technological;
 - transport;
 - navigating;
 - orientation systems;
 - life-support systems;
 - synchronizing system, etc.
2. The type of energy:
 - mechanical;
 - hydraulic;
 - pneumatic;
 - electric;
 - combined.
3. The control principle:
 - controlled by setting action;
 - controlled by disturbance;
 - controlled by deviation;
 - combined.

4. The number of feedbacks:
 - open-loop;
 - closed-loop;
 - single-loop system;
 - multiple-loop system.
5. The number of controlled variables:
 - single-input/single-output (SISO);
 - many-input/many-output (MIMO).
6. The degree of dependence which exists between several inputs and outputs:
 - separated;
 - multilinked.
7. The form of the setting action:
 - stabilization systems ($g(t) = const$);
 - program systems ($g(t)$ –function known beforehand);
 - tracking systems ($g(t)$ –time function unknown beforehand);
 - terminal systems (value $g(t_f)$ is given at the final time point t_f).
8. The amount of the prior information about the controlled object and disturbance:
 - ordinary (with the full information);
 - robust (with the incomplete information);
 - adaptive (with information minimum):
 - self-adjusting;
 - self-organizing;
 - self-learning (intellectual);
 - self-developing.
9. The character of the process:
 - continuous (analog);
 - discrete:
 - relay;
 - pulse;
 - digital.
10. The degree of definiteness of parameters and structures:
 - determined;
 - probabilistic (stochastic).
11. The steady-state accuracy:
 - static (by setting action, by disturbance);
 - astatic (by setting action, by disturbance).
12. The equation character:
 - linear;
 - nonlinear.

13. The availability of features:

- without delay;
- with delay;
- stationary;
- non-stationary;
- with concentrated parameters;
- with allocated parameters.

14. Stability:

- stable;
- unstable;
- neutral.

15. On other characteristics:

- controlled;
- uncontrolled;
- observable;
- non-observable;
- sensitive;
- rough;
- identifiable;
- non-identifiable;
- fault tolerance, etc.

Comprehension questions

1. What elements form a system functional diagram?
2. What is the feedback?
3. What is the block diagram?
4. What is the key diagram?
5. Itemize the classification ACS attributes.

Lecture №4

MATHEMATICAL DESCRIPTION OF ACS

Analysis is the essence the
intellectualism.

George Zimmel

Lecture outline:

1. Construction of ACS mathematical model.
2. ACS mathematical description.

4.1. Construction of ACS mathematical model

The mathematical model or mathematical description of ACS is the aggregate of mathematical correlations describing the dynamics of the system. As correlations we can treat algebraic, differential and difference equations, tables, graphs, plots, diagrams, algorithms, programs, matrices, transfer functions, operators.

The mathematical models of ACS are widely used to solve system analysis and synthesis tasks by mathematical methods. The type of mathematical model is determined by both system and current task. Different tasks of automatic control theory require the models of different accuracy to be applied. As it is impossible to create a model which will absolutely correspond to the system, we should choose which factors should be taken into account and which could be ignored in every case. This task is creative and cannot be solved unambiguously. The more factors are taken into account to increase the exactness of the model in the process of modeling the more cumbersome model becomes. Complicated equations are solved approximately, and therefore the more difficult the equation is, the less exact the decision can be found. In addition, cumbersome equations are more difficult to solve and to draw any conclusions.

The procedure of the input-output equation formation can be as follows:

1. Divide the system into elements. Determine its input and output signals (reference, disturbances, control actions, controlled variables).

2. Construct equations which connects the input and output signals using the physical laws of element operation or the experimental data.

3. Choose an operating point or the system operating mode. Construct equations of the steady-state element mode.

4. Enter the steady-state mode deviations. Formulate equations of element dynamics.

5. Subtract the equation of the steady-state mode from the dynamic equations; get the element dynamic equations for the operating point deviations.

6. If needed and possible, make linearization of the obtained equations.

7. Make up the element block diagram; find the linear elements transfer functions.

8. Make up the system block diagram. Exclude intermediate variables, get an input-output equation and transfer functions for the linearized systems.

9. Check up the adequacy of obtained got model, for example, by comparing experimental characteristics and estimated step responses (*adaequatys* (lat.) - equal).

Let us consider for linearization of equations in more detail.

If a nonlinear function $F(\cdot)$ does not contain the breaks, fractures or ambiguities, it can be linearized by a tangent or a Taylor series expansion (Fig. 4.1).

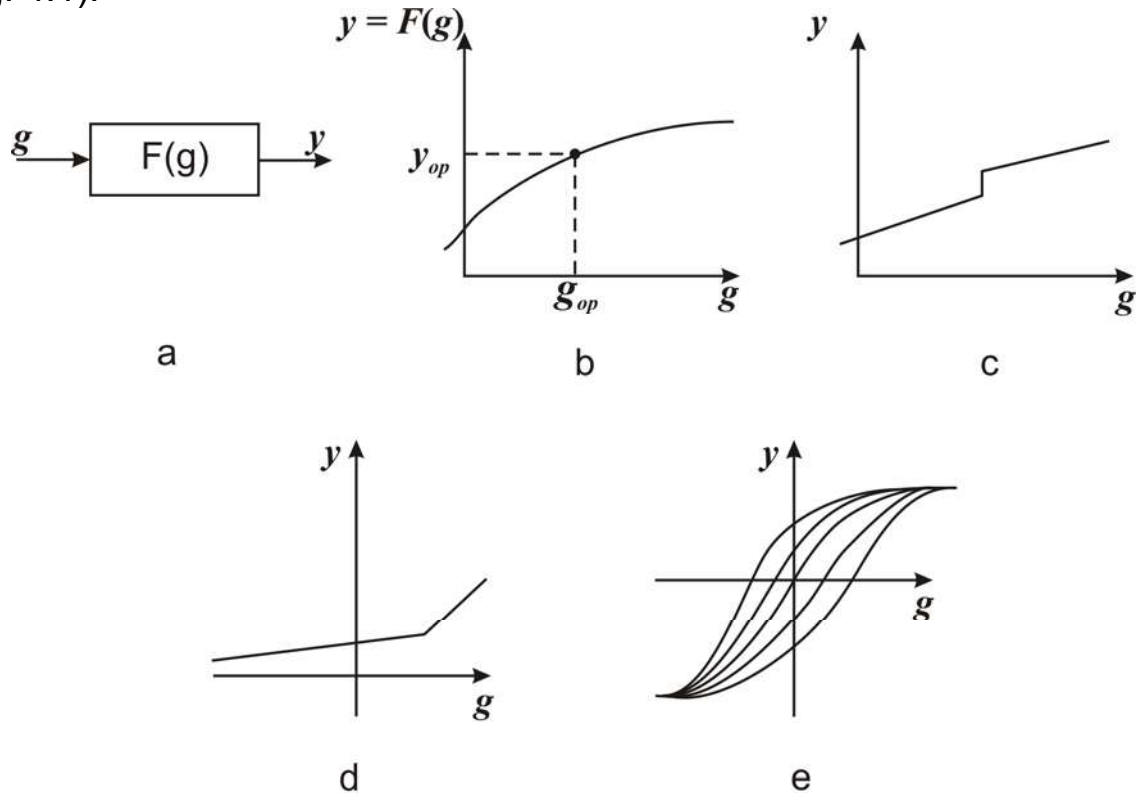


Figure 4.1– Static descriptions:
 a – link; b – smooth; c – break; d – fracture; e – ambiguity

It is possible to apply static linearization for the constant signals (Fig. 4.2).

$$F(g) \Big|_{g = g_{ss}} = k_{st}g, \quad k_{st} = \frac{y_{ss}}{g_{ss}}. \quad (4.1)$$

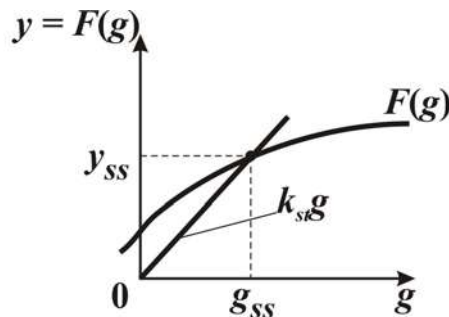


Figure 4.2 – Static linearization

These functions agree in the operating point.

In the case of small-deflection linearization we can apply a tangent linearization (Fig. 4.3)

$$F(g) = y_{ss} + k_{tang} \Delta g, \quad k_{tang} = \left. \frac{\partial F(g)}{\partial g} \right|_{g = g_{ss}}, \quad \Delta g = g - g_{ss}. \quad (4.2)$$

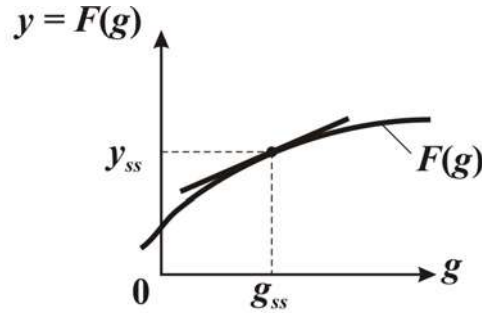


Figure 4.3 – Tangent linearization

If the function depends on multiple arguments $y = F(g_1, g_2, y, \dot{y}, \dots)$, it can be developed into Taylor series (Broke Taylor - 1685-1731),

$$y = F(g_{1ss}, g_{2ss}, y_{ss}, 0, \dots) + \left. \frac{\partial F(\dots)}{\partial g_1} \right|_{\substack{g_1 = g_{1ss} \\ g_2 = g_{2ss} \\ \dots}} \Delta g_1 +$$

$$+ \left. \frac{\partial F(\dots)}{\partial g_2} \right|_{\substack{g_1 = g_{1ss} \\ g_2 = g_{2ss} \\ \dots}} \Delta g_2 + \left. \frac{\partial F(\dots)}{\partial y} \right|_{\substack{g_1 = g_{1ss} \\ g_2 = g_{2ss} \\ \dots}} \Delta y + \left. \frac{\partial F(\dots)}{\partial \dot{y}} \right|_{\substack{g_1 = g_{1ss} \\ g_2 = g_{2ss} \\ \dots}} \Delta \dot{y} + \dots$$

Then we keep only the variables of the first infinitesimal order and neglect the infinitesimals of higher orders

$$y = y_{ss} + a \Delta g_1 + b \Delta g_2 + c \Delta y + d \Delta \dot{y} + \dots$$

Thus we obtain a linear equation.

If the working point deflection is considerable, the secant approximation (the least-squares method) will be applied (Fig. 4.4).

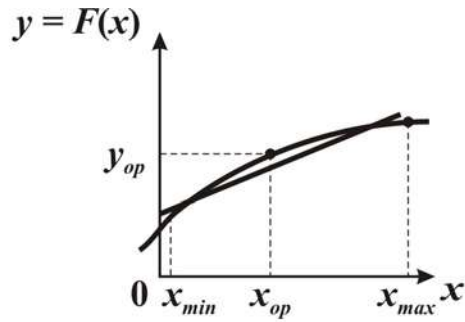


Figure 4.4 – Least-squares method

If any oscillating processes occur in the nonlinear systems, we should apply the harmonic linearization. For the random actions the statistic linearization is used.

4.2. ACS mathematical description

Fig. 4.5 represents the motor rotation speed stabilizing system.

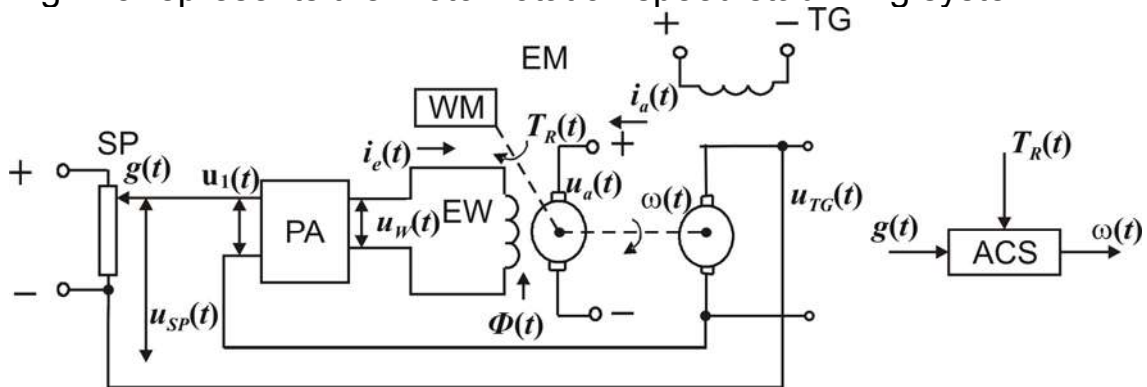


Figure 4.5 – The motor rotation speed stabilizing system

We divide the system into elements: a setting potentiometer (SP), a power amplifier (PA), an electrical motor (EM) with excitation winding (EW), a tachogenerator (TG), a working mechanism (WM) and designate inputs and outputs signals.

Then we find the dynamics element equations.

$$\text{SP: } u_s(t) = k_{sp} g(t),$$

$$\text{CD: } u_1(t) = u_{sp}(t) - u_{TG}(t),$$

$$\text{PA: } u_w(t) = k_{PA} u_1(t),$$

$$\text{EM: } u_w(t) = R_w i_w(t) + \omega(t) \frac{d\Phi}{dt},$$

$$u_a(t) = R_a i_a(t) + C_e \Phi(t) \omega(t) = \text{const},$$

$$T_r(t) = C_m \Phi(t) i_a(t) - J \frac{d\omega}{dt}.$$

$$\text{TG: } u_{TG}(t) = k_{TG} \omega(t).$$

The nonlinear function $\Phi(i_e)$ is shown on Fig. 4.6.

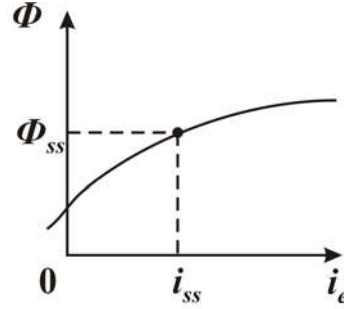


Figure 4.6 – Steady - state characteristic of electrical motor stator

We choose a steady-state mode operation point $\omega_{ss}, \Phi_{ss}, u_{TGss}, g_{ss}, u_{1ss}, i_{Wss}, i_{ass}, u_{Sss}, u_{Wss}, T_{ss}$. It corresponds to steady-state value of the input signal g_{ss}, T_{rss} .

We equate the steady-state mode

$$\begin{aligned} u_{Sss} &= k_{SP} g_{ss}; \quad u_{1ss} = u_{Sss} - u_{TGss}, \quad u_{Wss} = k_{PA} u_{1ss}, \quad u_{TGss} = k_{TG} \omega_{ss}; \\ u_{Wss} &= R_W i_{Wss}; \quad u_a = R_a i_{ass} + C_e \Phi_{ss} \omega_{ss}; \quad M_{CD} = C_M \Phi_{ss} i_{ass}; \\ \Phi_{ss} &= \Phi(i_{Wss}). \end{aligned} \quad (4.3)$$

We introduce the steady-state mode deflections

$$\begin{aligned} g &= g_{ss} + \Delta g, \quad \omega = \omega_{ss} + \Delta \omega, \quad \Phi = \Phi_{ss} + \Delta \Phi, \quad u_{TG} = u_{TGss} + \Delta u_{TG}, \quad u_1 = u_{1ss} + \Delta u_1, \\ i_{Sss} &= i_{Wss} + \Delta i_{Sss}, \quad i_a = i_{ass} + \Delta i_a, \quad u_{ss} = u_{Sss} + \Delta u_{ss}, \quad u_w = U_{Wss}, \\ T_R &= T_{rss} + \Delta T_R \end{aligned}$$

and obtain the element dynamic equations

$$\begin{aligned} u_{Sss} + \Delta u_S &= k_{SP} (g_{ss} + \Delta g), \quad u_{Wss} + \Delta u_W = R_W (i_{Wss} + \Delta i_W) + \omega \frac{d\Delta \Phi}{dt}, \\ u_{1ss} + \Delta u_1 &= u_{Sss} + \Delta u_S - u_{TG} = u_{TGss} + \Delta u_{TG}, \\ u_{ass} &= R_a (i_{ass} + \Delta i_a) + C_e (\Phi_{ss} + \Delta \Phi) (\omega_{ss} + \Delta \omega), \\ u_{Wss} + \Delta u_W &= k_{PA} (u_{1ss} + \Delta u_1), \end{aligned} \quad (4.5)$$

$$\begin{aligned}
T_{rSS} + \Delta T_R &= C_m (\Phi_{SS} + \Delta\Phi)(i_{aSS} + \Delta i_a) - J \frac{d\Delta\omega}{dt}, \\
u_{TGSS} + \Delta u_{TG} &= k_{TG}(\omega_{SS} + \Delta\omega), \\
\Phi_{SS} + \Delta\Phi &= \Phi(i_{WSS} + \Delta i_W).
\end{aligned}$$

We subtract the dynamic equations from the steady-state mode equations $\Delta u_S = k_{SP}\Delta g$,

$$\begin{aligned}
\Delta u_W &= R_W \Delta i_W + \omega \frac{d\Delta\Phi}{dt}, \\
\Delta u_1 &= \Delta u_S - \Delta u_{TG}, \quad \theta = R_a \Delta i_a + C_e \Phi_{SS} \Delta\omega + C_e \omega_{SS} \Delta\Phi + C_e \Delta\Phi \Delta\omega, \\
\Delta u_W &= k_{PA} \Delta u_1, \quad \Delta T_r = C_m \Phi_{SS} i_a + C_m i_{aSS} \Phi + C_m \Delta\Phi \Delta i_a - J \frac{d\Delta\omega}{dt}, \quad (4.6) \\
\Delta u_{TG} &= k_{TG} \Delta\omega, \quad \Delta\Phi = \Phi(i_{WSS} + \Delta i_W) - \Phi(i_{WSS}).
\end{aligned}$$

We linearized nonlinear equations, drop the sign Δ , and introduce the differential operator $p = \frac{d}{dt}$

$$\begin{aligned}
u_S &= k_{SP} g, & u_W &= R_W i_W + \omega p \Phi, \\
u_1 &= u_S - u_{TG} & \theta &= R_a i_a + C_e \Phi_{SS} \omega + C_e \omega_{SS} \Phi, \\
u_W &= k_{PA} u_1, & T_R &= C_m \Phi_{SS} i_a + C_m i_{aSS} \Phi - J p \omega, \quad (4.7) \\
u_{TG} &= k_{TG} \omega, & \Phi &= \Phi_{SS} + \left. \frac{\partial \Phi(i_W)}{\partial i_n} \right|_{i_W = i_{WSS}} \Delta i_W, \quad \Phi = a i_W.
\end{aligned}$$

We eliminate intermediate variables.

For the motor we get

$$\begin{cases}
u_W = R_W i_W + a \omega p_z i_W; \\
\theta = R_a i_a + C_e \Phi_{SS} \omega + C_e \omega_{SS} a i_W; \\
M_R = C_M \Phi_{SS} i_a + C_M i_{aSS} a i_W - J p \omega.
\end{cases} \quad (4.8)$$

Eliminate i_W, i_a

$$i_W = \frac{1}{R_W + a p \omega} u_W = \frac{1}{R_W (T_1 p + 1)} u_W, \text{ where } T_1 = \frac{a}{R_W} \text{ electric magnetic time}$$

constant of motor,

$$i_a = -\frac{C_e \Phi_{ss}}{R_a} \omega - \frac{C_e \omega_{ss} a}{R_a} \cdot \frac{1}{R_w (T_1 p + 1)} u_w,$$

$$T_r = -\frac{C_e C_m \Phi_{ss}^2}{R_a} \omega - \frac{C_e C_m \Phi_{ss} \omega_{ss} a}{R_a} \cdot \frac{1}{R_w (T_1 p + 1)} u_w + \frac{C_m i_{ass} a}{R_w (T_1 p + 1)} u_w - J p \omega.$$

Thus we obtained a single motor equation.
Let us put down it in more convenient form

$$R_a R_w (T_1 p + 1) M_R = -C_e C_m \Phi_{ss}^2 R_w (T_1 p + 1) \omega - C_e C_m \Phi_{ss} \omega_{ss} a u_w + C_m i_{ass} a R_a u_w - R_a R_w J (T_1 p + 1) p \omega. \quad (4.9)$$

We move the output signal and its derivatives to the left side of the deviation, and the input signal with its derivatives to the right side

$$(T_1 p + 1) (C_e C_m \Phi_{ss}^2 R_w + R_a R_w J p) \omega = (C_m i_{ass} a R_a - C_e C_m \Phi_{ss} \omega_{ss} a) u_w - R_a R_w (T_1 p + 1) T_r. \quad (4.10)$$

We divide both sides of equation by $(C_e C_m \Phi_{ss}^2 R_w)$. Finally we obtain the input-output equation for the engine in operator form

$$(T_1 p + 1) (T_2 p + 1) \omega(t) = k_1 u_w(t) - k_2 (T_2 p + 1) T_r(t), \quad (4.11)$$

where $T_2 = \frac{R_a J}{C_e C_m \Phi_{ss}^2}$ is the engine electromechanical time constant;

$k_1 = \frac{i_{ass} a R_a}{C_e \Phi_{ss}^2 R_w} - \frac{\omega_{ss} a}{\Phi_{ss} R_w}$ is the engine control action transient coefficient;

$k_2 = \frac{R_a}{C_e C_w \Phi_{ss}^2}$ is the engine disturbance transient coefficient.

We obtain the input-output equation by substituting the differential operator with the operation of differentiation. The differential form of the equation is as follows

$$T_1 T_2 \frac{d^2 \omega}{dt^2} + (T_1 + T_2) \frac{d\omega}{dt} + \omega(t) = k_1 u_w(t) - k_2 \left(T_1 \frac{dT_r}{dt} + T_r(t) \right). \quad (4.12)$$

The input-output element equations can be presented in diagram form after the following correlations are taken into account

$$u_s = k_{SP}g, \quad u_1 = u_s - u_{TG}, \quad u_W = k_{PA}u_1, \quad u_{TG} = k_{TG}\omega,$$

$$\omega = \frac{k_1}{(T_1p+1)(T_2p+1)}u_W - \frac{k_2}{(T_2p+1)}T_r.$$

We can draw an ACS block-diagram (Fig. 4.7).

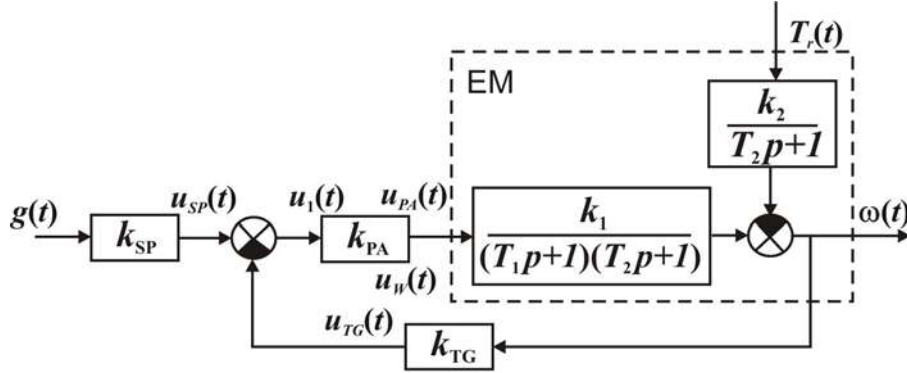


Figure 4.7 – Structural diagram of ACS

Then we eliminate other intermediate variables u_s, u_1, u_W, u_{TG} in order to obtain

$$\omega(t) = \frac{k_1 k_{PA} (k_{SP}g(t) - k_{TG}\omega(t))}{(T_1p+1)(T_2p+1)} - \frac{k_1}{T_2p+1}T_r(t), \quad (4.13)$$

or

$$((T_1p+1)(T_2p+1) + k_1 k_{PA} k_{TG})\omega(t) = k_1 k_{PA} k_{SP}g(t) - k_2(T_1p+1)T_r(t)$$

and in such form

$$\omega(t) = \frac{k_1 k_{PA} k_{SP}}{(T_1p+1)(T_2p+1) + k_1 k_{PA} k_{TG}}g(t) - \frac{k_2(T_1p+1)}{(T_1p+1)(T_2p+1) + k_1 k_{PA} k_{TG}}T_r(t). \quad (4.14)$$

The system input-output equation short form is

$$\omega(t) = \Phi_g(p)g(t) + \Phi_f(p)T_r(t) \quad (4.15)$$

and the differential form is

$$T_1 T_2 \frac{d^2 \omega}{dt^2} + (T_1 + T_2) \frac{d\omega}{dt} + (1 + k_1 k_{PA} k_{TG})\omega(t) = k_1 k_{PA} k_{SP}g(t) - k_2 \left(T_1 \frac{dT_r}{dt} + T_r(t) \right).$$

Terms:

- mathematical model;
- mathematical description;
- static linearization;
- operating point;
- tangent linearization;
- secant approximation;
- least-squares method;
- steady-state characteristic.

Comprehension questions

1. What is the mathematical model?
2. What kinds of linearization do you know?
3. What is harmonic linearization?
4. What is system working point?
5. Describe the procedure of obtaining the input - output equation.
6. What kinds of static characteristics do you know?

Lecture № 5**MATHEMATICAL DESCRIPTION OF ACS (ENDING)**

If we really know something, we know it due to the study of mathematics.

P. Gassendi

Lecture outline:

1. Forms of ACS equations recording.
2. Static and dynamic characteristics of ACS.

From here on we consider the simplest systems: linear, continuous, ordinary, stationary, etc.

5.1. Forms of ACS equations recording

The ACS mathematical model can be represented in several forms.

5.1.1. The standard form of the input – output equation recording

Let the system have a single setting action, a single disturbance and a single output (Fig. 5.1).

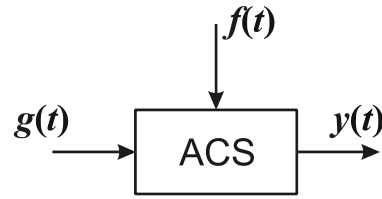


Figure 5.1 –The functional diagram of ACS

We set the output signals and their derivatives to the left side of the input-output equation and the input signals with its derivatives to the right side of the equation

$$\begin{aligned}
 a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y(t) = & b_0 \frac{d^m g}{dt^m} + b_1 \frac{d^{m-1} g}{dt^{m-1}} + \\
 & + \dots + b_m g(t) + c_0 \frac{d^l f}{dt^l} + c_1 \frac{d^{l-1} f}{dt^{l-1}} + \dots + c_l f(t). \quad (5.1)
 \end{aligned}$$

If this equation divided by factor a_n , the factors a_i/a_n will get the dimension sec^{n-i}

$$\begin{aligned}
 \frac{a_0}{a_n} \frac{d^n y}{dt^n} + \frac{a_1}{a_n} \frac{d^{n-1} y}{dt^{n-1}} + \dots + \frac{a_n}{a_n} y(t) = & \frac{b_0}{a_n} \frac{d^m g}{dt^m} + \frac{b_1}{a_n} \frac{d^{m-1} g}{dt^{m-1}} + \\
 & + \dots + \frac{b_m}{a_n} g(t) + \frac{c_0}{a_n} \frac{d^l f}{dt^l} + \frac{c_1}{a_n} \frac{d^{l-1} f}{dt^{l-1}} + \dots + \frac{c_l}{a_n} f(t).
 \end{aligned}$$

For the steady-state mode it will be

$$y_{ss} = \frac{b_m}{a_n} g_{ss} + \frac{c_l}{a_n} f_{ss}.$$

A coefficient $\frac{b_m}{a_n} = k_g$ is the system reference action transfer coefficient (gain) which has the dimension of $[y]/[g]$.

A coefficient $\frac{c_l}{a_n} = k_f$ is the system disturbance transfer coefficient (gain) which has the dimension of $[y]/[f]$.

The static equation can be easily obtained from the dynamic equation

$$y = k_g g + k_f f. \quad (5.2)$$

If n is an order of the equation, an order of the system, then the model feasibility conditions are: $m \leq n, l \leq n$.

The direct current motor input - output equation is as follows

$$T_1 T_2 \frac{d^2 \omega}{dt^2} + (T_1 + T_2) \frac{d\omega}{dt} + \omega(t) = k_u u_e(t) - k_{T_r} \left(T_1 \frac{dT_r}{dt} + T_r(t) \right).$$

5.1.2. The operator form of the input - output equation recording

Let us introduce the differential operator $p = \frac{d}{dt}$. Thus we obtain

$$a_0 p^n y(t) + a_1 p^{n-1} y(t) + \dots + a_n y(t) = b_0 p^m g(t) + \dots + b_m g(t) + c_0 p^l f(t) + \dots + c_l f(t)$$

or

$$A(p) y(t) = B(p) g(t) + C(p) f(t), \quad (5.3)$$

where

$$A(p) = a_0 p^n + a_1 p^{n-1} + \dots + a_n, \quad B(p) = b_0 p^m + b_1 p^{m-1} + \dots + b_m, \\ C(p) = c_0 p^l + c_1 p^{l-1} + \dots + c_l.$$

After solving the input - output equation (5.3), we get

$$y(t) = \frac{B(p)}{A(p)} g(t) + \frac{C(p)}{A(p)} f(t) = W_g(p) g(t) + W_f(p) f(t), \quad (5.4)$$

where $W_g(p)$ is the system reference action transfer operator;

$W_f(p)$ is the system disturbance transfer operator.

The engine equation is

$$\omega(t) = \frac{k_u}{(T_1 p + 1)(T_2 p + 1)} u_{EW}(t) - \frac{k_{T_r}}{T_2 p + 1} T_r(t). \quad (5.5)$$

5.1.3. The input - output equation image recording

There exist Laplace transformations (Pierre Simon Laplace, 1749 - 1827):

a direct transformation

$$Y(s) = L\{y(t)\} = \int_0^{\infty} y(t)e^{-st} dt, \quad (5.6)$$

an inverse transformation

$$y(t) = L^{-1}\{Y(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} Y(s)e^{st} ds, \quad s = \alpha + j\omega, \quad (5.7)$$

where $y(t)$ – is the original function, $Y(s)$ – is the image.

Let us apply Laplace transformations to the input - output equation (5.1) under the zero initial conditions. Thus we obtain the input - output image equation

$$\begin{aligned} a_0 s^n Y(s) + a_1 s^{n-1} Y(s) + \dots + a_n Y(s) = \\ = b_0 s^m G(s) + \dots + b_m G(s) + c_0 s^l F(s) + \dots + c_l F(s). \end{aligned} \quad (5.8)$$

In many books the same symbol is used both for the differential operator $p = \frac{d}{dt}$ and for the complex argument $p = \alpha + j\omega$.

To distinguish an operator from a complex argument we should identify whether we have the original or the image equation. Let us solve the image equation (5.8). We obtain

$$Y(s) = \frac{b_0 s^m + \dots + b_m}{a_0 s^n + \dots + a_n} G(s) + \frac{c_0 s^l + \dots + c_l}{a_0 s^n + \dots + a_n} F(s) \quad (5.9)$$

or
$$Y(s) = W_g(s)G(s) + W_f(s)F(s). \quad (5.10)$$

The equation (5.10) can be presented in diagram form (Fig. 5.2).

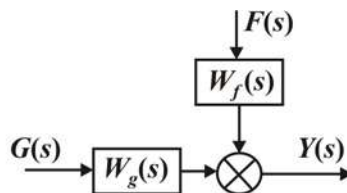


Figure 5.2 – The structural diagram of ACS

$W_g(s)$ being the system reference action transfer function,

$W_f(s)$ being the system disturbance transfer function.

If $f(t) \equiv 0$,

$$Y(s) = W_g(s)G(s), \quad (5.11)$$

then

$$W_g(s) = W(s) = \frac{Y(s)}{G(s)} \left| \begin{array}{l} f(t) \equiv 0 \\ y(0) = \dot{y}(0) = \dots = y^{(n-1)}(0) = 0 \end{array} \right. . \quad (5.12)$$

By analogy for $W_f(s)$

$$W_f(s) = \frac{Y(s)}{F(s)} \left| \begin{array}{l} g(t) \equiv 0 \\ y(0) = \dot{y}(0) = \dots = y^{(n-1)}(0) = 0 \end{array} \right. . \quad (5.13)$$

For the electrical motor we get

$$\Omega(s) = \frac{k_u}{(T_1s + 1)(T_2s + 1)} U_e(s) - \frac{k_{T_c}}{T_2s + 1} T_c(s). \quad (5.14)$$

The transfer function is the ratio of output signal Laplace transformation to the input signal Laplace transformation under zero initial conditions.

5.1.4. Cauchy normal form (Ogusten Cauchy, 1789-1857, a French mathematician)

Any differential equation of n -order can be replaced by n differential equations of the first order with the help of additional variables x_1, x_2, \dots, x_n , named state variables or phase variables.

For example, the engine equation at $T_r = 0$ looks like

$$T_1 T_2 \frac{d^2 \omega}{dt^2} + (T_1 + T_2) \frac{d\omega}{dt} + \omega(t) = k_1 u_{exc}(t), \quad (5.15)$$

or $a_0 \ddot{\omega} + a_1 \dot{\omega} + \omega = k_1 u_g$.

$$y = Cx + Dg + Mf . \quad (5.21)$$

The **Cauchy** normal form is convenient because of the differential state equation being of the first order and easily solved and the output equation being algebraic. If we know x, g and f , we can easily obtain y . The vector-matrix state equations and output equations have a compact form.

5.2. Static and dynamic characteristics of elements and systems

5.2.1. ACS static characteristics

Let us consider a system having scalar input and output (Fig. 5.3).

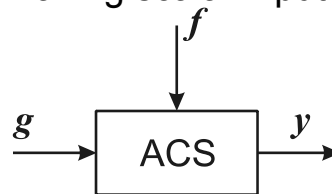


Figure 5.3 – The functional diagram of ACS with scalar inputs and output

The static equation for the system of this type is as follows: $y = k_g \cdot g + k_f \cdot f$, $k_f < 0$. For this equation we can obtain two sets of the characteristics (Fig. 5.4 and 5.5).

The element or system static characteristic is the dependence of an output signal on the input signal in the steady-state mode.

All the actual components have nonlinear static characteristics.

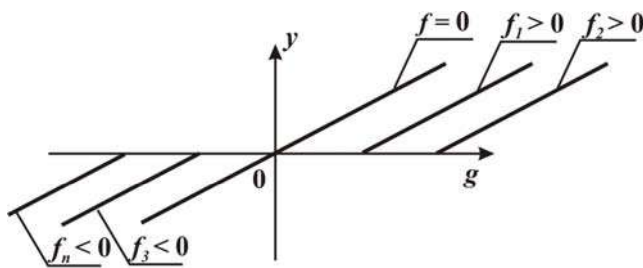


Figure 5.4 – The adjusting characteristics

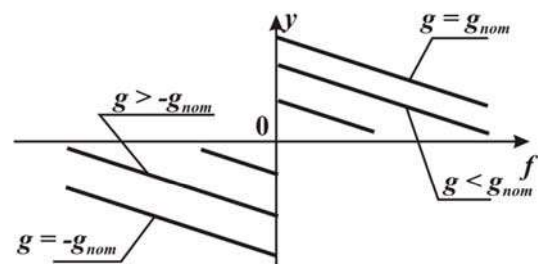


Figure 5.5 – Loading characteristics

5.2.2. ACS dynamic characteristics

There exist two types of the ACS dynamic characteristics, the time characteristics and the frequency characteristics.

The time characteristics include step response and pulse transitive (or weight) characteristics.

The frequency characteristics include the amplitude frequency characteristic (AFC), phase frequency characteristic (PhFC), amplitude phase frequency characteristic (APhFC), logarithmical amplitude frequency characteristic (LAFC), logarithmical phase frequency characteristic (LPhFC), etc.

5.2.2.1. Time characteristics

A step response is a reaction of an element or system to the unit step excitation $\mathbf{1}(t)$ under the zero initial conditions

$$h_g(t) = y(t) \left\{ \begin{array}{l} g(t) = \mathbf{1}(t), \\ f(t) = \mathbf{0}, \\ y(0), \dot{y}(0), \dots, y^{(n-1)}(0) = \mathbf{0}, \end{array} \right. \quad \text{is a reference step response;}$$

$$h_f(f) = y(t) \left\{ \begin{array}{l} g(t) = \mathbf{0}, \\ f(t) = \mathbf{1}(t), \\ y(0), \dots, y^{(n-1)}(0) = \mathbf{0}. \end{array} \right. \quad \text{is a disturbance step response.}$$

The step response can be found by the transfer function

$$Y(s) = W(s)G(s), \quad g(t) = \mathbf{1}(t), \quad G(s) = \frac{1}{s}, \quad (5.22)$$

$$H(s) = \frac{W(s)}{s}, \quad h(t) = L^{-1} \left\{ \frac{W(s)}{s} \right\} = \int_0^t w(t-\tau) \mathbf{1}(\tau) d\tau = \int_0^t w(\tau) \mathbf{1}(t-\tau) d\tau.$$

The impulse response is a reaction of an element or system on δ -function under the zero initial conditions $\mathbf{1}(t)$ is a Heaviside function (Oliver Heaviside, 1850 - 1925, an English physicist) (Fig. 5.6)

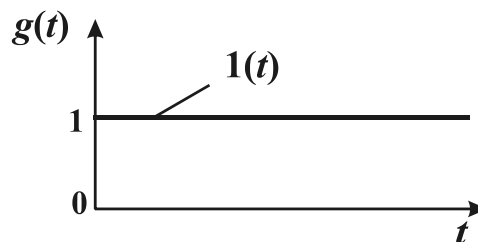


Figure 5.6 – Heaviside function

$$h'_g(t) = y(t) \left| \begin{array}{l} g(t) = \delta(t), \\ f(t) = 0, \\ y(0), \dots, y^{(n-1)}(0) = 0, \end{array} \right. \quad (5.23)$$

$$h'_f(t) = y(t) \left| \begin{array}{l} g(t) = 0, \\ f(t) = \delta(t), \\ y(0), \dots, y^{(n-1)}(0) = 0, \end{array} \right. \quad (5.24)$$

where $h'(t) = w(t)$ is a unit-pulse response or weight function.

$\delta(t) = \begin{cases} \infty, & \text{when } t = 0, \\ 0, & \text{when } t \neq 0 \end{cases}$ – is a Dirac function (Pol Dirac, 1902 – 1984, an English physicist) (Fig. 5.7).

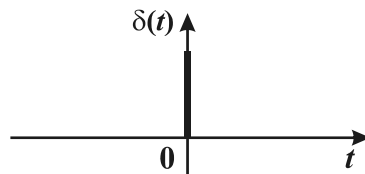


Figure 5.7 – Dirac function

The properties of Dirac function:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1(t), \quad \frac{d1(t)}{dt} = \delta(t), \quad \int_{-\infty}^{\infty} \varphi(t) \delta(t) dt = \varphi(0),$$

where $\varphi(t)$ is any kind of function.

We can find weight function using transfer function

$$Y(s) = W(s)G(s), \quad g(t) = \delta(t), \quad G(s) = 1,$$

$$H'(s) = W(s), \quad h'(t) = w(t) = L^{-1}\{W(s)\} = \frac{dh}{dt}. \quad (5.25)$$

With applying a weight function we can find the system reaction on any action

$$Y(s) = W(s)G(s), \quad (5.26)$$

therefore

$$y(t) = \int_0^t w(t - \tau)g(\tau)d\tau. \quad (5.27)$$

We use the original convolution theorem. The step response and the unit-pulse response can fully describe the ACS dynamics.

Step and pulse responses can be obtained experimentally (Fig. 5.8).

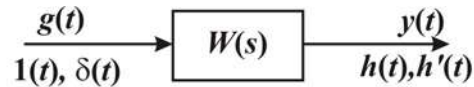


Figure 5.8 – Obtaining of step and pulse responses

5.2.2.2. ACS frequency characteristics

An amplitude-frequency characteristic (AFC, $A(\omega)$) is a dependence of the output signal amplitude ratio to an input signal amplitude on frequency (Fig. 5.9, 5.10)

$$g(t) = a \sin \omega t, \quad y(t) = b(\omega) \sin(\omega t + \varphi(\omega)),$$

$$A(\omega) = \frac{b(\omega)}{a},$$

$$A(\omega) = |W(j\omega)| = \sqrt{U^2(\omega) + V^2(\omega)}.$$

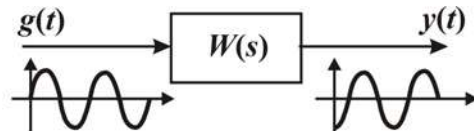


Figure 5.9 – Obtaining of frequency characteristics

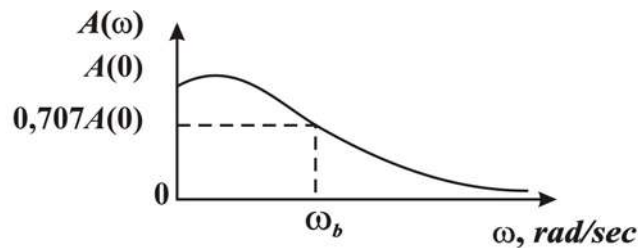


Figure 5.10 – Amplitude-frequency characteristic

A phase-frequency characteristic (PhFC, $\varphi(\omega)$) is a dependence of the output-to-input phase shift on the frequency (Fig. 5.11)

$$\varphi(\omega) = \arg W(j\omega) = \operatorname{arctg} \frac{V(\omega)}{U(\omega)}$$

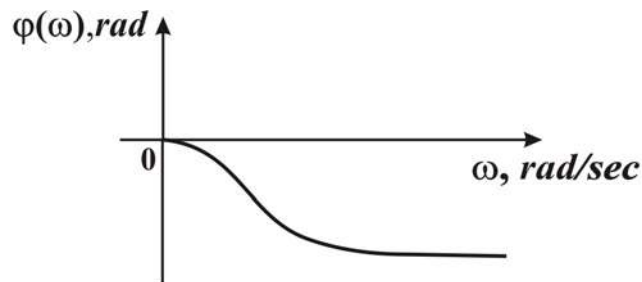


Figure 5.11 – Phase-frequency characteristic

When using this formula to calculate the phase we should take into account signs of imaginary $V(\omega)$ and real $U(\omega)$ parts of $W(j\omega)$.

The amplitude-phase-frequency characteristic (APhFC, $W(j\omega)$) is a trajectory of the vector $W(j\omega)$ movement at frequency ω change (usually ranged from 0 up to $+\infty$) (Fig. 5.12).

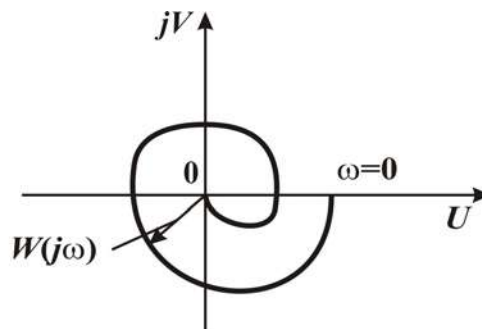


Figure 5.12 – Amplitude-phase-frequency characteristic

The frequency is logarithmically scaled in rad/sec on the abscissa axis. The segment of axis corresponding to the 10 times frequency changes is called a decade. The input-to-output signal relation of the device is expressed in decibels when constructing LAFC on a y-axis:

$$L(\omega) = 20 \lg A(\omega), dB.$$

LAFC and LPhFC are logarithmic AFC and PhFC (Fig. 5.13).

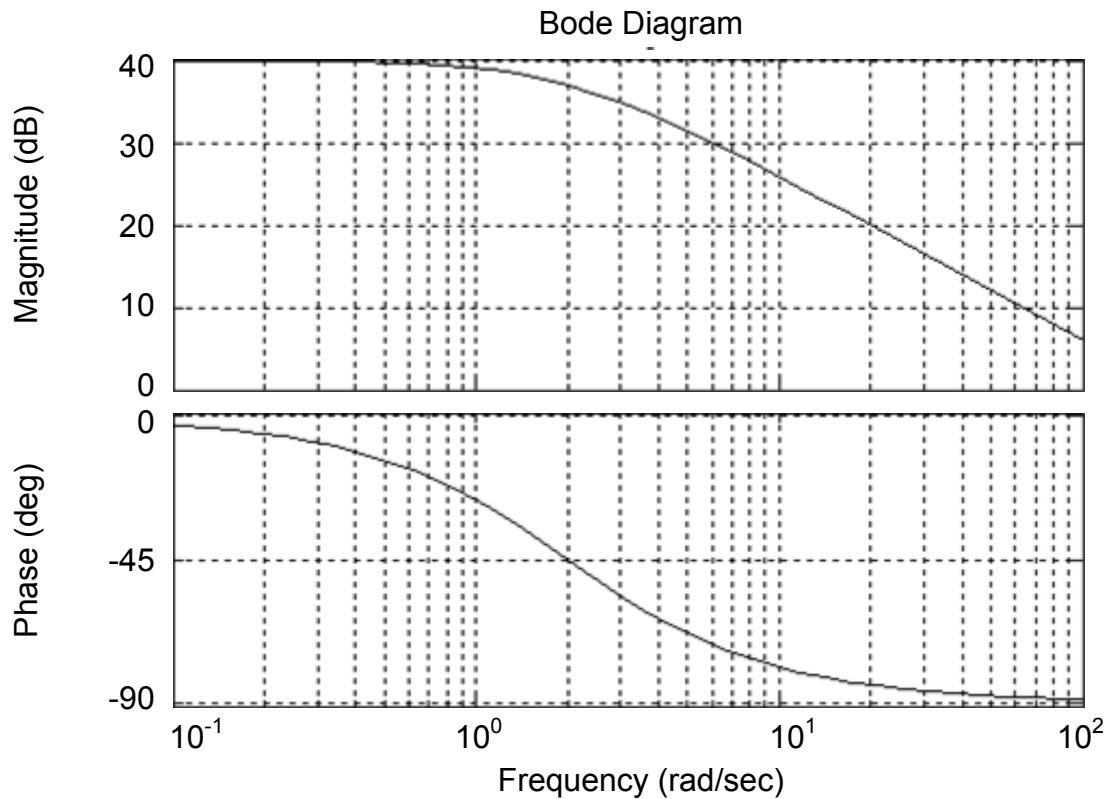


Figure 5.13 – LAFC and LPhFC

John Neper (1550 - 1617) was the Scottish nobleman. In 1614 he issued the book «The Description of the wonderful logarithms table» in Edinburgh.

Terms:

- transfer function,
- state variables,
- output equation,
- static characteristic,
- step response,
- unit-pulse response,
- amplitude-frequency characteristic,
- phase-frequency characteristic,
- amplitude-phase-frequency characteristic.

Comprehension questions

1. What forms of ACS equation recording do you know?
2. How can we obtain the static equation from the dynamic one?
3. What is a Laplace transformation?

4. How can we replace a differential equation of n -th order by n differential equations of the first order?
5. What is a step response and pulse response?
6. What are of the properties of Dirac function?
7. What kinds of frequency characteristics do you know? What are their peculiarities?

Lecture №6

TRANSFER FUNCTIONS OF CONTINUOUS LINEAR STATIONARY ACS

As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.

Albert Einstein

Lecture outline:

1. Transfer functions of series, parallel and feedback link connection.
2. ACS block-diagram transformation rules.
3. The system transfer function getting by means of Mason's formula.
4. The ACS reference and disturbance transfer functions, in respect to error.

6.1. Transfer functions of series, parallel and feedback links connection

6.1.1. Series links connection

A series links connection is a type of links connection when the output signal of the previous link becomes the input signal of the following link $y_i = g_{i+1}$ (Fig. 6.1).

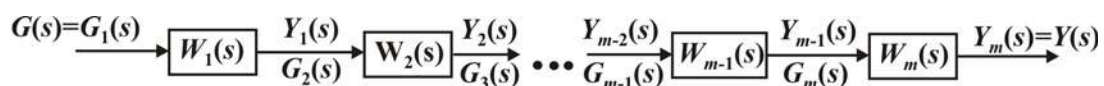


Figure 6.1 – Series links connection

Let us find a transfer function of the series links connection. The output signal image is connected with the input signal image by the transfer function

$$\begin{aligned}
Y(s) &= W_m(s) \cdot Y_{m-1}(s) = W_m(s) \cdot W_{m-1}(s) \cdot Y_{m-2}(s) = \dots = \\
&= W_m(s) \cdot W_{m-1}(s) \dots W_1(s) \cdot G(s).
\end{aligned} \tag{6.1}$$

According to the definition we have $W(s) = \frac{Y(s)}{G(s)}$. From here

$$W(s) = \prod_{i=1}^m W_i(s). \tag{6.2}$$

In the case of series links connection these transfer functions are multiplied.

Let us consider the series link connection frequency characteristics. We substitute $s = j\omega$ in transfer function: $W(j\omega) = \prod_{i=1}^m W_i(j\omega)$. If to present the transfer complex coefficient as

$$W_i(j\omega) = A_i(\omega) e^{j\varphi_i(\omega)}, \tag{6.3}$$

then we get $W(j\omega) = \prod_{i=1}^m A_i(\omega) e^{j\varphi_i(\omega)} = \prod_{i=1}^m A_i(\omega) e^{j \sum_{i=1}^m \varphi_i(\omega)} = A(\omega) e^{j\varphi(\omega)}$.

Thus,

$$A(\omega) = \prod_{i=1}^m A_i(\omega), \tag{6.4}$$

$$\varphi(\omega) = \sum_{i=1}^m \varphi_i(\omega). \tag{6.5}$$

In the case of the series links connection the magnitude-frequency characteristics are multiplied and the phase-frequency characteristics are summed. We pass on to the logarithmic magnitude-frequency characteristics

$$L(\omega) = 20 \lg A(\omega) = 20 \lg \prod_{i=1}^m A_i(\omega) = \sum_{i=1}^m L_i(\omega), \tag{6.6}$$

$$L(\omega) = \sum_{i=1}^m L_i(\omega). \tag{6.7}$$

Thus we get the logarithmic magnitude-frequency characteristics equally summed. This property provides a simple method of LMFC construction for the series links connection. We should remember that the ACT is engaged in directed action links, but a transfer function is a characteristic of the process of the data transformation conducted by the defined link that is completely different from the electric key diagrams which are engaged in elements transforming electric processes. For example, for the circuit containing two series resistors (Fig. 6.2) the correlation $R = R_1 + R_2$ is correct.

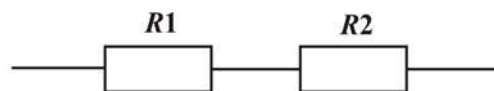


Figure 6.2 – Series connection of resistors

This circuit doesn't include the directed action links.

On the other hand, if two amplifiers which gain factors are $k_1 = 2$, $k_2 = 5$ connected in series, they will amplify a signal $k = 10$ times.

6.1.2. Parallel links connection

A parallel links connection is a type of links connection when the input signals of all links are equal, and the connection output signal is equal to the sum of element output signals (Fig. 6.3).

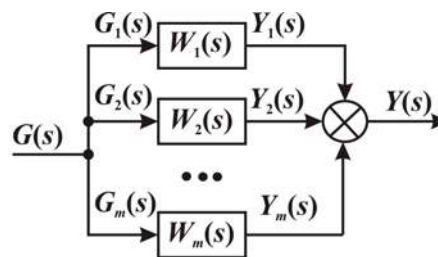


Figure 6.3 – Parallel links connection

Let us find the transfer function of the parallel link connection. We will express the output signal image in terms of the input signal image

$$Y(s) = Y_1(s) + Y_2(s) + \dots + Y_m(s) = W_1(s)G_1(s) + W_2(s)G_2(s) + \dots + W_m(s)G_m(s) = (W_1(s) + W_2(s) + \dots + W_m(s))G(s).$$

This implies

$$W(s) = \sum_{i=1}^m W_i(s). \quad (6.8)$$

In the case of parallel links connection the transfer functions are summed.

In the case of parallel links connection it is more useful to express the frequency characteristics in terms of the rectangular coordinates

$$W(j\omega) = \sum_{i=1}^m W_i(j\omega) = \sum_{i=1}^m (U_i(\omega) + jV_i(\omega)) = \sum_{i=1}^m U_i(\omega) + j \sum_{i=1}^m V_i(\omega) = U(\omega) + jV(\omega),$$

where

$$U(\omega) = \sum_{i=1}^m U_i(\omega), \quad (6.9)$$

$$V(\omega) = \sum_{i=1}^m V_i(\omega). \quad (6.10)$$

For this type of connection the Nyquist system diagram are obtained by the summation of the Nyquist element diagrams. Do not confuse the parallel link connection up the parallel element connection.

If two amplifiers which gain factors are $k_1 = 2$, $k_2 = 5$ are connected in parallel, they will amplify the signal seven times.

6.1.3. Feedback links connection

The first link $W_1(s)$ is enveloped in the feedback (FB) by means of the second link $W_2(s)$ (Fig. 6.4).

The connection has two paths of a signal transmission: a forward path which goes from input to output and the feedback line passing a signal from output to input.

A two-link feedback connection is a type of connection when the output signal of the first link becomes an input signal of the second link, and the input signal of the first link is the algebraic sum of an input connection signal and an output signal of the second link.

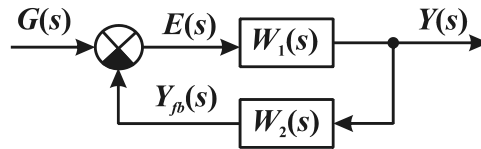


Figure 6.4 – Feedback links connection

A feedback is considered positive, if $\varepsilon = g + z$, and negative, if $\varepsilon = g - z$.

Let us express the connections of all signals images with the link transfer function

$$\begin{aligned} E(s) &= G(s) \pm Y_{fb}(s), \\ Y(s) &= W_1(s) \cdot E(s), \\ Y_{fb}(s) &= W_2(s) \cdot Y(s). \end{aligned}$$

Here on the "+" stands for the positive feedback (PFB), and the "-" means the negative (NFB).

We will exclude the intermediate variables

$$\begin{aligned} Y(s) &= W_1(s)[G(s) \pm Y_{fb}(s)] = W_1(s)[G(s) \pm W_2(s)Y(s)], \\ [1 \mp W_1(s)W_2(s)] \cdot Y(s) &= W_1(s) \cdot G(s). \end{aligned}$$

Here on the "-" is for PFB and the "+" is for NFB.

We solve for the transfer function of the feedback connection. The closing formula is as follows

$$W(s) = \frac{Y(s)}{G(s)} = \frac{W_1(s)}{1 \mp W_1(s) \cdot W_2(s)}. \quad (6.11)$$

Negative feedbacks are widely used in amplifiers in order to stabilize the transfer coefficient.

For example, the transfer coefficient of the amplifier changes in the range $k_1 = 100 \div 200$. If it is enveloped in the feedback with the factor $k_2 = 0,1$, we obtain

$$k_{min} = \frac{100}{1 + 100 \cdot 0,1} = 9,09; \quad k_{max} = \frac{200}{1 + 200 \cdot 0,1} = 9,52.$$

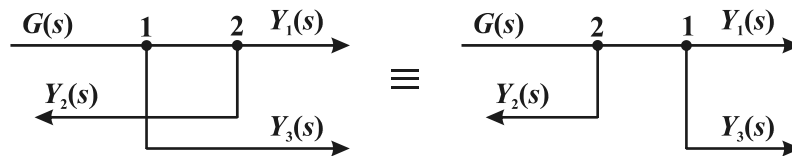
Thus, the 50% gain instability is reduced to less than 5 % but it is attained at the expense of gain decrease.

6.2. ACS block-diagram transformation rules

Two diagrams are equivalent if the identical input signals cause identical output signals.

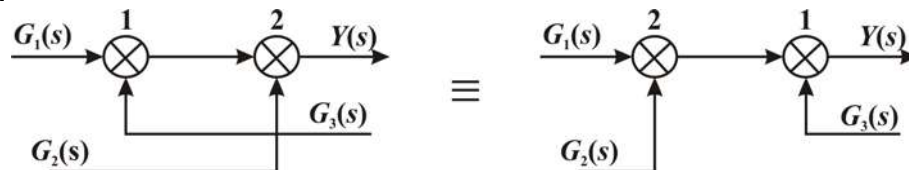
If it is difficult to find pure typical connections, we can apply structural transformation in order to simplify the complex ACS construction. These are some rules of block diagram transformation.

Rule 1:



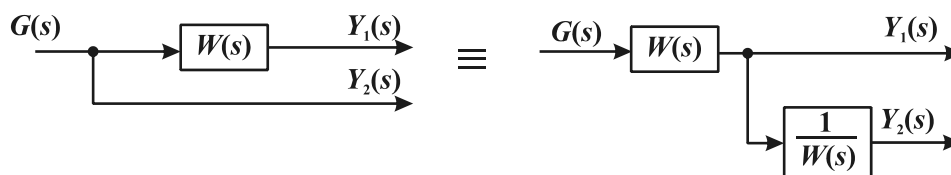
a

Rule 2:



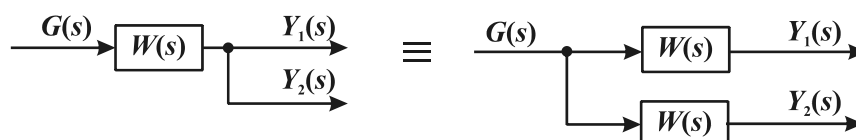
b

Rule 3:



c

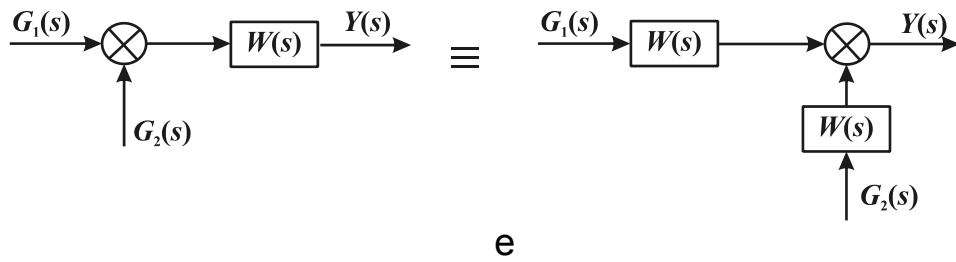
Rule 4:



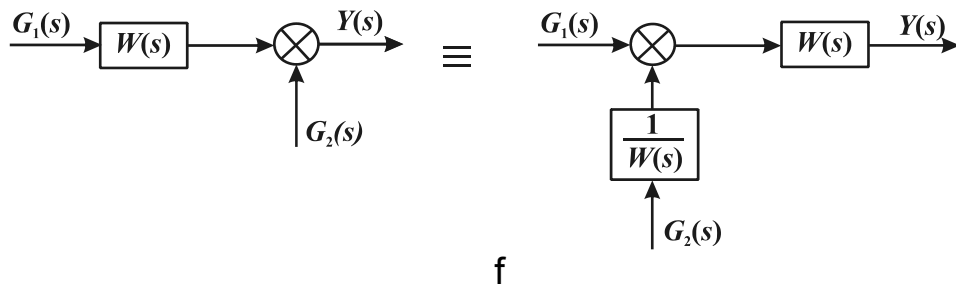
d

Figure 6.5 – Block-diagram transformation rules

Rule 5:



Rule 6:



Rule 7:

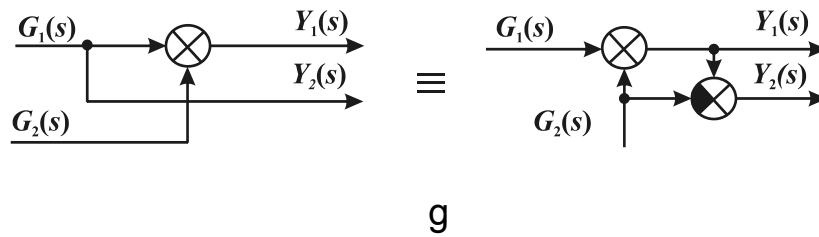


Figure 6.5. Ending

Nodes can be transposed (Fig. 6.5, a)

Adders can be transposed (Fig. 6.5, b).

If a node is transferred through a link with a signal, the inverse link should be added (Fig. 6.5, c).

If a link is transferred through a node with a signal, the same link should be added (Fig. 6.5, d).

If a summer is transferred through a link with a signal, the same link should be added (Fig. 6.5, e).

If a link is transferred through a summer with a signal, the inverse link should be added (Fig. 6.5, f).

If a node is transferred through a summer with a signal, the one more summer should be added Fig. 6.5, g).

Example: we obtain a diagram (Fig. 6.6).

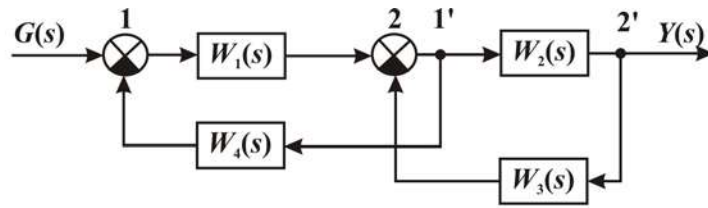


Figure 6.6 – Block diagram of the control system with crossing loops

Let us find the transfer function.

The typical links connections formulae can not be applied as the loops are crossing. We transfer the node 1' through the link $W_2(s)$ and through the node 2' with a signal course. Thus the link $\frac{1}{W_2(s)}$ should be introduced (Fig. 6.7).

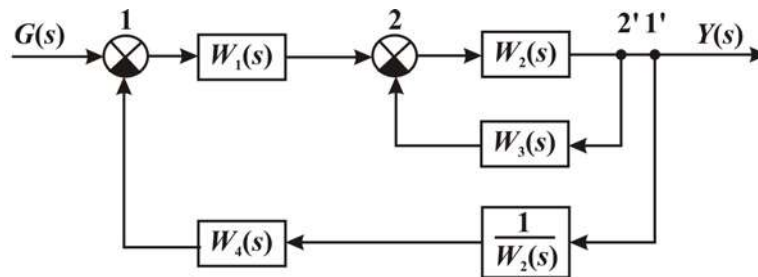


Figure 6.7 – Block diagram of the control system with feedbacks

Now we can apply the closing formula and the series connection formula in series:

$$W(s) = \frac{W_1(s) \frac{W_2(s)}{1 + W_2(s)W_3(s)}}{1 + \frac{W_1(s)W_2(s)}{1 + W_2(s)W_3(s)} \cdot W_4(s) \frac{1}{W_2(s)}} = \frac{W_1(s)W_2(s)}{1 + W_2(s)W_3(s) + W_1(s)W_4(s)}. \quad (6.12)$$

6.3. The system transfer function derivation by means of Mason's formula

The graph theory allows finding linear system transfer functions by any input and output of system without any structural transformations. Let us scrutinize this method with the example. The system is presented on the Fig. 6.8.

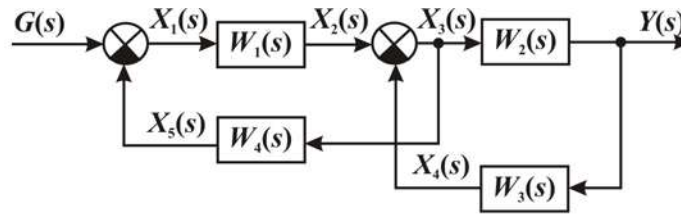


Figure 6.8 – Block diagram of the control system with crossing loops

A graph is a set of nodes and branches (edges, arches).

Let us build the system signal flow graph (Fig. 6.9), each signal assigned the graph node, each link assigned the directed graph branch the weight of which equals to its transfer function.

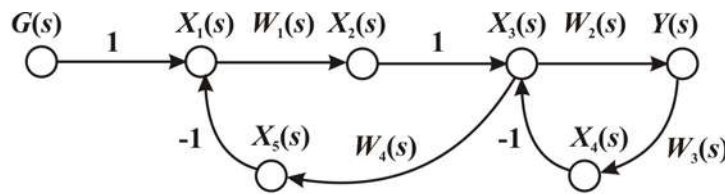


Figure 6.9 – Control system graph

The nodes vanish from the graph because we have the same signal in both input and output. The graph has no summers because of the node signal being equal to the sum of incoming signals.

The forward path is an open input-output circuit of graph branches.

The loop is a closed circuit of branches in which each node is passed only once.

The nontouching loops have no common nodes.

The system transfer function between an input g and an output y can be found by means of Mason's formula

$$\Phi_{yg}(s) = \frac{\sum_{i=1}^{n_{fp}} W_{fp,i}(s) \Delta_i(s)}{\Delta(s)}, \quad (6.13)$$

where $W_{fp,i}(s)$ is a transfer function of i -th forward path;

n_{fp} is a number of forward paths between an input g and an output y ;

$\Delta(s) = 1 - \sum_{j=1}^{m_1} W_{1j}(s) + \sum_{k=1}^{m_2} W_{2k}(s) - \sum_{l=1}^{m_3} W_{3l}(s) + \dots$ is a graph determinant;

$\Delta_i(s)$ is a determinant of graph complement to i -th forward path;

n_1, n_2, n_3, \dots is a number of loops, nontouching pairs, nontouching threes, etc;

$W_{1j}(s)$ is a j -th loop transfer function;

$W_{2k}(s)$ is a transfer function product of k -th nontouching pair;

$W_{3l}(s)$ is a transfer functions product of l -th nontouching three.

In the example we had

$$n_{fp} = 1, n_2 = 0, n_3 = 0 \dots,$$

$$W_{11}(s) = -W_1(s) \cdot W_4(s),$$

$$W_{12}(s) = -W_2(s) \cdot W_3(s),$$

$$W_{fp1}(s) = W_1(s) \cdot W_2(s),$$

$$\Delta(s) = 1 + W_1(s) \cdot W_4(s) + W_2(s) \cdot W_3(s),$$

$$\Delta_1(s) = 1,$$

$$\Phi_{yg}(s) = \frac{Y(s)}{G(s)} = \frac{W_1(s) \cdot W_2(s)}{1 + W_1(s) \cdot W_4(s) + W_2(s) \cdot W_3(s)}. \quad (6.14)$$

Thus we obtain the same answer as in the previous example.

6.4. The ACS reference and disturbance transfer functions, in respect to error

Let us consider the system with the single unit negative feedback (Fig. 6.10).

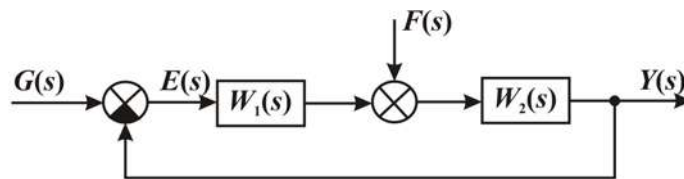


Figure 6.10 – Control system with unit feedback

We can distinguish six transfer functions:

1. The open-loop transfer function by the setting action

$$W_{yg}(s) = \left. \frac{Y(s)}{G(s)} \right|_{f \equiv 0} = W_1(s) \cdot W_2(s) = W_{fp}(s) = W(s). \quad (6.15)$$

2. The open-loop transfer function by the disturbance action

$$W_{yf}(s) = \left. \frac{Y(s)}{F(s)} \right|_{g \equiv 0} = W_2(s). \quad (6.16)$$

3. The feedback transfer function by the reference action

$$\Phi_{yg}(s) = \left. \frac{Y(s)}{G(s)} \right|_{f \equiv 0} = \frac{W_1(s)W_2(s)}{1 + W_1(s)W_2(s)} = \frac{W_{fp}(s)}{1 + W_{fp}(s)W_{fb}(s)}. \quad (6.17)$$

4. The feedback transfer function by the disturbance action

$$\Phi_{yf}(s) = \left. \frac{Y(s)}{F(s)} \right|_{g \equiv 0} = \frac{W_2(s)}{1 + W_1(s)W_2(s)} = \frac{W_f(s)}{1 + W(s)}. \quad (6.18)$$

5. The feedback transfer function in respect to the error caused by reference action

$$\Phi_{\varepsilon g}(s) = \Phi_{\varepsilon}(s) = \left. \frac{E(s)}{G(s)} \right|_{f \equiv 0} = 1 - \Phi(s) = \frac{1}{1 + W_1(s)W_2(s)} = \frac{1}{1 + W_{op}(s)}. \quad (6.19)$$

6. The feedback transfer function in respect to the error, caused by the disturbance (for the given diagram it coincides with the closed-loop transfer function by the disturbance action to a single sign)

$$\Phi_{\varepsilon f}(s) = \frac{-W_2(s)}{1 + W_1(s)W_2(s)}. \quad (6.20)$$

On the basis of the superposition principle we can put down

$$\begin{aligned} Y(s) &= \Phi(s) \cdot G(s) + \Phi_f(s) \cdot F(s); \\ E(s) &= \Phi_{\varepsilon}(s) \cdot G(s) - \Phi_f(s) \cdot F(s). \end{aligned} \quad (6.21)$$

The (6.21) helps to construct input-output and error original equations.

Terms:

- series links connection,
- parallel links connection,
- feedback links connection,

- graph,
- Mason's formula,
- nontouching loops,
- phase-frequency characteristic,
- open-loop system,
- feedback system.

Comprehension questions

1. What is a series links connection?
2. What is a parallel links connection?
3. How can we find a transfer function of series links connection?
4. How can we find a transfer function of parallel links connection?
5. What is a purpose of the negative feedback?
6. Which diagrams are equivalent?
7. Which are the ACS block-diagram transformation rules?
8. Why do we use the Mason's formula?
9. What transfer functions has the closed-loop system?

Lecture №7

THE APPLICATION OF THE STATE SPACE METHOD FOR CONTINUOUS LINEAR ACS

Lecture outline:

1. The state and output equations for continuous linear automatic control systems.
2. The solving of the state and output equations.
3. The vector-matrix block diagram and the matrix transfer functions for continuous linear ACS.

The method belongs to the modern automatic control theory. It allows finding response to any action under any initial conditions and evaluating stability, controllability and observability.

7.1. The state and output equations for continuous linear automatic control systems

The continuous linear system with r inputs and l outputs can be described by n differential state equations of the 1st order (Fig. 7.1)

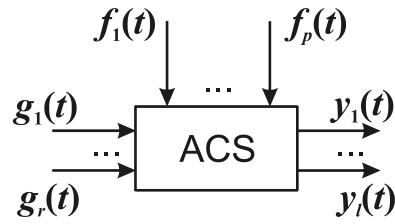


Figure 7.1– ACS

$$\begin{cases} \dot{x} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}g_1 + \dots + b_{1r}g_r + m_{11}f_1 + \dots + m_{1p}f_p, \\ \dots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}g_1 + \dots + b_{nr}g_r + m_{n1}f_1 + \dots + m_{np}f_p, \end{cases} \quad (7.1)$$

where $x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ is a state vector;

$g = \begin{bmatrix} g_1 \\ \dots \\ g_r \end{bmatrix}$ is a reference action vector;

$f = \begin{bmatrix} f_1 \\ \dots \\ f_p \end{bmatrix}$ is a disturbance vector;

and by the algebraic output equations

$$\begin{cases} y_1 = c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n + d_{11}g_1 + \dots + d_{1r}g_r + n_{11}f_1 + \dots + n_{1p}f_p, \\ \dots \\ y_l = c_{l1}x_1 + \dots + c_{ln}x_n + d_{l1}g_1 + \dots + d_{lr}g_r + n_l f_1 + \dots + n_{lp}f_p, \end{cases} \quad (7.2)$$

where $y = \begin{bmatrix} y_1 \\ \dots \\ y_l \end{bmatrix}$ is an output vector.

These equations in the vector-matrix form look like

$$\begin{aligned} \dot{x} &= Ax + Bg + Mf, \quad x(t_0), \\ y &= Cx + Dg + Nf, \end{aligned} \tag{7.3}$$

where $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ is the state n -by- n matrix,

$B = \begin{bmatrix} b_{11} & \dots & b_{1r} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nr} \end{bmatrix}$ is a reference input n -by- r matrix,

$C = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{l1} & \dots & c_{ln} \end{bmatrix}$ is an output l -by- n matrix,

$D = \begin{bmatrix} d_{11} & \dots & d_{1r} \\ \dots & \dots & \dots \\ d_{l1} & \dots & d_{ln} \end{bmatrix}$ is a reference l -by- r by-pass matrix,

$M = \begin{bmatrix} m_{11} & \dots & m_{1p} \\ \dots & \dots & \dots \\ m_{n1} & \dots & m_{np} \end{bmatrix}$ is a disturbance input n -by- p matrix,

$N = \begin{bmatrix} n_{11} & \dots & n_{1p} \\ \dots & \dots & \dots \\ n_{l1} & \dots & n_{lp} \end{bmatrix}$ is a disturbance l -by- p by-pass matrix.

Eventually the state vector x moves through n -dimensional Euclidean space and describes the phase trajectory. $x(t)$ is the representation point which characterizes the ACS state or the certain phase of system movement at any instant (Fig. 7.2).

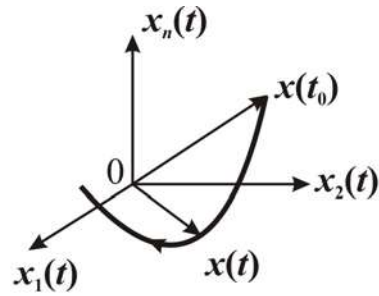


Figure 7.2 – State space

The state space method performs the ACS dynamics by the representation point motion in the n -dimensional Euclidean space.

The state variable characterizes the system element energy content. We can choose the variables corresponding to the element energy content or their linear combinations in the capacity of state variables (e.g. capacitor voltage or charge, inductor current, angular speed and linear velocity, mass shift, spring expansion, etc. as well as their linear combinations).

So, if $x_1 = u_{c1}$, $x_2 = u_{c2}$, we can choose and other sets of state variables $x'_1 = u_{c1} + u_{c2}$, $x'_2 = 2u_{c1} + 3u_{c2}$, etc.

The state vector formal definition: **The state vector** is the minimal set of variables x_1, x_2, \dots, x_n knowledge of which in the initial time point t_0 together with knowledge of input actions $g_1(t), \dots, g_r(t), f_1(t), \dots, f_p(t)$, in an interval $t_0 \leq t \leq t_1$, allows to determine all output signals $y_1(t), \dots, y_l(t)$ at any moment t in the range (t_0, t_1) .

7.2. The solving of the state and output equations

Let $f(t) = \mathbf{0}$, then the state and output equations took the form of

$$\begin{aligned} \dot{x} &= A x + B g, \quad x(t_0), \\ y &= C x + D g. \end{aligned} \quad (7.4)$$

Let us solve the homogeneous state equation with specified initial conditions

$$\dot{x} = A x, \quad x(t_0). \quad (7.5)$$

Let $t_0 = 0$. The scalar differential equation $\dot{x} = a x$, $x(0)$ has the solution

$$x(t) = C e^{\lambda t},$$

where λ is the characteristic equation root $\lambda = a$, C is the constant of integration determined by the initial conditions $C = x(0)$.

Thus $x(t) = x(0)e^{at}$.

The exponential function can be expanded in power series

$$x(t) = x(0) \sum_{i=0}^{\infty} \frac{(at)^i}{i!}.$$

We assume that the solution of the vector-matrix state equation is of the following form by the analogy to the scalar case

$$x(t) = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} x(0). \quad (7.6)$$

We check the assumption by substituting the prospective solution into the homogeneous state equation $\dot{x} = Ax$

$$\sum_{i=1}^{\infty} \frac{A^i t^{i-1}}{i!} x(0) = A \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} x(0) = A \sum_{i=0}^{\infty} \frac{(At)^i}{i!} x(0).$$

We introduce a new variable $k = i - 1$.

Thus we obtain the identity. The solution of the homogeneous differential equation is as follows

$$x(t) = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} x(0).$$

We introduce a designation

$$\Phi(t) = \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = e^{At} \quad (7.7)$$

then $x(t) = \Phi(t)x(0)$.

$\Phi(t)$ is the **fundamental** or **transitive matrix**.

If $t = 0$ it becomes a unit matrix $\Phi(0) = 1$.

We solve the equation (7.5) with the help of Laplace transformation

$$\begin{aligned} \dot{x} = Ax, x(0) & & (sI-A)X(s) &= x(0), \\ sX(s) - x(0) &= AX(s), & X(s) &= (sI-A)^{-1} x(0), \\ sX(s) - AX(s) &= x(0), & x(t) &= L^{-1} \left\{ (sI-A)^{-1} \right\} \times x(0). \end{aligned}$$

If we compare both solutions, we obtain an equality

$$\Phi(t) = L^{-1} \left\{ (sI - A)^{-1} \right\}, \quad \Phi(s) = (sI - A)^{-1}. \quad (7.8)$$

The fundamental matrix can be determined with the help Sylvester formula (for aliquant roots)

$$\Phi(t) = \sum_{i=1}^n F_i e^{\lambda_i t}, \quad \text{where } F_i = \prod_{\substack{j=1 \\ j \neq i}}^n \frac{A - \lambda_j I}{\lambda_i - \lambda_j}, \quad (7.9)$$

where λ_i and λ_j being the roots of the system characteristic equation

$$\det(\lambda I - A) = 0.$$

Now we solve the state equation (7.3) with the help of Laplace transformation

$$\begin{aligned} \dot{x} &= Ax + Bg, \quad x(0), \\ sX(s) - x(0) &= AX(s) + BG(s), \\ sX(s) - AX(s) &= x(0) + BG(s), \\ (sI - A)X(s) &= x(0) + BG(s), \\ X(s) &= (sI - A)^{-1} x(0) + (sI - A)^{-1} BG(s), \\ X(s) &= \Phi(s)x(0) + \Phi(s)BG(s), \\ x(t) &= \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Bg(\tau)d\tau. \end{aligned}$$

The output equation $y = Cx + Dg$ has the solution

$$y(t) = C\Phi(t)x(0) + C \int_0^t \Phi(t - \tau)Bg(\tau)d\tau. \quad (7.10)$$

The free component does not depend on the reference action $g(t)$ and is defined by initial conditions $x(0)$ and properties of system alone. The forced component depends on the input action $g(t)$ and the system properties and does not depend on initial conditions. This formula (7.10) allows finding an output signal at any initial conditions and any input actions.

For instance, we can find the step response

$$h(t) = C \int_0^t \Phi(t - \tau)Bd\tau + D1(t). \quad (7.11)$$

7.3. The vector-matrix block diagram and the matrix transfer functions for continuous linear ACS

The state and output equations have the following form

$$\begin{aligned} \dot{x} &= Ax + Bg + Mf, x(t_0), \\ y &= Cx + Dg + Nf. \end{aligned} \quad (7.12)$$

We can present (7.12) with the block diagram (Fig. 7.3).

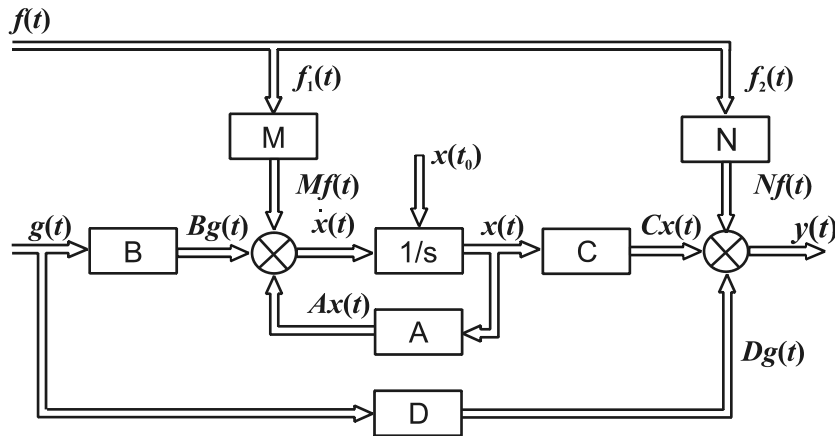


Figure 7.3– Block diagram of ACS

The circuit contains the n -dimensional integrator with the specified initial conditions.

Double lines indicate the vector signal paths. This circuit can be submitted in the scalar form as well.

Example 1. Create a system block diagram presented by equations

$$\begin{cases} \dot{x}_1 = 2x_1 + 3x_2 + 4g_1, \\ \dot{x}_2 = 5x_1 + 6x_2 + 7g_2, \end{cases} \quad \begin{cases} y_1 = 3x_1 + g_1, \\ y_2 = 4x_2. \end{cases}$$

Let us write out values of matrixes A, B, C, D:

$$\begin{aligned} A &= \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}; B = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}; \\ C &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}; D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

ACS block diagram corresponding to the set of equations is shown on Fig. 7.4.

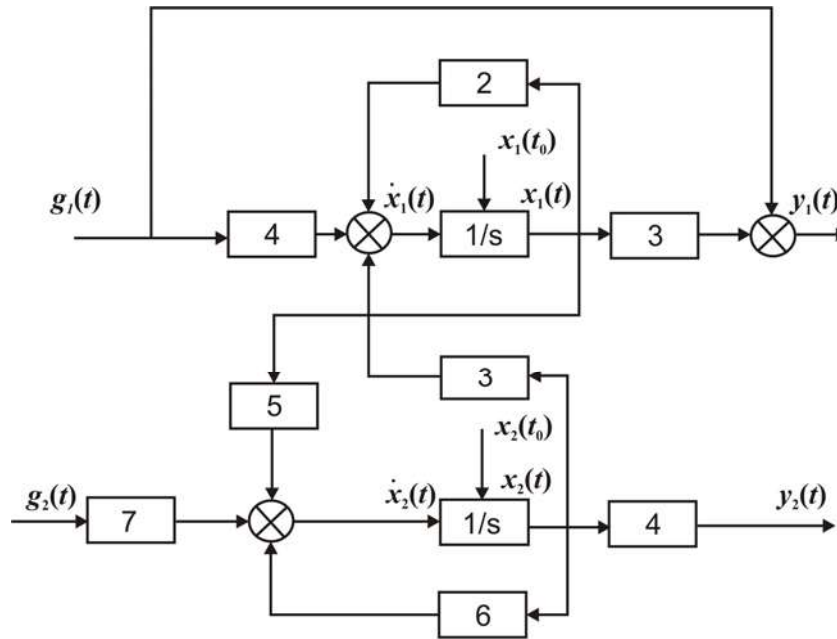


Figure 7.4 – Example of block diagram

Example 2. Using the block diagram find the state and output equations (Fig. 7.5).

The integrator outputs become state variables x_1 and x_2 , these variables characterizing the block energy content. Other elements (summers, amplifiers) do not store the energy.

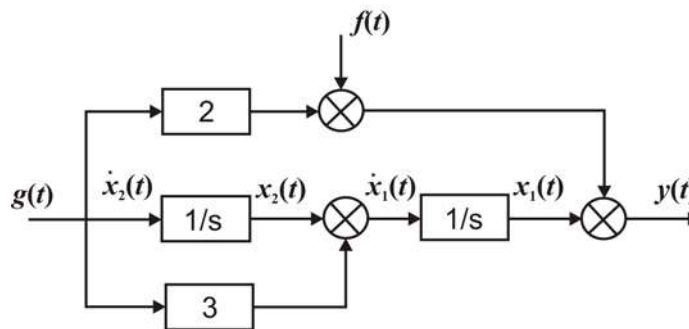


Figure 7.5 – Block diagram for example 2

Thus

$$\begin{cases} \dot{x}_1 = x_2 + 3g, \\ \dot{x}_2 = g, \end{cases} \quad y = x_1 + 2g + f,$$

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

$$C = [1 \quad 0], \quad D = [2],$$

$$M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad N = [1].$$

Now we can find transfer functions (e.g. using Mason formula).

We apply Laplace transformations to the state and output equations under zero initial conditions and solve them

$$sX(s) = AX(s) + BG(s) + MF(s),$$

$$X(s) = \Phi(s)BG(s) + \Phi(s)MF(s),$$

$$Y(s) = CX(s) + DG(s) + NF(s),$$

$$Y(s) = (C\Phi(s)B + D)G(s) + NF(s).$$

Thus $W_x(s) = \Phi(s)B = (sI - A)^{-1}B$ is a system matrix transfer function of the state caused by reference action,

$$W_y(s) = C\Phi(s)B + D = C[(sI - A)^{-1}]B + D$$

is a system matrix output transfer function,

$W_{xij}(s)$ is a system transfer function between j -th an input and i -th state variable,

$W_{yij}(s)$ is a system transfer function between j -th an input and i -th output.

Terms:

- state variable,
- state equation,
- output equation,
- state matrix,
- input matrix,
- output matrix,
- by-pass matrix,
- fundamental matrix,
- block diagram,
- matrix transfer function.

Comprehension questions

1. What is state variable?
2. What is state?
3. What is state equation?
4. What is output equation?
5. What is state matrix?
6. What is input matrix?
7. What is output matrix?
8. What is by-pass matrix?
9. What is fundamental matrix?
10. What is block diagram?
11. What is matrix transfer function?

Lecture № 8

THE APPLICATION OF THE STATE SPACE METHOD FOR CONTINUOUS LINEAR ACS (ENDING)

The mind can be almighty if it will not serve for humanity.

Sophocles

Lecture outline:

1. The derivation of the state and output equations for continuous linear ACS with the help of a controlled canonic representation method.
2. The controllability and observability of continuous linear ACS.

8.1. The getting of the state and output equations for continuous linear ACS with the help of a controlled canonic representation method

During the previous lecture we specified, that having chosen corresponding state variables we can describe system by the n differential state equations of the first order and the l algebraic output equations. If we have the ACS basic circuit, the signals describing the ACS element energy content or the linear combinations of the elements can be chosen as state variables. We shall consider the problem of choosing state variable without the ACS basic circuit but with the transfer function

$$W(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}, \quad m \leq n \quad (8.1)$$

or the input – output equation

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y(t) = b_0 \frac{d^m g}{dt^m} + b_1 \frac{d^{m-1} g}{dt^{m-1}} + \dots + b_m g(t),$$

$$m \leq n.$$

The announced problem is considered by the example of the ACS having single input and single output (Fig. 8.1).

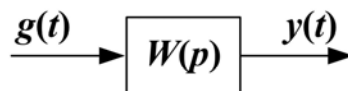


Figure 8.1 – Block diagram of the ACS with one input and one output

This problem has a number of solutions, e.g. to decompose the transfer function into simple fractions and then to connect the elements corresponding with the certain fractions in parallel (a parallel programming method); to decompose the transfer function numerator and denominator into simple factors and then to connect the elements corresponding to the first order fractions in series (a consecutive programming method); identification canonic representation of the system, etc.

Let us consider one of these methods, namely the controlled canonic representation method (CCRM).

The output Laplace image is equal to

$$Y(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} \cdot G(s), \quad m \leq n.$$

If we introduce an auxiliary variable $z(t)$, we get

$$Z(s) = \frac{G(s)}{a_0 s^n + a_1 s^{n-1} + \dots + a_n},$$

$$Y(s) = (b_0 s^m + \dots + b_m) Z(s)$$

or

$$a_0 z^{(n)}(t) + a_1 z^{(n-1)}(t) + \dots + a_n z(t) = g(t).$$

We solve the last equation relative to the senior derivative

$$z^{(n)}(t) = -\frac{a_1}{a_0} z^{(n-1)}(t) - \frac{a_2}{a_0} z^{(n-2)}(t) - \dots - \frac{a_n}{a_0} z(t) + \frac{1}{a_0} g(t). \quad (8.2)$$

The dependence of the output $y(t)$ on the variable $z(t)$ is expressed by the following equality

$$y(t) = b_0 z^{(m)}(t) + b_1 z^{(m-1)}(t) + \dots + b_m z(t). \quad (8.3)$$

On the basis of the equations (8.2) and (8.3) we build the circuit for the state variables in the most common case $m = n$ (Fig. 8.2). The output integrator is chosen for the state variable x_i .

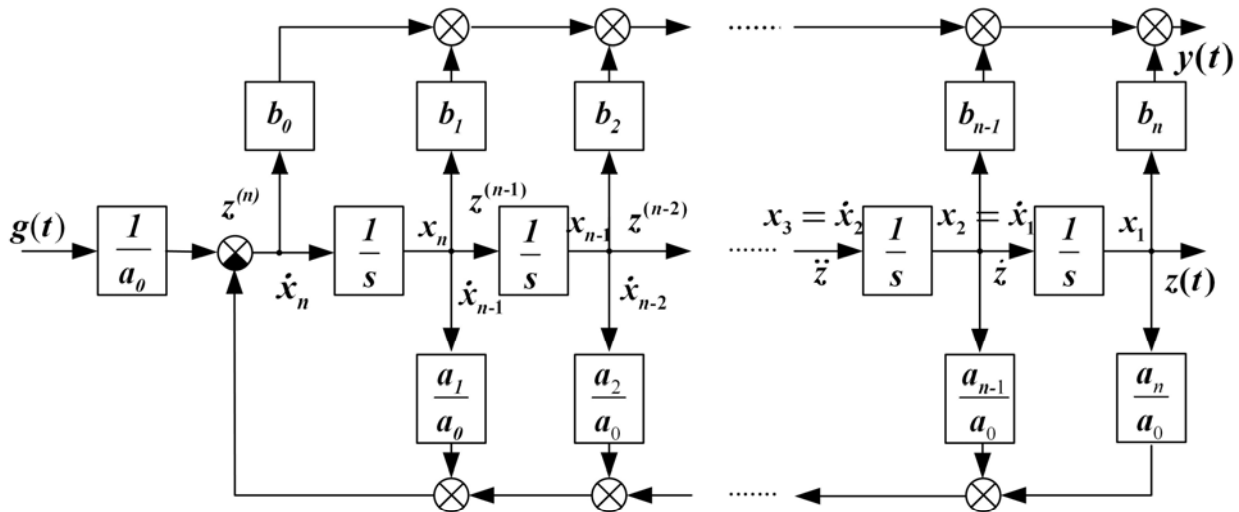


Figure 8.2 – The block diagram in state variables

Now we get integrators inputs are equal to the corresponding derivatives.

Let us construct the state equations for $a_0 = 1, b_0 = 0$

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dots\dots\dots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots - a_1 x_n + g; \end{cases} \quad (8.4)$$

or

$$\dot{x} = Ax + Bg,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-1} & \dots & -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad (8.5)$$

There is a unit matrix I in a right top corner of the matrix A . The last line of the matrix A contains transfer function denominator coefficient, signed "-". Such matrix A is called the Frobenius matrix. The matrix B contains a single nonzero element – the last one.

Now we write the output equation

$$y = b_n x_1 + b_{n-1} x_2 + \dots + b_2 x_{n-1} + b_1 x_n.$$

We get a single equation because the considered system has only one output. The matrix form of this equation is

$$y = Cx + Dg, \quad (8.6)$$

where $C = [b_n \ b_{n-1} \ \dots \ b_2 \ b_1]$, $D = [0]$.

The particular case:

If the transfer function numerator does not contain s or the input – output differential equation has no derivatives of an input signal, the block diagram (Fig.8.2) has a single amplifier with factor b_n in the top line. Then the output signal y is equal to $b_n x_1$, the other state variables are proportional to derivatives of an output y . In this case an output signal ($x_1 = y$) and $(n - 1)$ its derivatives ($x_2 = \dot{y}, \dots, x_n = y^{(n-1)}$) can be chosen for the state variables.

Example: To find $A, B, C, D, \Phi(t)$ for the electrical motor.

The motor input – output equation goes as follows

$$T_1 T_2 \frac{d^2 \omega}{dt^2} + (T_1 + T_2) \frac{d\omega}{dt} + \omega(t) = ku, \quad (8.7)$$

or

$$\ddot{\omega}(t) + a_1 \dot{\omega}(t) + a_2 \omega(t) = bu(t),$$

where
$$a_1 = \frac{T_1 + T_2}{T_1 T_2}, \quad a_2 = \frac{1}{T_1 T_2}, \quad b = \frac{k}{T_1 T_2}.$$

We choose an output signal ω and its derivative $\dot{\omega}$ for state variables

$$x_1 = \omega, \quad x_2 = \dot{\omega}.$$

Let us find the state and output equations using variables x_1 and x_2 and the input – output equation

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = -a_2 x_1 - a_1 x_2 + bu; \\ y = x_1 = \omega. \end{cases}$$

Then we get

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C = [1 \quad 0], \quad D = [0].$$

The block diagram (Fig. 8.3) corresponds to the state and output equations.

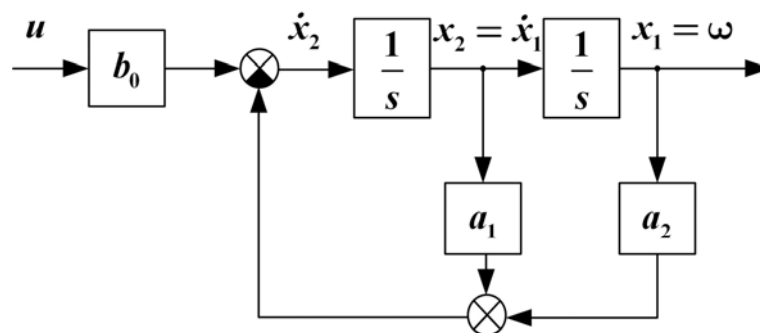


Figure 8.3 – The motor block diagram in state variables

Let us find a fundamental matrix $\Phi(t)$ with the help of the Sylvester formula.

$$\begin{aligned} \Phi(t) &= \sum_{i=1}^n F_i e^{\lambda_i t}, \\ F_i &= \prod_{\substack{j=1 \\ j \neq i}}^n \frac{A - \lambda_j I}{\lambda_i - \lambda_j}, \end{aligned} \quad (8.8)$$

where $\lambda_{i,j}$ - are roots of the characteristic equation $\det(\lambda I - A) = 0$.

The characteristic equation of the engine has the form

$$\lambda^2 + a_1\lambda + a_2 = 0.$$

We find its roots using the formula

$$\lambda_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}.$$

Let $a_1 = 6$, $a_2 = 5$, $b = 0.1$. Then $\lambda_1 = -1$, $\lambda_2 = -5$.

$$\text{Let } i=1, j=2 \text{ then } F_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{\begin{bmatrix} 0 & I \\ -5 & -6 \end{bmatrix} - (-5) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}{-1 - (-5)} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ -\frac{5}{4} & -\frac{1}{4} \end{bmatrix}.$$

$$\text{For } i=2, j=1 \text{ } F_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{\begin{bmatrix} 0 & I \\ -5 & -6 \end{bmatrix} - (-1) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}{-5 - (-1)} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{5}{4} & \frac{5}{4} \end{bmatrix}.$$

Now we can write down a motor fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t} & \frac{1}{4}e^{-t} - \frac{1}{4}e^{-5t} \\ -\frac{5}{4}e^{-t} + \frac{5}{4}e^{-5t} & -\frac{1}{4}e^{-t} + \frac{5}{4}e^{-5t} \end{bmatrix}.$$

It is natural, that at $t = 0$ it is equal to unit matrix

$$\Phi(t) = e^{At} = I.$$

The home task: find the step response $h(t) = C \int_0^t \Phi(t - \tau) B d\tau$.

8.2. The controllability and observability of continuous linear ACS

The system, the state of which can be changed from any initial state $x(t_0)$ to any final state $x(t_f)$ within a limited period of time $(t_f - t_0) < \infty$ and at the limited input action $|g| < \infty$ (Fig. 8.4), is called a system controlled on the state.

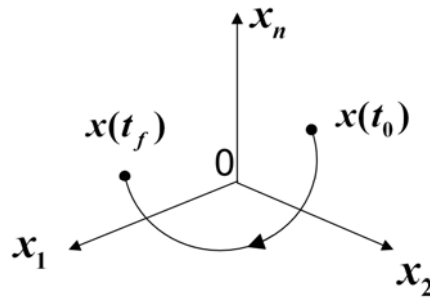


Figure 8.4 – The state space

The system, the output of which can be changed from any initial value $y(t_0)$ to any final value $y(t_f)$ within a limited period of time $t_f - t_0 < \infty$ and at the limited input action $|g(t)| < \infty$, is called a system controlled on the output.

Let us consider the ACS (Fig. 8.5).



Figure 8.5 – The functional diagram of the ACS

For it we have

$$\begin{aligned} \dot{x} &= Ax + Bg, \\ y &= Cx + Dg. \end{aligned}$$

Now we form a matrix of controllability on the state

$$U_x = [B : AB : \dots : A^{n-1}B].$$

The theorem. For the continuous linear ACS to be thoroughly controlled on the state the rank of a controllability matrix on the state should be equal to the order of the system $\text{rank}(U_x) = n$.

Now we form a matrix of controllability on the output

$$U_y = [CB : CAB : \dots : CA^{n-1}B].$$

The theorem. For the ACS to be thoroughly controlled on the output the rank of a controllability matrix on the output should be equal to the number of outputs $\text{rank}(U_y) = l$.

System observability is an opportunity to define all the state variables x_1, x_2, \dots, x_n on the measured input and output signals.

We can form a system observability matrix

$$N = \left[C^T : A^T C^T : \dots : (A^T)^{n-1} C^T \right],$$

where T is a matrix transposition symbol.

The theorem. The system is thoroughly observable if the observability matrix rank is equal to the order of the system $rank(N) = n$.

Example. We have a system (Fig. 8.6). Estimate the system controllability and observability if the order of the system is $n = 2$.

We introduce state variables x_1 and x_2 .

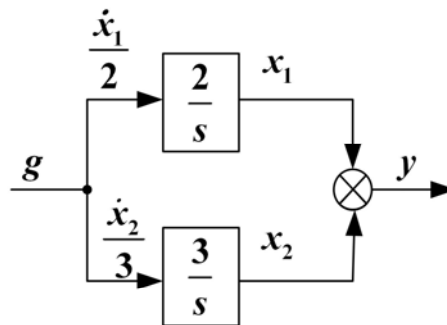


Figure 8.6 - The block diagram in state variables

Then we set up state and output equations

$$\begin{cases} \dot{x}_1 = 2g, \\ \dot{x}_2 = 3g, \\ y = x_1 + x_2 \end{cases}$$

and get matrixes

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = [1 \quad 1], \quad D = [0],$$

$$U_x = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \quad U_y = [5 \quad 0], \quad N = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now we can determine the matrixes U_x, U_y, N ranks

$$\text{rank}U_x = 1 \neq n, \quad \text{rank}U_y = 1 = l, \quad \text{rank}N = 1 \neq n.$$

The system is not thoroughly controlled on the state, is controlled on the output and is not thoroughly observable.

Terms:

- a system controlled on the state,
- a system controlled on the output,
- a system observability.

Comprehension questions

1. Name the methods to solve the problem of choosing the state variables for the ACS transfer function.
2. What kind of system is called a system controlled on the state?
3. What kind of system is called a system controlled on the output?
4. Name the necessary and sufficient conditions of the controllability on the state.
5. Name the necessary and sufficient conditions of the controllability on the output.
6. Name the definition of system observability.

Lecture № 9

THE ACS STABILITY

Let be strong not versus truth, but stronger than truth.

The Apostle Paul

Lecture outline:

1. Concept of ACS stability.
2. The estimation of continuous linear system stability by the roots of characteristic equations.
3. Routh criterion of stability.
4. Hurwitz criterion of stability.
5. Mikhaylov criterion of stability.

9.1. Concept of ACS stability

In this lecture we will discuss the dynamic but not the structural stability.

A stable system is a system which being deranged returns to the initial state or the undisturbed movement after the elimination of the disturbance action. The stability is the main ACS attribute. The unstable system is disabled.

We distinguish stability of the balance state, stability of undisturbed movement and the system stability (Fig. 9.1).

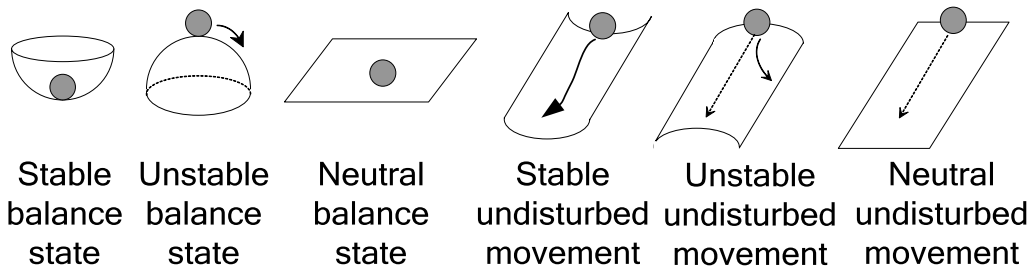


Figure 9.1 - Examples of stability

System dynamics can be described by the state equations

$$\begin{cases} \dot{\vartheta}_1 = f_1(\vartheta_1, \vartheta_2, \dots, \vartheta_n), \\ \dots\dots\dots \\ \dot{\vartheta}_n = f_n(\vartheta_1, \vartheta_2, \dots, \vartheta_n), \end{cases} \vartheta(0).$$

A solution $\vartheta(t)$ corresponds to some initial condition $\vartheta(0)$ (Fig. 9.2).

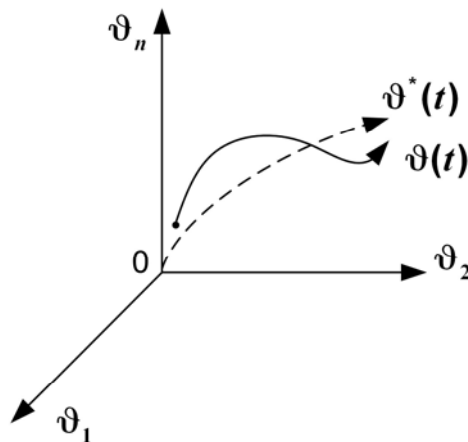


Figure 9.2 – The disturbed movements

One of the solutions (either rated or required) is called an undisturbed movement $\mathcal{G}^*(t)$. All other solutions $\mathcal{G}(t)$ are disturbed movements.

We introduce the deviation $x(t)$ from the undisturbed movement:

$$x(t) = \mathcal{G}(t) - \mathcal{G}^*(t).$$

ACS is described by the deviation state equations.

$$\begin{cases} \dot{x}_1 = \varphi_1(x_1, x_2, \dots, x_n), \\ \dots\dots\dots, \\ \dot{x}_n = \varphi_n(x_1, x_2, \dots, x_n), \quad \varphi(0). \end{cases}$$

The deviation undisturbed movement $x^*(t) = \mathcal{G}^*(t) - \mathcal{G}^*(t) = 0$ goes into the state space zero point.

The state of object is represented on Fig. 9.3.

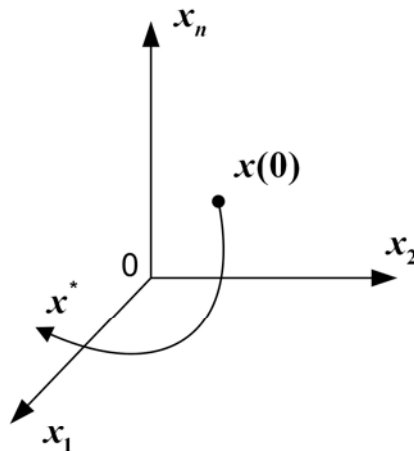


Figure 9.3 – State of object

The Lyapunov's definition of the undisturbed movement stability

The undisturbed movement $x^*(t)$ is stable if for any arbitrarily small $\varepsilon > 0$ there is the $\delta > 0$ depending on ε , so that if the condition $\|x(t_0)\| < \delta$ is satisfied, the inequality $\|x(t)\| < \varepsilon$ is performed for any instant t from the range $t_0 \leq t \leq \infty$.

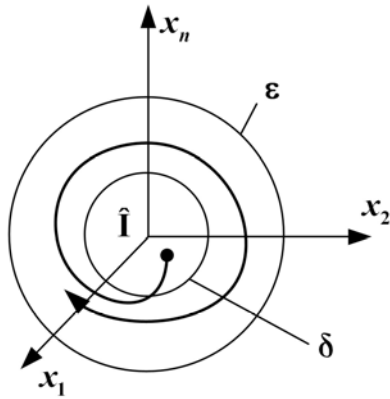


Figure 9.4 – Stability of system

If besides the given conditions the $\|x(t)\| \rightarrow 0$ satisfied at $t \rightarrow \infty$, then the undisturbed movement $x^*(t)$ is called asymptotically stable. The system stability is shown on Fig. 9.4.

The system is considered stable if all its undisturbed movements are asymptotically stable. We can distinguish a stability «in the small», «in the large», «as a whole» and an absolute stability.

For linear systems the problem of the stability evaluation becomes easier, because even if one undisturbed movement is stable the rest of the movements are stable also.

9.2. The estimation of continuous linear system stability by the roots of characteristic equations

Let the system be described by the input – output equation

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y(t) = b_0 \frac{d^m g}{dt^m} + b_1 \frac{d^{m-1} g}{dt^{m-1}} + \dots + b_m g(t). \quad (9.1)$$

We shall derange it ($y(0) \neq 0$) and remove the input action ($g(t) \equiv 0$). Now the system is described by the homogeneous equation

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y(t) = 0, \quad y(0), \dot{y}(0), \dots, y^{(n-1)}(0). \quad (9.2)$$

The solution has the form

$$y(t) = \sum_{i=1}^n C_i e^{\lambda_i t}, \quad (9.3)$$

where C_i is an integration constant determined by initial conditions, λ_i is an nonmultiple root of the system characteristic equation.

If i -th root is r -multiple, then

$$y_i(t) = (C_{i1} + C_{i2}t + \dots + C_{ir}t^{r-1})e^{\lambda_i t}. \quad (9.4)$$

The solution $y_i(t) \rightarrow 0$ at $t \rightarrow \infty$, if all the constituents $y_i(t) \rightarrow 0$ at $t \rightarrow \infty$. Now we can present roots $\lambda_i = \alpha_i + j\omega_i$ and substitute them into (9.4).

Let us consider all possible variants of values λ_i .

1) A real root:

$$\lambda_i = \alpha_i \quad y_i(t) = \tilde{N}_i e^{\alpha_i t}. \quad (9.5)$$

The solution $y_i(t) \rightarrow 0$ at $t \rightarrow \infty$, if $\alpha_i < 0$ (Fig. 9.5).

2) A complex root:

$$\lambda_i = \alpha_i + j\omega_i. \quad (9.6)$$

Any complex root $\lambda_i = \alpha_i + j\omega_i$ corresponds to the conjugate root $\lambda_{i+1} = \alpha_i - j\omega_i$.

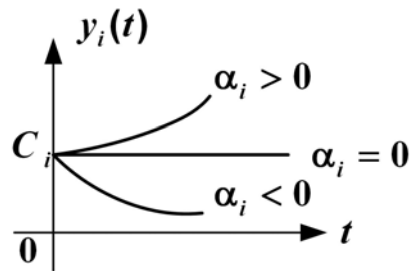


Figure 9.5 – The solution of the equation

If we have the pair of complex-conjugate roots, the solution is

$$y_i(t) = C_i e^{(\alpha_i + j\omega_i)t} + C_{i+1} e^{(\alpha_i - j\omega_i)t} = C e^{\alpha_i t} \sin(\omega_i t + \varphi_0). \quad (9.7)$$

The solution $y_i(t) \rightarrow 0$ at $t \rightarrow \infty$, if $\alpha_i < 0$ (Fig. 9.6).

For the linear continuous ACS to be stable it is necessary and sufficient for all the roots of characteristic equation (pole transfer function) to lie in the left half plane. The system can be stable, unstable or neutral.

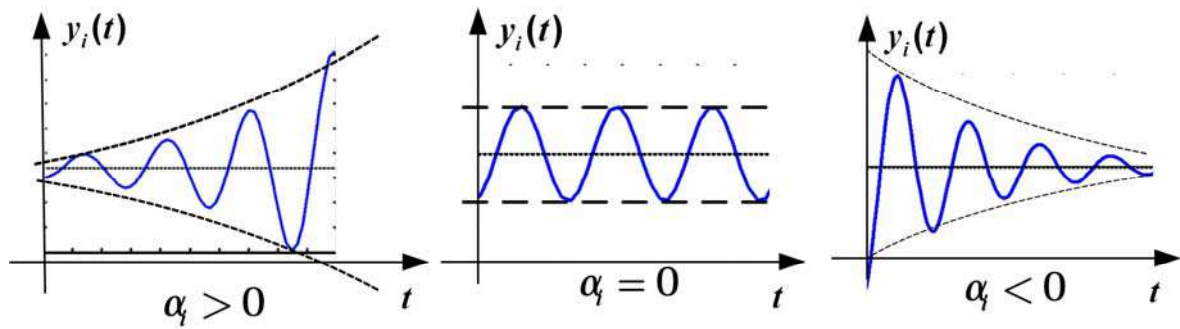


Figure 9.6 – Complex – conjugate root

$\eta = |\alpha|_{min}$ is the degree of stability for the stable system (Fig. 9.7).

To estimate the ACS stability we should find the roots of its characteristic equation.

9.3. Routh criterion of stability

There is the characteristic of the ACS stability estimation which does not demand the root calculation. Such characteristic is called a stability criterion.

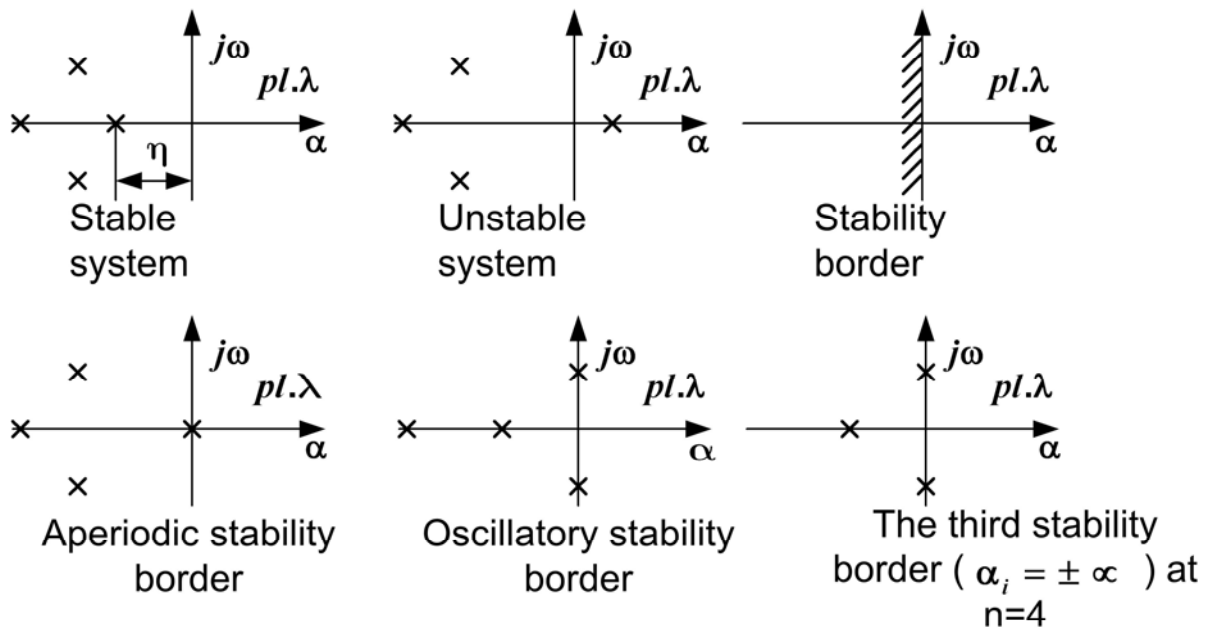


Figure 9.7 – Degree of stability

There are algebraic and frequency criteria.

The algebraic Routh criterion was developed in 1877 by an English mathematician.

The Routh tables are constructed on the basis of the input-output equations or the transfer function. If $a_0 < 0$, then all the factors are multiplied by -1.

In the first table line we write factors with even indexes starting with a_0 .

In the second table line we write factors with odd indexes starting with a_1 .

Non-existent factors are substituted by zeros

$$\begin{array}{cccccc|c} a_0 & a_2 & a_4 & a_6 & \dots & \dots & 0 \\ a_1 & a_3 & a_5 & a_7 & \dots & \dots & 0 \\ c_{31} & c_{32} & \dots & \dots & \dots & \dots & 0 \\ c_{41} & c_{42} & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{array} .$$

All the next lines are filled by the factors calculated by the formula

$$c_{ij} = \frac{c_{i-1,1}c_{i-2,j+1} - c_{i-2,1}c_{i-1,j+1}}{c_{i-1,1}}, \quad (9.8)$$

e.g.

$$c_{31} = \frac{a_1 a_2 - a_0 a_3}{a_1} .$$

The table is filled until it has only one nonzero element in the line.

For system to be stable it is necessary and sufficient if all the first elements in any line of the Routh table were positive: $c_{i1} > 0, i = 1, 2, \dots$.

9.4. Hurwitz criterion of stability

The algebraic criterion was developed in 1895 by the German mathematician A. Hurwitz.

The criterion consists of two conditions.

In the first place the positive property of all the characteristic polynomial factors should be verified (Stodola criterion)

$$D(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n; \quad a_i > 0, i = 0, 1, \dots, n. \quad (9.9)$$

If this condition is not satisfied, the system cannot be stable. If this condition is satisfied, the Hurwitz n -by- n matrix will be formed:

$$\begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ 0 & a_1 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n \end{bmatrix}.$$

The characteristic polynomial factors from a_1 to a_n we write on the matrix main diagonal. The factors with increasing indexes we write over the diagonal, the factors with decreasing indexes we write under the diagonal. Non-existent factors are substituted by zeros.

In the second place all the diagonal minors of the Hurwitz matrix should be positive

$$\Delta_i > 0, i = 1, \dots, n,$$

$$\Delta_1 = a_1, \Delta_2 = a_1 a_2 - a_0 a_3, \dots, \Delta_n = a_n \Delta_{n-1}.$$

For the ACS to be stable it is necessary and sufficient for all the characteristic polynomial factors $a_i > 0, i = 0, 1, \dots, n$ and all the diagonal minors $\Delta_i > 0, i = 1, \dots, n$ to be positive.

For the first- and second-order systems it is sufficient to check only the first condition $a_0 > 0, a_1 > 0, a_2 > 0$. If it is satisfied, the second condition $\Delta_1 = a_1 > 0, \Delta_2 = a_1 a_2 > 0$ is satisfied automatically. For the third-order systems the Hurwitz criterion has a form:

1. $a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0$.
2. $\Delta_2 = a_1 a_2 - a_0 a_3 > 0$, or $a_1 a_2 > a_0 a_3$.

If both these conditions are satisfied, the system is stable, as all the other conditions $\Delta_1 = a_1 > 0, \Delta_3 = a_3 \Delta_2 > 0$ are also satisfied automatically.

In general for systems of any order it is sufficient to check the positivity of barely even or odd minors in general. If one of these conditions is satisfied the other condition is satisfied automatically (Lyenar-Shipar criterion, 1914).

Example. We have a system (Fig. 9.8) with transfer function

$$W(s) = \frac{Y(s)}{G(s)} = \frac{k}{s(T_1s + 1)(T_2s + 1)},$$

$$k = 10; T_1 = 0.1; T_2 = 0.01.$$

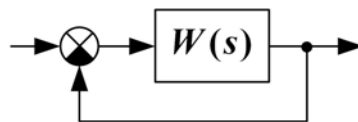


Figure 9.8 – The given system

Determine the stability of the open-loop and closed loop systems.

Solution. The open-loop system stability is easily estimated by the roots of its characteristic equation (Fig. 9.9)

$$s_1 = 0, \quad s_2 = -\frac{1}{T_1}, \quad s_3 = -\frac{1}{T_2}.$$

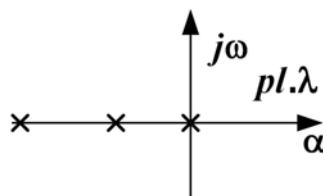


Figure 9.9 – Root of the characteristic equation

Open-loop system is on the aperiodic stability border.

Let us find the transfer function of the closed loop system

$$\Phi(s) = \frac{k}{s(T_1s + 1)(T_2s + 1) + k}. \quad (9.10)$$

The characteristic equation looks like

$$D(s) = s(T_1s + 1)(T_2s + 1) + k = 0. \quad (9.11)$$

It is not so easy to find the roots of the equation. We apply the Hurwitz criterion and write down the characteristic polynomial

$$D(s) = T_1 T_2 s^3 + (T_1 + T_2) s^2 + s + k. \quad (9.12)$$

Then we calculate the characteristic polynomial factors

$$a_0 = T_1 T_2 = 0,001, a_1 = T_1 + T_2 = 0,11, a_2 = 1, a_3 = k = 10. \quad (9.13)$$

The first condition of the Hurwitz criterion is satisfied

$$a_i > 0, i = 0, 1, 2, 3.$$

The system is stable because $a_1 a_2 > a_0 a_3$, $0,11 > 0,01$.

It is obvious from the last formula, that the system can become unstable if k grows.

If we introduce the characteristic polynomial $D(s)$ with the new variable $D(s - \eta)$, we can find out, whether the given system possesses the degree of stability η .

9.5. Mikhaylov criterion of stability

The algebraic criterion was developed in 1938 year.

We have the closed loop ACS transfer function

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{B(s)}{D(s)}. \quad (9.14)$$

We build a Mikhaylov curve for $0 \leq \omega \leq \infty$

$$D(j\omega) = a_0 (j\omega)^n + a_1 (j\omega)^{n-1} + \dots + a_n = X(\omega) + jY(\omega), \quad (9.15)$$

$$D(0) = a_n.$$

A hodograph is the motion trajectory of the terminus of $D(j\omega)$ in the complex plane (X, jY) under the frequency variation $0 \leq \omega \leq \infty$ (Fig. 9.10).

For the ACS to be stable it is necessary and sufficient for the Mikhaylov curve $D(j\omega)$ to begin on positive real semi axis (at $\omega = 0$) and to bypass

successively in a positive direction at $\omega \rightarrow \infty$ n quadrants under the frequency variation $0 \leq \omega \leq \infty$.

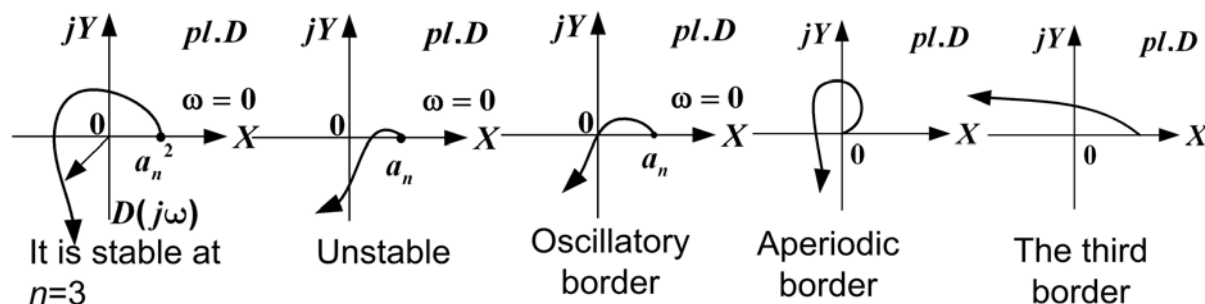


Figure 9.10 – Movement trajectory of the end of vector

It is possible not to build a curve $D(j\omega)$ but only to check the root alternation of $X(\omega) = 0$ and $Y(\omega) = 0$ equations.

The characteristic polynomial $D(s)$ can be expressed by the roots

$$D(s) = a_0(s - s_1)(s - s_2)...(s - s_n). \quad (9.16)$$

If we go from s to $j\omega$, we obtain

$$D(j\omega) = a_0(j\omega - s_1)(j\omega - s_2)...(j\omega - s_n). \quad (9.17)$$

Let us estimate the argument (phase) change of a vector $D(j\omega)$ under the frequency variation $-\infty \leq \omega \leq \infty$.

The increment of the $j\omega - s_i$ vector in the case of s_j lying in the left half plane is equal to

$$\Delta \arg_{-\infty \leq \omega \leq \infty} (j\omega - s_i) = \pi. \quad (9.18)$$

The increment of the $j\omega - s_i$ vector in the case of s_i lying in the right half plane is equal to

$$\Delta \arg_{-\infty \leq \omega \leq \infty} (j\omega - s_j) = -\pi. \quad (9.19)$$

If the right half plane has l roots, and the left one has $n - l$ roots, then

$$\Delta \arg_{-\infty \leq \omega \leq \infty} D(s) = (n - l)\pi - l\pi = (n - 2l)\pi. \quad (9.20)$$

If all the n roots are in the left half of the plane, we get

$$\Delta \arg_{-\infty \leq \omega \leq \infty} D(j\omega) = n\pi \quad (9.21)$$

or

$$\Delta \arg_{0 \leq \omega \leq \infty} D(j\omega) = n \frac{\pi}{2}. \quad (9.22)$$

All these factors mean that roots either lie on the real axis, or are located in pairs symmetrically to the real axis.

Terms:

- stability,
- stability criterion,
- stable state,
- unstable state,
- neutral state,
- stable motion,
- unstable motion,
- neutral motion,
- stable system,
- neutral system,
- Routh criterion,
- Hurwitz criterion,
- Mikhaylov's criterion,
- aperiodical border,
- oscillatory border.

Comprehension questions

1. What is the stability? List the kinds of stability.
2. How to estimate stability by the characteristic equation?
3. What is Routh criterion?
4. What is the degree of stability?
5. What is Hurwitz criterion?
6. What is Mikhaylov's criterion?

Lecture №10

THE ACS STABILITY (ENDING)

Only the first step is difficult.
Voltaire (1694 -1778)

Lecture outline:

1. Nyquist stability criterion.
2. The logarithmic stability criterion.
3. Zubov stability criterion.

10.1. Nyquist stability criterion

It is developed by the American amplifier expert in 1932. It is applied to the unit negative feedback systems. It allows assessing the closed system stability at the open-loop systems MPFC.

The closed ACS block diagram should be reduced to the Fig. 10.1: it should have a unit feedback.

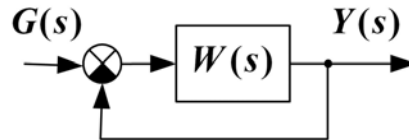


Figure 10.1 - The block diagram of the closed ACS

Let

$$W(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^m + a_1 s^{m+1} + \dots + a_n} = \frac{B(s)}{A(s)}. \quad (10.1)$$

From the closing formula we obtain

$$\Phi(s) = \frac{W(s)}{1+W(s)} = \frac{\frac{B(s)}{A(s)}}{1 + \frac{B(s)}{A(s)}} = \frac{B(s)}{A(s) + B(s)} = \frac{B(s)}{D(s)},$$

where

$$D(s) = A(s) + B(s).$$

Now we proceed to the Fourier transformation. For this purpose we do the substitution $s = j\omega$

$$\Phi(j\omega) = \frac{W(j\omega)}{1+W(j\omega)}. \quad (10.2)$$

Let us consider the change of the $1+W(j\omega)$ vector argument changes provided by the frequency ω change from 0 to ∞

$$\begin{aligned} \Delta_{0 \leq \omega \leq \infty} \arg(1+W(j\omega)) &= \Delta_{0 \leq \omega \leq \infty} \arg\left(1 + \frac{B(j\omega)}{A(j\omega)}\right) = \\ &= \Delta_{0 \leq \omega \leq \infty} \arg \frac{A(j\omega) + B(j\omega)}{A(j\omega)} = \Delta_{0 \leq \omega \leq \infty} \arg D(j\omega) - \Delta_{0 \leq \omega \leq \infty} \arg A(j\omega). \end{aligned} \quad (10.3)$$

According to the Mikhaylov stability criterion for the closed-loop system it should be

$$\Delta \arg D(j\omega) = n\pi/2, \quad 0 \leq \omega \leq \infty \quad (10.4)$$

The open-loop system can have l roots in the right half-plane. Then

$$\Delta \arg A(j\omega) = (n - 2l)\pi/2, \quad 0 \leq \omega \leq \infty \quad (10.5)$$

We can observe the possible variants.

1) The open-loop system is stable, so it has no poles in the right half-plane, ($l = 0$). Then

$$\Delta \arg (1+W(j\omega)) = \Delta \arg D(j\omega) - \Delta \arg A(j\omega) = n\pi/2 - n\pi/2 = 0. \quad (10.6)$$

The vector $1+W(j\omega)$ should not cover the origin of coordinates and the $W(j\omega)$ vector should not cover the point $(-1, j0)$ (Fig. 10.2).

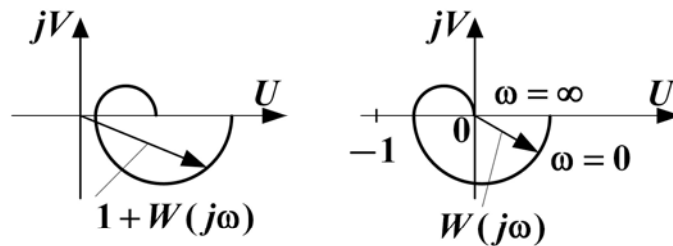


Figure 10.2 – The APFC of open-loop system

Theorem.

If the open-loop system has no poles in right half-plane then for the closed-loop system to be stable it is necessary and sufficient that the open-loop system APFC does not cover the point $(-1, j0)$ under the frequency change from 0 to ∞ .

If the open-loop system is neutral and lies at the aperiodic stability border then we get

$$W(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1}} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s(a_0 s^{n-1} + a_1 s^{n-2} + \dots + a_{n-1})}$$

and $a_n = 0$, $A(s)$ has one zero root, $W(s)$ has a zero pole, the system contains an integrator. It is astatic by the reference action (Fig. 10.3).

If the open-loop system has ν integrators then the system is said to have the ν -th order of astatism and $a_n = a_{n-1} = \dots = a_{n-\nu+1} = 0$, $A(s)$ has ν zero roots and the open-loop system transfer function can be represented as

$$W(s) = \frac{W_0(s)}{s^\nu} = \lim_{\alpha \rightarrow 0} \frac{W_0(s)}{(s + \alpha)^\nu}. \quad (10.7)$$

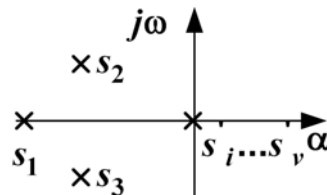


Figure 10.3 – The roots of the open-loop system

We consider the integrators being an extreme case of aperiodic links. We go to $j\omega$

$$W(j\omega) = \frac{W_0(j\omega)}{(j\omega)^\nu} = \lim_{\alpha \rightarrow 0} \frac{W_0(j\omega)}{(j\omega + \alpha)^\nu}. \quad (10.8)$$

The open-loop system APFC has a gap at the frequency $\omega = 0$. In this case we supplement the APFC by the part of the infinite radius arch, beginning from the real axis to the angle $-\nu\pi/2$, as every integrator introduces the phase delay for $-\pi/2$ radians.

The Nyquist diagram is represented on the Fig. 10.4.

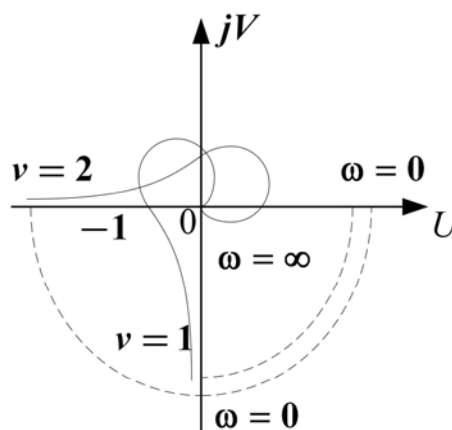


Figure 10.4 – The APFC of then open-loop system with delay $-\nu\pi/2$ radians

If the open-loop system is neutral and lies on the oscillatory limit of stability then the system APFC has a gap at the frequency ω_{res} . In this case we do the similar limiting process which causes the necessity to supplement the APFC with the infinite radius arch for the angle $-\pi$ at the frequency $\omega = \omega_{res}$. After the supplementation we use the same Nyquist formulation (Fig. 10.5).

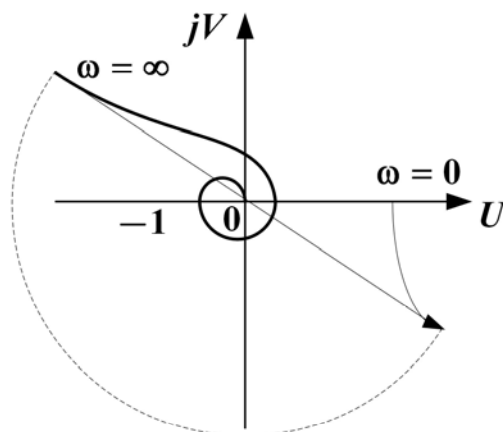


Figure 10.5 – The APFC of then open-loop system

2) If the open-loop system is not stable then $l \neq 0$ and

$$\Delta_{0 \leq \omega \leq \infty} \arg(1+W(j\omega)) = \Delta_{0 \leq \omega \leq \infty} \arg D(j\omega) - \Delta_{0 \leq \omega \leq \infty} \arg A(j\omega) = n\pi/2 - (n-2l)\pi/2 = 2\pi \frac{l}{2}.$$

Consequently, the vector $1+W(j\omega)$ should cover the origin of coordinates $\frac{l}{2}$ times in the positive direction.

Theorem.

For the closed-loop system to be stable in case of the open-loop system instability it is necessary and sufficient if at the frequency ω changing from 0 to ∞ the open-loop system APFC (the supplementary APFC) $W(j\omega)$ covers the point with coordinates $(-1, j0)$ $\frac{l}{2}$ times in the positive direction, where l is a number of the poles of the open-loop system lying in the right half-plane.

The advantage of the Nyquist criterion is that we can estimate the system stability by the experimental open-loop system APFC without building its mathematical model (Fig. 10.6)

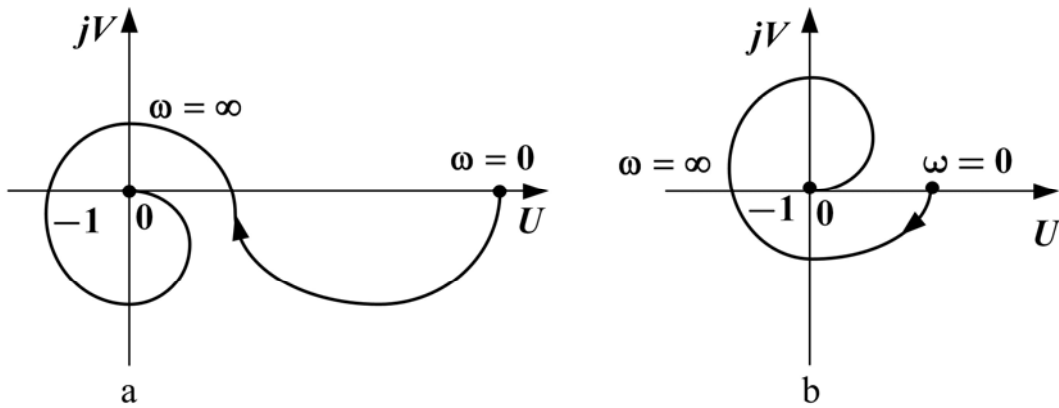
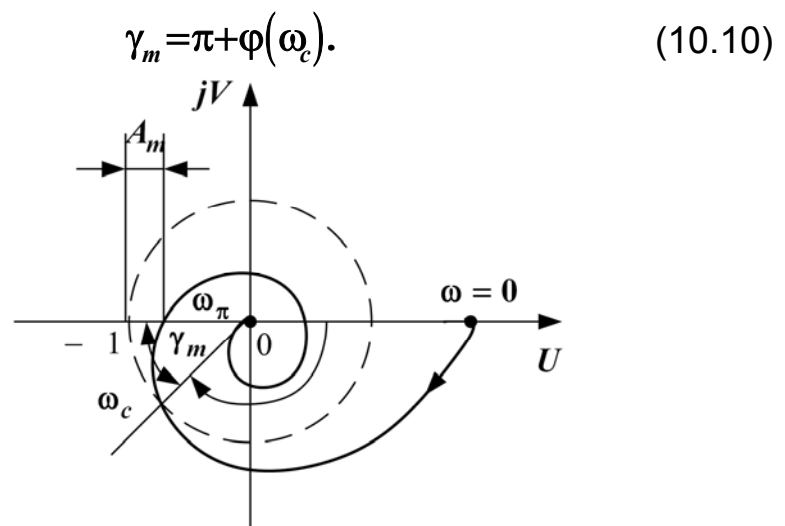


Figure 10.6 – The APFC of closed -loop system at $l=2$, a – the closed-loop system is stable, b – the closed-loop system is unstable

If we have a stable system, we can introduce the margin of stability at the amplitude A_m and at the phase γ_m . The margin of stability at the amplitude A_m is determined at the frequency $\omega_{-\pi}$, at which $\varphi(\omega_{-\pi}) = -\pi$ (Fig. 10.7)

$$A_m = 1 - |W(j\omega_{-\pi})| = 1 - A(\omega_{-\pi}). \quad (10.9)$$

The margin of stability at the phase γ_m is determined at the cutoff frequency ω_c , at which $A(\omega_c) = 1$



$$\gamma_m = \pi + \varphi(\omega_c). \quad (10.10)$$

Figure 10.7 – The gain and phase margins of the stable system

10.2. The logarithmic stability criterion

The logarithmic stability criterion follows from the Nyquist criterion.

If the open-loop system is stable, the APFC should not cover the point $(-1, j0)$ (Fig. 10.8, a).

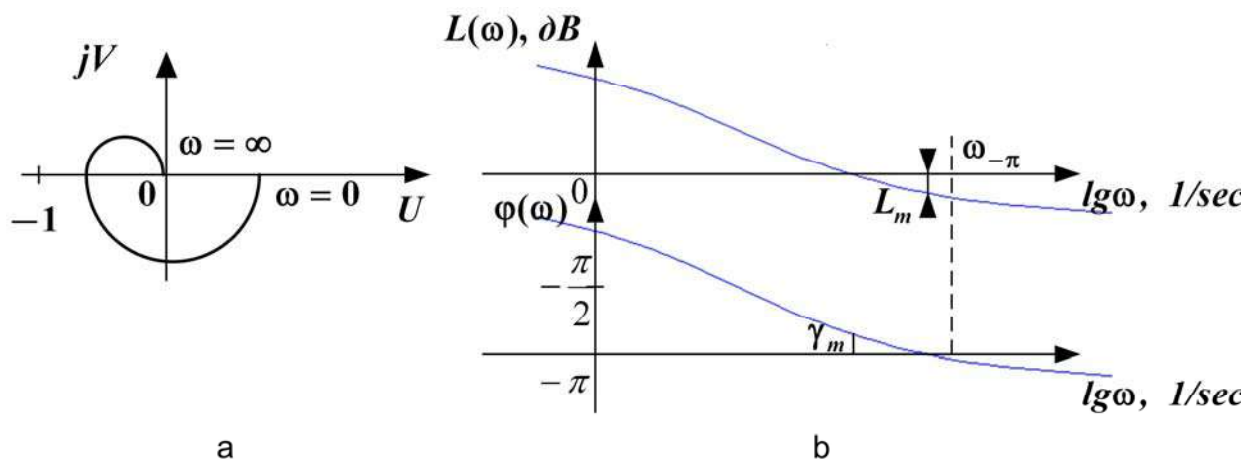


Figure 10.8 – The APFC and LPFC of the stable open-loop system

Generally it is necessary to determine how many times the point $(-1, j0)$ has been covered (Fig. 10.9).

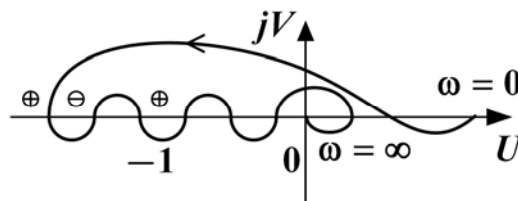


Figure 10.9 – Nyquist diagram of the open-loop system

This is calculated by the number of positive π_+ and negative π_- crossings of the open-loop system APFC with the real axis segment $(-1, -\infty)$. The crossing is considered positive if the point moves through the APFC downwards and negative if the point moves upwards. For the ACS to be stable it is necessary and sufficient for the equality to be satisfied

$$\pi_+ - \pi_- = \frac{l}{2}. \tag{10.11}$$

Theorem.

For the closed-loop system to be stable it is necessary and sufficient if at $L(\omega) > 0$ the open-loop system LPFC (the supplementary LPFC) crosses the lines $-\pi \pm 2k\pi$, $k = 0, 1, \dots$ downwards $\frac{l}{2}$ times less than

upwards (l being the number of poles of the open-loop system lying in right half-plane (Fig. 10.10).

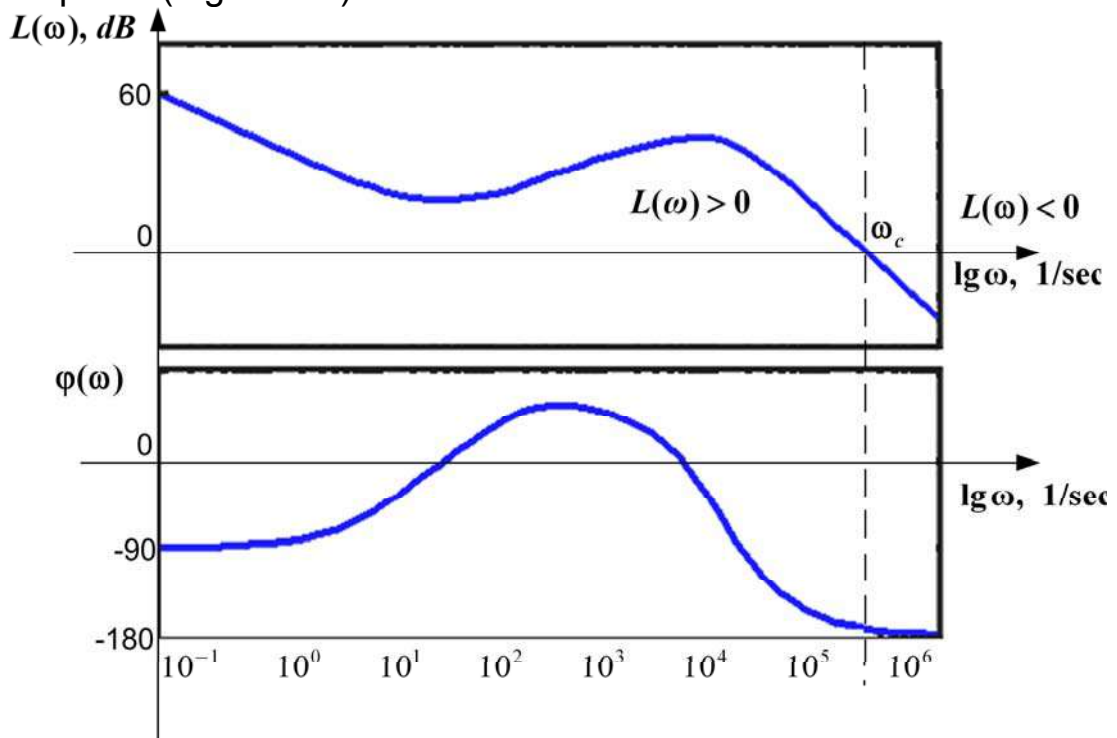


Figure 10.10 – The LAFC and LPFC of open-loop system

10.3. Zubov stability criterion

This algebraic criterion is applied for the matrix description of the ACS

$$\begin{aligned} \dot{x} &= Ax + Bg; \\ y &= Cx + Dg. \end{aligned} \quad (10.12)$$

The auxiliary matrix $V = I + 2(A - I)^{-1}$ is introduced.

For the ACS to be stable it is necessary and sufficient if

$$\|V^k\| \rightarrow 0 \text{ at } k \rightarrow \infty. \quad (10.13)$$

Terms:

- Mikhailov criterion,
- the stable open-loop system,
- the poles of the open-loop system,
- an integrator,
- the infinite radius arch,
- the gain margin and phase margin of stability,
- Nyquist criterion,

- Zubov stability criterion.

Comprehension questions

1. What is the Nyquist stability criterion?
2. How can we find the zero roots of open-loop system?
3. What is the logarithmic stability criterion?
4. List the Fourier transformation formulae.
5. What is the Zubov stability criterion?
6. How can we find the gain and the phase margins for the closed-loop system?

Lecture № 11

THE METHOD OF D -DIVIDING

Lecture outline:

1. Finding of the ACS stability field with the help of the method of one parameter plane D -dividing.
2. Finding of the ACS stability field with the help of the method of two parameter plane D -dividing.

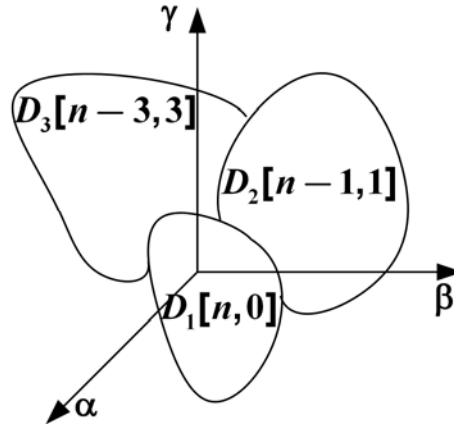
11.1. Finding of the ACS stability field with the help of the method of D -dividing one parameter plane

We should know what parameter values make the system stable. It can be found out if in the parameter space we mark out the areas which have the same number of the characteristic equation roots located in the left half plane.

As the parameters $\alpha, \beta, \gamma, \dots$ we can choose element or system transfer coefficients k_i , time constants T_i , etc. Such division of the parameter space is called D -dividing. The $D_i [n-l, l]$ note means, that in D_i area at any $\alpha, \beta, \gamma, \dots$ parameter value the characteristic polynomial $D(s)$ has $n-l$ roots in the left half plane and l roots in the right half plane. The stability domain is designated by $D[n, 0]$.

The D -dividing of the parameter space can be executed if we reflect the stability border of the plane s (Fig. 11.1) into the parameter space. We should form the border using the equations $D(j\omega) = 0$ at $\omega = 0$, i.e. $a_n = 0$ is the aperiodic stability border, $D(j\omega) = 0$ at $\omega \neq 0$ is the oscillatory

stability border, $a_0 = 0$ is the third stability border. If these equations contains parameters α, β, γ , etc., the corresponding borders will appear in the parameter space. If the equation contains only one adjustable parameter α , we shall D -divide the plane of one parameter, the parameter is considered complex $\alpha = x + jy$.



Let the system transfer function have the following form

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{\hat{A}(s)}{D(s)}, m \leq n. \quad (11.1)$$

The characteristic polynomial $D(s)$ coefficients depend on the system parameters $a_i = a_i(\alpha, \beta, \dots)$. Let $\alpha = var, \beta, \gamma, \dots = const$.

Any parameter α is included to the polynomial $D(j\omega)$ so that

$$D(j\omega) = a_0(j\omega)^n + a_1(j\omega)^{n-1} + \dots + a_n = 0. \quad (11.2)$$

We choose the components which contain the parameters of our interest

$$D(j\omega) = \alpha Q(j\omega) + R(j\omega) = 0 \Rightarrow \alpha = -\frac{R(j\omega)}{Q(j\omega)} = X(\omega) + j \cdot Y(\omega) = \alpha(\omega), \quad (11.3)$$

where $X(\omega)$ is the real component and even function, $Y(\omega)$ is the odd function.

We build a curve $\alpha(\omega)$ for the frequency ω changing from $-\infty$ to $+\infty$. The D -dividing curve $\alpha(\omega)$ reflects an imaginary axis of the s plane, therefore we shade its left part as well as the imaginary axis for the

frequency changing from $-\infty$ to $+\infty$. The D -dividing curve is symmetrical against the X axis. This curve divides the α parameter plane into three areas. The most shaded area D_2 can be a stability domain. We choose any α parameter value (e.g. $\alpha = 0$) from this domain and check the system stability by one of the criteria. If the system is stable at the given α parameter value, it is stable at the any parameter value from this domain. The domain under consideration is the stability domain. We designate it by $D_2 [n,0]$. Other domains are designated as $D_1 [n-2,2]$, $D_3 [n-1,1]$ (Fig. 11.2).

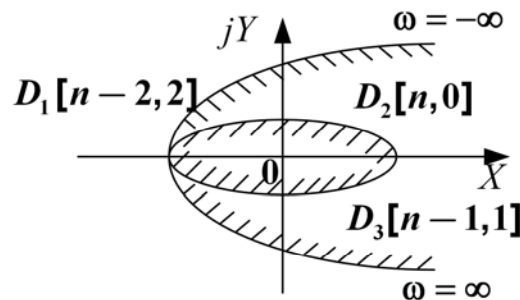


Figure 11.2 – D-dividing of parameter plane

If the system turns out unstable at the chosen parameter value, it is unstable at any parameter value. These systems are called structurally unstable. The maintenance of their stability demands not only to change the parameter value but also to introduce a corrector, i.e. to change structure of the system.

Example 1

We have a follow-up system (Fig. 11.3),

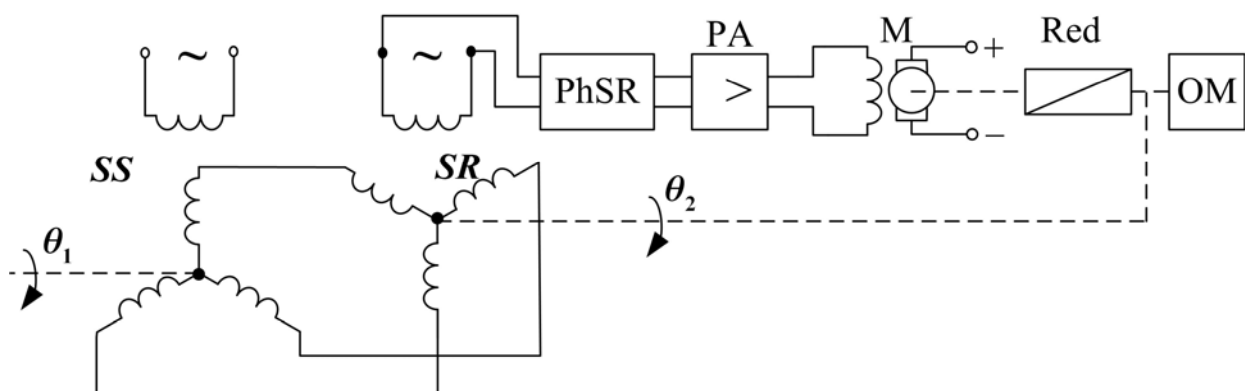


Figure 11.3 – Follow-up system

where

- SS – a setting selsyn;
- SR – a receiving selsyn ;

PhSR – a phase-sensitive rectifier;
PA – a power amplifier;
M – a motor;
Red – a reducer;
OM – an operational mechanism.

The structural diagram of this system has the following form (Fig. 11.4).

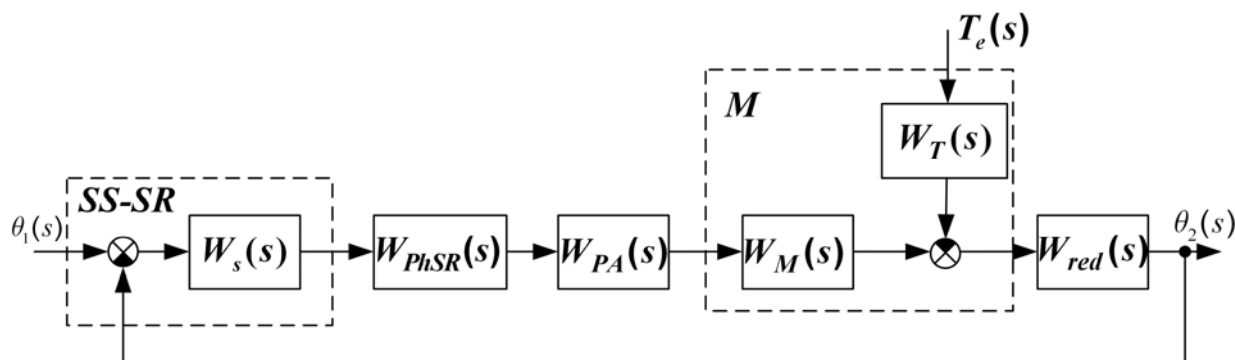


Figure 11.4 – Structural diagram of the system

Let us find the transfer functions of this system:

$$W(s) = \frac{\theta_2(s)}{\theta_1(s)} = \frac{k}{(T_1s + 1)(T_2s + 1)s} \text{ is an open-loop system transfer}$$

function,

$$\Phi(s) = \frac{\theta_2(s)}{\theta_1(s)} = \frac{k}{(T_1s + 1)(T_2s + 1)s + k} \text{ is a closed-loop system transfer}$$

function.

k is a variable, $T_1 = 0,1 \text{ sec}$, $T_2 = 0,6 \text{ sec}$.

We should obtain the parameter values k at which the system is stable

$$D(s) = T_1T_2s^3 + (T_1 + T_2)s^2 + s + k = 0, \quad (11.4)$$

$$a_0 = T_1T_2, a_1 = T_1 + T_2, a_2 = 1, a_3 = k.$$

We have three equations for three stability borders

$$D(j\omega) = T_1T_2(j\omega)^3 + (T_1 + T_2)(j\omega)^2 + j\omega + k = 0 \quad (11.5)$$

Therefore,

$$k = \underbrace{(T_1 + T_2)\omega^2}_{X(\omega)} + j \underbrace{(T_1T_2\omega^3 - \omega)}_{Y(\omega)} = k(\omega).$$

Now we proceed to calculating the D -dividing curve (Table 11.1).

Table 11.1

ω	0	0.1	1	...	∞
x ₌	0	0,007	0,7	...	∞
y ₌	0	-0,1	-0,94	...	∞

As the D -dividing curve is symmetrical against the real axis X , the calculation is provided only for positive values ω and we can build the second part of a curve symmetrically (Fig. 11.5).

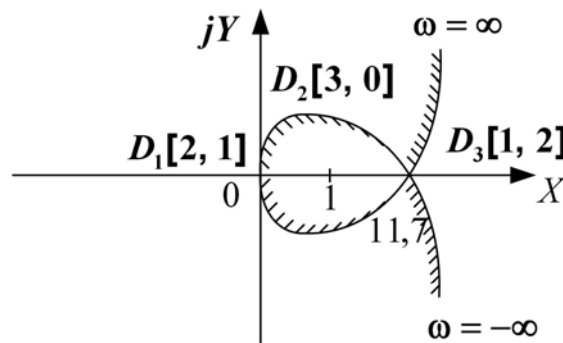


Figure 11.5 – D-dividing of k-plane

We can find the intersection of D -dividing curve and the X axes

$$Y(\omega) = 0, \quad \omega^2 = \frac{1}{T_1 T_2},$$

$$X(\omega) = (T_1 + T_2) \cdot \frac{1}{T_1 T_2} = \frac{0.1 + 0.6}{0.1 \cdot 0.6} = 11.7.$$

We shade the left part of the curve.

We check the system stability at $k = 1$ using Hurwitz criterion

1. $a_0 = T_1 T_2 > 0, a_1 = T_1 + T_2 > 0, a_2 = 1 > 0, a_3 = k > 0.$

The first Stodola theorem is satisfied.

2. Hurwitz matrix is

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix}.$$

Let us check

$$\begin{aligned}\Delta_2 &= a_1 a_2 - a_0 a_3 > 0 - ? \\ (T_1 + T_2) \cdot 1 &> T_1 T_2 k - ? \\ 0.1 + 0.6 &> 0.1 * 0.6 .\end{aligned}$$

The second Hurwitz condition also is satisfied.

The system is stable at $k = 1$ and at any k from the range $0 < k < 11.7$.

Let us designate the domains: $D_1[2,1], D_2[3,0], D_3[1,2]$.

Finding of the ACS stability field with the help of the method of two parameter plane D -dividing

We have a closed-loop system transfer function

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{B(s)}{D(s)}, m \leq n. \quad (11.6)$$

α, β are variables, γ, δ, \dots are constants.

To find the α and β parameter values at which the system is stable we write the stability border equations

$$a_0 = 0, a_n = 0, D(j\omega) = 0. \quad (11.7)$$

The parameter plane (α, β) contains the curves reflecting the stability borders. The oscillatory border equation has the following form

$$D(j\omega) = a_0 (j\omega)^n + a_1 (j\omega)^{n-1} + \dots + a_n = 0. \quad (11.8)$$

We choose the components which contain the parameters of our interest. At first we write components which have the first, e.g. α

$$D(j\omega) = \alpha P(j\omega) + \beta Q(j\omega) + R(j\omega) = 0. \quad (11.9)$$

Then we divide this equation into real and imaginary parts

$$\begin{cases} \alpha P_1(\omega) + \beta Q_1(\omega) + R_1(\omega) = 0, & P_1, Q_1, R_1 - \text{real parts}, \\ \alpha P_2(\omega) + \beta Q_2(\omega) + R_2(\omega) = 0, & P_2, Q_2, R_2 - \text{imaginary parts}. \end{cases} \quad (11.10)$$

We solve the obtained equations set against the parameters with the help of the Kramer rule (Gabriel Kramer (1704 – 1752), a Swiss mathematician)

$$\alpha = \frac{\Delta_{\alpha}(\omega)}{\Delta(\omega)} = \frac{\begin{vmatrix} -R_1(\omega) & Q_1(\omega) \\ -R_2(\omega) & Q_2(\omega) \end{vmatrix}}{\begin{vmatrix} P_1(\omega) & Q_1(\omega) \\ P_2(\omega) & Q_2(\omega) \end{vmatrix}} = \alpha(\omega), \quad (11.11)$$

$$\beta = \frac{\Delta_{\beta}(\omega)}{\Delta(\omega)} = \frac{\begin{vmatrix} P_1(\omega) & -R_1(\omega) \\ P_2(\omega) & -R_2(\omega) \end{vmatrix}}{\begin{vmatrix} P_1(\omega) & Q_1(\omega) \\ P_2(\omega) & Q_2(\omega) \end{vmatrix}} = \beta(\omega). \quad (11.12)$$

We build the curve $\beta(\alpha)$ on the α and β parameter plane at the frequency change - $-\infty \leq \omega \leq \infty$.

The functions $\Delta_{\alpha}(\omega)$, $\Delta_{\beta}(\omega)$, $\Delta(\omega)$ are determinants, and $\Delta(\omega)$ is the main determinant. The equations set will have a solution if determinants do not vanish simultaneously (Fig. 11.6).

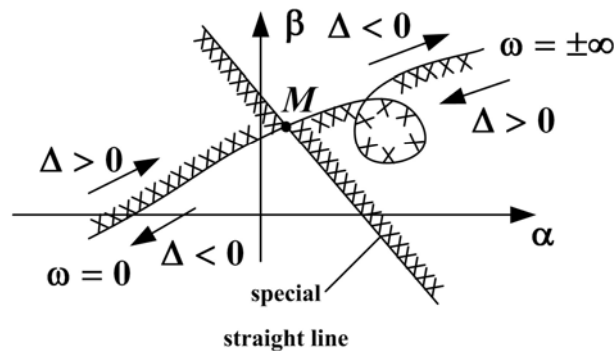


Figure 11.6 – D-dividing of parameters plane

The functions $\alpha(\omega)$ and $\beta(\omega)$ are even, therefore the curve $\beta(\alpha)$ is covered twice, once for both positive and negative frequencies, and double-hatched. If $\Delta(\omega) > 0$, the D -dividing curve is left-hatched. If $\Delta(\omega) < 0$, the D -dividing curve is right-hatched. At the point M there $\Delta_{\alpha} = \Delta_{\beta} = \Delta = 0$ we get a specific straight line. If in critical point M the determinant $\Delta(\omega)$ reverses its sign, the specific straight line is hatched otherwise it is not. If the specific straight line appears at $\omega = 0$, it is hatched once. If it the specific strait line appears at $\omega \neq 0$, it is double-hatched. The specific line hatching direction coincides with the hatching direction of the D -dividing curve in the critical point M .

Example 2

For the same follow-up system (Example 1) we have

$$D(s) = T_1 T_2 s^3 + (T_1 + T_2) s^2 + s + k, \quad k, T_1$$

are the variables, $T_2 = 0.6$ is a constant. At what parameter values k and T_1 the system is stable?

Let k be the first parameter, and T_1 be the second one. We write the equations of three stability borders

$$a_0 = T_1 T_2 = 0, \quad a_3 = 0, \quad D(j\omega) = T_1 T_2 (j\omega)^3 + (T_1 + T_2) (j\omega)^2 + j\omega + k = 0.$$

Then we choose parameters

$$D(j\omega) = k \cdot \underbrace{1}_{P(j\omega)} + T_1 \underbrace{(T_2 (j\omega)^3 + (j\omega)^2)}_{Q(j\omega)} + \underbrace{T_2 (j\omega)^2 + j\omega}_{R(j\omega)} = 0.$$

We divide this equation into real and imaginary parts

$$\begin{cases} k - T_1 \omega^2 - T_2 \omega^2 = 0, \\ k \cdot 0 - T_1 T_2 \omega^3 + \omega = 0, \end{cases} \Rightarrow \Delta(\omega) = \begin{vmatrix} 1 & -\omega^2 \\ 0 & -T_2 \omega^3 \end{vmatrix} = -T_2 \omega^3,$$

$$k = \frac{\Delta_k(\omega)}{\Delta(\omega)} = \frac{T_2^2 \omega^2 + 1}{T_2} = \frac{1}{T_2} + \omega^2 T_2, \quad \Delta_k(\omega) = \begin{vmatrix} T_2 \omega^2 & -\omega^2 \\ -\omega & -T_2 \omega^3 \end{vmatrix} = -T_2^2 \omega^5 - \omega^3,$$

$$T_1 = \frac{\Delta_{T_1}(\omega)}{\Delta(\omega)} = \frac{1}{T_2 \omega^2}, \quad \Delta_{T_1}(\omega) = \begin{vmatrix} 1 & T_2 \omega^2 \\ 0 & -\omega \end{vmatrix} = -\omega.$$

These equations are easily solved without determinants. But main determinant $\Delta(\omega)$ is necessary to draw the hatching. We build curves $T_1(k)$ for all three stability borders.

At $\omega = 0$ the determinants vanish

$$\Delta_k(0) = \Delta_{T_1}(0) = \Delta(0) = 0. \quad (11.13)$$

There is a special straight line ($k = 0$) at the frequency $\omega = 0$. The special straight line coincides with the aperiodic stability border (Fig. 11.7).

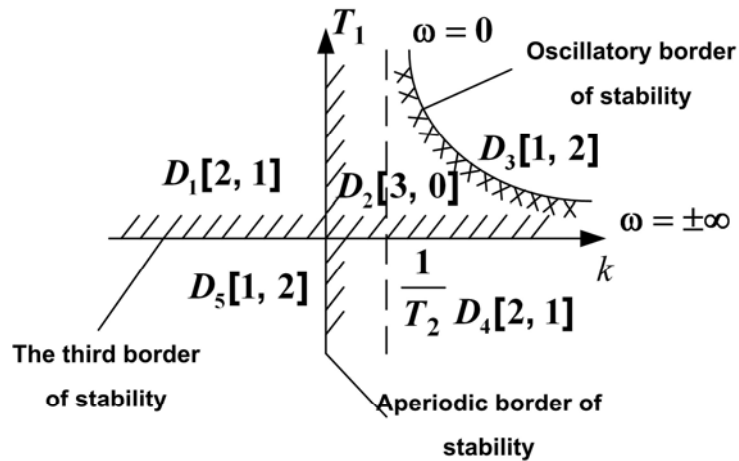


Figure 11.7 – D-dividing of parameters plane

We hatch the right side of the D -dividing curve at the frequency change from 0 up to ∞ because of $\Delta(\omega) = -T_2\omega^3 < 0$. We hatch the left side of the D -dividing curve at the frequency change from $-\infty$ up to 0 because of $\Delta(\omega) = -T_2\omega^3 > 0$, i.e. the curve is double-hatched.

The specific straight line is single-hatched as it appears at $\omega = 0$ and the determinant $\Delta(\omega)$ reverses the sign in the critical point. The third border is above-hatched as the system is stable only at $a_i > 0, i = 0, 1, 2, 3$. $a_0 = T_1T_2 > 0 \rightarrow T_1 > 0$. We designate the domains D_1, \dots, D_5 .

We choose the most hatched domain D_2 and check the system stability at $k = 1, T_1 = 0.1$. Consequently $D_2[3, 0]$ is the stability domain. We mark other domains by transition from D_2 in D_1 , in D_3 and in D_4 , from D_4 we go to D_5 .

Terms:

- D -dividing method,
- one parameter plane D -dividing,
- two parameter plane D -dividing,
- parameter space,
- stability domain,
- D -dividing curve,
- specific straight line.

Comprehension questions

1. What is the meaning of the D -dividing method?
2. What is the meaning of one parameter plane D -dividing?
3. What is the sense of the two parameter plane D -dividing?

Lecture № 12

THE CONTROL QUALITY EVALUATION

Lecture outline:

1. The control quality evaluation by the step response.
2. The control quality evaluation by the frequency characteristics.
3. The control quality evaluation by the poles and zeros of transfer function.

12.1. The control quality evaluation by the step response

There are direct and indirect control quality measures. The direct measures are introduced by the step response and the indirect are introduced by other characteristics.

Let us consider the direct quality measures.

We can distinguish monotonous, aperiodic and oscillatory step responses (Fig. 12.1).

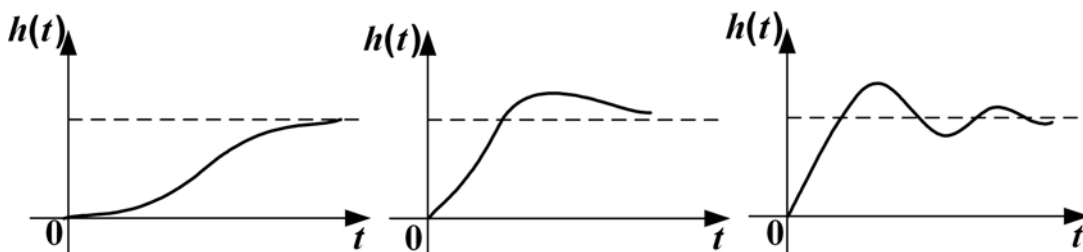


Figure 12.1 – Kinds of step responses

Let

$$g(t) = 1(t), \quad G(s) = \frac{1}{s}, \quad (12.1)$$

$$W(s) = \frac{Y(s)}{G(s)} = \frac{b_0 s^m + \dots + b_m}{a_0 s^n + \dots + a_n} = \frac{W_0(s)}{s^\nu}. \quad (12.2)$$

We shall study the direct measures by the example of the oscillatory step response at the reference action (Fig. 12.2).

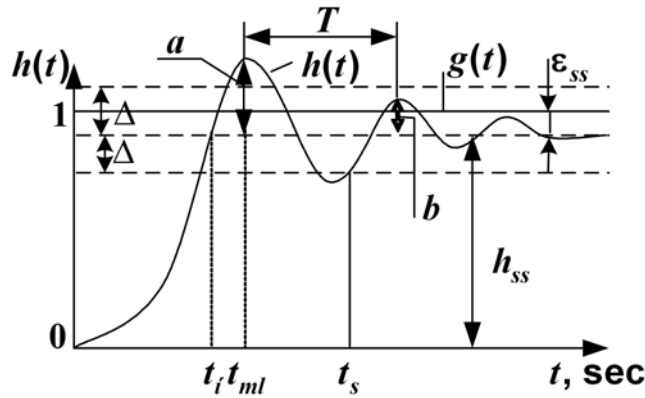


Figure 12.2 – Oscillatory step response

The direct measures are divided into the basic and additional ones. The basic measures include:

- 1) The steady-state value $h_{ss} = \lim_{t \rightarrow \infty} h(t)$.

This value can be read off from the principle limiting value theorem

$$h_{ss} = \lim_{s \rightarrow 0} H(s)s = \lim_{s \rightarrow 0} \Phi(s)G(s)s = \lim_{s \rightarrow 0} \frac{W(s)}{1+W(s)} \frac{1}{s} s = \begin{cases} \frac{k}{1+k}, & \text{if ACS is static } (\nu=0), \\ 1, & \text{if ACS is astatic } (\nu \neq 0). \end{cases}$$

As

$$W(s) = \frac{W_0(s)}{s^\nu}, \text{ then } \lim_{s \rightarrow 0} \frac{W(s)}{1+W(s)} = \lim_{s \rightarrow 0} \frac{W_0(s)}{s^\nu + W_0(s)} = \lim_{s \rightarrow 0} \frac{k}{s^\nu + k}. \quad (12.3)$$

At $\nu = 0$ we obtain $\lim_{s \rightarrow 0} \frac{W_0(s)}{s^\nu + W_0(s)} = \frac{k}{1+k}$,

And at $\nu \neq 0$ $\lim_{s \rightarrow 0} \frac{W_0(s)}{s^\nu + W_0(s)} = 1$,

where $k = \lim_{s \rightarrow 0} W(s)s^\nu = \frac{b_m}{a_n}$

or $\frac{b_m}{a_{n-1}}$ (at $a_n = 0$),

or $\frac{b_m}{a_{n-2}}$ (at $a_n = a_{n-1} = 0$).

2) The steady-state static error

$$\varepsilon_{SS} = \varepsilon_{st} = \lim_{t \rightarrow \infty} \varepsilon(t). \quad (12.4)$$

This error can be read off from the limiting value theorem as well

$$\begin{aligned} \varepsilon_{SS} &= \lim_{s \rightarrow 0} E(s)s = \lim_{s \rightarrow 0} \Phi_{\varepsilon}(s)G(s)s = \lim_{s \rightarrow 0} \frac{1}{1+W(s)} = \\ &= \lim_{s \rightarrow 0} \frac{s^{\nu}}{s^{\nu} + W_0(s)} = \begin{cases} \frac{1}{1+k} & , \nu = 0 \text{ (static system) ,} \\ 0 & , \nu \neq 0 \text{ (astatic system) .} \end{cases} \end{aligned}$$

The steady-state error can be obtained by the step response

$$\varepsilon(t) = g(t) - y(t) \Rightarrow \varepsilon_{SS} = 1 - h_{SS} = \begin{cases} \frac{1}{1+k} & , \text{if } \nu = 0, \\ 0 & , \text{if } \nu \neq 0. \end{cases}$$

3) The overshoot: $\sigma = \frac{h_{max} - h_{SS}}{h_{SS}} \cdot 100\%$.

It is usually considered normal $\sigma = 10 \div 30\%$. However the overshoot is not admitted for some systems, e.g. the docking system of the spacecraft and the orbital station, the cutter to the lathe supply control system, etc.

4) The setting time t_s is the shortest period of time after which the deviation $h(t)$ from h_{SS} does not exceed the given value Δ . Usually $\Delta = 0.05h_{SS}$, for precisional systems $\Delta = 0.01h_{SS}$.

The additional measures include:

1. The number of oscillations N during the period of time t_s . Usually $N = 1,5 \div 2$.

2. The rise time t_r is a time of the first steady-state value achievement.

3. Time of the first maximum t_{m1} .

4. The degree of damping

$$\chi = \frac{a}{b} = \frac{h_{max1} - h_{SS}}{h_{max2} - h_{SS}}. \quad (12.5)$$

5. The oscillation period T .

6. The angular oscillation frequency $\omega = 2\pi / T$, the cyclic frequency $f = 1/T$, etc.

The step response can be obtained experimentally, according to the input – output equation, by the transfer function, by the weight function, by the of state space method matrixes, by the Heaviside formula, by the state equation, by the output equations, etc.

Having the transfer function

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{B(s)}{D(s)}, \quad (12.6)$$

we can calculate the output by solving the input – output equation

$$D(p)y(t) = B(p)g(t). \quad (12.7)$$

At the zero conditions and the single step input we get

$$g(t) = 1, y(0) = \dots = y^{(n-1)}(0). \quad (12.8)$$

The input-output equation can be solved by the Laplace transformation

$$D(s)Y(s) - Y_0(s) = B(s)G(s). \quad (12.9)$$

We represent the input image as

$$G(s) = \frac{G_1(s)}{G_2(s)}, \quad (12.10)$$

then we obtain

$$Y(s) = \frac{B(s)G_1(s)}{D(s)G_2(s)} + \frac{Y_0(s)}{D(s)} \quad (12.11)$$

and

$$y(t) = \sum_{i=1}^n \frac{B(s_i)G_1(s_i)}{D'(s_i)G_2(s_i)} e^{s_i t} + \sum_{j=1}^l \frac{B(s_j)G_1(s_j)}{D(s_j)G_2'(s_j)} e^{s_j t} + \sum_{k=1}^n \frac{Y_0(s_k)}{D'(s_k)} e^{s_k t}, \quad (12.12)$$

where s_i, s_k are the transfer function poles;

s_j are the poles of an input action image,

$$D'(s_i) = \left. \frac{dD(s)}{ds} \right|_{s=s_i}, \quad G'_2(s_j) = \left. \frac{dG_2(s)}{ds} \right|_{s=s_j}. \quad (12.13)$$

For the step response $h(t) = y(t) \Big|_{g(t)=1}$ we have:

$$G(s) = \frac{1}{s}, G_1(s) = 1, G_2(s) = s, l = 1, s_1 = 0, G'_2(s) = 1; y(0) = \dots = y^{(n-1)}(0) = 0,$$

$$h(t) = \sum_{i=1}^n \frac{B(s_i)}{s_i D'(s_i)} e^{s_i t} + \frac{B(0)}{D(0)} \quad (12.14)$$

is the Heaviside expansion for nonmultiple roots.

The step response can be obtained:

- by the state space method

$$h(t) = C \int_0^t \Phi(t-\tau) B d\tau + D \cdot 1(t); \quad (12.15)$$

- by the weight function

$$h(t) = \int_0^t w(t-\tau) d\tau = \int_0^t w(\tau) \cdot 1(t-\tau) d\tau; \quad (12.16)$$

- by the Laplace transformation

$$h(t) = L^{-1} \left\{ \frac{\Phi(s)}{s} \right\}. \quad (12.17)$$

In addition to reproducing the reference action $g(t)$ any ACS should suppress the disturbance, therefore the control quality is estimated also by the step response at the disturbance action $h_f(t)$ (Fig. 12.3).

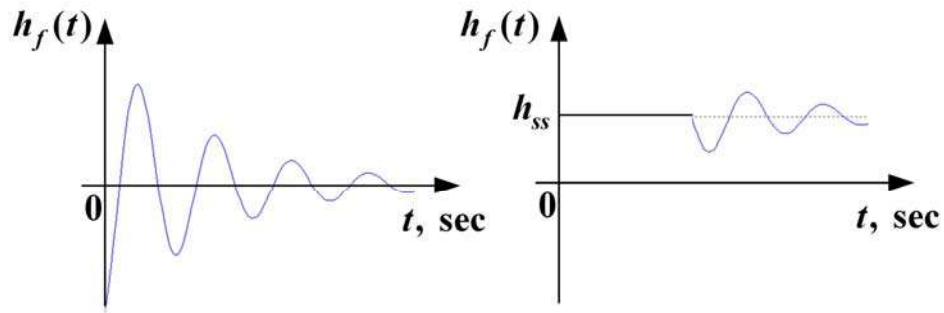


Figure 12.3 - Step response on disturbance action

12.2. The control quality evaluation by the frequency characteristics

The closed-loop system estimation by AFCh (Fig. 12.4):

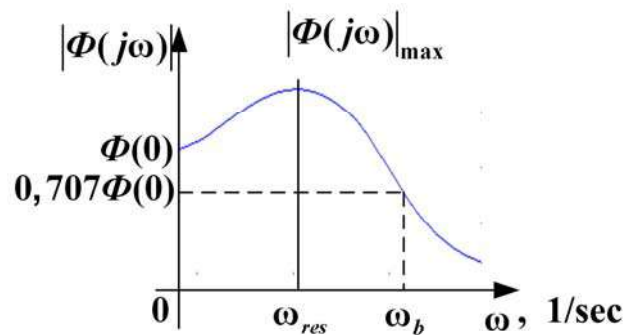


Figure 12.4 – AFCh of the closed - loop system

1. The steady-state value can be calculated by the limiting value theorem

$$h_{SS} = \Phi(0) = \lim_{s \rightarrow 0} \frac{W(s)}{1+W(s)} = \begin{cases} \frac{k}{k+1} & \text{for static system;} \\ 1 & \text{for astatic system} \end{cases} \quad (12.8)$$

or

$$\varepsilon_{SS} = 1 - \Phi(0). \quad (12.9)$$

2. The oscillatory measure is

$$M = \frac{|\Phi(j\omega)|_{max}}{\Phi(0)}. \quad (12.10)$$

$M = 1,2 \div 1,7$ is usually admitted.

3. The frequency range $(0, \omega_b)$ is a bandwidth. The wider is the system bandwidth, the higher is its swiftness. On the other hand, the wider is the bandwidth, the more disturbances pass through the system.

4. The resonance frequency ω_{res} . It should not coincide with the aircraft resonance frequency.

The open-loop system AFCh estimation of the gain and phase margins A_m, γ_m and setting time estimation are shown in Fig. 12.5, a, b.

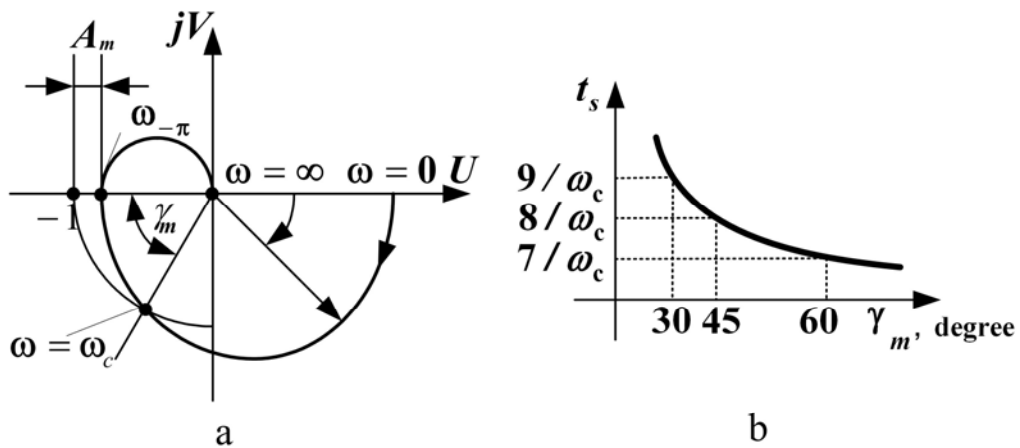


Figure 12.5 – Stability estimation by AFCh of open-loop system

At the cutoff frequency $A(\omega_c) = 1$; $t_s = (1...2)2\pi/\omega_c$; $t_{m1} \approx \frac{\pi}{\omega_c}$.

The open-loop system Bode diagram (Fig. 12.6) allows to get estimation of gain and phase margins of the closed loop system.

By the inclination of LAFCh low-frequency asymptote we can determine ν (quantity of integrators). If $\nu = 0$, the system is static and $\varepsilon_{st} = \frac{1}{1+k}$. If $\nu \neq 0$, then $\varepsilon_{st} = 0$. Value $20 \log(k)$ is at $\omega = 1$, cutoff frequency is at $L(\omega) = 0$.

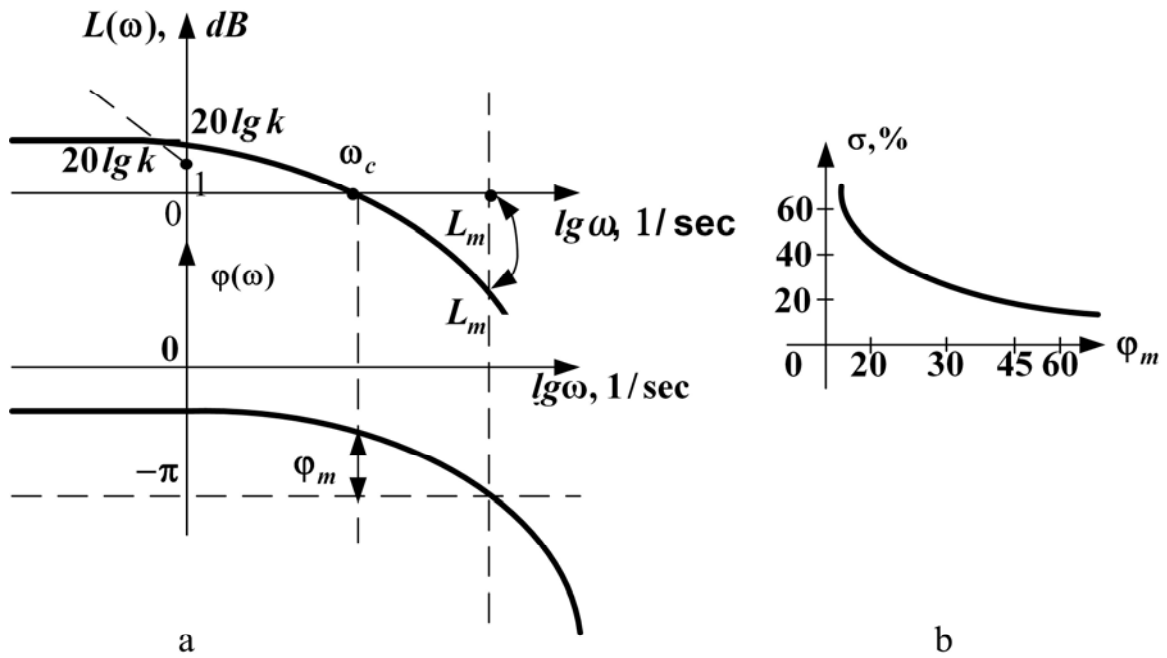


Figure 12.6 - Estimation by open - loop system diagramm

12.3. The control quality evaluation by the poles and zeros of transfer function

Let the closed-loop system transfer function is

$$\Phi(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{b_0 (s - s_{0j}) \dots (s - s_{0m})}{a_0 (s - s_1) \dots (s - s_n)}, \quad m \leq n, \quad (12.11)$$

where s_i are the transfer function poles, s_{0j} are the zeros of transfer function.

The free component $y_i(t) = C_i e^{s_i t}$ corresponds to each pole. The poles nearest to the imaginary axis are called the dominating poles.

Value η is the degree of stability (Fig. 12.7).

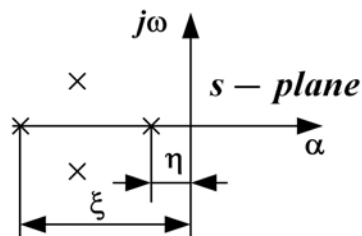


Figure 12.7 - Control quality estimation by poles and zeros of transfer function

If the dominating pole is real, then it corresponds to the constituent $C_i e^{\alpha_i t}$.

Example: $W(s) = \frac{k}{Ts + 1}$,

$$h(t) = k(1 - e^{\alpha t}) = k(1 - e^{-\eta t}) = \left|_{\eta = \frac{1}{T}} = k(1 - e^{-t/T}).$$

For $t = t_s$ $h(t_s) = 0.95 \cdot h_{SS}$,

$$k(1 - e^{-\eta t_s}) = 0.95k, \quad e^{-\eta t_s} = 0.05, \quad t_s = \frac{\ln 20}{\eta} \approx \frac{3}{\eta} = 3T.$$

For the aperiodic link $t_s \approx 3T$. If the dominating poles are complex, $t_s \leq \frac{3}{\eta}$. The desired zone of the pole disposition is shown on Fig. 12.8.

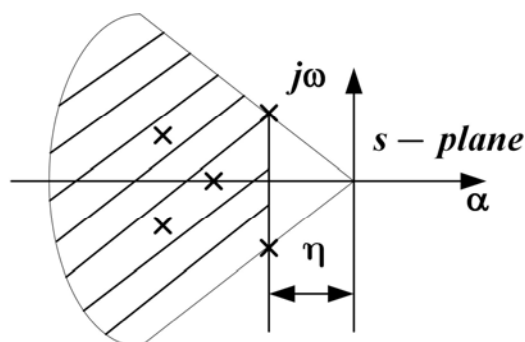


Figure 12.8 - Transfer function poles desired zone

We can find out, whether the system possesses the given degree of stability, in the following way. We take the transfer function denominator

$$D(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n, \quad (12.12)$$

then we introduce the substitution $s = q - \eta$ and obtain

$$D(q) = a'_0 q^n + a'_1 q^{n-1} + \dots + a'_n. \quad (12.13)$$

Further we check whether the new system is stable. If the system is stable, the initial system possesses the given degree of stability.

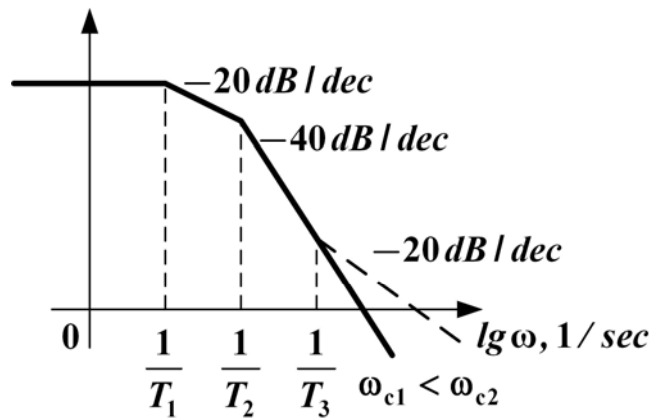
The system oscillatory measure is the ratio

$$\mu = \left| \frac{\omega_i}{\alpha_i} \right|_{max} . \quad (12.14)$$

If the system includes the zeros s_{0j} , it results in the bandwidth expansion, the swiftness increase, the gain margin L_m and phase margin φ_m increase, if $|s_{0j}| < \omega_b$.

Fig. 12.9 presents the LAFCh for two systems described by the transfer functions

$$W_1(s) = \frac{k}{(T_1s+1)(T_2s+1)} \text{ and } W_2(s) = \frac{k(T_3s+1)}{(T_1s+1)(T_2s+1)} .$$



12.9 LAFCh

Zeros, which are located near poles, compensate the influence of these poles. Let transfer function of the closed-loop system have the following form

$$\Phi(s) = \frac{b_0(s-s_{01}) \dots (s-s_{0j}) \dots (s-s_{0m})}{a_0(s-s_1) \dots (s-s_i) \dots (s-s_n)} . \quad (12.15)$$

Then, if $s_{0j} \approx s_i$ the multipliers $(s-s_{0j})$ and $(s-s_i)$ are reduced.

The geometric mean root is

$$\sqrt[n]{s_1 s_2 \dots s_n} = \sqrt[n]{\frac{a_n}{a_0}} . \quad (12.16)$$

Terms:

- steady-state value,
- steady-state static error,
- overshoot,
- setting time,
- oscillatory measure,
- bandwidth,
- cutoff frequency,
- resonance frequency,
- gain margin,
- phase margin,
- stability degree.

Comprehension questions

1. How can we read off the steady-state value?
2. How can we read off the steady-state static error?
3. How can we obtain the overshoot?
4. How can we obtain the setting time?
5. How can we obtain the oscillatory measure?
6. How can we obtain the bandwidth?
7. How can we obtain the cutoff frequency?
8. How can we obtain the resonance frequency?
9. How can we obtain the gain margin?
10. How can we obtain the phase margin?
11. How can we obtain the stability degree?

Lecture №13**THE ANALYSIS OF THE ACS ACCURACY**

When you study sciences, the examples are more useful than the rules.

I. Newton (1643 – 1727)

Lecture outline:

1. The control quality analysis by the integral estimates.
2. The steady state ACS accuracy estimation at the power-mode input actions.

3. The steady state ACS accuracy estimation at the input actions changing slowly.
4. The ACS accuracy estimation at the harmonic input actions.

13.1. The control quality analysis by the integral estimates

Let us consider the following system (Fig. 13.1).

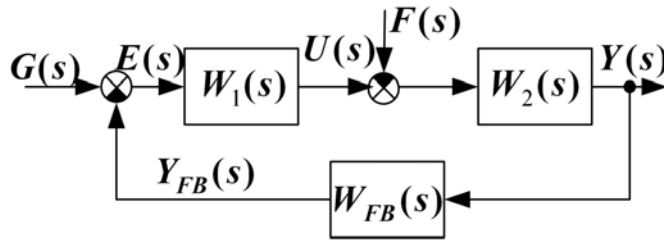


Figure 13.1 – Feedback system

We have the loop transfer functions $W_l(s) = W_1(s)W_2(s)W_{FB}(s)$.

At $W_{FB}(s) = 1$; $y_{FB}(t) = y(t)$; $g(t)$ is a reference action,
 $g = y_{FBreq} = y_{FBdes}$;

f is a disturbance action,

u is a control action,

$\varepsilon = g - y_{FB} = y_{FBreq} - y_{FB}$ is an error (a deviation),

y_{FBreq} is the required value of y_{FB} ,

y_{FBdes} is the desirable value of y_{FB}

The error ε depends both on the reference action g and on the disturbance f . Applying the superposition principle for linear systems we can present the error ε as the sum of two constituents, ε_g being an error caused by the reference action and ε_f being an error caused by the disturbance

$$\varepsilon = \varepsilon_g + \varepsilon_f, \quad (13.1)$$

that is in the image form

$$E(s) = E_g(s) + E_f(s) = \Phi_{\varepsilon_g}(s)G(s) + \Phi_{\varepsilon_f}(s)F(s). \quad (13.2)$$

Let $f(t) = 0$. So we consider an error caused by the reference action only. Let the transient function be of the monotonous character. An error can be represented as the sum (Fig. 13.2)

$$\varepsilon(t) = \underbrace{\varepsilon_{tr}(t)}_{\substack{\text{transient} \\ \text{component}}} + \underbrace{\varepsilon_{ss}}_{\substack{\text{steady-state} \\ \text{component}}} ; \quad \varepsilon_{tr}(t) = \varepsilon(t) - \varepsilon_{ss}.$$

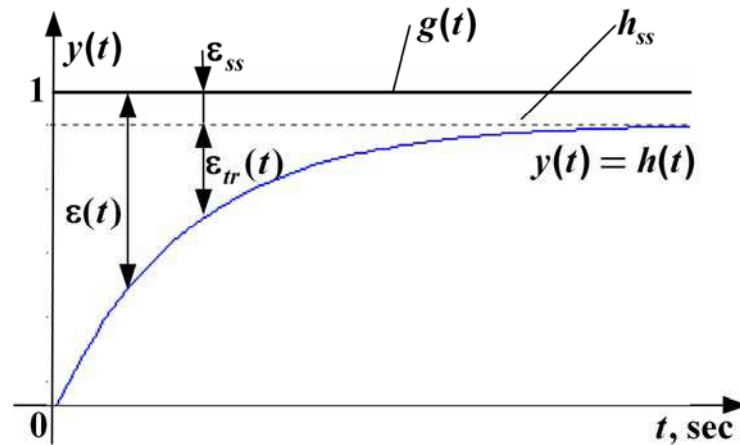


Figure 13.2 Presentation an error as the sum

If we introduce the linear integral estimates for the process quality regulation $I_{0m} = \int_0^{\infty} \varepsilon_{tr}(t) t^m dt$, $m = 0, 1, 2, \dots$, then at $m=0$ we receive

$$I_{00} = \int_0^{\infty} \varepsilon_{tr}(t) dt, \text{ that is an area under the curve } \varepsilon_{tr}(t) \text{ (Fig. 13.3).}$$

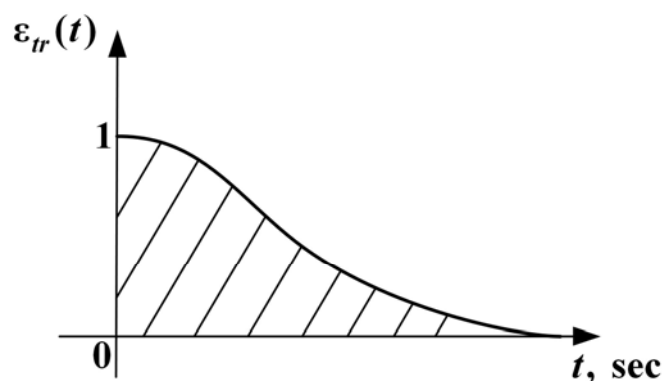


Figure 13.3 The curve $\varepsilon_{tr}(t)$

The smaller this area is, the more accurate and swift the system becomes. Having synthesized the systems, we achieve the I_{00} estimation decrease. To increase the system swiftness the integral estimates I_{01}, I_{02} , etc. are minimized. The integral linear quality estimation of I_{00} is convenient because it can be calculated without the $h(t)$ or $\varepsilon_{tr}(t)$ step response computation.

The Laplace transformation of the error $\varepsilon_{tr}(t)$ transient component has the form of

$$E_{tr}(s) = \int_0^{\infty} \varepsilon_{tr}(t) e^{-st} dt. \quad (13.3)$$

At $s = 0$ this formula coincides with the linear integral quality estimation

$$I_{00} = E_{tr}(s)|_{s=0} = E_{tr}(0) = [\Phi_{\varepsilon}(s)G(s) - E_{ss}(s)]|_{s=0}. \quad (13.4)$$

Thus,

$$I_{00} = \lim_{s \rightarrow 0} \left[\frac{\Phi_{\varepsilon}(s)}{s} - E_{ss}(s) \right] - \text{for static system,}$$

$$I_{00} = \lim_{s \rightarrow 0} \left[\frac{\Phi_{\varepsilon}(t)}{s} \right] - \text{for astatic system,} \\ \text{because } \varepsilon_{ss} = 0 \text{ and } v \neq 0.$$

However the linear integral estimations cannot be applied to the oscillatory step responses. In this case the area under the curve $\varepsilon_{tr}(t)$ can be small because of the positive and negative values compensation, and the system regulation quality can become unacceptable. For the oscillatory step responses the quadratic integral quality estimations can be applied (Fig.13.4).

$$I_{1m} = \int_0^{\infty} \left[\varepsilon_{tr}^2(t) + \tau_1^2 \left(\frac{d\varepsilon_{tr}}{dt} \right)^2 + \dots + \tau_m^{2m} \left(\frac{d^m \varepsilon_{tr}}{dt^m} \right)^2 \right] dt. \quad (13.5)$$

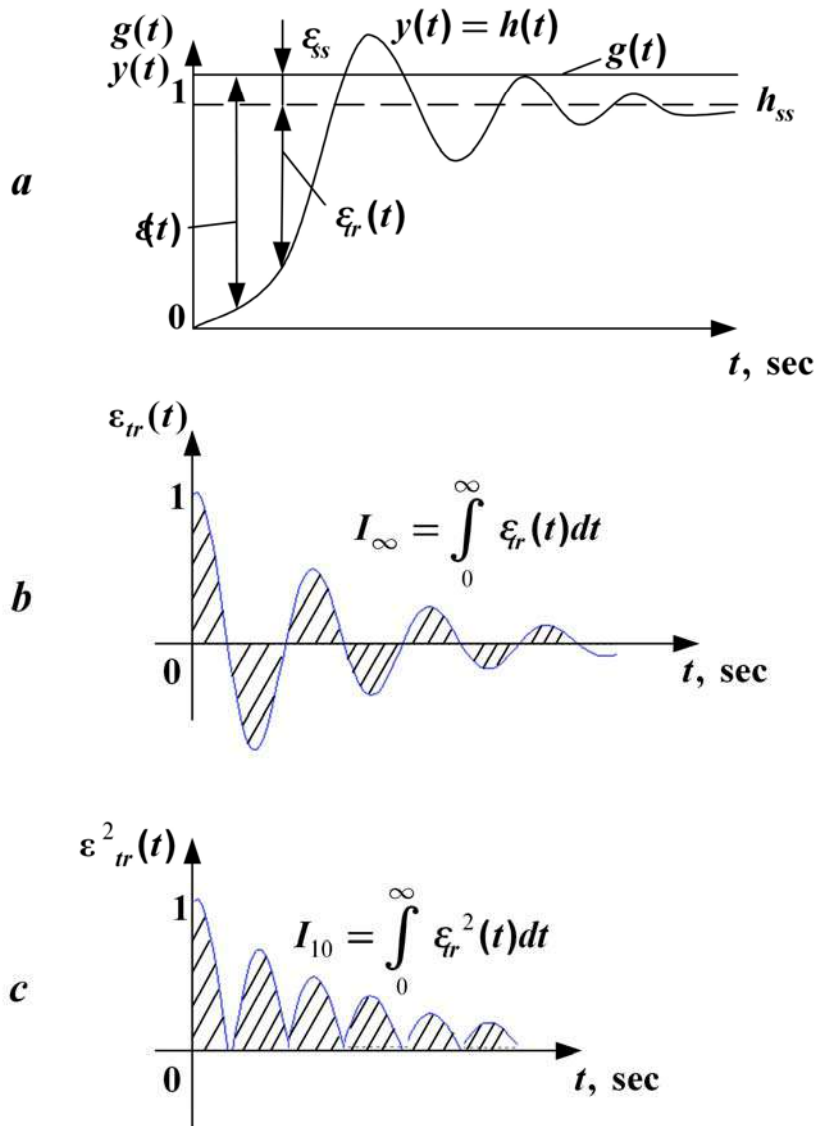


Figure 13.4 – Quadratic integral estimations of quality

At $m = 0$ we get

$$I_{10} = \int_0^{\infty} \epsilon_{tr}^2(t) dt . \quad (13.6)$$

The integral square estimate (ISE) I_{10} also can be calculated without the step response $h(t)$. Let us write down the Fourier image for transient component of the error $\epsilon_{tr}(t)$

$$E_{tr}(j\omega) = \int_0^{\infty} \varepsilon_{tr}(t) e^{-j\omega t} dt. \quad (13.7)$$

The original function is of the following form

$$\varepsilon_{tr}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{tr}(j\omega) e^{j\omega t} d\omega. \quad (13.8)$$

If we substitute the last expression into the integral square estimate (13.6), we get

$$I_{10} = \int_0^{\infty} \varepsilon_{tr}(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{tr}(j\omega) e^{j\omega t} d\omega dt. \quad (13.9)$$

After simple transformations we obtain

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{tr}(j\omega) \times \underbrace{\int_0^{\infty} \varepsilon_{tr}(t) e^{j\omega t} dt}_{\substack{\text{Fourier transformation} \\ \text{for argument } (-j\omega)}} \times d\omega, \quad (13.10)$$

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{tr}(j\omega) E_{tr}(-j\omega) d\omega. \quad (13.11)$$

Finally we can write the result formula down

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |E_{tr}(j\omega)|^2 d\omega. \quad (13.12)$$

The error image can be expressed in terms of the transfer function

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_{\varepsilon}(j\omega) G(j\omega)|^2 d\omega \quad - \text{ for astatic systems, } (13.13)$$

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_{\varepsilon}(j\omega) G(j\omega) - E_{ss}(j\omega)|^2 d\omega \quad - \text{ for static systems, } (13.14)$$

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{1+W(j\omega)} \frac{1}{j\omega} - E_{ss}(j\omega) \right|^2 d\omega, \quad (13.15)$$

where $W(j\omega) = W_1(j\omega)W_2(j\omega)W_{fb}(j\omega)$.

To calculate the integral square estimate we use the standard integral

$$I_{10} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b_0(j\omega)^{2n-2} + b_1(j\omega)^{2n-4} + \dots + b_n}{|a_0(j\omega)^n + a_1(j\omega)^{n-1} + \dots + a_n|^2} d\omega. \quad (13.16)$$

The book [4] presents the formulae for $n = 1, 2, \dots, 10$.

In general case

$$I_{10} = \frac{(-1)^{n-1} \det(N)}{2a_0 \det(M)}, \quad (13.17)$$

$$M = \begin{bmatrix} a_1 & a_3 & \dots & 0 \\ a_0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & a_n \end{bmatrix}, \quad N = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-1} \\ a_0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & a_n \end{bmatrix}.$$

13.2. The steady state ACS accuracy estimation at the power-mode input actions

We consider the same system (Fig. 13.1).

Let $f(t) \equiv 0$, $g(t) = g_r t^r$, r is the order of the input action.

At $r = 0$: $g(t) = g_0 = \text{const}$.

At $r = 1$: $g(t) = g_1 t = Vt = \Omega t$ – ramp input.

At $r = 2$: $g(t) = g_2 t^2 = \frac{at^2}{2}$ – quadratic input.

The Laplace representation of the input power-mode action has the following form

$$G(s) = L\{g_r t^r\} = \frac{g_r r!}{s^{r+1}}. \quad (13.18)$$

On the basis of the original limiting value theorem we have

$$\lim_{t \rightarrow \infty} \varepsilon(t) = \lim_{s \rightarrow 0} E(s)s = \lim_{s \rightarrow 0} \Phi_{\varepsilon}(s)G(s)s = \lim_{s \rightarrow 0} \frac{1}{1+W(s)}G(s)s, \quad (13.19)$$

where $W(s) = \frac{Y(s)}{E(s)} = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_n} = \frac{W_0(s)}{s^\nu}$,

$\nu = 0, 1, 2, \dots$ is the number of integrators in the open-loop system transfer function at the reference action.

After the substitution we obtain

$$\lim_{t \rightarrow \infty} \varepsilon(t) = \lim_{s \rightarrow 0} \frac{1}{1 + \frac{W_0(s)}{s^\nu}} \frac{g_r r!}{s^{r+1}} s = \lim_{s \rightarrow 0} \frac{s^\nu}{s^\nu + W_0(s)} \frac{g_r r!}{s^r}$$

or

$$\lim_{t \rightarrow \infty} \varepsilon(t) = \lim_{s \rightarrow 0} \frac{s^{\nu-r} g_r r!}{s^\nu + k}, \quad (13.20)$$

where k is the open-loop system transfer coefficient at the reference action;

ν is the ACS astatism order at the reference action;

r is the input action order.

As follows from the formula

1) At $\nu > r$ (the ACS astatism order is higher than the input signal order), $\varepsilon_{ss} = 0$;

$$2) \text{ At } \nu = r, \quad \varepsilon_{ss} = \lim_{s \rightarrow 0} \frac{g_r r!}{s^\nu + k} = \begin{cases} \frac{g_0}{1+k} & \text{of } \nu = 0 \text{ (static system);} \\ \frac{g_r r!}{k} & \text{of } \nu \neq 0 \text{ (astatic system);} \end{cases}$$

3) At $\nu < r$, $\varepsilon_{ss} = \infty$.

Note: in all cases ν is the ACS astatism order at the reference action.

Examples of the transient characteristics are presented at Fig. 13.5.

Now we shall estimate the ACS accuracy under the disturbance. Let $g(t) \equiv 0$, $f(t) \neq 0$.

Let $f(t) = f_r t^r$ be a power-mode disturbance, r be the order of the disturbance.

Then

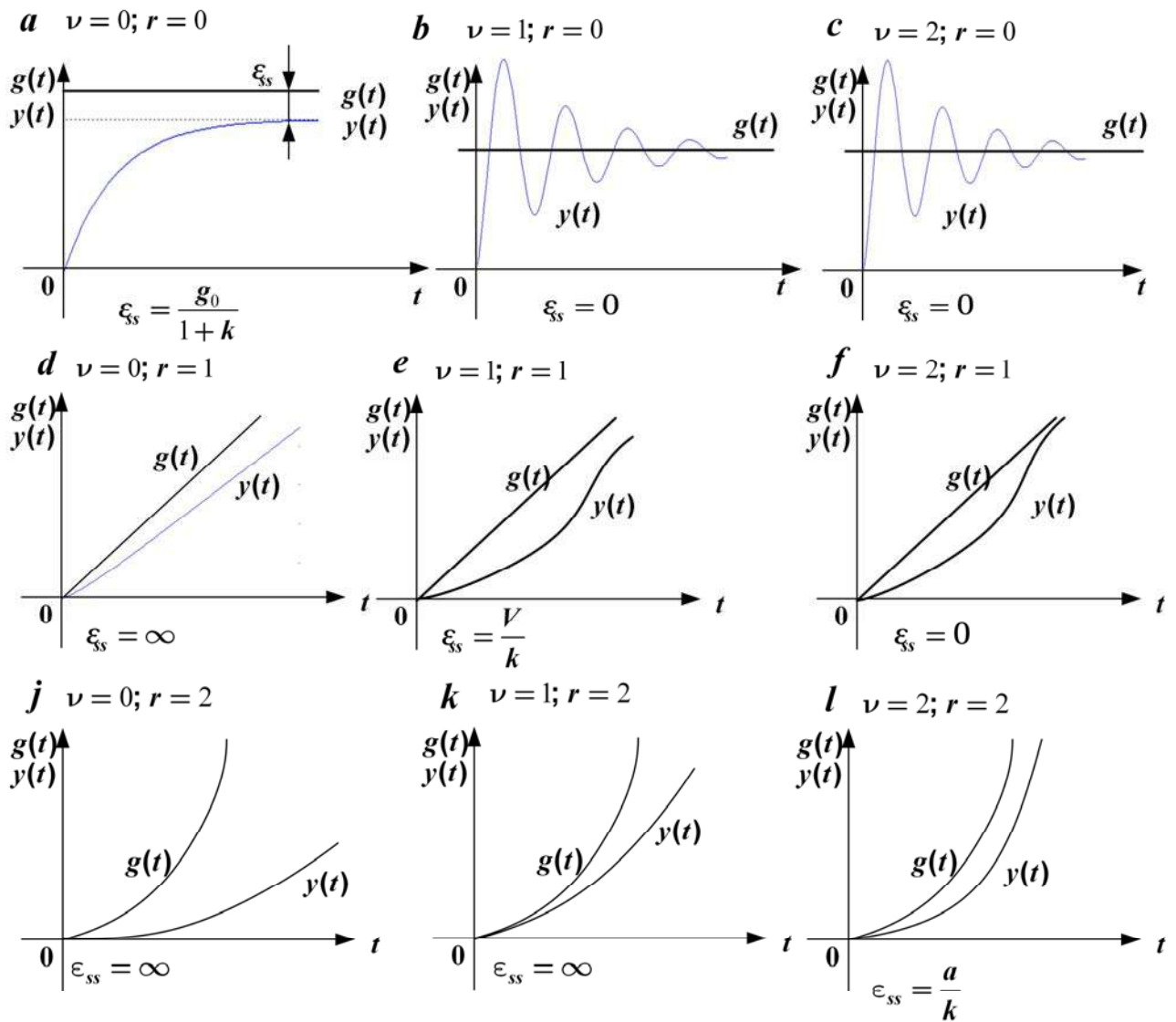


Figure 13.5 – Examples of transient characteristics

$$\lim_{t \rightarrow \infty} \varepsilon_f(t) = \lim_{s \rightarrow 0} E_f(s)s = \lim_{s \rightarrow 0} \Phi_\varepsilon(s)F(s)s = \lim_{s \rightarrow 0} \frac{-W_f(s)W_{fb}(s)}{1+W(s)}F(s)s. \quad (13.22)$$

Both transfer functions $W_f(s) = W_2(s)$ and $W(s)$ can have integrators. We shall allocate them

$$W_f(s) = \frac{W_{f0}(s)}{s^\mu}, \quad W(s) = \frac{W_0(s)}{s^\nu},$$

where μ is a number of the integrators of the open-loop system transfer function at the disturbance;

ν is a number of integrators of the open-loop system transfer function at the reference action.

After the substitution we obtain

$$\lim_{t \rightarrow \infty} \varepsilon_f(t) = \lim_{s \rightarrow 0} \frac{s^{\nu-\mu-r} k_f k_{fb} f_r r!}{s^\nu + k}, \quad (13.23)$$

where k_f is the open-loop system transfer coefficient at the disturbance.

As follows from this formula:

- 1) At $\nu - \mu > 0$ system is astatic at the disturbance, i.e. $\varepsilon_{fst} = 0$;
- 2) At $\nu - \mu = 0$ system is static at the disturbance

$$\varepsilon_{fss} = \lim_{s \rightarrow 0} \frac{k_{fb} k_f f_r r!}{s^\nu + k} = \begin{cases} \frac{k_{fb} k_f f_0}{1+k}, & \text{at } \nu = \mu = r = 0; \\ \frac{k_{fb} k_f f_r r!}{k}, & \text{at } \nu \neq 0; \end{cases}$$

- 3) At $\nu - \mu < r$ the steady state error $\varepsilon_{fss} = \infty$.

The ACS astaticism order at the disturbance is equal to $\nu - \mu$. For the system to become astatic at the disturbance, $\nu > \mu$.

13.3. The steady state ACS accuracy estimation at the input actions changing slowly

We consider the same system. Let $g(t)$ be the slowly changing signal, $g' \approx 0$, $f(t) \equiv 0$.

Let us expand the image $\Phi_\varepsilon(s)$ in the formula $E(s) = \Phi_\varepsilon(s)G(s)$ into the Taylor series in the neighbourhood of $s = 0$.

Thus we get

$$E(s) = \left[\Phi_\varepsilon(0) + \frac{d\Phi_\varepsilon(s)}{ds} \Big|_{s=0} s + \frac{1}{2!} \frac{d^2\Phi_\varepsilon(s)}{ds^2} \Big|_{s=0} s^2 + \dots \right] G(s). \quad (13.24)$$

Now we go to the original functions

$$\varepsilon(t) = \Phi_\varepsilon(0)g(t) + \frac{d\Phi_\varepsilon(s)}{ds} \Big|_{s=0} \frac{dg}{dt} + \frac{1}{2!} \frac{d^2\Phi_\varepsilon(s)}{ds^2} \Big|_{s=0} \frac{d^2g}{dt^2} + \dots,$$

or

$$\varepsilon(t) = c_0 g(t) + c_1 \frac{dg}{dt} + \frac{1}{2!} c_2 \frac{d^2 g}{dt^2} + \dots + \frac{1}{i!} c_i g^{(i)}(t) + \dots, \quad (13.25)$$

where $c_i = \left. \frac{d^i \Phi_\varepsilon(s)}{ds^i} \right|_{s=0}$ are error coefficients;

c_0 is a position error coefficient;

c_1 is a velocity error coefficient;

c_2 is an acceleration error coefficient, etc.

In this series there is a final number of components for the slowly changing signal $g(t)$.

Often we can introduce the error constants:

$D_\Omega = \frac{1}{c_1}$ is a velocity error constant,

$D_\varepsilon = \frac{2}{c_2}$ is an acceleration error constant,

$D_{\varepsilon} = \frac{3!}{c_3}$ is the 3th differential error constant, etc.

The error coefficients c_i can be calculated in the following way.

If

$$W(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{B(s)}{A(s)}, \quad m \leq n,$$

then

$$\Phi_\varepsilon(s) = \frac{1}{1+W(s)} = \frac{A(s)}{A(s)+B(s)}.$$

Let us equate this expression with the Taylor series.

$$\Phi_\varepsilon(s) = \frac{A(s)}{A(s)+B(s)} = c_0 + c_1 s + \frac{1}{2!} c_2 s^2 + \dots = C(s).$$

As follows from this equation $A(s) = [A(s) + B(s)]C(s)$. The factors at the identical powers s^i should be equal in the expression

$$a_0 s^n + a_1 s^{n-1} + \dots + a_n = (a_0 s^n + a_1 s^{n-1} + \dots + a_n + b_0 s^m + b_1 s^{m-1} + \dots + b_m) \times \left(c_0 + c_1 s + \frac{c_2}{2!} s^2 + \dots \right).$$

At $i = 0$: $a_n = (a_n + b_m)c_0$, whence

$$c_0 = \frac{a_n}{a_n + b_m} = \frac{1}{1 + b_m / a_n} = \begin{cases} 1 / (1 + k) \text{ if } a_n \neq 0 \text{ (static system),} \\ 0 \text{ if } a_n = 0 \text{ (astatic system).} \end{cases}$$

At $i = 1$: $a_{n-1} = (a_n + b_m)c_1 + (a_{n-1} + b_{m-1})c_0$, whence

$$c_1 = \frac{a_{n-1} - (a_{n-1} + b_{m-1})c_0}{a_n + b_m} = \begin{cases} \frac{a_{n-1}(a_n + b_m) - (a_{n-1} + b_{m-1})a_n}{(a_n + b_m)^2} \text{ at } a_n \neq 0, (\nu = 0), \\ a_{n-1} / b_m = 1 / k_V \text{ at } a_n = 0, (\nu = 1), \text{ where } k_V = W_0(s) \Big|_{s=0} \\ 0 \text{ at } a_n = a_{n-1} = 0, (\nu = 2). \end{cases}$$

At $i = 2$: if $\nu = 2$, $a_n = a_{n-1} = 0$ and

$$a_{n-2} = b_m \frac{1}{2} c_2 + b_{m-1} c_1 + (a_{n-2} + b_{m-2}) c_0;$$

$$c_2 = 2a_{n-2} / b_m = 2 / k_a .$$

At $\nu = 1$: $D_\Omega = k_\nu$ is the velocity error constant. At $\nu = 2$: $D_\varepsilon = k_\varepsilon$ is the acceleration error constant.

Example: $g(t) = Vt$, $\nu = 1$.

$$\varepsilon(t) = c_0 g(t) + c_1 \frac{dg}{dt} + \frac{1}{2!} c_2 \frac{d^2 g}{dt^2} + \dots, \quad \varepsilon(t) = c_1 \frac{dg}{dt},$$

that's why $c_0 = 0$ and $\frac{d^2 g}{dt^2} = \frac{d^3 g}{dt^3} = \dots = 0$,

$$c_1 = \frac{1}{k}, \quad \frac{dg}{dt} = V,$$

$$\varepsilon(t) = \varepsilon_{ss} = \frac{V}{k} - \text{result is coincide with example of previous part.}$$

13.4. The ACS accuracy estimation at the harmonic input actions

Let $g(t) = g_m \sin \omega_1 t$, $f(t) \equiv 0$.

Then

$$y_{ss}(t) = b_m \sin(\omega_1 t + \varphi),$$

and

$$\varepsilon_{ss}(t) = \varepsilon_m \sin(\omega_1 t + \varphi_0),$$

where $\varepsilon_m = |\Phi_\varepsilon(j\omega_1)| g_m = \left| \frac{1}{1+W(j\omega_1)} \right| g_m \approx \frac{g_m}{|W(j\omega_1)|}$.

If we need $\varepsilon_m \leq \delta$ at $\omega = \omega_1$, we choose LAFC in the following way: $L(\omega)$ should go above the point $L' = (\omega_1, 20 \lg(g_m / \delta))$ (Fig.13.6). The error amplitude at the frequency is ω_1

$$\delta_m(\omega) \approx g_m / 10^{\frac{L(\omega_1)}{20}}.$$

As $\varepsilon_m = \frac{g_m}{|W(j\omega_1)|} \leq \delta$, $L(\omega_1) = 20 \lg |W(j\omega_1)| \geq 20 \lg \frac{g_m}{\delta}$.

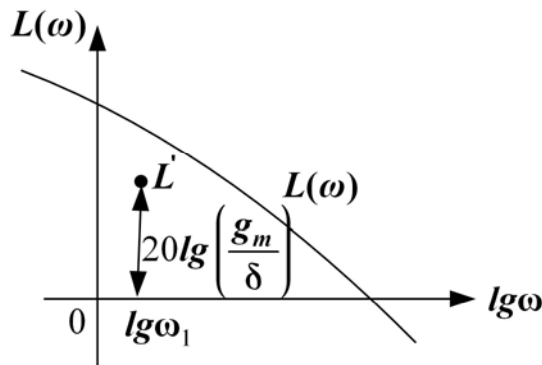


Figure 13.6 – LAFC

Terms:

- integral square estimate (ISE),
- integral linear estimation,
- the astaticism order.

Comprehension questions

1. Give the superposition principle.
2. Why is the integral linear estimation convenient?

3. When is it impossible to use the integral linear estimations? What should we do in this case?
4. Give the limiting value theorem.
5. How can we produce the steady state ACS accuracy estimation at the power-mode input actions?
6. How can we produce the steady state ACS accuracy estimation at the input actions changing slowly?
7. How can we produce the ACS accuracy estimation at the harmonic input actions?

Lecture №14

THE ANALYSIS OF THE ACS ACCURACY (ENDING)

Life requires motion.

Aristotle

Lecture outline:

1. The ACS accuracy estimation at the random input actions.
2. The example of the ACS accuracy estimation at the random input actions.

14.1. The ACS accuracy estimation at the random input actions

Let us consider the ACS (Fig. 14.1)

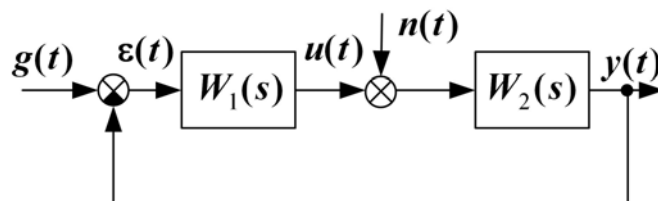


Figure 14.1 – The ACS with random actions

Let $g(t)$, $\varepsilon(t)$, $n(t)$, $y(t)$ be centered stationary random processes ($n(t)$ - noise).

A random process is the time change of a variable, it being a random variable at any specific instant.

Let us give the basic characteristics of the random processes.

A centered random process is the process, which mathematical expectation is equal to zero ($\overset{\circ}{x}, \overset{\circ}{g}$).

A mathematical expectation

$$M[x(t)] = m_x(t) = \int_{-\infty}^{\infty} x(t) f(x,t) \approx \frac{1}{n} \sum_{i=1}^n x_i(t) \quad (14.1)$$

is the realization assembly average value.

For stationary processes we have

$$M[x(t)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt \quad (14.2)$$

is the time average.

The random process distribution function is

$$F(x(t)) = P(x(t) \leq x_1). \quad (14.3)$$

The random process distribution density is

$$f(x,t) = \frac{dF(x(t))}{dx}. \quad (14.4)$$

The random process correlation function is

$$R_x(t, \tau) = M \left[\overset{\circ}{x}(t) \overset{\circ}{x}(\tau) \right]. \quad (14.5)$$

A stationary random process is the process which mathematical expectation does not depend on time $M[x(t)] = m_x = \text{const}$ and the correlation function depends on a single argument $R_x(t, \tau) = R_x(t - \tau) = R_x(\lambda)$.

A variance is $D_x = M \left[\overset{\circ}{x}^2(t) \right]$.

A standard deviation

$$\sigma_x = \sqrt{D_x} \quad (14.6)$$

characterizes a deviation of a random variable from the average value (Fig. 14.2).

A spectral density is

$$S_x(\omega) = F \{ R_x(\tau) \},$$

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau,$$

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau} d\omega,$$

$$D_x = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega.$$

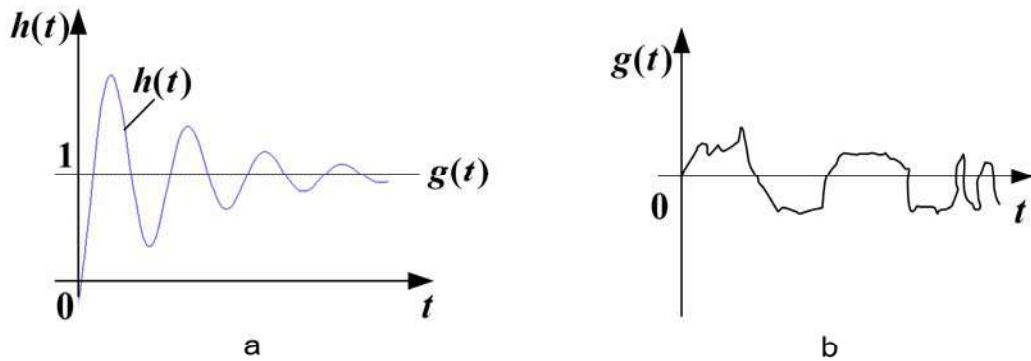


Figure 14.2 – Determined (a) and Random (b) processes

The spectral density characterizes the frequency distribution variance.

To estimate the ACS at the determined actions we can introduce the parameters ε_{ss} , σ , t_s , M , ω_r , γ_m , A_m , ... At the random actions these parameters become pointless.

In this case they apply such parameter, as the **standard error deviation**.

The problem of estimating the ACS accuracy at the random actions includes the following. We get the $W_1(s)$, $W_2(s)$ system transfer functions, the probabilistic characteristics of the setting $g(t)$ and disturbance $n(t)$ actions, $R_g(\tau)$, $S_g(\omega)$, $R_n(\tau)$, $S_n(\omega)$, $R_{ng}(\tau)$, $S_{gn}(\omega)$, $R_{ng}(\tau)$, $S_{ng}(\omega)$. To find the probabilistic characteristics of the output signal $y(t)$ and the error $\varepsilon(t)$: $R_y(\tau)$, $S_y(\omega)$, $S_\varepsilon(\omega)$, σ_ε .

The solution.

According to the superposition property which is possessed by any linear system, we consider the output signal to be the sum of the system responses to the input signals

$$Y(s) = \Phi(s)G(s) + \Phi_n(s)N(s),$$

where

$$\Phi(s) = \frac{W_1(s)W_2(s)}{1 + W_1(s)W_2(s)} \quad (14.11)$$

and

$$\Phi_n(s) = \frac{W_2(s)}{1 + W_1(s)W_2(s)}.$$

Let $L^{-1}\{\Phi(s)\} = k(t)$, $L^{-1}\{\Phi_n(s)\} = k_n(t)$.

We find the correlation characteristic of an output signal

$$R_y(\tau) = M[y(t), y(t+\tau)] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y(t) y(t+\tau) dt, \quad (14.12)$$

$$y(t) = \int_{-\infty}^{\infty} g(t-\lambda)k(\lambda)d\lambda + \int_{-\infty}^{\infty} n(t-\lambda)k_n(\lambda)d\lambda, \quad (14.13)$$

$$y(t+\tau) = \int_{-\infty}^{\infty} g(t+\tau-\eta)k(\eta)d\eta + \int_{-\infty}^{\infty} n(t+\tau-\eta)k_n(\eta)d\eta. \quad (14.14)$$

Let's substitute these expressions into the correlation function (14.12)

$$\begin{aligned} R_y(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} g(t-\lambda)k(\lambda)d\lambda \int_{-\infty}^{\infty} g(t+\tau-\eta)k(\eta)d\eta dt + \\ &+ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} n(t-\lambda)k_n(\lambda)d\lambda \int_{-\infty}^{\infty} n(t+\tau-\eta)k_n(\eta)d\eta dt + \\ &+ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} g(t-\lambda)k(\lambda)d\lambda \int_{-\infty}^{\infty} n(t+\tau-\eta)k_n(\eta)d\eta dt + \\ &+ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} n(t-\lambda)k_n(\lambda)d\lambda \int_{-\infty}^{\infty} g(t+\tau-\eta)k(\eta)d\eta dt \quad (14.15) \end{aligned}$$

and change the order of integration

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t-\lambda)g(t+\tau-\eta)dt \cdot k(\lambda)k(\eta)d\lambda d\eta +$$

$$\begin{aligned}
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T n(t-\lambda)n(t+\tau-\eta)dt \cdot k_n(\lambda)k_n(\eta)d\lambda d\eta + \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t-\lambda)n(t+\tau-\eta)dt \cdot k(\lambda)k_n(\eta)d\lambda d\eta + \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T n(t-\lambda)g(t+\tau-\eta)dt \cdot k_n(\lambda)k(\eta)d\lambda d\eta. \quad (14.16)
\end{aligned}$$

From this follows

$$\begin{aligned}
R_y(\tau) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_g(\lambda+\tau-\eta)k(\lambda)k(\eta)d\lambda d\eta + \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_n(\lambda+\tau-\eta)k_n(\lambda)k_n(\eta)d\lambda d\eta + \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{gn}(\lambda+\tau-\eta)k(\lambda)k_n(\eta)d\lambda d\eta + \\
& + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ng}(\lambda+\tau-\eta)k_n(\lambda)k(\eta)d\lambda d\eta. \quad (14.17)
\end{aligned}$$

If the reference and disturbance inputs are independent, the last two components disappear from the expression (14.17).

Now we shall find the output signal spectral density. To do this we apply the Fourier transformation to the previous expression

$$S_y(\omega) = F\{R_y(\tau)\} = \Phi(j\omega)\Phi(-j\omega)S_g(\omega) + \Phi_n(j\omega)\Phi_n(-j\omega)S_n(\omega) \quad (14.18)$$

or

$$S_y(\omega) = |\Phi(j\omega)|^2 S_g(\omega) + |\Phi_n(j\omega)|^2 S_n(\omega). \quad (14.19)$$

By analogy we obtain the error spectral density

$$S_\varepsilon(\omega) = |\Phi_\varepsilon(j\omega)|^2 S_g(\omega) + |\Phi_n(j\omega)|^2 S_n(\omega). \quad (14.20)$$

If the disturbance $n(t)$ is imposed to the system input,

$$S_{\varepsilon}(\omega) = |\Phi_{\varepsilon}(j\omega)|^2 S_g(\omega) + |\Phi(j\omega)|^2 S_n(\omega). \quad (14.21)$$

Now we find the error standard deviation

$$\sigma_{\varepsilon} = \sqrt{D_{\varepsilon}} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[|\Phi(j\omega)|^2 S_g(\omega) + |\Phi_n(j\omega)|^2 S_n(\omega) \right] d\omega}. \quad (14.22)$$

If the spectral density $S_{\varepsilon}(\omega)$ is presented in the form of

$$S_{\varepsilon}(\omega) = \frac{b_0(j\omega)^{2n-2} + b_1(j\omega)^{2n-4} + \dots + b_{n-1}}{\left| a_0(j\omega)^n + a_1(j\omega)^{n-1} + \dots + a_n \right|^2}, \quad (14.23)$$

the dispersion $D_{\varepsilon} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\varepsilon}(\omega) d\omega$ corresponds to the standard integral

$$D_{\varepsilon} = \frac{(-1)^{n-1} \det N_n}{2a_0 \det M_n}, \quad (14.24)$$

where

$$N_n = \begin{vmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \\ a_0 & a_2 & a_4 & \dots & \mathbf{0} \\ \mathbf{0} & a_1 & a_3 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & a_n \end{vmatrix} \quad \begin{array}{l} \text{- Hurwitz's matrix, in} \\ \text{which instead of first row} \\ \text{factors } b_0, b_1, \dots, b_{n-1} \text{ are;} \end{array}$$

$$M_n = \begin{vmatrix} a_1 & a_3 & \dots & \mathbf{0} \\ a_0 & a_2 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & a_n \end{vmatrix} \quad \text{- Hurwitz's matrix.}$$

Thus, for $n=1$ we get $D = \frac{b_0}{2a_0a_1}$.

For $n=2$ we get $D = \frac{a_0b_1 - b_0a_2}{2a_0a_1a_2}$.

For $n=3$ we get $D = \frac{a_2a_3b_0 - a_0a_3b_1 + a_0a_1b_2}{2a_0a_3(a_1a_2 - a_0a_3)}$.

"The ACS Designing Atlas" by Topcheev J.I. gives the formulae to provide the calculation of this integral up to $n = 10$.

14.2. The example of the ACS accuracy estimation at the random input actions

We have the aerial target tracking follow-up system (Fig.14.3) and the probabilistic characteristics (Fig 14.4):

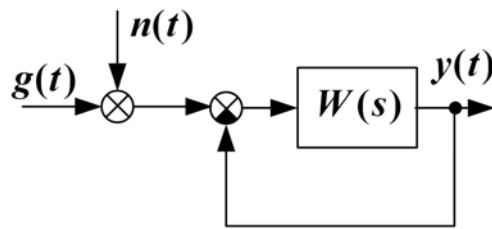


Figure 14.3 – Tracking system

$R_g(\tau) = De^{-\alpha\tau}$ is a correlation function of the useful signal,

$S_g(\omega) = \frac{2D\alpha}{\alpha^2 + \omega^2}$ is a spectral density of the useful signal,

$R_n(\tau) = c^2\delta(\tau)$ is a correlation function of the noise,

$S_n(\omega) = c^2$ is a spectral density of the a white noise.

We have

$$M[g(t)] = 0, \quad R_{gn}(\tau) = R_{ng}(\tau) = 0, \quad \varepsilon = g - y,$$

$$M[n(t)] = 0, \quad S_{gn}(\omega) = S_{ng}(\omega) = 0, \quad W(s) = \frac{k}{s(Ts + 1)}.$$

We should find the mathematical expectation, the $\varepsilon(t)$ system error signal dispersion at the steady-state mode and the σ_ε error standard deviation.

The solution.

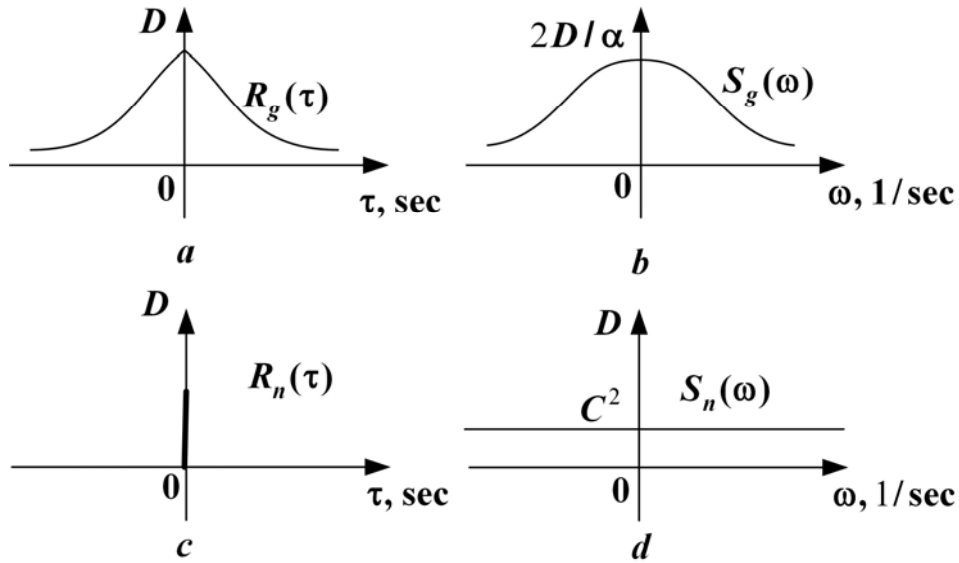


Figure 14.4 – Probabilistic characteristics

We find the Laplace image of the error $E(s)$

$$E(s) = \Phi_{\varepsilon}(s)G(s) - \Phi_n(s)N(s) = \frac{1}{1+W(s)}G(s) - \frac{W(s)}{1+W(s)}N(s) = E_g(s) + E_n(s).$$

If $M[g(t)] = M[n(t)] = 0$, then $M[\varepsilon(t)] = 0$.

Then we find the signal variance $\varepsilon_g(t)$

$$D_{\varepsilon g} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\varepsilon g}(\omega) d\omega,$$

where

$$S_{\varepsilon g}(\omega) = |\Phi_{\varepsilon}(j\omega)|^2 S_g(\omega),$$

or

$$S_{\varepsilon g}(\omega) = \left| \frac{j\omega(Tj\omega + 1)}{j\omega(Tj\omega + 1) + k} \right|^2 \cdot \frac{2D\alpha}{\alpha^2 + \omega^2} = \frac{2D\alpha\omega^2(T^2\omega^2 + 1)}{|j\omega(Tj\omega + 1) + k|^2 \cdot |\alpha + j\omega|^2}.$$

We can present the error spectral density $S_{\varepsilon g}(\omega)$ caused by the reference action $\varepsilon_g(t)$ as

$$S_{\varepsilon g}(\omega) = \frac{2D\alpha \left[T^2(j\omega)^4 - (j\omega)^2 \right]}{\left| T(j\omega)^3 + (\alpha T + 1)(j\omega)^2 + (\alpha + k)(j\omega) + \alpha k \right|^2} = \frac{2D\alpha \left[b_0(j\omega)^4 + b_1(j\omega)^2 \right]}{\left| a_0(j\omega)^3 + a_1(j\omega)^2 + a_2(j\omega) + a_3 \right|^2},$$

where

$$b_0 = T^2, b_1 = -1, b_2 = 0, a_0 = T, a_1 = \alpha T + 1, a_2 = \alpha + k, a_3 = \alpha k, n = 3.$$

Applying the standard integral we obtain

$$\begin{aligned} D_{\varepsilon g} &= \frac{a_2 a_3 b_0 - a_0 a_3 b_1 + a_0 a_1 b_2}{2a_0 a_3 (a_1 a_2 - a_0 a_3)} 2D\alpha = \frac{2D\alpha \left[(\alpha + k)\alpha k T^2 + T\alpha k + T(\alpha T + 1) \cdot 0 \right]}{2T\alpha k \left[(\alpha T + 1)(\alpha + k) - T\alpha k \right]} = \\ &= \frac{D\alpha(1 + \alpha T + Tk)}{\alpha^2 T + \alpha + k}. \end{aligned}$$

The error variance caused by the disturbance is calculated by analogy

$$\begin{aligned} S_{\varepsilon n}(\omega) &= |\Phi(j\omega)|^2 S_n(\omega) = \frac{k^2}{|j\omega(Tj\omega + 1) + k|^2} c^2 = \\ &= \frac{k^2 c^2}{|T(j\omega)^2 + j\omega + k|^2} = \frac{b_1}{|a_0(j\omega)^2 + a_1(j\omega) + a_2|^2}, \end{aligned}$$

where $b_0 = 0, b_1 = k^2 c^2, a_0 = T, a_1 = 1, a_2 = k, n = 2$.

$$\text{At } n = 2 \text{ we obtain } D_{\varepsilon n} = \frac{a_0 b_1 - b_0 a_2}{2a_0 a_1 a_2} = \frac{T k^2 c^2}{2Tk} = \frac{kc^2}{2}.$$

As the signals $g(t)$ and $n(t)$ are noncorrelated, then the system error variance D_{ε} is equal to the sum of the signal variances ε_g and ε_n ,

$$D_{\varepsilon} = \frac{D\alpha(1 + \alpha T + kT)}{\alpha^2 T + \alpha + k} + \frac{kc^2}{2}.$$

The error standard deviation (ESD) is

$$\sigma_{\varepsilon} = \sqrt{D_{\varepsilon}} = \sqrt{\frac{D\alpha(1 + \alpha T + kT)}{\alpha^2 T + \alpha + k} + \frac{kc^2}{2}}.$$

To reduce the ESD ε_g we should increase k .

To reduce the ESD ε_g we should reduce k .

To minimize the full error standard deviation we choose k_{opt} but thus we get the problem of the ESD optimization (Fig. 14.5).

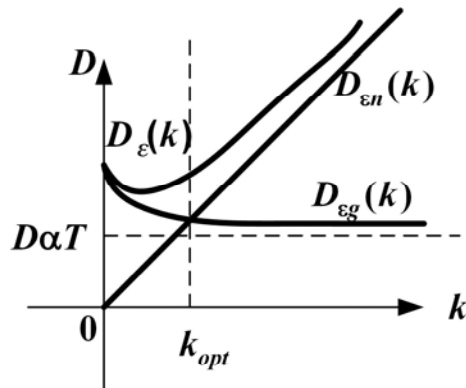


Figure 14.5 – The ACS dispersion as a function of the gain

Terms:

- random process,
- centered random process,
- mathematical expectation,
- distribution function,
- distribution density,
- correlation function,
- stationary random process,
- variance (dispersion),
- standard deviation,
- spectral density.

Comprehension questions

1. How can we estimate the output signal spectral density?
2. How can we estimate the error spectral density?
3. How can we estimate the output signal standard deviation?
4. How can we estimate the error standard deviation?

Lecture №15

THE CONNECTION BETWEEN THE LINEAR ACS FREQUENCY AND TIME CHARACTERISTICS

The planet is a cradle of the human mind but it is impossible to live in a cradle eternally.

K. E. Tsiolkovsky

Lecture outline:

1. The connection between the open and closed-loop ACS frequency characteristics.
2. The calculation of the ACS time characteristics by its frequency characteristics.
3. The estimation of the ACS step response properties by the closed-loop system RFC.

15.1. The connection between the open and closed-loop ACS frequency characteristics

Let us consider the closed-loop system (Fig. 15.1).

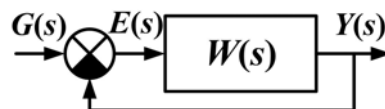


Figure 15.1 – Closed-loop system

The input and output are connected by the equation

$$\Phi(s) = \frac{W(s)}{1+W(s)} = \frac{Y(s)}{G(s)}, \quad (15.1)$$

$$Y(s) = \Phi(s)G(s), \quad Y(j\omega) = \Phi(j\omega)G(j\omega).$$

Thus, the closed-loop system AFFC can be constructed by the given open-loop system AFFC (Fig. 15.2.), having applied the formula

$$\Phi(j\omega) = \frac{W(j\omega)}{1+W(j\omega)}. \quad (15.2)$$

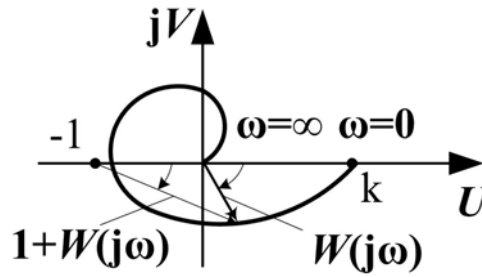


Figure 15.2 – The open-loop system AFCC

At $\omega = 0$ we obtain

$$\Phi(0) = \begin{cases} \frac{k}{1+k}, & \text{if } \nu = 0 \text{ (static system),} \\ 1, & \text{if } \nu \neq 0 \text{ (astatic system).} \end{cases} \quad (15.3)$$

At $\omega \neq 0$ $|W(j\omega)|$ is divided by $|1+W(j\omega)|$ and a phase of the vector $1+W(j\omega)$ is subtracted from a phase of the vector $W(j\omega)$.

The closed-loop system AFCC can be constructed in a different way

$$\begin{aligned} \Phi(j\omega) &= \frac{U(\omega) + jV(\omega)}{1 + U(\omega) + jV(\omega)} = \frac{U(\omega)(1 + U(\omega)) + V^2(\omega)}{(1 + U(\omega))^2 + V^2(\omega)} + j \frac{V(\omega)}{(1 + U(\omega))^2 + V^2(\omega)} = \\ &= P(\omega) + jQ(\omega) = |\Phi(j\omega)| e^{j\psi(\omega)}, \end{aligned} \quad (15.4)$$

where

$$P(\omega) = |\Phi(j\omega)| \cos \psi(\omega);$$

$$Q(\omega) = |\Phi(j\omega)| \sin \psi(\omega);$$

$$\psi(\omega) = \operatorname{arctg} \frac{Q(\omega)}{P(\omega)}.$$

At $\omega = 0$ we obtain $\Phi(0) = P(0)$, $Q(0) = 0$, $\psi(0) = 0$.

At $\omega = \infty$ we get $P(\infty) = Q(\infty) = 0$, at $m < n$ $\psi(\infty) = (m - n) \cdot \pi / 2$.

The examples of the frequency characteristics are shown in Fig. 15.3.

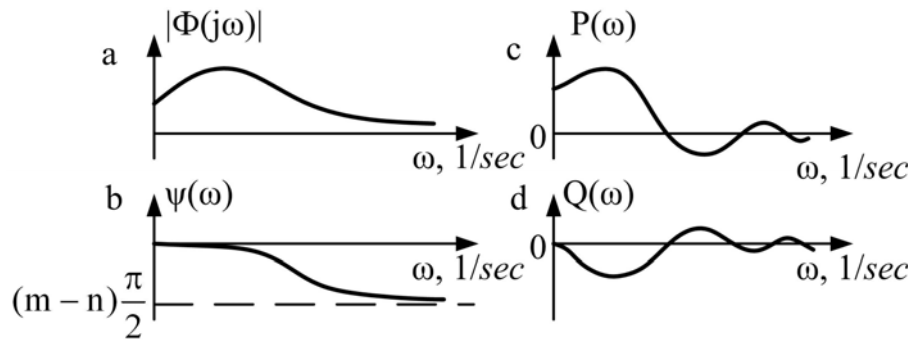


Figure 15.3 – Frequency characteristics of the closed-loop system

15.2. The calculation of the ACS time characteristics by its frequency characteristics

The system output signal can be calculated by its image applying the inverse Laplace transformation

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{\Phi(s)G(s)\} \text{ as } Y(s) = \Phi(s)G(s). \quad (15.5)$$

At $g(t) = \delta(t)$ we get $G(s) = 1$ and $y(t) = L^{-1}\{\Phi(s)\} = h'(t)$.

At $g(t) = 1(t)$ we obtain $G(s) = \frac{1}{s}$ and $y(t) = L^{-1}\left\{\frac{\Phi(s)}{s}\right\} = h(t)$.

The result of the inverse Laplace transformation depends on the poles of the image $Y(s)$ which can lie both in the left and in the right half plane (Fig. 15.4). We present $Y(s)$ as the sum of two components, $Y_r(s)$ being the regular component (the poles lie only in left the half plane), and $Y_n(s)$ being the irregular component (the poles lie on the imaginary axis and in the right half plane): $Y(s) = Y_r(s) + Y_n(s)$.

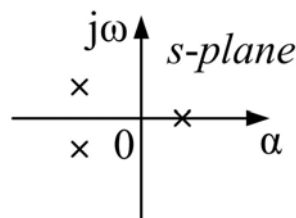


Figure 15.4 – An s-plane pole plot

Correspondingly the original consists of two components

$$y(t) = y_r(t) + y_n(t).$$

The inverse Fourier transformation can be applied to the regular component $Y_r(s)$, and the Laplace transformation can be applied to the irregular one

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_r(j\omega) e^{j\omega t} d\omega + \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} Y_n(s) e^{st} ds. \quad (15.6)$$

Let us consider the regular component $Y_r(j\omega) = R(\omega) + jS(\omega)$, where
 $R(\omega) = \text{Re}[Y_r(j\omega)]$, $S(\omega) = \text{Im}[Y_r(j\omega)]$.

We obtain

$$\begin{aligned} y_r(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [R(\omega) + jS(\omega)] e^{j\omega t} d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [R(\omega) + jS(\omega)] (\cos\omega t + j\sin\omega t) d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{[R(\omega)\cos\omega t - S(\omega)\sin\omega t]}_{\text{even function}} d\omega + \\ &+ j \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{[R(\omega)\sin\omega t + S(\omega)\cos\omega t]}_{\text{odd function}} d\omega = \\ &= \frac{1}{\pi} \int_0^{\infty} [R(\omega)\cos\omega t - S(\omega)\sin\omega t] d\omega. \end{aligned} \quad (15.7)$$

At $t < 0$

$$y_r(t) = 0. \quad y_r(t) = 0 = \frac{1}{\pi} \int_0^{\infty} [R(\omega)\cos\omega t + S(\omega)\sin\omega t] d\omega. \quad (15.8)$$

By summing and subtracting two last expressions we obtain

$$y_r(t) = \frac{2}{\pi} \int_0^{\infty} R(\omega)\cos\omega t d\omega \quad \text{or} \quad y_r(t) = -\frac{2}{\pi} \int_0^{\infty} S(\omega)\sin\omega t d\omega. \quad (15.9)$$

Thus, there exists the unequivocal communication of the output signal with the ACS frequency characteristics for minimal-phase systems.

Let us find the pulse transitive characteristic $h'(t)$. For this we put $g(t) = \delta(t)$. Thus $G(s) = 1$ and $Y(s) = H'(s) = \Phi(s)$.

For the stable system we have $\Phi(s) = \Phi_r(s)$ as all the poles lie in the left half plane. Therefore the irregular component is absent

$$H'(j\omega) = Y(j\omega) = Y_r(j\omega) = \Phi(j\omega) = P(\omega) + jQ(\omega) = R(\omega) + jS(\omega). \quad (15.10)$$

By substituting the values $R(\omega)$ and $S(\omega)$ into expressions for $y_r(t)$ (15.9), we get

$$h'(t) = \frac{2}{\pi} \int_0^{\infty} P(\omega) \cos \omega t d\omega, \quad (15.11)$$

$$h'(t) = -\frac{2}{\pi} \int_0^{\infty} Q(\omega) \sin \omega t d\omega. \quad (15.12)$$

Thus, the system weight characteristic $h'(t)$ can be calculated by these formulae, if the closed-loop system RFC or IFC are given.

The minimal-phase system RFC and IFC cohere by the Hilbert double transformation

$$P(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Q(u)}{u - \omega} du, \quad Q(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(u)}{u - \omega} du. \quad (15.13)$$

To calculate the step response $h(t)$ we put $g(t) = 1(t)$ then $G(s) = 1/s$ and $G(j\omega) = 1/j\omega$. We have

$$H(s) = Y(s) = \Phi(s)G(s) = \Phi(s)/s = Y_r(s) + Y_n(s). \quad (15.14)$$

The image of $G(s) = 1/s$ has a zero pole lying on the imaginary axis. Therefore we have the irregular component. We shall find it.

The steady-state signal value corresponds to the zero pole

$$\lim_{t \rightarrow \infty} y(t) = y_n(t) = \lim_{s \rightarrow 0} Y(s)s = \lim_{s \rightarrow 0} \frac{\Phi(s)}{s} s = \Phi(0) = P(0) = \begin{cases} \frac{k}{k+1} \text{ at } v=0, \\ 1 \text{ at } v \neq 0. \end{cases} \quad (15.15)$$

So $y_n(t) = P(0)$ and $Y_n(s) = P(0)/s$. As $Y(s) = Y_r(s) + Y_n(s)$ then

$$Y_r(s) = Y(s) - Y_n(s) = \frac{\Phi(s)}{s} - \frac{P(0)}{s} = \frac{\Phi(s) - P(0)}{s}. \quad (15.16)$$

We can apply the Fourier transformation to the irregular component

$$Y_r(j\omega) = \frac{\Phi(j\omega) - P(0)}{j\omega} = \frac{P(\omega) + jQ(\omega) - P(0)}{j\omega} = \frac{Q(\omega)}{\omega} + j \frac{P(0) - P(\omega)}{\omega} =$$

$$= R(\omega) + jS(\omega). \quad (15.17)$$

Thus we obtain $R(\omega) = \frac{Q(\omega)}{\omega}$, $S(\omega) = \frac{P(0) - P(\omega)}{\omega}$. We substitute these expressions in the $y_r(t)$ formula (15.9) and we add the irregular component to calculate $h(t)$

$$h(t) = P(0) + \frac{2}{\pi} \int_0^{\infty} \frac{Q(\omega)}{\omega} \cos \omega t \, d\omega, \quad (15.18)$$

$$h(t) = P(0) - \frac{2}{\pi} \int_0^{\infty} \frac{P(0) - P(\omega)}{\omega} \sin \omega t \, d\omega =$$

$$= P(0) - \frac{2}{\pi} P(0) \int_0^{\infty} \underbrace{\frac{\sin \omega t}{\omega t}}_{\frac{\pi}{2}} \, d\omega t + \frac{2}{\pi} \int_0^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega. \quad (15.19)$$

Thus,

$$h(t) = \frac{2}{\pi} \int_0^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega, \quad (15.20)$$

i.e. the step response is unequivocally determined by the real frequency characteristic (RFC) $P(\omega)$ of the closed-loop system.

15.3. The estimation of the ACS step response properties by the closed-loop system RFC

The formulae

$$h(t) = \frac{2}{\pi} \int_0^{\infty} \frac{P(\omega)}{\omega} \sin \omega t \, d\omega,$$

$$H(s) = \frac{\Phi(s)}{s}.$$

Imply the following properties of the step response:

1) The step response finite value (Fig. 15.5) is equal to the initial RFC value $h(\infty) = P(0)$.

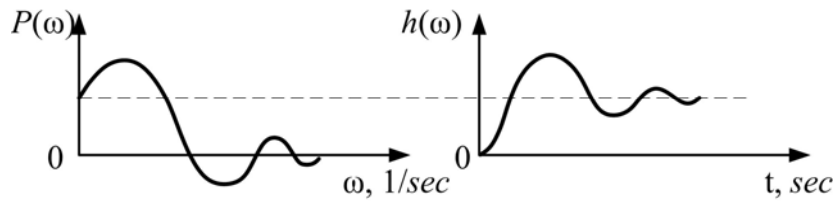


Figure 15.5 – The step response finite value

The proving:

$$h(\infty) = \lim_{t \rightarrow \infty} h(t) = \lim_{s \rightarrow 0} H(s)s = \lim_{s \rightarrow 0} \frac{\Phi(s)s}{s} = \Phi(0) = P(0).$$

2) The RFC finite value is equal to the step response initial value $h(0) = P(\infty) = 0$ at $m < n$.

The proving:

$$h(0) = \lim_{t \rightarrow 0} h(t) = \lim_{s \rightarrow \infty} H(s)s = \lim_{s \rightarrow \infty} \frac{\Phi(s)s}{s} = \Phi(\infty) = P(\infty) \text{ at } m < n, \\ \Phi(\infty) = 0.$$

3) The approximately identical step responses correspond to the approximately identical RFC. It follows from the integral property to smooth functions. Under this property we can replace the complexed exact RFC by the piecewise linear polyline which integral can be easier calculated (Fig. 15.6).

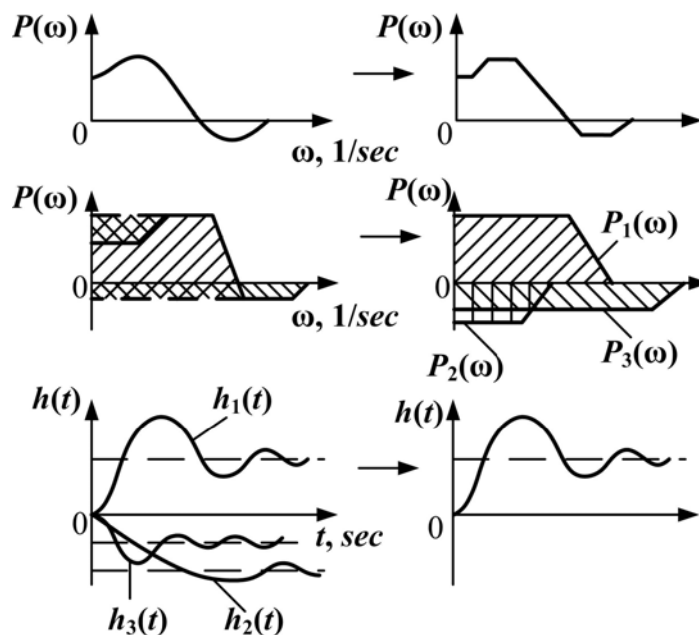


Figure 15.6 – Ratio between $P(\omega)$ and $h(t)$

4) If the RFC is as the sum of several characteristics, then $h(t)$ also becomes the sum of the step responses corresponding to the certain RFC

$$P(\omega) = \sum_{i=1}^m P_i(\omega) \rightarrow h(t) = \sum_{i=1}^m h_i(t).$$

This is implied by the integration linearity which is the basis for the step responses construction method elaborated by V.V.Solodovnikov (trapezium method)

$$P(\omega) = P_1(\omega) + P_2(\omega) + P_3(\omega) \rightarrow h(t) = h_1(t) + h_2(t) + h_3(t).$$

5) If two RFC differ only in the ordinate axis scale, then the corresponding $h(t)$ differ only in the ordinate axis scale as well (Fig. 15.7).

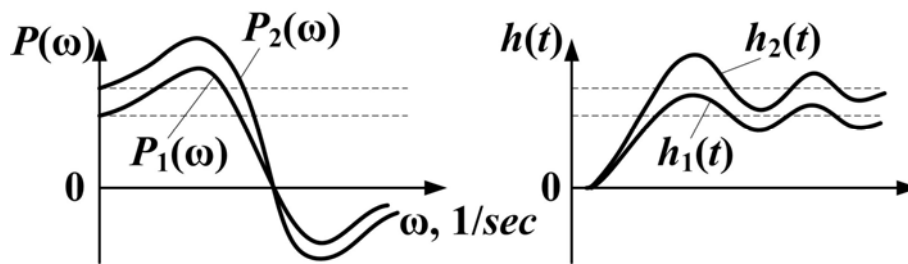


Figure 15.7 – The scaling of the RFC value

$$P_1(\omega), P_2(\omega) = mP_1(\omega),$$

$$h_2(t) = \frac{2}{\pi} \int_0^{\infty} \frac{P_2(\omega)}{\omega} \sin \omega t \, d\omega =$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{mP_1(\omega)}{\omega} \sin \omega t \, d\omega = mh_1(t).$$

6) If two RFC differ only in the abscissa axis scale, then the corresponding $h(t)$ also differ only in the abscissa axis scale (Fig. 15.8). The wider is the RFC, the narrower the $h(t)$ becomes

$$P_1(\omega), P_2(\omega) = P_1(m\omega),$$

$$h_2(t) = \frac{2}{\pi} \int_0^{\infty} \frac{P_2(\omega)}{\omega} \sin \omega t \, d\omega = \frac{2}{\pi} \int_0^{\infty} \frac{P_1(m\omega)}{m\omega} \sin(m\omega \frac{t}{m}) \, dm\omega = \frac{2}{\pi} \int_0^{\infty} \frac{P_1(\lambda)}{\lambda} \sin \lambda \frac{t}{m} \, d\lambda = h_1\left(\frac{t}{m}\right)$$

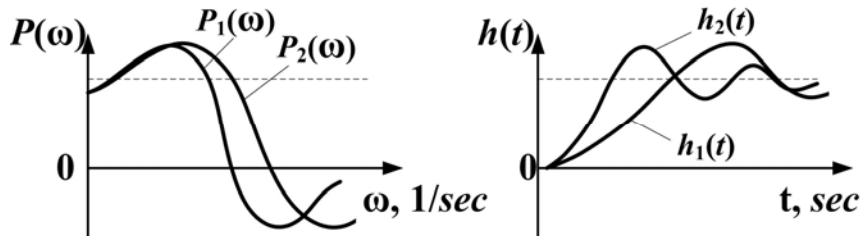


Figure 15.8 – The scaling of the RFC width

The wider is the bandwidth, the swifter is the system. There exist the h - function tables in which the step responses calculated for unit trapeze (Fig. 15.9) are given.

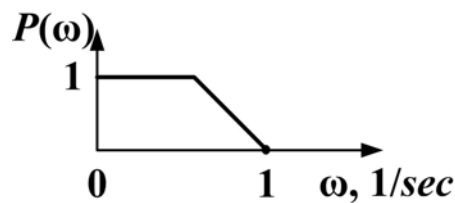


Figure 15.9 – The unit RFC

These characteristics make it possible to calculate the step responses corresponding to the trapezia of any height and width, applying the properties 4 and 5.

7) If RFC is a positive decreasing convex function, i.e. $P(\omega) > 0, \frac{dP(\omega)}{d\omega} < 0, \frac{d^2P(\omega)}{d\omega^2} > 0$, then $h(t)$ is the monotonous function, i.e. $\sigma = 0$ (Fig. 15.10).

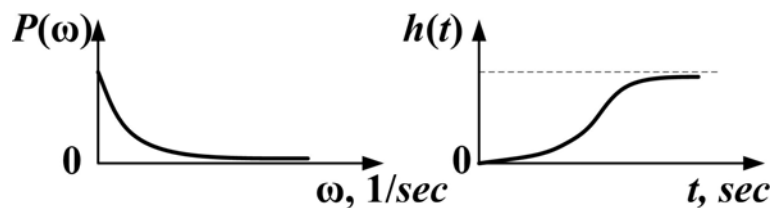


Figure 15.10 – Monotonous $h(t)$

8) If RFC can be replaced by the trapezium, then $\sigma < 18\%$ and $\frac{\pi}{\omega_n} < t_s < \frac{4\pi}{\omega_p}$, where ω_p is the positivity frequency (Fig. 15.11).

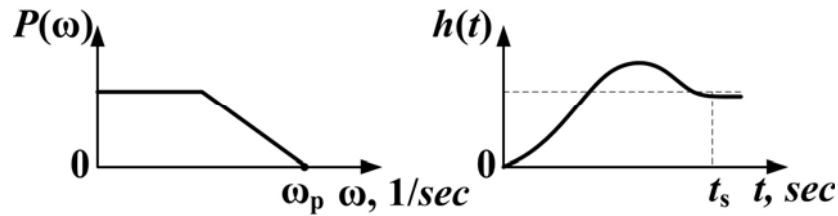


Figure 15.11 – Positive RFS

9) If the positive RFC has the $P_{max} > P(0)$ overshoot, $\sigma = \frac{1.18P_{max} - P(0)}{P(0)} 100\%$. The setting time t_s is calculated by the schedules.

10) If RFC has break at $\omega = 0$, $h(t)$ tends to ∞ (Fig. 15.12). The system lies on the aperiodic stability border.

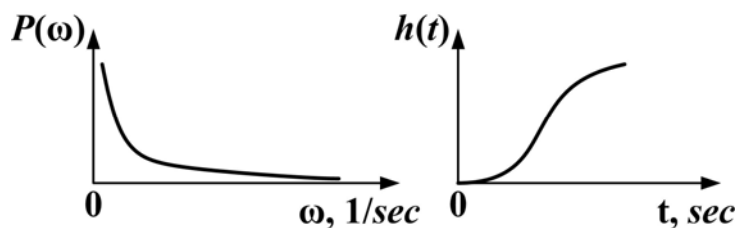


Figure 15.12 – Break at $\omega=0$

11) If RFC has break at $\omega \neq 0$, the system lies on the oscillatory stability border (Fig. 15.13).

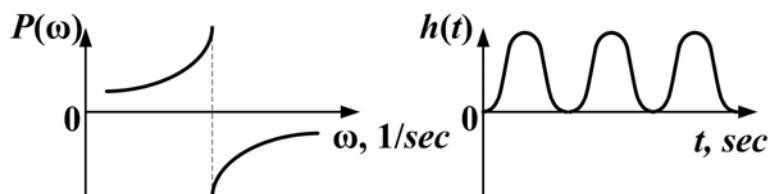


Figure 15.13 – Break at $\omega \neq 0$

Terms:

- real frequency characteristic RFC,
- imaginary frequency characteristic IFC.

Comprehension questions

1. Explain the AFFC connection of the closed-loop and open systems.
2. Calculate the initial and final values of the RFC, IFC, AFC and FFC for the closed-loop system.
3. How can we calculate the step response having the RFC?
4. What properties of the closed-loop system step response can be estimated by the RFC?
5. What is the Solodovnikov method based on?

Lecture № 16**THE ROOT LOCUS METHOD****Lecture outline:**

1. The bases of the root locus method.
2. The example of the root locus building.

16.1. The bases of the root locus method

The root locus is a set of the trajectories of the closed-loop system characteristic equation roots on complex s -plane when one of the system parameters is changed, e.g. $0 \leq k \leq \infty$ or $-\infty \leq k \leq \infty$.

The method has suggested by the Russian scientists Teodorchik K. F., Bendrikov G. A., Strelkov S. P. (1948 – 1949) and the US scientist W.R. Evans (1948).

The root locus method allows to carry out the system analysis and to choose the parameter k , providing the most profitable location of poles.

Let us consider the system (Fig. 16.1).

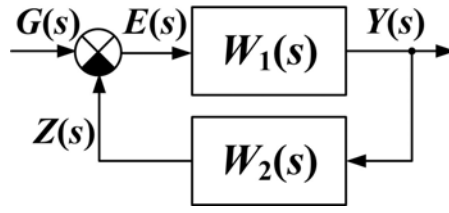


Figure 16.1 – The closed-loop system

The transfer function of this system is

$$\Phi(s) = \frac{Y(s)}{G(s)} = \frac{W_1(s)}{1 + W_1(s)W_2(s)} = \frac{W_1(s)}{1 + W(s)}. \quad (16.1)$$

The open-loop transfer function

$$W(s) = \frac{Z(s)}{G(s)} = W_1(s)W_2(s) = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_n}, \quad m \leq n \quad (16.2)$$

can be presented as

$$W(s) = \frac{B(s)}{A(s)} = \frac{kB_1(s)}{A(s)} = \frac{ka(s - s_{01})(s - s_{02}) \dots (s - s_{0m})}{(s - s_1)(s - s_2) \dots (s - s_n)} = \frac{ka \prod_{i=1}^m (s - s_{0i})}{\prod_{j=1}^n (s - s_j)}, \quad (16.3)$$

where $s_{0i}, i = 1, 2, \dots, m$ are the zeroes of the open-loop system transfer function;

$s_j, j = 1, 2, \dots, n$ are the poles of the open-loop system transfer function.

The closed-loop system characteristic equation is

$$1 + W(s) = 0. \quad (16.4)$$

The roots of this equation $s_k, k = 1, 2, \dots, n$ are the poles of the closed-loop system transfer function. When the parameter k changes, the poles move through the complex s -plane and describe the paths (Fig. 16.2). The set of n of such paths (trajectories) forms root locus.

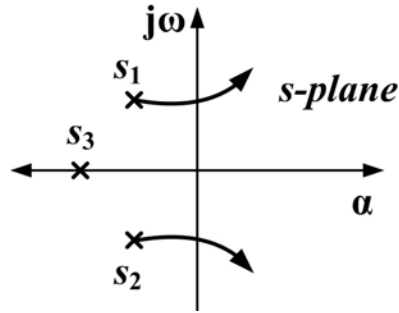


Figure 16.2 – The root locus

From the characteristic equation (16.4) we get $W(s) = -1$. This complex expression can be divided into two expressions

$$|W(s)| = 1, \quad (16.5)$$

$$\arg W(s) = \pm\pi(2i+1), i=0,1,2,\dots \quad (16.6)$$

We write expression for module $|W(s)|$

$$\frac{ka \prod_{i=1}^m |s - s_{0i}|}{\prod_{j=1}^n |s - s_j|} = 1 \text{ or } \frac{ka \prod_{i=1}^m l_{0i}}{\prod_{j=1}^n l_j} = 1, \quad (16.7)$$

where $l_{0i} = |s_k - s_{0i}|, l_j = |s_k - s_j|$ (Fig. 16.3). We select the parameter k from the previous expression

$$k = \frac{\prod_{j=1}^n l_j}{a \prod_{i=1}^m l_{0i}}. \quad (16.8)$$

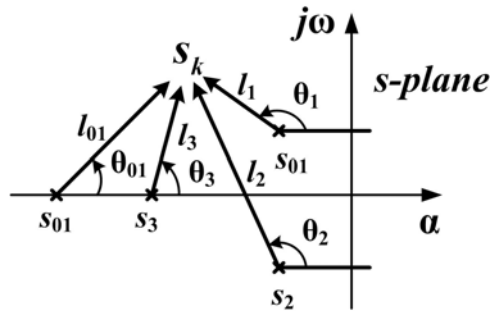


Figure 16.3 – The calculation of the $W(s)$ modulus and phase

Let us write an expression for the vector $W(s)$ argument

$$\arg W(s) = \arg \prod_{i=1}^m (s - s_{0i}) - \arg \prod_{j=1}^n (s - s_j) = \pm(2i+1)\pi, i = 0, 1, 2, \dots$$

This implies

$$\sum_{i=1}^m \theta_{0i} - \sum_{j=1}^n \theta_j = \pm\pi(2i+1), i = 0, 1, 2, \dots \quad (16.9)$$

Any point Δ on s -plane which satisfies the equation (16.9), belongs to the root locus, so it is a pole of the closed-loop system for a certain value of the parameter k . This value is calculated by the formula (16.8).

θ_{0i} is a phase of the vector which is drawn from i -th zero of the open-loop system to the researched point s_k .

θ_j is a phase of the vector which is drawn from j -th pole of the open-loop system to the researched point s_k .

The equation (16.9) is convenient because it doesn't include the value of the parameter k .

The equation (16.9) makes it possible to build the root locus. We choose the most suitable location of the closed-loop system poles and using the equation (16.8) estimate the value of the parameter which provides the required pole locations.

When building the root locus manually, we choose the point Δ on s -plane and check whether it satisfies the equation (16.9). If yes, then this point belongs to the root locus. If no, we move the point horizontally or vertically and check the condition (16.9) again. To build the root locus in the MATLAB you should use the command *rlocus*.

Let us consider the root locus properties, resulting from the equation (16.9) and the characteristic equation.

1. Root locus has n branches, n being the order of the system.
2. The root locus branches are symmetrical against the real axis.
3. The root locus branches at $k = 0$ begin from the poles of the open-loop system.

The closed-loop system characteristic equation is

$$1 + \frac{kB_1(s)}{A(s)} = 0 \text{ or } A(s) + kB_1(s) = 0. \quad (16.10)$$

If $k = 0$ then $A(s) = 0$.

So the closed-loop system characteristic equation at $k = 0$ coincides with the open-loop system characteristic equation. Therefore the poles of the closed-loop system at $k = 0$ coincide with poles of the open-loop system.

$$s_k = s_j, \quad j = 1, 2, \dots, n.$$

So the poles of the open-loop system s_j are the initial points for the root locus of the closed-loop system.

4. When $k = \infty$ m branches of the root locus end in the zeros of the open-loop system.

The equation

$$\frac{A(s)}{k} + B_1(s) = 0, \quad (16.11)$$

follows from the characteristic equation $A(s) + kB_1(s) = 0$, hence at $k = \infty$ we have $B_1(s) = 0$, $s_k = s_{0i}, k = 1, 2, \dots, m$ and the closed-loop system characteristic equation has a roots, coinciding with the zero of the open-loop system, so zeroes of the open-loop system s_{0i} are the finite points of the root locus.

5. Other $n - m$ branches of the root locus at $k = \infty$ go to infinity along the asymptotes.

6. As the branches are symmetrical the asymptote centre lies on the real axis and its coordinates (Fig. 16.4) can be found by the formula

$$a^* = \frac{\sum_{j=1}^n s_j - \sum_{i=1}^m s_{0i}}{n - m}. \quad (16.12)$$

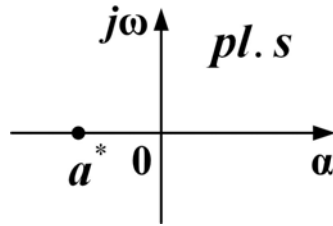


Figure 16.4 – The asymptote centre

7. The asymptotes divergence angle are calculated by the formula

$$\Theta_i = \frac{\pi}{n-m}(2i+1), i = 0, 1, \dots \quad (16.13)$$

8. Any point of the real axis which has the odd number of the zeroes and the poles of the open-loop system on its right belongs to the root locus (Fig. 16.5).

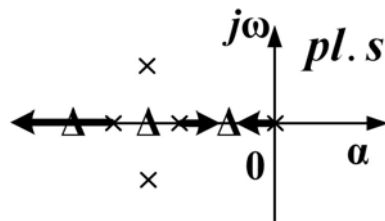


Figure 16.5 – Parts of the real axis, belonging to the root locus

This property follows from the equation (16.9). We choose the parts of the real axis which belong to the root locus, and put the arrows from the pole to the zero.

9. The poles can coincide at some value of the parameter k . Thus appears a multiple point which coordinates can be calculated by the equation (16.9), written for the angle tangents (Fig. 16.6)

$$\sum_{i=1}^m \frac{\omega}{|\alpha - s_{0i}|} - \sum_{j=1}^n \frac{\omega}{|\alpha - s_j|} = 0 \text{ or } \sum_{i=1}^m \frac{1}{|\alpha - s_{0i}|} - \sum_{j=1}^n \frac{1}{|\alpha - s_j|} = 0. \quad (16.14)$$

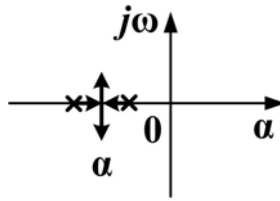


Figure 16.6 – Coincide point α

10. When k increases, the poles disperse from multiple point at the angles

$$\theta_i = \frac{\pi}{2}(2i + 1), i = 0, 1, \dots \quad (16.15)$$

11. The pole angles of departure θ_l from the complex initial points s_j and the branch angles of the entry θ_r to the complex final points s_{0i} are calculated from expressions which follow from the equation (16.9)

$$\theta_l = \sum_{i=1}^m \theta_{0i} - \sum_{\substack{j=1 \\ j \neq l}}^n \theta_j - \pi(2i + 1), \quad \theta_r = \sum_{j=1}^n \theta_j - \sum_{\substack{i=1 \\ i \neq r}}^m \theta_{0i} - \pi(2i + 1). \quad (16.16)$$

12. The intersection of the root locus branches with the imaginary axis of plane s can be calculated by one of the stability criteria.

13. The step response corresponds to the given location of the closed-loop system poles on the root locus

$$h(t) = \frac{B(0)}{D(0)} + \sum_{k=1}^n \frac{B(s_k)}{s_k D'(s_k)} e^{s_k t} = \frac{B(0)}{D(0)} + \sum_{k=1}^n \frac{b_0 \prod_{i=1}^m |s_k - s_{0i}| e^{j \sum_{i=1}^m \theta_{0i}}}{|s_k| e^{j\theta_k} a_0 \prod_{\substack{j=1 \\ j \neq k}}^n |s_k - s_j| e^{j \sum_{j=1}^n \theta_j}} e^{s_k t}. \quad (16.17)$$

16.2. The example of the root locus building

We have the system (Fig.16.7) which transfer functions are

$$W(s) = \frac{k(s+3)}{s(s+4)(s^2+2s+2)} = \frac{kB_1(s)}{A(s)},$$

$$\Phi(s) = \frac{k(s+3)}{s(s+4)(s^2+2s+2)+k(s+3)} = \frac{k(s+3)}{s^4+6s^3+10s^2+(8+k)s+3k}.$$

The closed-loop system characteristic polynomial is

$$D(s) = s^4 + 6s^3 + 10s^2 + (8+k)s + 3k.$$

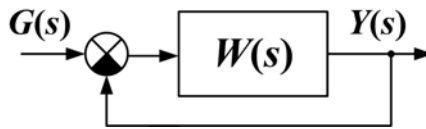


Figure 16.7 – Closed-loop system

The open-loop system characteristic polynomial is

$$A(s) = s(s+4)(s^2+2s+2).$$

We build the root locus for the closed-loop system at the parameter k changing within the range of $0 \leq k \leq \infty$ (Fig. 16.8).

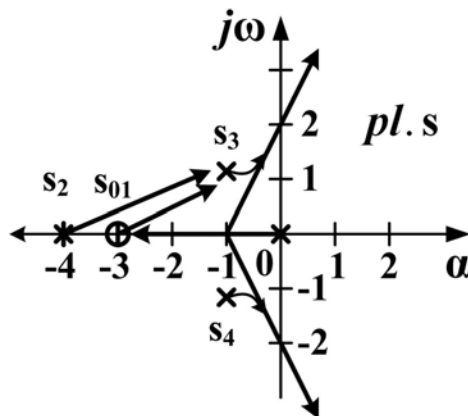


Figure 16.8 – Root locus

We find the initial and finite points of the root locus and put them to the plane s

$$B_1(s) = 0 \rightarrow s_{0i} = -3, \quad A(s) = 0 \rightarrow s_1 = 0, s_2 = -4, s_3 = -1 + j, s_4 = -1 - j.$$

We mark the parts of the real axis α which belong to the root locus. The arrows go from the poles to the zero. Then we find the asymptote star centre coordinate

$$a^* = \frac{\sum_{j=1}^n s_j - \sum_{i=1}^m s_{0i}}{n-m} = \frac{0 + (-4) + (-1 + j) + (-1 - j) - (-3)}{4-1} = -1.$$

From the centre $n-m=3$ asymptotes disperse at the angles $\theta_i^* = \frac{\pi}{n-m}(2i+1), i=0,1,\dots$, at $i=0$ $\theta_0^* = \frac{\pi}{3}$, at $i=1$, $\theta_1^* = \pi$, at $i=2$, $\theta_2^* = \frac{5\pi}{3}$. At the further growth of i the angles will repeat. The angles of the root locus branches coming out of the complex initial points s_3 and s_4 can be calculated by the formula

$$\theta_3 = \sum_{i=1}^m \theta_{0i} - \sum_{\substack{j=1, \\ j \neq 3}}^n +\pi = 25^\circ - 135^\circ - 20^\circ - 90^\circ + 180^\circ = -40^\circ, \quad \theta_4 = 40^\circ.$$

We form the Hurwitz matrix and the equation of the oscillatory stability border

$$\begin{vmatrix} 6 & 8+k & 0 & \cdot & 0 \\ 1 & 10 & 3k & \cdot & 0 \\ 0 & 6 & 8+k & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 10 & \cdot & 3k \end{vmatrix},$$

$$\Delta_{T-1} = \Delta_3 = 0,$$

$$-3k(6 \cdot 6) + (8+k)(6 \cdot 10 - (8+k)) = 0 \rightarrow k = 5.95.$$

We substitute the obtained 'k' to the polynomial $D(j\omega)$

$$D(j\omega) = (j\omega)^4 + 6(j\omega)^3 + 10(j\omega)^2 + 13.95j\omega + 17.85 = 0$$

And find $\omega = 1.525$, $s_1^* = -1.87$, $s_2^* = -4.13$, $s_3^* = j1.525$, $s_4^* = -j1.525$.

Terms:

- root locus,
- asymptote star,
- multiple point.

Comprehension questions

1. What are properties of the root locus?
2. What is the root locus procedure?
3. Where do the initial points of the root locus lie?
4. Where do the finite points of the root locus lie?
5. Where does the star of the asymptotes lie?
6. How can we find the center of the asymptote star?
7. How can we calculate the angles of the asymptotes?

Lecture № 17

THE SENSITIVITY OF THE AUTOMATIC CONTROL SYSTEM

The sensitivity is the system attribute which describes the changes of its output, transfer function, the gain, a state vector, the quality criterion under the influence of the parametric variation and the changes of characteristic and environment conditions.

The ACS parametric variation can cause the loss of the system stability and serviceability.

Let us consider the process described by the transfer function $W_{pr}(s)$ (Fig. 17.1).

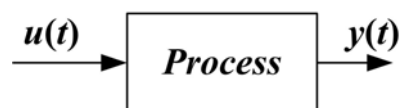


Figure 17.1 – The process

The output image in the open-loop system is

$$Y(s) = W_{pr}(s) \cdot U(s). \quad (17.1)$$

The parameters and the transfer function $W_{pr}(s)$ change under the influence of the environmental alterations and the operational component deterioration. Let the transfer function possess the value

$$W'_{pr}(s) = W_{pr}(s) + \Delta W(s). \quad (17.2)$$

Then the output image is equal

$$Y(s) = (W_{pr}(s) + \Delta W(s)) \cdot U(s). \quad (17.3)$$

We get the output image deviation

$$\Delta Y(s) = Y'(s) - Y(s) = \Delta W(s) \cdot U(s). \quad (17.4)$$

Thus, the parametric variation directly affects the output change. Let us close the system (Fig. 17.2).

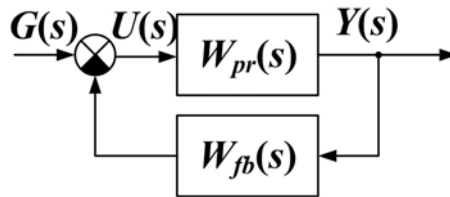


Figure 17.2 – The closed-loop system

At the nominal parameters the output image has the form

$$Y(s) = \frac{W_{pr}(s)}{1 + W_{pr}(s) \cdot W_{fb}(s)} \cdot G(s). \quad (17.5)$$

And after the parametric variation we obtain

$$Y'(s) = \frac{W_{pr}(s) + \Delta W(s)}{1 + (W_{pr}(s) + \Delta W(s)) \cdot W_{fb}(s)} \cdot G(s). \quad (17.6)$$

Let us find the image of the output changes at

$$1 + W_{pr}(s)W_{fb}(s) \gg 1,$$

$$\begin{aligned}
\Delta Y(s) &= Y'(s) - Y(s) = \\
&= \left(\frac{W_{pr}(s) + \Delta W(s)}{1 + (W_{pr}(s) + \Delta W(s)) \cdot W_{fb}(s)} - \frac{W_{pr}(s)}{1 + W_{pr}(s) \cdot W_{fb}(s)} \right) \cdot G(s) = \\
&= \frac{W_{pr}(s) + \Delta W(s) + W_{pr}^2(s) \cdot W_{fb}(s) + \Delta W(s) \cdot W_{pr}(s) \cdot W_{fb}(s) - W_{pr}(s) -}{(1 + W_{pr}(s) \cdot W_{fb}(s))^2} \\
&\frac{-W_{pr}^2(s)W_{fb}(s) - \Delta W(s) \cdot W_{pr}(s) \cdot W_{fb}(s)}{(1 + W_{pr}(s) \cdot W_{fb}(s))^2} \cdot G(s) = \frac{\Delta W(s)}{(1 + W_{pr}(s) \cdot W_{fb}(s))^2} \cdot G(s). \quad (17.7)
\end{aligned}$$

To compare the qualities of closed- and open-loop systems we find relative changes of their output images.

For the open-loop system we get

$$\frac{\Delta Y(s)}{Y(s)} = \frac{\Delta W(s) \cdot U(s)}{W_{pr}(s) \cdot U(s)} = \frac{\Delta W(s)}{W_{pr}(s)}. \quad (17.8)$$

For the closed-loop system we obtain

$$\frac{\Delta Y(s)}{Y(s)} = \frac{\Delta W(s) \cdot G(s) \cdot (1 + W_{pr}(s) \cdot W_{fb}(s))}{(1 + W_{pr}(s) \cdot W_{fb}(s))^2 \cdot W_{pr}(s) \cdot G(s)} = \frac{\Delta W(s) / W_{pr}(s)}{(1 + W_{pr}(s) \cdot W_{fb}(s))}. \quad (17.9)$$

Thus, the relative change of the output image for the closed-loop system is $(1 + W_{pr}(s) \cdot W_{fb}(s))$ times less than for the open-loop system.

So the closed-loop system is less sensitive to the parametric variations than the open-loop system.

To reduce the closed-loop system sensitivity to the parametric variation we the following condition be satisfied in the whole range of the operating frequencies

$$(1 + W_{pr}(s) \cdot W_{fb}(s)) \gg 1. \quad (17.10)$$

How can the sensitivity be estimated in terms of quantity?

The sensitivity function V was first formally determined in 1945 by the American researcher H.W. Bode in order to estimate the influence of the resistor R resistance value on the amplifier gain k

$$V_R^K = \frac{\partial K}{\partial R}. \quad (17.11)$$

The sensitivity function can become the quantitative estimation of the ACS as the sensitivity function is a particular derivative of the observable quality parameter J at the parameter p , which changes under the environmental influence

$$V_p^J = \frac{\partial J}{\partial p}. \quad (17.12)$$

Along with the sensitivity function V the relative sensitivity S determined by the formula

$$S_p^J = \frac{\Delta J/J}{\Delta p/p} \quad (17.13)$$

can be used.

At small deviations of the parameter p and at the transition to the limits we obtain

$$S_p^J = \frac{\partial J/p}{\partial p/J}. \quad (17.14)$$

Now we calculate the sensitivity of the open-loop system transfer function to the parametric variation

$$S_{W_{pr}}^{W_{pr}} = \frac{\partial W_{pr}(s) \cdot W_{pr}(s)}{\partial W_{pr}(s) \cdot W_{pr}(s)} = 1. \quad (17.15)$$

Thus, all the open-loop system parametric variations affect its output directly. To compensate the open-loop system output variations we should know its parameters (both old and new) precisely.

Now we calculate the sensitivity of the closed-loop system transfer function

$$\Phi(s) = \frac{W_{pr}(s)}{1 + W_{pr}(s) \cdot W_{fb}(s)} \quad (17.16)$$

to the parametric variations $S_{W_{pr}}^\Phi$ by the definition

$$S_{W_{pr}}^\Phi = \frac{\partial \Phi(s)/\Phi(s)}{\partial W_{pr}(s)/W_{pr}(s)} =$$

$$\begin{aligned}
&= \frac{\left(1 + W_{pr}(s) \cdot W_{fb}(s) - W_{fb}(s) \cdot W_{pr}(s)\right) \cdot W_{pr}(s) \cdot \left(1 + W_{pr}(s) \cdot W_{fb}(s)\right)}{\left(1 + W_{pr}(s) \cdot W_{fb}(s)\right)^2 \cdot W_{pr}(s)} = \\
&= \frac{1}{1 + W_{pr}(s) \cdot W_{fb}(s)}. \tag{17.17}
\end{aligned}$$

We can see from the last expression, that the closed-loop system sensitivity $S_{W_{pr}}^{\Phi}$ can be reduced in comparison with the open-loop system sensitivity $S_{W_{pr}}^{W_{pr}}$ by increasing $W_{pr}(s), W_{fb}(s)$ in the whole range of the operating frequencies. The process parameter values can be determined less precisely for the closed-loop system.

Let us calculate the closed-loop system sensitivity to the change of the feedback transfer function

$$S_{W_{fb}}^{\Phi} = \frac{\partial \Phi(s) / \Phi(s)}{\partial W_{fb}(s) / W_{fb}(s)} \approx -1. \tag{17.18}$$

The feedback changes affect the output changes directly. Thus it is very important to use such feedback components which change little during the utilization.

If transfer function is submitted as a fraction

$$\Phi(s) = \frac{B(s)}{D(s)}, \tag{17.19}$$

then the sensitivity to the p parametric variation can be calculated as follows

$$\begin{aligned}
S_p^{\Phi} &= \frac{\partial \Phi(s) / \Phi(s)}{\partial p(s) / p(s)} = \frac{\partial \ln \Phi(s, p)}{\partial \ln p} = \\
&= \frac{\partial \ln B(s, p)}{\partial \ln p} - \frac{\partial \ln D(s, p)}{\partial \ln p} = S_p^B - S_p^D. \tag{17.20}
\end{aligned}$$

Example 1

We have the inverting amplifier based on the operation amplifier (OA) (Fig. 17.3).

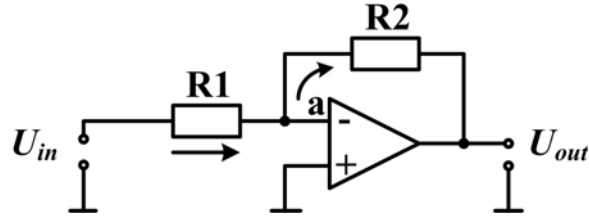


Figure 17.3 – The inverting amplifier

The gain of the modern operational amplifier is $A \geq 10^4$.

The OA input current is negligibly small because of the high input resistance. Therefore for the inverting input of OA we can write down

$$\frac{U_{in} - U_a}{R_1} = \frac{U_a - U_{out}}{R_2}, \quad U_{out} = -AU_a.$$

We exclude the potential of a point U_a from these equations

$$\frac{U_{in}}{R_1} + \frac{U_{out}}{A \cdot R_1} = -\frac{U_{out}}{A \cdot R_2} - \frac{U_{out}}{R_2}.$$

The output signal can be expressed in terms of the input

$$R_2 \cdot U_{out} + R_1 \cdot U_{out} + A \cdot R_1 \cdot U_{out} = -A \cdot R_2 \cdot U_{in}$$

or

$$U_{out} = -\frac{A \cdot R_2}{R_1 + R_2 + A \cdot R_1} \cdot U_{in} \rightarrow U_{out} \approx -\frac{R_2}{R_1} \cdot U_{in},$$

$$U_{out} = -\frac{A \cdot \frac{R_2}{R_1 + R_2}}{1 + \frac{A \cdot R_1}{R_1 + R_2}} \cdot U_{in}, \quad \Phi = -\frac{A \cdot R_2}{R_1 + R_2 + A \cdot R_1}.$$

The equivalent block diagram of the inverting amplifier looks like (Fig. 17.4).

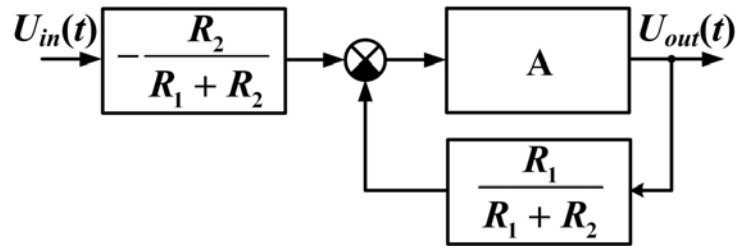


Figure 17.4 – The equivalent diagram

We can calculate the system sensitivity to the parameter A at

$$A = 10^4; R_1 = 0,1 \text{ MOhm}; R_2 = 1 \text{ MOhm}.$$

$$S_A^\Phi = \frac{\partial\Phi/\Phi}{\partial A/A} = \frac{1}{1 + W_{pr} \cdot W_{fb}} =$$

$$= \frac{1}{1 + \frac{A \cdot R_1}{R_1 + R_2}} = \frac{1}{1 + \frac{10^4 \cdot 10^5}{10^5 + 10^6}} \approx 0,001.$$

Thus, the inverting amplifier is almost tolerant to the change of the parameter A .

We calculate the system sensitivity to the change of the feedback element R_2 by the formula

$$S_{R_2}^\Phi = \frac{\partial\Phi/\Phi}{\partial R_2/R_2} = \frac{(-A \cdot (R_1 + R_2 + A \cdot R_1) + A \cdot R_2) R_2}{(R_1 + R_2 + A \cdot R_1)^2 \cdot (-A \cdot R_2)} = \frac{R_1 + A \cdot R_2}{R_1 + R_2 + A \cdot R_1} \approx 1.$$

The feedback resistance R_2 change affect the inverting amplifier output change directly.

The concept of sensitivity is now applied to the ACS analysis, identification and synthesis. We can consider the sensitivity of the output, the transfer function, the quality criterion to the parametric, disturbance and controlled variations.

We form the sensitivity equation to the change of the parameter p for the system

$$\dot{x} = f(x, u, t, p). \quad (17.21)$$

We introduce the sensitivity function $V_p^x = \frac{\partial x}{\partial p}$ and differentiate the state equation of system by the parameter p

$$\frac{\partial dx}{\partial p \cdot dt} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial p} + \frac{\partial f}{\partial p}, \quad (17.22)$$

$$\dot{V}_p^x = \frac{\partial f}{\partial x} \cdot V + \frac{\partial f}{\partial p}. \quad (17.23)$$

From this equation we can find V .

We write down the scalar form of the system equation

$$\dot{x}_i = f_i(x, u, t, p), i = 1, 2, \dots, n. \quad (17.24)$$

Then we introduce the sensitivity function to the vector parameter

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \dots \\ p_m \end{bmatrix}, V_{p_k}^{x_i} = \frac{\partial x_i}{\partial p_k}, i = 1, 2, \dots, n; k = 1, 2, \dots, m. \quad (17.25)$$

The scalar form of the sensitivity equation is the following

$$\dot{V}_{p_k}^{x_i} = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} V_{p_k}^{x_j} + \frac{\partial f_i}{\partial p_k}, i = 1, 2, \dots, n; k = 1, 2, \dots, m. \quad (17.26)$$

These equations are linear against the sensitivity function $V_{p_k}^{x_i}$. By solving them we obtain the sensitivity function of any state variable x_i ($i = 1, 2, \dots, n$) to any parameter p_k ($k = 1, 2, \dots, m$).

For linear systems we have

$$\dot{x} = A \cdot x + B \cdot u, \dot{V}_p^x = A \cdot V_p^x + \frac{\partial A}{\partial p} x + \frac{\partial B}{\partial p} u. \quad (17.27)$$

Or in the scalar form

$$\dot{x}_i = \sum_{j=1}^n a_{ij} x_j + \sum_{l=1}^r b_{il} u_l, \dot{V}_{p_k}^{x_i} = \sum_{j=1}^n a_{ij} V_{p_k}^{x_j} + \sum_{j=1}^n \frac{\partial a_{ji}}{\partial p_k} x_j + \sum_{l=1}^r \frac{\partial b_{il}}{\partial p_k} u_l. \quad (17.28)$$

Example 2.

To find the output sensitivity function to the change of parameters k and T for the object which has the transfer function

$$W(s) = \frac{Y(s)}{U(s)} = \frac{k}{T_s + 1}.$$

We write the input-output equation

$$T \frac{dy}{dt} + y(t) = kU(t).$$

Then $\frac{dy}{dt} = -\frac{1}{T}y(t) + \frac{k}{T}u(t)$ or $\dot{x} = -\frac{1}{T}x + \frac{k}{T}u$, $A = -\frac{1}{T}$; $B = \frac{k}{T}$.

We introduce two sensitivity functions

$$V_k^x = V_k^y = \frac{\partial y}{\partial k} = \frac{\partial x}{\partial k}, \quad V_T^x = V_T^y = \frac{\partial y}{\partial T} = \frac{\partial x}{\partial T}.$$

We differentiate the object equation by the parameter k
 $\dot{V}_k^x = -\frac{1}{T}V_k^x + \frac{1}{T}U$ and get $V_k^x(t) = \left(1 - e^{-\frac{t}{T}}\right)U_0$ at $U(0) = U_0$.

We differentiate the object equation on the parameter T :

$$\dot{V}_T^x = -\frac{1}{T} \cdot V_T^x(t) - \frac{k}{T^2} \cdot U(t) + \frac{1}{T^2} \cdot y(t)$$

and get

$$V_T^x(t) = \left(1 - e^{-\frac{t}{T}}\right) \cdot \left(\frac{y}{T} - \frac{k \cdot U}{T}\right).$$

The output deviation at the change of both parameters is

$$\Delta y = V_k^y(t) \cdot \Delta k + V_T^y(t) \cdot \Delta T.$$

The frequency characteristics has the following form

$$A(\omega) = \frac{k}{\sqrt{T^2 \cdot \omega^2 + 1}}; \quad \varphi(\omega) = -\text{arctg} T \cdot \omega; \quad V_k^A = \frac{\partial A}{\partial k} = \frac{1}{\sqrt{T^2 \cdot \omega^2 + 1}}; \quad V_k^\varphi = \frac{\partial \varphi}{\partial k} = 0.$$

The sensitivity functions of these characteristics to the parameter T are as follows

$$V_T^A = \frac{\partial A(\omega)}{\partial T} = -\frac{k\omega^2}{(T^2 \cdot \omega^2 + 1)^{3/2}},$$

$$V_T^\varphi = \frac{\partial \varphi(\omega)}{\partial T} = \frac{\omega}{T^2 \cdot \omega^2 + 1},$$

$$\Delta A(\omega) = \frac{1}{\sqrt{T^2 \cdot \omega^2 + 1}} \cdot \Delta k - \frac{k\omega^2}{(T^2 \cdot \omega^2 + 1)^{3/2}} \cdot \Delta T, \quad \Delta \varphi(\omega) = \frac{\omega}{T^2 \cdot \omega^2 + 1} \cdot \Delta T.$$

Terms:

- sensitivity,
- relative sensitivity,
- sensitivity function.

Comprehension questions

1. What is the sensitivity of the system?
2. How can we estimate the sensitivity quantitatively?
3. How can we estimate the relative sensitivity quantitatively?

Lecture № 18**METHODS OF PROVIDING THE REQUIRED ACS ACCURACY****Lecture outline:**

1. The providing of the required ACS astatism order.
2. The selection of the required open-loop system gain.
3. The scaling of the reference action and the feedback signal.
4. The combined control.

18.1. The providing of the required ACS astatism order

The accuracy of the automatic control system is determined by its structure, parameters and input actions $g(t)$ and $f(t)$.

The error formula caused by the reference and disturbance actions have been earlier obtained for the systems which have one input, one output and the single negative feedback (Fig. 18.1)

$$\varepsilon_g = \lim_{s \rightarrow 0} \frac{s^{\nu-r} g_r r!}{s^{\nu} + K}, \quad (18.1)$$

$$\varepsilon_f = \lim_{s \rightarrow 0} \frac{s^{\nu-M-r} K_f f_r r!}{s^{\nu} + K}. \quad (18.2)$$

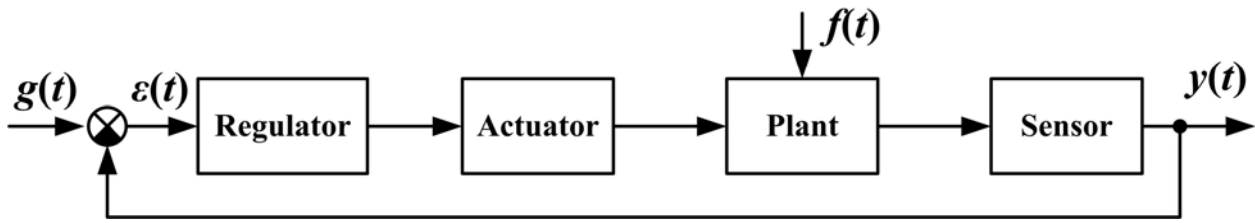


Figure 18.1 – A system with one reference input, one output and the unit negative feedback

These formulae imply the possibility to increase the accuracy by increasing the astatism order, i.e. by the introduction of integrators. The integrators should be introduced to the system till the point of the disturbance application is reached in order not to raise the μ -number that is the number of integrators in the open-loop system transfer function at the reference action. ν_{req} is defined by the permissible static ε_{st} and velocity ε_{vel} errors:

- if $\varepsilon_{st.perm} \neq 0$, then $\nu_{req} = 0$;
- if $\varepsilon_{st.perm} = 0$, then $\nu_{req} = 1$;
- if $\varepsilon_{vel.perm} = 0$, then $\nu_{req} = 2$.

Let the astatism order ν be equal to zero, i.e. the system has no integrators. We do not change the transfer coefficient K , $\nu_{req} = 1$. We introduce the integrator $\frac{1}{s}$ $\nu = 1$. The logarithmic phase-frequency characteristic lowers on $-\frac{\pi}{2}$. The inclination of the logarithmic amplitude-frequency characteristic raises on $-20dB/dec$. The bandwidth decreases. The line $\varphi(\omega) = -\pi$ intersects with the logarithmic phase-frequency characteristic at the inclination $-60 \div -40dB/dec$. The process becomes oscillating. The setting time t_s increases. The quality parameters deteriorate. The system can become unstable (Fig.18.2).

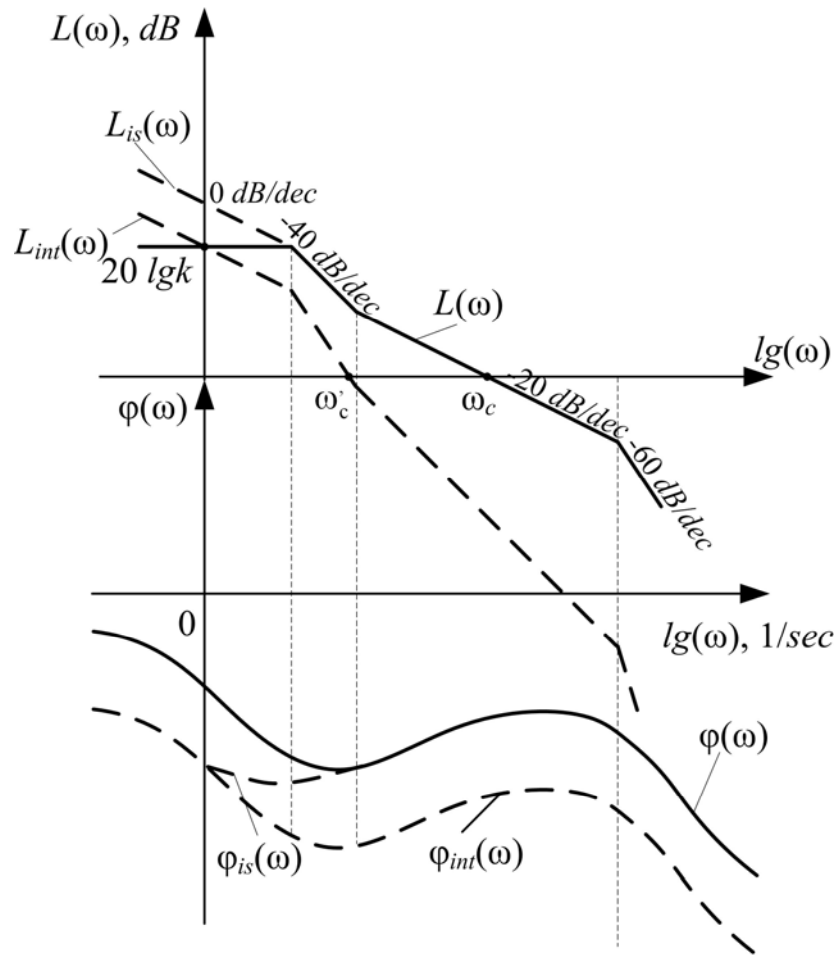


Figure 18.2 – Bode diagrams

The introduction of an isodrome link is a more suitable way to increase the astatism order

$$W_C(s) = k_{is} \frac{T_{is}s + 1}{s}. \quad (18.3)$$

This link changes the system logarithmic amplitude-frequency characteristic and logarithmic phase-frequency characteristic at the low frequencies and does not introduce changes at the middle and high frequencies, what allows raising the steady-state ACS accuracy without aggravating the system quality parameters in a transient mode (Fig. 18.3). To eliminate the velocity error ε_{vel} one more isodrome link should be introduced.

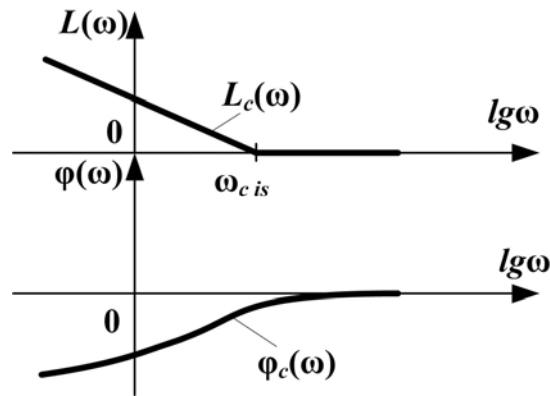


Figure 18.3 – Bode diagram for the isodrome link

18.2. The selection of the required open-loop system gain

The simplest way to increase the ACS accuracy is increase the open-loop system gain k . As it follows from the error formula, the transfer coefficient should be increased for those blocks which are located between the comparing device and the point of the disturbance application $f(t)$.

The required value k_{req} can be obtained from the conditions $\varepsilon_g \leq \varepsilon_{perm}$ and $\varepsilon_f \leq \varepsilon_{f perm}$ and (18.1), (18.2). The greater value k_{req} is chosen from the two obtained.

If the k is increased, the open-loop system LAFC rises, the ω_{cut} , the bandwidth and the swiftness of system are enhanced. But this results in the increase of σ , M and the reduction of γ_m , L_m . The system can become unstable. The correcting device should be introduced to provide the required quality measures σ, M, γ_m, L_m .

18.3. The scaling of the reference action and the feedback signal

18.3.1. The scaling of the reference action

We introduce the correcting device $W_c(s)$ to scale the reference action (Fig.18.4). In this case the comparing device output signal is not equal to the error

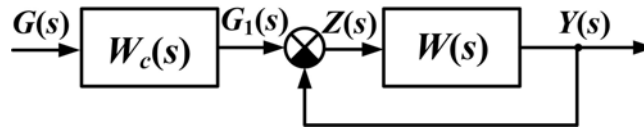


Figure 18.4 – The scaling of the reference action

$$z = g_1 - y \neq \varepsilon. \quad (18.4)$$

We calculate the transfer function of the correcting device $W_c(s)$ which provides the increase of the ACS accuracy

$$E(s) = G(s) - Y(s), \quad (18.5)$$

$$\Phi_\varepsilon(s) = \frac{E(s)}{G(s)} = 1 - \Phi(s),$$

$$\Phi_\varepsilon(s) = 1 - \frac{W_c(s)W(s)}{1+W(s)} = \frac{1+W(s) - W_c(s)W(s)}{1+W(s)}. \quad (18.6)$$

If we choose

$$W_c(s) = \frac{1+W(s)}{W(s)} = 1 + \frac{1}{W(s)}, \quad (18.7)$$

then $\Phi_\varepsilon(s) = 0$ and the system has no error ε at any reference action. The system, for which $\Phi_\varepsilon(s) = 0$, is called absolutely invariant at the reference action.

However, the transfer function $1 + \frac{1}{W(s)}$ cannot always be realized.

So if

$$W(s) = \frac{b_0s^m + b_1s^{m-1} + \dots + b_m}{a_0s^n + a_1s^{n-1} + \dots + a_n}, \quad m < n, \quad (18.8)$$

then the order of the $W_c(s)$ transfer function numerator turns out is higher than that of the denominator, and the function is unrealizable. In this case transfer function $W_c(s)$ is realized approximately so that the order of numerator was not higher than the denominator. In this case we say that the system is invariant to ε , i.e. it has an error. In the elementary case we choose

$$W_c(s) = 1 + \frac{1}{k} \quad (18.9)$$

then the steady-state error ε_{st} at the constant action $g = const$ will be absent. The system became astatic without introducing the integrator.

18.3.2. The scaling of the feedback signal

We introduce the correcting device $W_c(s)$ to the feedback (Fig. 18.5). In this case the comparing device output signal is not equal to the error ε because $z = g - y_1 \neq \varepsilon$.

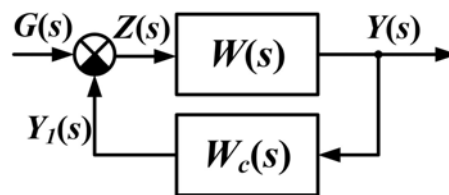


Figure 18.5 – The scaling of the feedback signal

We calculate the system transfer function by the error

$$\begin{aligned} E(s) &= G(s) - Y(s), \\ \Phi_\varepsilon(s) &= \frac{E(s)}{G(s)} = 1 - \Phi(s) = 1 - \frac{W(s)}{1 + W(s)W_c(s)} = \\ &= \frac{1 + W(s)W_c(s) - W(s)}{1 + W(s)W_c(s)}. \end{aligned} \quad (18.10)$$

If we choose

$$W_c(s) = \frac{-1 + W(s)}{W(s)} = 1 - \frac{1}{W(s)}, \quad (18.11)$$

then $\Phi_\varepsilon(s) = 0$.

The system will be absolutely invariant at the reference action. Such correcting device can be realized at $m = n$ and becomes unrealizable at $m < n$.

The transfer function can be realized accurately within ε

$$W_c(s) \approx 1 - \frac{1}{W(s)}. \quad (18.12)$$

In the elementary case we choose $W_c(s) = 1 - \frac{1}{k}$ to eliminate the static error ε_{st} .

Example.

We introduce the correcting device $W_c(s)$ to the ACS which structure is shown at the Fig. 18.6 for the static and velocity errors to be absent in the system, i.e. $\varepsilon_{st} = 0$ and $\varepsilon_{vel} = 0$. It is equivalent to the requirement of providing the astatism order $\nu = 2$ without introducing the integrators.

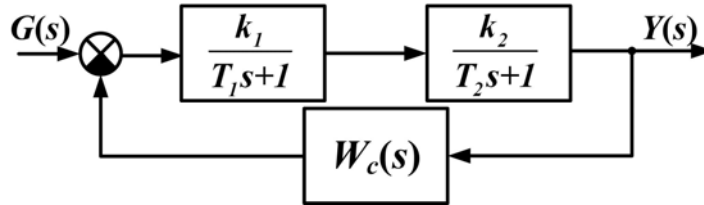


Figure 18.6 – The scaling of the feedback

We write down the system transfer function by the error

$$\Phi_\varepsilon(s) = \frac{E(s)}{G(s)} = 1 - \Phi(s) = 1 - \frac{W_1(s)W_2(s)}{1 + W_1(s)W_2(s)W_c(s)}$$

and choose $W_c(s) = \frac{k_c}{T_c s + 1}$.

Then

$$\begin{aligned} \Phi_\varepsilon(s) &= 1 - \frac{k_1 k_2 (T_c s + 1)}{(T_1 s + 1)(T_2 s + 1)(T_c s + 1) + k_1 k_2 k_c} = \\ &= \frac{(T_1 s + 1)(T_2 s + 1)(T_c s + 1) - k_1 k_2 (T_c s + 1) + k_1 k_2 k_c}{(T_1 s + 1)(T_2 s + 1)(T_c s + 1) + k_1 k_2 k_c}. \end{aligned}$$

The absolutely invariant system in this case is not realized, as $m < n$, $m = 0$ and $n = 2$.

To provide $\varepsilon_{st} = 0$ it is necessary to secure the equality $c_n = 0$ in the transfer function

$$\Phi_\varepsilon(s) = \frac{c_0 s^n + c_1 s^{n-1} + \dots + c_{n-1} s + c_n}{a_0 s^n + a_1 s^{n-1} + \dots + a_n},$$

where $c_n = 1 - k_1 k_2 + k_1 k_2 k_c$.

We choose $k_c = 1 - \frac{1}{k_1 k_2}$.

To provide $\varepsilon_{vel} = 0$ we secure the equality $c_{n-1} = 0$.

Therefore, we choose T_c from the condition

$$c_{n-1} = T_1 + T_2 + T_c - k_1 k_2 T_c = 0.$$

Thus we get $T_c = \frac{T_1 + T_2}{k_1 k_2 - 1}$.

18.3.3. The scaling of both the reference and feedback signals

We introduce two compensating devices $W_{c1}(s)$ and $W_{c2}(s)$ (Fig.18.7).

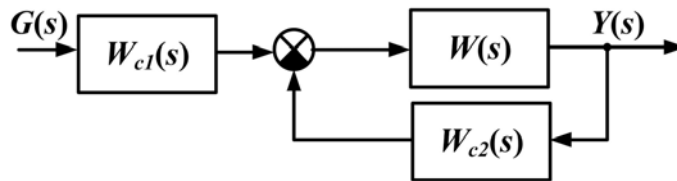


Figure 18.7 – The scaling of both the reference and feedback signals

We calculate the system transfer function by the error

$$\begin{aligned} \Phi_\varepsilon(s) &= 1 - \Phi(s) = 1 - \frac{W_{c1}(s)W(s)}{1 + W(s)W_{c2}(s)} = \\ &= \frac{1 + W(s)W_{c2}(s) - W_{c1}(s)W(s)}{1 + W(s)W_{c2}(s)}. \end{aligned} \quad (18.13)$$

If we choose the correcting devices $W_{c1}(s)$ and $W_{c2}(s)$ which satisfy the equality

$$W_{c2}(s) = \frac{W(s)W_{c1}(s) - 1}{W(s)} = W_{c1}(s) - \frac{1}{W(s)}, \quad (18.14)$$

then the system is absolutely invariant at the reference action. To eliminate the static error we choose

$$k_{c2} = k_{c1} - \frac{1}{k}.$$

18.4. The combined control

18.4.1. The combined control at the reference action

Let us introduce the parallel correcting device (Fig. 18.8) and calculate the system transfer function by the error

$$\Phi_{\varepsilon}(s) = \frac{1 - W_c(s)W_2(s)}{1 + W_1(s)W_2(s)}. \quad (18.15)$$

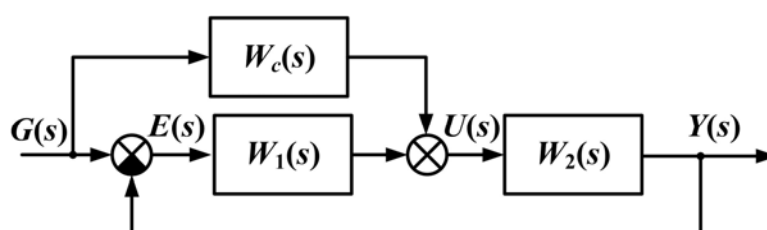


Figure 18.8 – The parallel correcting

We introduce the correcting device $W_c(s)$ to provide the control at both the deviation and at the reference action, i.e. $u = u(\varepsilon, g)$.

If we choose $W_c(s) = \frac{1}{W_2(s)}$, the system will be absolutely invariant at the reference action. In the elementary case we choose $k_c = \frac{1}{k_2}$ to secure $\varepsilon_{st} = 0$.

18.4.2. The combined control at the disturbance action

We introduce the compensating device $W_c(s)$ to secure the equality $u = u(\varepsilon, f)$ (Fig. 18.9).

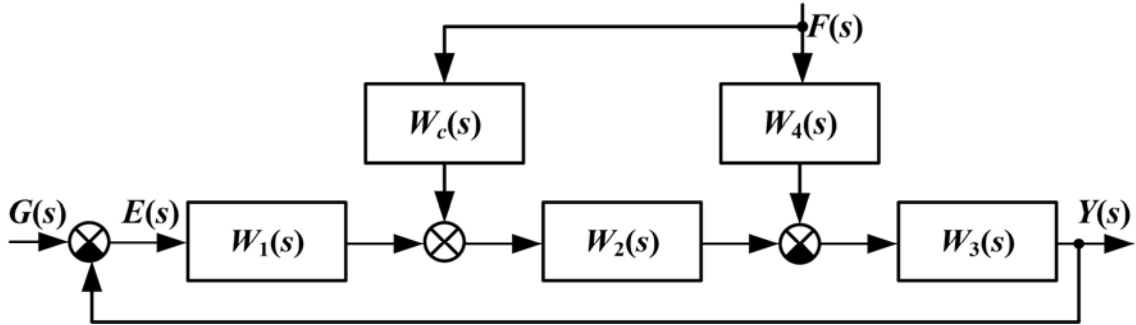


Figure 18.9 – The compensation control

To get the absolutely invariant system at the disturbance action we should secure $\Phi_f(s) = 0$.

Let us calculate the system transfer function at the disturbance

$$\Phi_{f\,inv}(s) = \frac{Y(s)}{F(s)} = \frac{-W_4(s)W_3(s) + W_c(s)W_2(s)W_3(s)}{1 + W_1(s)W_2(s)W_3(s)} = 0. \quad (18.16)$$

This implies the following condition

$$W_c(s) = \frac{W_4(s)}{W_2(s)}. \quad (18.17)$$

If it is impossible or difficult to realize this condition, we choose $k_c = \frac{k_4}{k_2}$ to secure $\varepsilon_{f\,st} = 0$.

Let us estimate the system error ε_f if $W_c(s) \neq \frac{W_4(s)}{W_2(s)}$ by calculating the closed-loop system transfer function at the disturbance

$$\Phi_{f\,inv}(s) = \frac{-W_4(s)W_3(s) + W_c(s)W_2(s)W_3(s)}{1 + W_1(s)W_2(s)W_3(s)} =$$

$$= -\frac{W_4(s)W_3(s)}{1+W_1(s)W_2(s)W_3(s)} \left[1 - \frac{W_c(s)W_2(s)}{W_4(s)} \right] = \Phi_f(s)\Phi_c(s). \quad (18.18)$$

We designate $L^{-1}\{\Phi_c(s)\} = \varphi_c(t)$.

I.e. the system can be presented as at the Fig. 18.10.

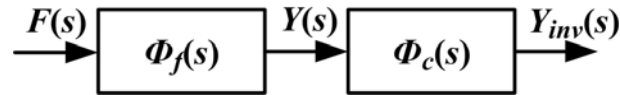


Figure 18.10 – The equivalent presentation of the system

Then

$$Y_{inv}(s) = \Phi_f(s)\Phi_c(s)F(s) = \Phi_c(s)Y(s). \quad (18.19)$$

We go to the originals

$$y_{inv}(t) = \int_0^t \varphi_c(\tau)y(t-\tau)d\tau = \int_0^t \left[y(t) + \dot{y}\tau + \frac{1}{2!}\ddot{y}\tau^2 + \dots + \frac{1}{r!}y^{(r)}\tau^r + R \right] \varphi_c(\tau)d\tau \quad (18.20)$$

and introduce the coefficients

$$c_i = \frac{1}{i!} \int_0^t \tau^i \varphi_c(\tau)d\tau. \quad (18.21)$$

Subject to these designations we get

$$y_{inv}(t) = \sum_{i=0}^r c_i y^{(i)} + \int_0^t R \varphi_c(\tau)d\tau. \quad (18.22)$$

If all $c_i = 0, i = 0, 1, 2, \dots$, then $y_{inv}(t) = 0$ and the system is absolutely invariant at the disturbance action.

If not every $c_i = 0$, system is invariant to ε .

At $c_0 = 0 \quad \varepsilon_{fst} = 0$.

At $c_0 = c_1 = 0 \quad \varepsilon_{fst} = \varepsilon_{fvel} = 0$.

It is convenient to increase of the astatism order by introducing the combined control as this method does not change the poles of the initial system but adds the pole of the compensating device. So the introduction of the stable compensating device preserves the stability of the initial system.

Terms:

- isodrome link,
- invariant system,
- scaling of the reference action,
- scaling of the feedback signal,
- combined control.

Comprehension questions

1. What devices should be introduced to increase the accuracy?
2. What permissible errors the required astatism order is determined by?
3. How should we change the system gain in order to increase its accuracy?
4. How does the LAFC change at the increase of the coefficient k ?
5. What device should be applied for scaling the reference action?
6. Is the comparing device output signal equal to error?

Lecture № 19**THE ACS CORRECTION****Lecture outline:**

1. Correcting devices.
2. Synthesis of the serial correcting devices with help of logarithmic amplitude - frequency characteristic.

19.1. Correcting devices**19.1.1. Kinds of correcting devices**

We introduce the correcting devices to the system in order to secure the ACS stability and the required quality parameters (Fig. 19.1).

According to the point of connection the correcting devices are divided into following kinds:

- serial (CD1);

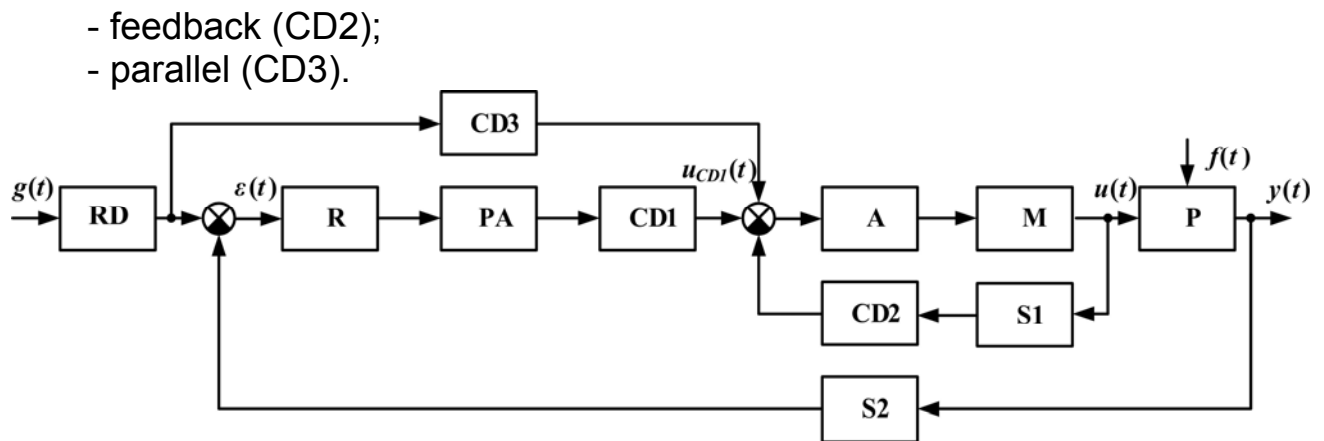


Figure19.1 – The automatic control system

In Fig. 19.1 the following marks are used:

RD is a reference device;

R is a regulator;

PA is a preliminary amplifier;

CD1, CD2, CD3 are correcting devices;

A is an amplifier;

M is a motor;

P is a plant;

S1, S2 are sensors.

A serial CD1 is connected after the comparing device or after the preliminary amplifier.

A feedback CD2 covers the power amplifier A and/or the motor M

$$W_{eq}(s) = \frac{U(s)}{U_{CD1}(s)} = \frac{W_A(s)W_M(s)}{1 + W_A(s)W_M(s)W_{CD2}(s)W_{S1}(s)} \approx \frac{1}{W_{CD2}(s)W_{S1}(s)}. \quad (19.1)$$

By the form of describing equations correcting devices are divided into linear and nonlinear.

The correcting devices can be passive (without the additional source of energy) and active (with the additional source of energy), continuous and discrete (depending on the nature of process).

By the type of the current correcting devices are divided into the devices of direct current and the devices of alternating current. The active

linear serial and feedback continuous devices of direct current are used the most widely.

19.1.2. The ACS stabilization and damping by introducing the phase lead

Let us consider the system (Fig.19.2) which has the transfer function

$$W(s) = \frac{Y(s)}{G(s)} = \frac{k}{S(T_1s + 1)(T_2s + 1)}. \quad (19.2)$$

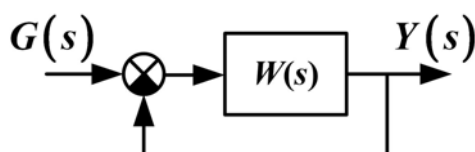


Figure 19.2 – Considered system

This system has the frequency characteristics shown at the Fig. 19.3.

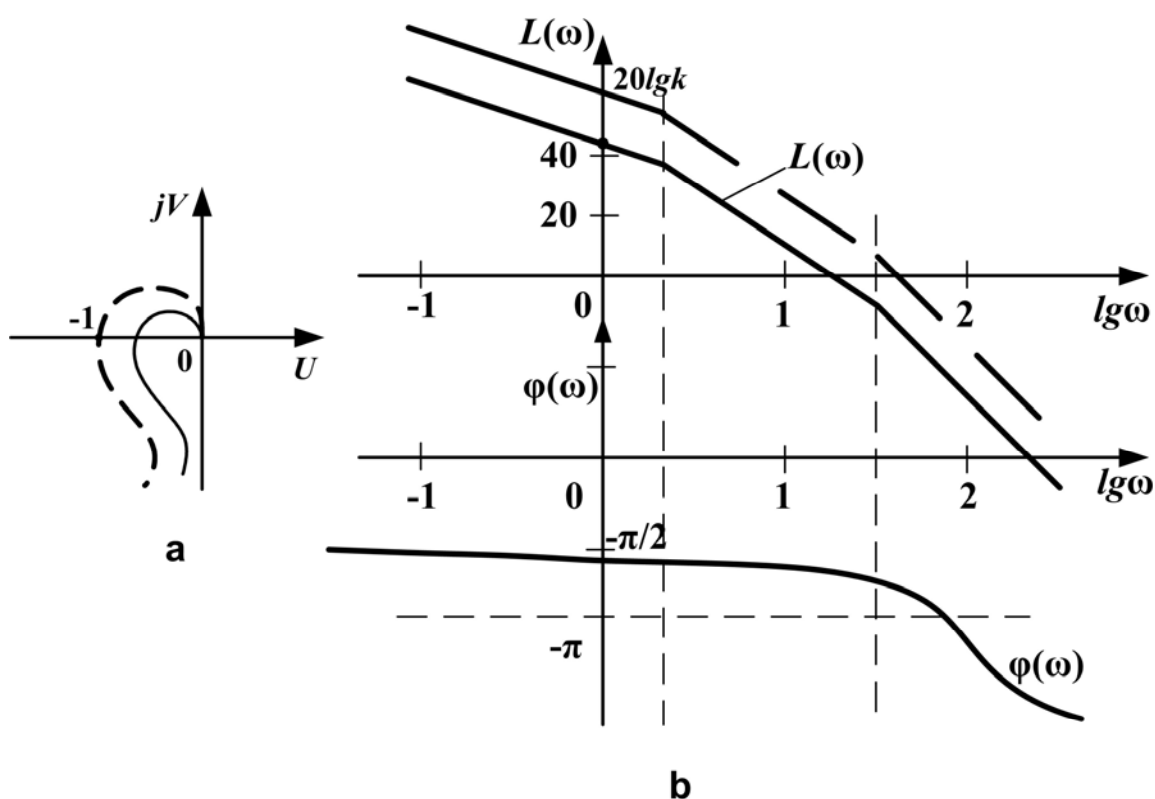


Figure 19.3 – Nyquist and Bode diagrams

This system has $\varepsilon_{st} = 0$, $\varepsilon_{vel} = V/k_V$. To decrease the velocity error ε_{vel} we should increase the system gain. Thus, the amplitude-phase frequency characteristic extends and can cover the point $(-1, j_0)$. The system can become unstable. To provide the required stability we introduce the phase lead correcting element, e.g. the phase lead proportional-differential link (Fig. 19.4).

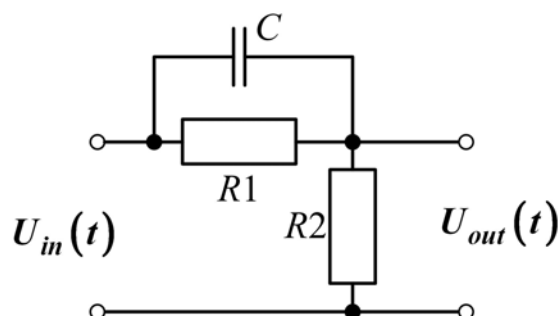


Figure 19.4 – Phase leading link

The correcting device transfer function is determined by the formula

$$\begin{aligned}
 W_{cd}(s) &= \frac{U_{out}(s)}{U_{in}(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{R_2}{\left(R_1 \cdot \frac{1}{Cs}\right) / \left(R_1 + \frac{1}{Cs}\right) + R_2} = \\
 &= \frac{R_2(R_1Cs + 1)}{(R_1 + R_2)(R_1Cs + 1)} = \frac{k_{cd}(T_{cd1}s + 1)}{T_{cd2}s + 1}, \quad (19.13)
 \end{aligned}$$

where $k_{cd} = \frac{R_2}{R_1 + R_2}$, $T_{cd1} = R_1C$, $T_{cd2} = \frac{R_1R_2C}{R_1 + R_2}$, $T_{cd1} > T_{cd2}$.

The Bode diagram of this link is shown at the Fig. 19.5.

The system regains stability. Thus we increase the cutoff frequency ω_c and the system swiftness but the noise immunity goes down.

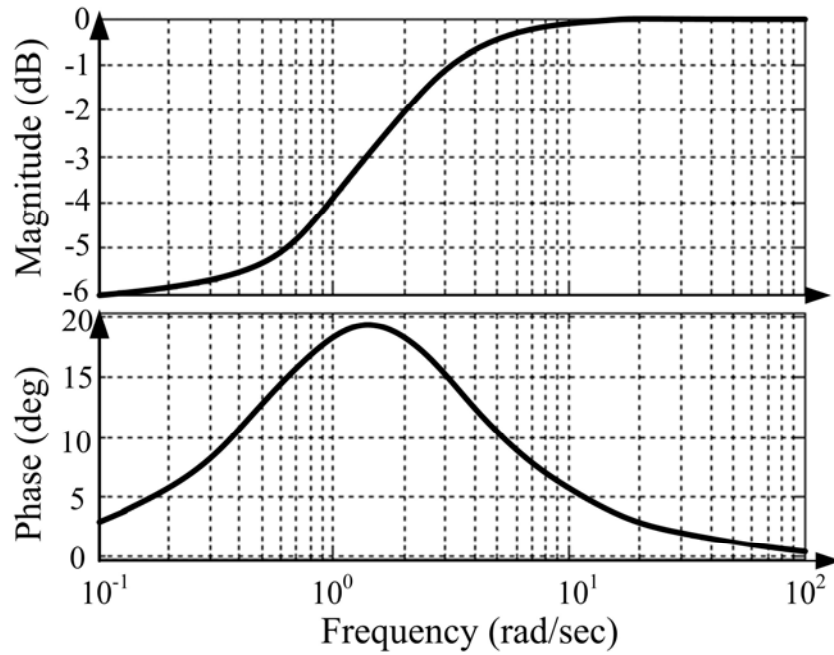


Figure 19.5 – Bode diagram of the phase-leading link

19.1.3. The ACS stabilization and damping by suppressing the middle frequencies

We can secure stability with the help of proportional integral-differential link (Fig. 19.6)

$$W_{CD}(s) = \frac{(T_1s + 1)(T_2s + 1)}{(T_3s + 1)(T_4s + 1)}. \quad (19.4)$$

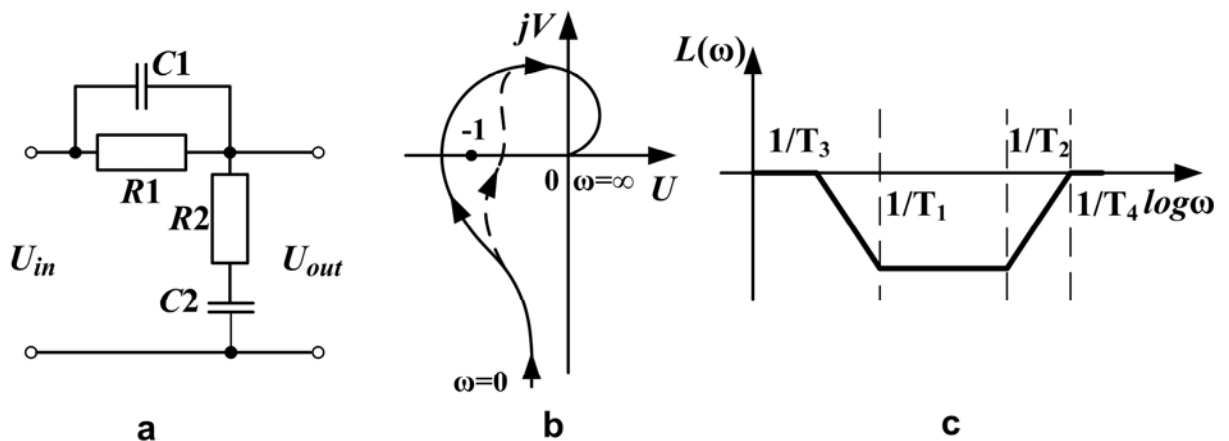


Figure 19.6 - Proportional integral-different link

Let $T_3 > T_1 > T_2 > T_4$.

This link suppresses the middle frequencies and provides the noncoverage of the points $(-1, j0)$ by the amplitude – phase frequency characteristic.

19.1.4. The increase of the ACS swiftness by applying the feedback to the inertial elements

We consider the aperiodic link (Fig. 19.7),

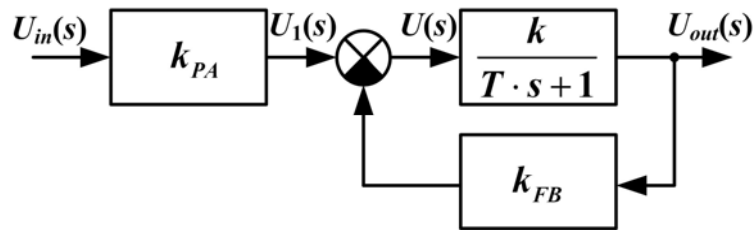


Figure 19.7 – Feedback using

which has the transfer function

$$W(s) = \frac{U_{out}(s)}{U(s)} = \frac{k}{Ts + 1}.$$

Let us cover it by the negative feedback with the help of the proportional link k_{fb} . After the correction we get

$$W_1(s) = \frac{U_{out}(s)}{U_1(s)} = \frac{W(s)}{1 + W(s)k_{fb}} = \frac{k}{Ts + 1 + k_{fb}k}. \quad (19.5)$$

Let us present the transfer function of the correcting link in the standard form

$$W_1(s) = \frac{k_1}{T_1s + 1}, \quad k_1 = \frac{k}{1 + k_{fb}k}, \quad T_1 = \frac{T}{1 + k_{fb}k}. \quad (19.6)$$

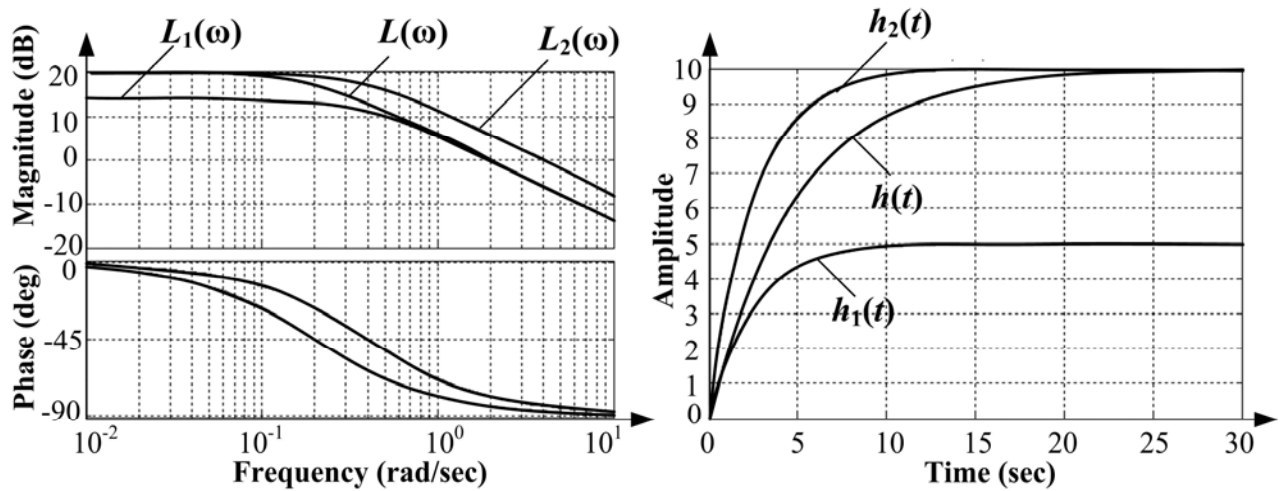


Figure 19.8 – Results of feedback using

During such correction the nature of the link isn't changed. The link remains aperiodic but the time constant and the gain of the link decreased when the negative feedback was introduced. To compensate the gain decrease we use the preliminary amplifier with $k_{pa} = 1 + k_{fb}k$.

The link swiftness is increased at the expense of the gain increase or the input increase. The coverage of the aperiodic link by the negative feedback is equal to introducing the serial proportionally - differential link. With the help of the feedback we can change not only parameters but the nature of the link as well. For example, the integrator can be transformed into the aperiodic link.

19.2. Synthesis of the serial compensating devices with help of logarithmic amplitude - frequency characteristic

The part of system (the controlled object, the power amplifier, the actuator, sensors, etc.) is usually given before designing the ACS. The set of the quality parameters $\varepsilon_{st}, \varepsilon_{vel}, \sigma, M, \gamma_m, L_m, t_s$ is made the demand to the system. We should determine, what compensating device should be introduced to the system to implement the requirements to the control quality. There are some methods of solving this problem.

Let's consider the method of synthesizing the serial correcting devices with help of the open-loop system logarithmical amplitude frequency characteristics (LAFC).

1. Using the preset values $\varepsilon_{st}, \varepsilon_{vel}, \varepsilon_{harm}$ which describe the system accuracy at the steady-state mode we determine the required order of astatism and ν_{req} the open-loop system gain k_{req} .

The required astatism order ν_{req} at the reference action is provided by introducing the necessary number of integrators to $W(s)$. The required astatism order at the disturbance action ($\nu - \mu$) is provided by introducing integrators up to the point of the disturbance application. The required gain k_{req} is secured by the preliminary amplifier located before the point of the disturbance application.

If the input action $g(t) = a \sin(\omega_k t)$ and permissible harmonical error ε_{harm} are given, we should find the control point L' on the open-loop system LAFC (Fig.19.9). The system of the required astatism order and the required gain is called the available system. Its transfer function is

$$W_{av}(s) = \frac{k_{req}(\dots)}{s^{\nu_{req}}(\dots)}. \quad (19.7)$$

2. Knowing the transfer function $W_{av}(s)$ we build the asymptotic LAFC of the available system $L_{av}(\omega)$. It should go over the control point L' .

3. According to the requirements σ, t_s, M , which determine the transient performance, we build the middle frequency segment of the desirable LAFC $L_d(\omega)$.

A low-frequency segment of the desirable LAFC is superposed with the low-frequency segment of the available LAFC. The high-frequency segment of the LAFC lies far below the axis $lg \omega$. The transfer coefficient at the high frequencies is small; therefore the signal does not pass through system. Consequently the LAFC form at the high frequencies does not influence the ACS quality parameters. We draw a high-frequency segment of the desirable LAFC combining it with $L_{av}(\omega)$ or parallel to the $L_{av}(\omega)$ in order to simplify the required correcting device.

4. We find the LAFC of the serial correcting device $L_{CD}(\omega)$ by subtracting the LAFC of the available system $L_{av}(\omega)$ from the LAFC of the desirable system $L_d(\omega)$. According to $L_{CD}(\omega)$ we write down the transfer function (Fig. 19.10).

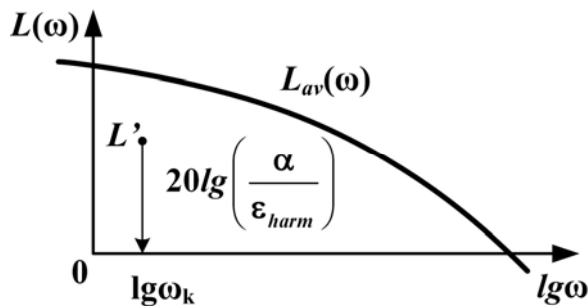


Figure 19.9 – Control point

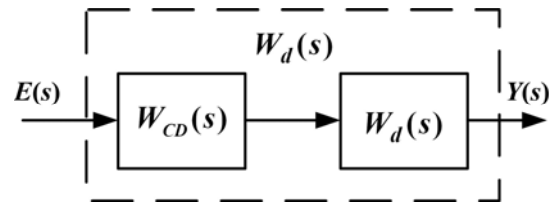


Figure 19.10 – Desired system

$$W_d(s) = W_{cd}(s)W_{av}(s), W_{cd}(s) = \frac{W_d(s)}{W_{av}(s)}, \quad (19.8)$$

$$L_{cd}(\omega) = L_d(\omega) - L_{av}(\omega). \quad (19.9)$$

5. In the tables of the correcting links we find the circuit of the correcting device which has the required LAFC $L_{CD}(\omega)$ and calculate the parameters of the resistors and the capacitors which secure the necessary time constants.

6. We build the step responses of the corrected closed-loop system and check whether the system satisfies the requirements. If it does not, we find out, which exactly parameter does not satisfy and change the correcting device.

Example

We have

$$W(s) = \frac{k}{(T_1s + 1)(T_2s + 1)}, \quad k = 50, T_1 = 0.5 \text{ sec}, T_2 = 0.05 \text{ sec},$$

$$\varepsilon_{st} = 0, \varepsilon_{vel} \leq 0.01V \text{ if } g = Vt, V = 1V/sec, \varepsilon_1 \leq 0.02V \text{ if}$$

$$g(t) = a \sin(\omega_1 t),$$

$$a = 0.04, \omega_1 = 5 rad/sec, \sigma \leq 30\%, t_s \leq 1 sec.$$

We should find the circuit and the parameters of the serial correcting device which secures the given quality parameters.

The solution.

To provide the required accuracy in the steady-state mode the system must have one integrator: $v_{req} = 1$.

The required gain k_{req} is calculated by the relation

$$\varepsilon_{vel} = \frac{V}{k} \leq \varepsilon_{perm}, k_{req} \geq \frac{V}{\varepsilon_{perm}} = \frac{1}{0.01} = 100 1/sec.$$

We introduce the preliminary amplifier with $k_{pa} = 2$, as $\frac{k_{req}}{k} = \frac{100}{50} = 2$

and obtain $W_{av}(s) = \frac{k_{req}}{s(T_1s + 1)(T_2s + 1)}$.

We build the asymptotic LAFC of the available system $L_{av}(\omega)$ (Fig. 19.11).

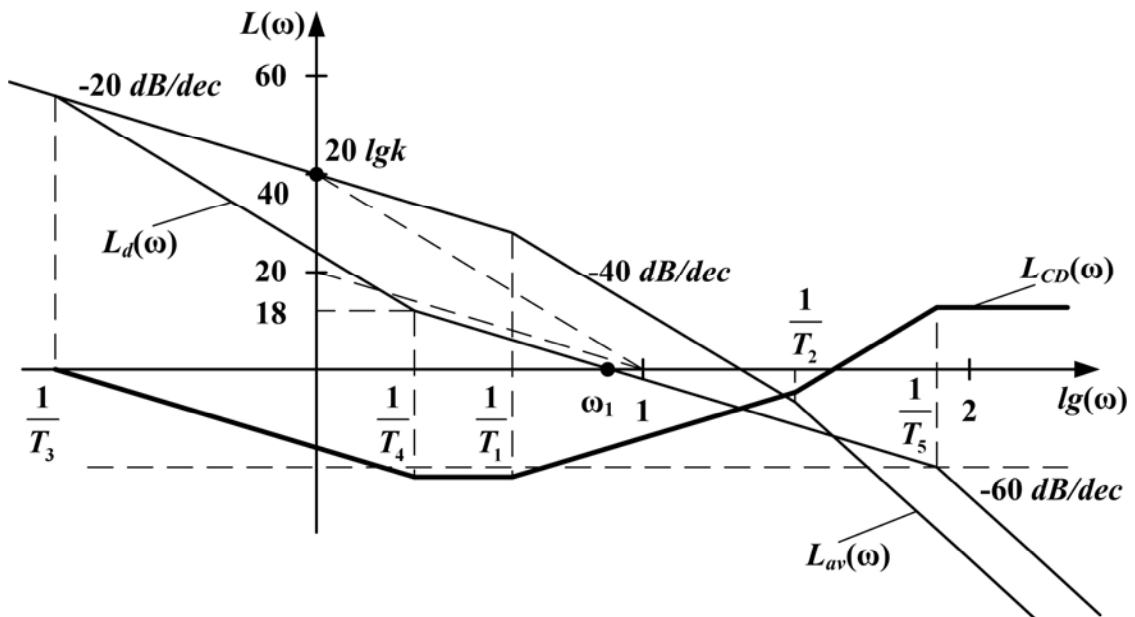


Figure 19.11 - LAFC

We construct the desirable LAFC using the Solodovnikov nomograms (Fig.19.12) and check it. If it does not satisfy us, we lift the LAFC.

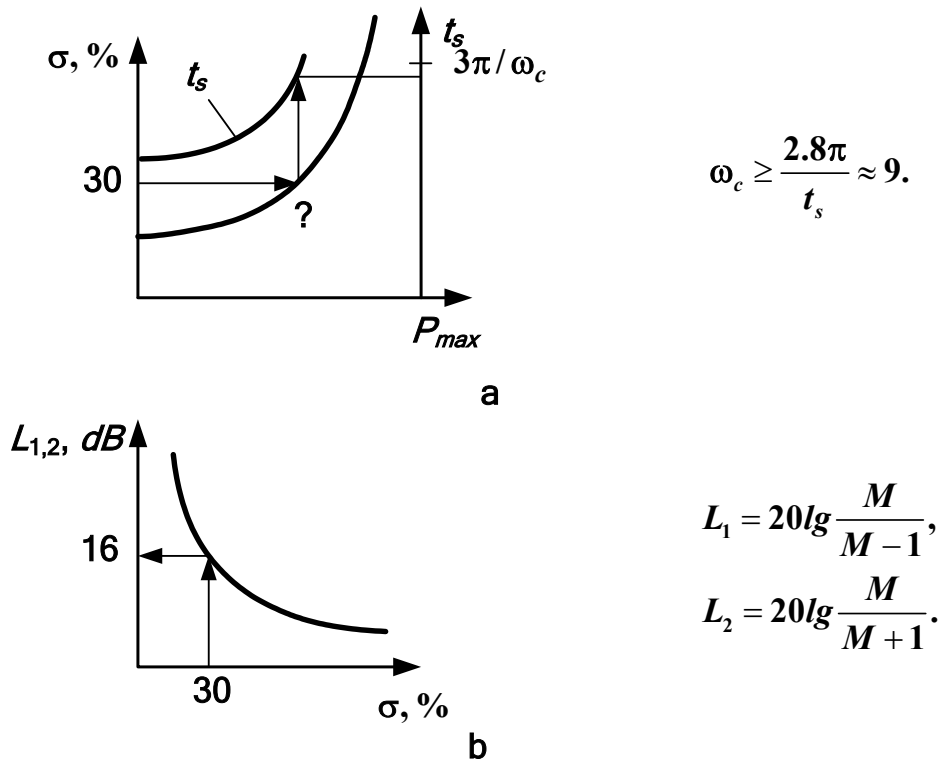


Figure 19.12 – Nomograms of Solodovnikov

We subtract $L_{at}(\omega)$ from $L_d(\omega)$ and find $L_{CD}(\omega)$. Thus, we get

$$W_{CD}(s) = \frac{(T_4s + 1)(T_1s + 1)(T_2s + 1)}{(T_3s + 1)(T_5s + 1)^2}.$$

The circuit of such device is not in the table. We piece it of two and connect them in series through the preliminary amplifier (Fig. 19.13).

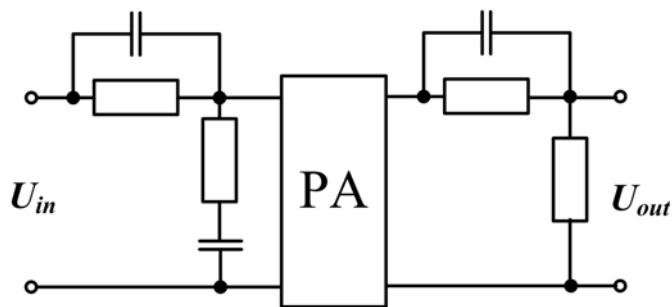


Figure 19.13 – Correcting device

Terms:

- correcting device,
- serial, parallel, feedback correcting devices,
- differential link,
- aperiodic link,
- preliminary amplifier.

Comprehension questions

1. Why do we introduce the correcting devices?
2. What kinds of correcting devices do you know?
3. Why do we need to increase the gain of the system?
4. Which link can provide stability?
5. What system is called available?
6. How can we calculate the required number of the integrators?
7. How can we calculate the required the open-loop system gain?
8. How can we build the available LAFC?
9. How can we build the desirable LAFC?
10. How can we build the LAFC of the required correcting serial device?
11. How can we find the transfer function of the correcting device?

Lecture №20**THE SYNTHESIS OF THE MODAL CONTROL SYSTEMS****Lecture outline:**

1. The standard forms of the ACS characteristic polynomial.
2. The modal control at the measurable state vector.
3. The modal control by the part of the poles at the partly measurable state vector.

1. The standard forms of the ACS characteristic polynomial

One of the ACS designing methods is the method of modal control. The point of method is that the regulator is executed as a set of proportional connections at every state variable (Fig. 20.1). The coefficients k_i are chosen so that the poles of the closed-loop system are arranged as it is desired. A mode is the system constituent, determined by a certain pole. The

unchangeable part (UP) of the system consist of the plant, actuator, the power amplifier and the sensors.

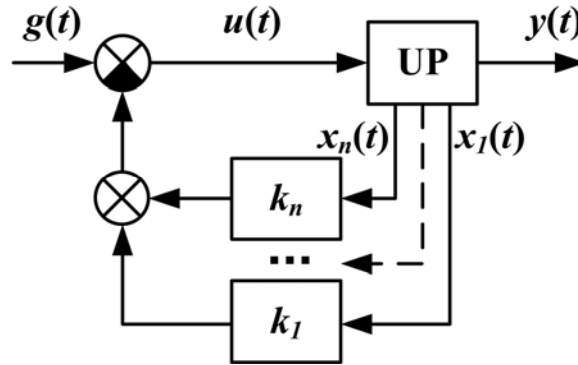


Figure 20.1 – Modal control system

Thus, the desired characteristic polynomial of the synthesized system will have the known coefficients, e.g.

$$D_{des}(s) = a_0 s^n + a_1 \omega_0 s^{n-1} + \dots + a_n \omega_0^n = \sum_{i=0}^n a_i \omega_0^i s^{n-i},$$

where ω_0 is a parameter which determines the real time of transitive process. The greater is ω_0 , the faster the process progresses.

There are different approaches to choose the desired location of the roots of the closed-loop system characteristic equation. If the chosen roots are real and identical ($s_i = -\omega_0$), the characteristic polynomial becomes the Newton binomial

$$D_{des}(s) = (s + \omega_0)^n = \sum_{i=0}^n a_i \omega_0^i s^{n-i}, a_i = c_n^i, i = \overline{0, n}. \quad (20.1)$$

The binomial coefficients are determined by the formula $c_n^i = \frac{n!}{i!(n-i)!}$. In particular, $a_0 = a_n = 1$.

The standard forms of this kind are called binomial.

The closed-loop system transfer function is in this case as follows

$$\Phi(s) = \frac{Y(s)}{U(s)} = \frac{A(s)}{D_{des}(s)}. \quad (20.2)$$

The step responses of such systems for the relative time $\tau = \omega_0 t$ are presented at the Fig. 20.2.

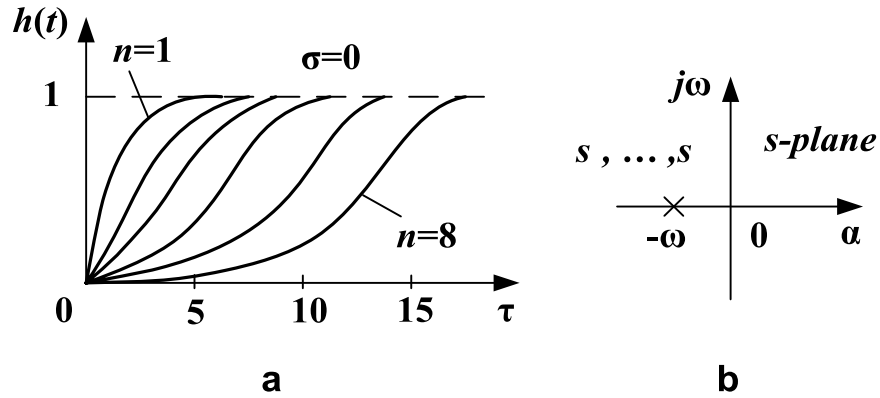


Figure 20.2 – Standard step responses (a), Newton root location (b)

The transfer function determining algorithm.

1. To determine the system order n .
2. To find the setting relative time τ_s for n .
3. To calculate

$$\omega_0 \geq \frac{\tau_s}{t_{s \text{ perm}}} \quad (20.3)$$

by the permissible setting time $t_{s \text{ perm}}$.

4. To find $a_i, D_{des}(s), \Phi(s)$ (Table 20.1).

The other well-known root location (suggested by Butterworth) is shown at the Fig. 20.3. The poles are disposed in the semicircumference of the radius ω_0 in the left half plane (Fig. 20.3). The angle between the imaginary axis and the ray which goes through the nearest root φ_1 is equal to the half of angle between the neighbour rays φ_2 : $\varphi_1 = \varphi_2 / 2$.

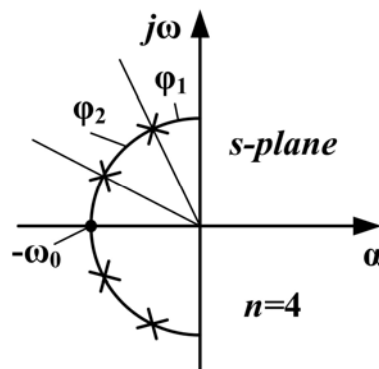


Figure 20.3 – Butterworth root location

The coefficients of the standard characteristic polynomials are shown at the Table 20.1, 20.2.

Table 20.1 – Binomial standard coefficient

Order of systems n	Coefficients of binomial form								
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1	1	-	-	-	-	-	-	-
2	1	2	1	-	-	-	-	-	-
3	1	3	3	1	-	-	-	-	-
4	1	4	6	4	1	-	-	-	-
5	1	5	10	10	5	1	-	-	-
6	1	6	15	20	15	6	1	-	-
7	1	7	21	35	35	21	7	1	-
8	1	8	28	56	70	56	28	8	1

Table 20.2 – Butterworth standard coefficient

Order of systems n	Coefficients of the Butterworth form								
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1	1	-	-	-	-	-	-	-
2	1	1.4	1	-	-	-	-	-	-
3	1	2	2	1	-	-	-	-	-
4	1	2.6	3.4	2.6	1	-	-	-	-
5	1	3.24	5.24	5.24	3.24	1	-	-	-
6	1	3.86	7.46	9.13	7.46	3.86	1	-	-
7	1	4.5	10.1	14.6	14.6	10.1	4.5	1	-
8	1	5.18	13.14	21.84	25.69	21.84	13.14	5.18	1

Butterworth standard step responses are presented in Fig. 20.4.

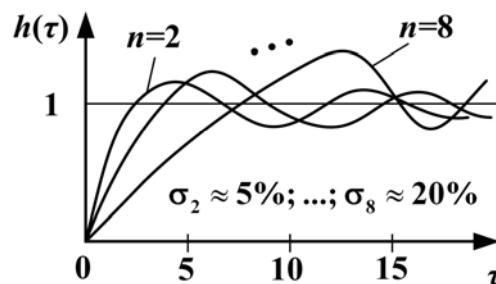


Figure 20.4 – Butterworth standard step responses

The great number of other standard root locations is developed.

20.2. The modal control at the measurable state vector

All state variables can be measured. We should remove all n poles to the desired points.

We have the unchangeable part described by the equations

$$\dot{x} = Ax + Bu, y = Cx, \quad (20.4)$$

$$[g] = [u] = r; [y] = m.$$

Then

$$D(s) = \det[sI - A] = s^n + a_1s^{n-1} + \dots + a_n. \quad (20.5)$$

Let us suppose the dynamic matrix A has the controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \quad (20.6)$$

If we close the system $u = -kx + g$. The desired characteristic polynomial is determined by the desired root location

$$D_{des}(s) = s^n + \gamma_1s^{n-1} + \dots + \gamma_n. \quad (20.7)$$

We should calculate K , which will secure the required coefficients of the characteristic polynomial (Fig. 20.5).

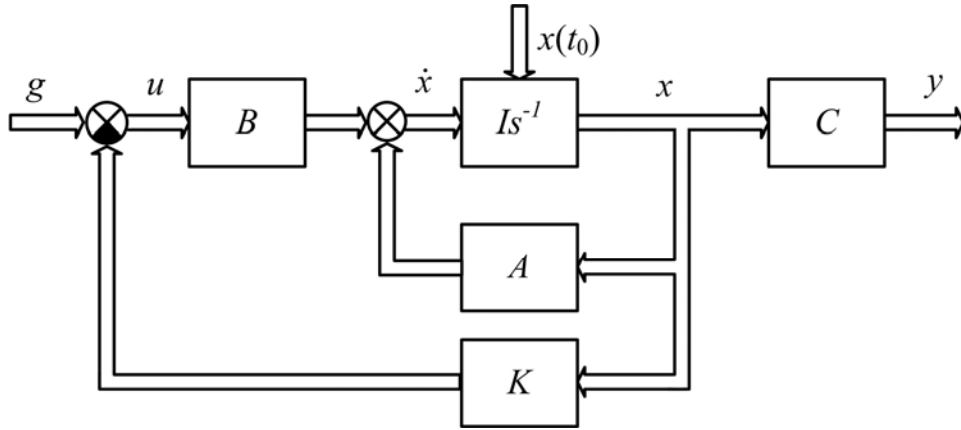


Figure 20.5 – Block diagram of the modal control system

If $r = 1$, the problem has a single solution.

If, $r > 1$, there are several solutions.

We will present the closed system in the form of controllable canonical presentation (see Lecture №7)

$$\dot{x} = Ax - Bkx + Bg = (A - Bk)x + Bg = A_{cl}x + Bg, \quad (20.8)$$

where $A_{cl} = A - Bk$.

Let's find desired determinant

$$D_{des}(s) = D_{cl}(s) = \det(sI - A + Bk) = \det(sI - A_{cl}), \quad (20.9)$$

$$A_{cl} = A - Bk = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad \dots \quad k_n] =$$

$$= \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 1 \\ -a_n - k_1 & -a_{n-1} - k_2 & \dots & -a_1 - k_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 1 \\ -\gamma_n & -\gamma_{n-1} & \dots & -\gamma_1 \end{bmatrix} = A_{des}. \quad (20.10)$$

Thus,

$$-a_n - k_1 = -\gamma_n, \dots, -a_1 - k_n = -\gamma_1; k_1 = \gamma_n - a_n, \dots, k_n = \gamma_1 - a_1$$

or

$$k_i = \gamma_{n-i+1} - a_{n-i+1}, i = 1, 2, \dots, n. \quad (20.11)$$

Example

We have

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -2x_1 + 3x_2 + u, \\ a_1 &= -3; a_2 = 2; \\ D(s) &= s^2 - 3s + 2; \\ s_1 &= 1, s_2 = 2. \end{aligned}$$

The location of the UP poles does not satisfy the requirements as the system is unstable (Fig. 20.6).

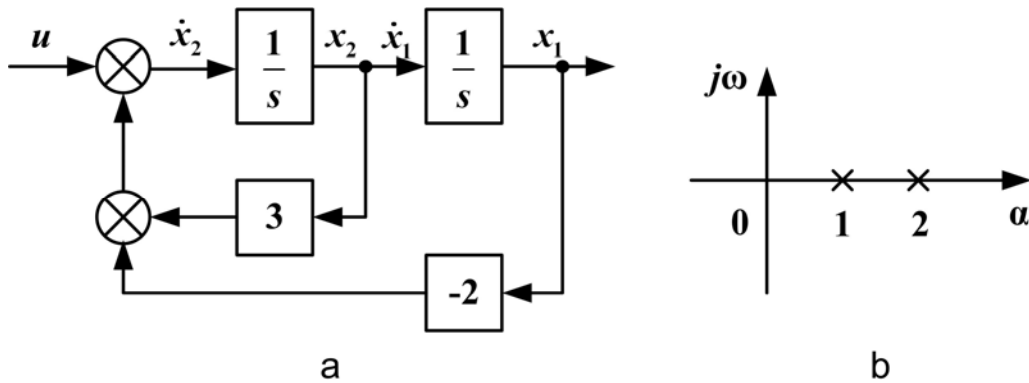


Figure 20.6 – Given system (a) and its poles (b)

We should find k_1, k_2 which secure the poles of the closed-loop system $\lambda_1 = -3$ and $\lambda_2 = -1$.

We determine the desired characteristic polynomial

$$D_{des}(s) = D_{cl}(s) = (s - (-3))(s - (-1)) = s^2 + 4s + 3.$$

Its coefficients are $\gamma_1 = 4, \gamma_2 = 3$.

We calculate the coefficients of the regulator

$$k_1 = \gamma_2 - a_2 = 3 - 2 = 1, k_2 = \gamma_1 - a_1 = 4 - (-3) = 7.$$

The system which provides the required location of the poles is shown in Fig. 20.7.

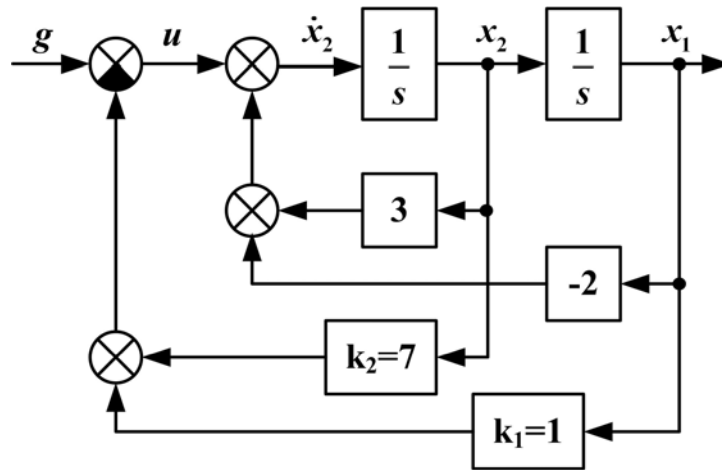


Figure 20.7 – Synthesized system

20.3. The modal control by the part of the poles at the partly measurable state vector

It is possible to control all the poles using the state feedback if all n state variables are measurable. If some of them are not, the special devices are applied to determine nonmeasurable state variables by the measured signals u and y . These devices solve the task of observing and are called observers, assessment filters or estimators

We consider the system (Fig. 20.8)

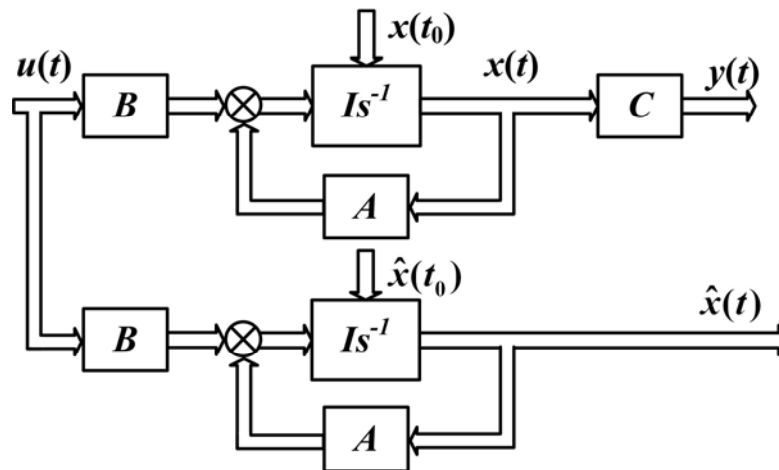


Figure 20.8 – Given system and simplest observer

$$\dot{x} = Ax + Bu; y = Cx, \quad (20.12)$$

where u is an r -metrical vector, y is an m -metrical vector. Both of them are accessible for measuring.

Matrices A, B, C are known.

We shall construct the model of the system with all the state variables $\hat{x}(t)$ measurable.

The input signal $u(t)$ is the same. We have

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu. \quad (20.13)$$

The state variables in a model $\hat{x}(t_0)$ correspond to the state variables $x(t)$ if the initial conditions are established properly ($x(t_0) = \hat{x}(t_0)$).

The disadvantage of this simplest observer is that it is operated in the open-loop mode. As the A, B, C system matrices and its initial state are not known exactly, and the system input deformations can not be introduced to the model accurately, the estimation of the state vector $\hat{x}(t)$ after a certain period of model operation will strongly differ from the vector $x(t)$.

The simplest estimator does not use all present information about the system. The output vector $y(t)$ does not participate in the $\hat{x}(t)$ formation.

We will improve the simplest estimator (Fig. 20.9).

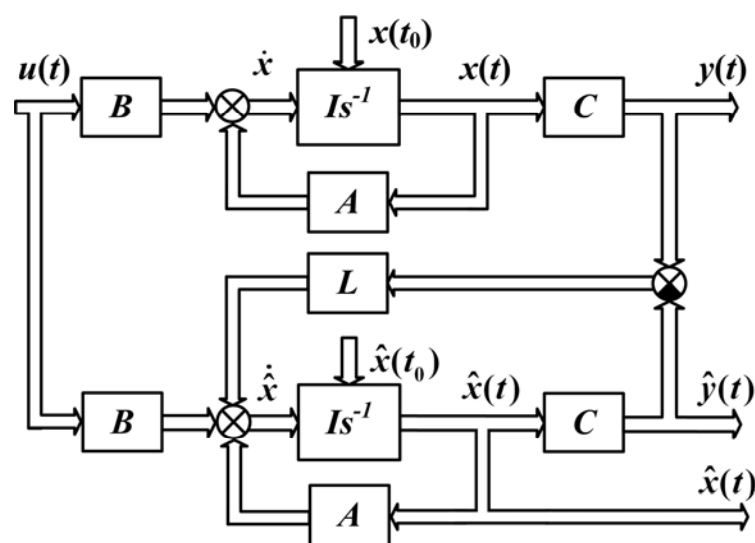


Figure 20.9 – System with Kalman observer

The system output signal $y(t)$ is compared with the output signal of the estimator $\hat{y}(t)$ and their difference is given to the estimator input

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L[y(t) - \hat{y}(t)], \\ \dot{\hat{x}}(t) &= [A - LC]\hat{x}(t) + Bu(t) + LCx(t), \\ \dot{x}(t) &= Ax(t) + Bu(t).\end{aligned}\tag{20.14}$$

This filter is called Ljueberger, or Kalman, filter.

We obtain the dynamic equation of the state vector evaluation error by subtracting the estimator equation from the system equation

$$\begin{aligned}\tilde{x}(t) &= x(t) - \hat{x}(t), \\ \dot{\tilde{x}}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) = A(x(t) - \hat{x}(t)) - LC(x(t) - \hat{x}(t))\end{aligned}$$

or

$$\dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t).\tag{20.15}$$

If the proper values of the $A - LC$ matrix have negative real parts then $\hat{x}(t)$ will be equal to $x(t)$ after the end of the estimator transitive process

$$\lim_{t \rightarrow \infty} \hat{x}(t) = x(t).\tag{20.16}$$

The estimator dynamic properties depend on the L matrix selection. By choosing the L matrix elements, we can provide any desired set of the proper poles s_1, s_2, \dots, s_n of the matrix observer $A - LC$.

If the initial system is observed, the estimator evaluates all n state variables $x_i(t)$. If system is not fully observed, we can estimate only the observed part of the state vector.

An estimating filter, in which $x(t) - \hat{x}(t) \rightarrow 0$, is called asymptotic, its dimension equal to n . Therefore, it is called the filter of complete order. If some state variables are measurable, and we should estimate only the missing state vector variables, we can construct the estimating filter of incomplete order.

Subject to the estimator application, the system of modal control has the following form (Fig. 20.10).

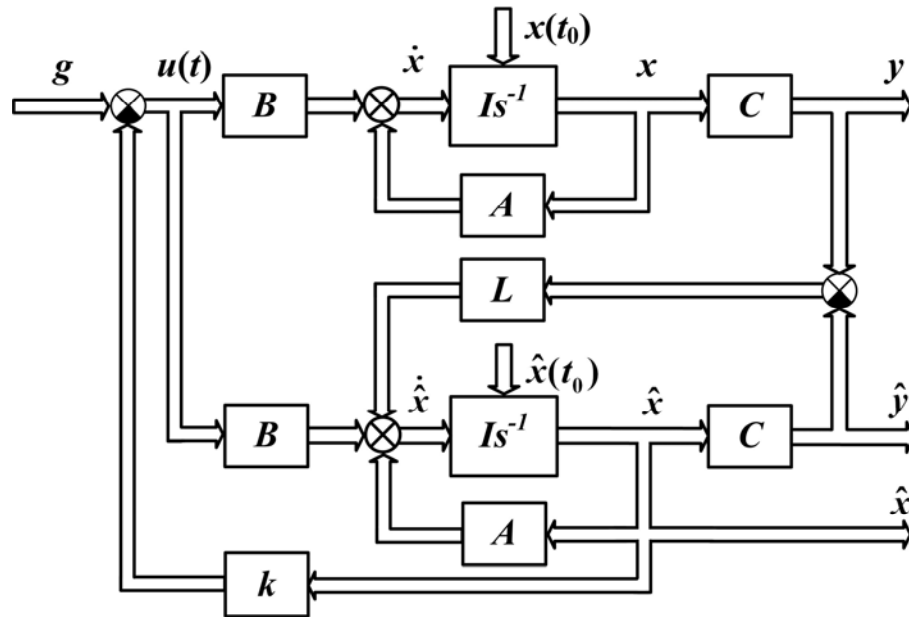


Figure 20.10 – Closed-loop system

The feedback matrix k is chosen so that the characteristic polynomial of the matrix $(A - Bk)$ coincides with the desired

$$D_{des}(s) = s^n + \gamma_1 s^{n-1} + \dots + \gamma_n.$$

The estimator is as follows

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu. \quad (20.17)$$

Its characteristic polynomial

$$D_{est}(s) = s^n + \beta_1 s^{n-1} + \dots + \beta_n = D_{est des}(s)$$

can have any desired form.

The control $u = g - k\hat{x}$.

We shall consider the questions: Is it possible to obtain the desired characteristic polynomial in the closed-loop system? How does the estimator affect the system?

We choose the matrix k so that characteristic polynomial $D_{[A-Bk]}(s) = D_{des}(s)$, and we choose the L matrix so that $D_{[A-LC]}(s) = D_{est des}(s)$. Then the characteristic polynomial of the closed-loop system

$$\begin{aligned}\dot{x} &= Ax + Bg - Bk\hat{x}, \\ \dot{\hat{x}} &= (A - LC - Bk)\hat{x} + Bg + LCx\end{aligned}$$

has the form $D_{cl}(s) = D_{des}(s)D_{est}(s)$.

To prove this we go to the new state variable $\tilde{x} = x - \hat{x}$,

$$\begin{aligned}\dot{x} &= [A - Bk]x + Bk\tilde{x} + Bg, \\ \dot{\tilde{x}} &= [A - LC]\tilde{x}\end{aligned}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - Bk & Bk \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} g.$$

Therefore,

$$\begin{aligned}D_{cl}(s) &= \det(sI - A_{cl}) = \det \begin{bmatrix} sI - A + Bk & -Bk \\ 0 & sI - A + LC \end{bmatrix} = \\ &= \det(sI - A + Bk) \cdot \det(sI - A + LC) = D_{[A-Bk]}(s) D_{[A-LC]}(s) = D_{des}(\lambda) D_{est}(\lambda).\end{aligned}$$

Thus, the feedback chain and the estimator can be constructed independently.

Example. We have the system (Fig. 20.11)

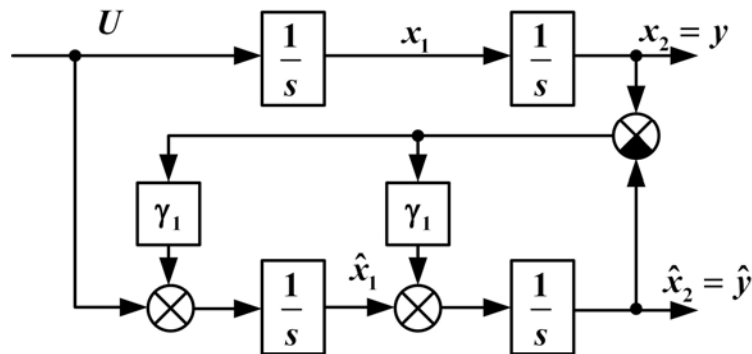


Figure 20.11 – Example of unchangeable part

$$\begin{aligned}\dot{x} &= Ax + Bu, \\ y &= Cx,\end{aligned}$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1 \ 0]$. We reduce the system to the canonical form of the identification canonic presentation by the transformation matrix

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1; \end{cases}$$

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \text{ --?}, \alpha_1 = 0, \alpha_2 = 0. D_{UP}(s) = s^2.$$

If the characteristic polynomial is given, we get

$$l_1 = \gamma_2 - \alpha_2 = \gamma_2, l_2 = \gamma_1 - \alpha_1 = \gamma_1,$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + L(y) + Bu$$

or

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\gamma_2 \\ 1 & -\gamma_1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}.$$

Terms:

- standard characteristic polynomial,
- Newton root location,
- Butterworth root location,
- modal control,
- observer (estimator).

Comprehension questions

1. What is the Newton root location?
2. What is the Butterworth root location?
3. What is a modal control?
4. How can we find the feedback coefficients?
5. What is an estimator?
6. How can we find the Kalman estimator coefficients?

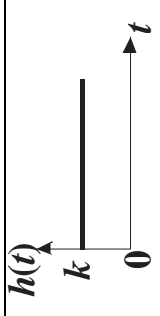
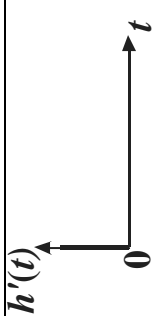
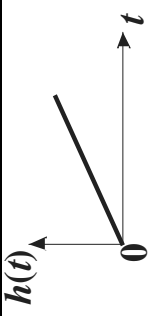
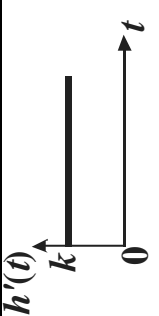
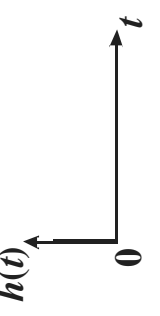
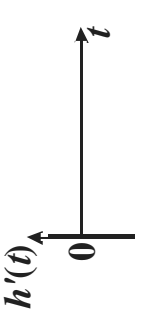
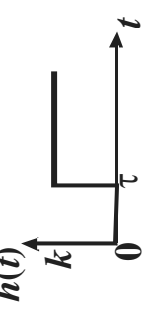
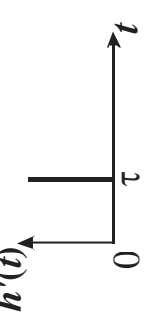
Appendix A. Table of Laplace transform theorems

Number	Theorem	Original	Image
1	Linearity	$\sum_{i=1}^m a_i x_i(t)$	$\sum_{i=1}^m a_i X_i(s)$
2	Lagging	$x(t - \tau), \tau > 0$	$X(s)e^{-\tau s}$
3	Multiplication by t	$x(t)t$	$-\frac{dX(s)}{ds}$
4	Multiplication by e^{-at}	$x(t)e^{-at}, a > 0$	$X(s + a)$
5	Scaling	$x(at), a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$
6	Differentiation	$\frac{dx}{dt}$	$sX(s) - x(0)$
7	Integrating	$\int_0^t x(\tau) d\tau$	$\frac{X(s)}{s}$
8	Initial value	$x(0) = \lim_{t \rightarrow 0} x(t) = \lim_{s \rightarrow \infty} sX(s)$	
9	Final value	$x(\infty) = \lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$	
10	Convolution	$\int_0^t x_1(\tau) x_2(t - \tau) d\tau$	$X_1(s) X_2(s)$

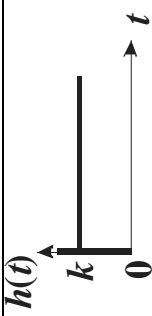
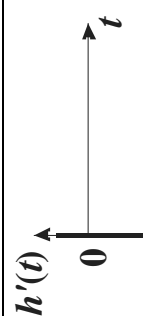
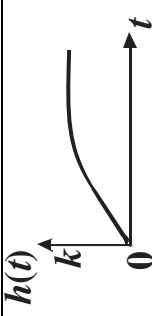
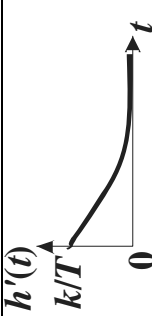
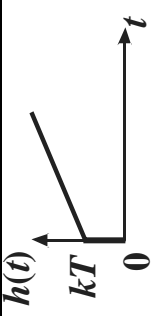

Appendix B. Table of Laplace transforms

Number	Time function $x(t), t > 0$	Laplace transform $X(s)$
1	$\delta(t)$	1
2	$1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	t^n	$\frac{n!}{s^{n+1}}$
5	e^{-at}	$\frac{1}{s+a}$
6	te^{-at}	$\frac{1}{(s+a)^2}$
7	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
8	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
9	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
10	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
11	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
12	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$

Appendix C. Table of typical links

Number	Name of the link	Input-output equation	Transfer function $W(s) = \frac{Y(s)}{X(s)}$	Step response $h(t)$	Pulse response $h'(t) = \omega(t)$
1	Amplifying	$y(t) = kx(t)$	k	 $h(t) = k1(t)$	 $h'(t) = k\delta(t)$
2	Integrating	$\frac{dy}{dt} = kx(t)$ $sY(s) = kX(s)$	$\frac{k}{s}$	 $h(t) = kt$	 $h'(t) = k$
3	Differentiating	$y(t) = k \frac{dx}{dt}$ $Y(s) = ksX(s)$	ks	 $h(t) = k\delta(t)$	 $h'(t) = k\delta'(t)$
4	Delaying	$y(t) = kx(t - \tau)$ $Y(s) = ke^{-\tau s} X(s)$	$ke^{-\tau s}$	 $h(t) = k1(t - \tau)$	 $h'(t) = k\delta(t - \tau)$

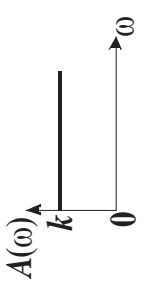
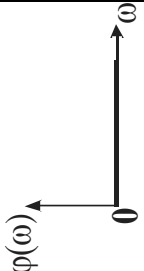

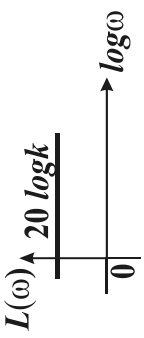
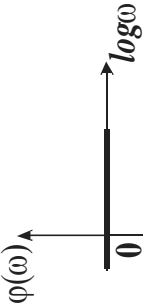
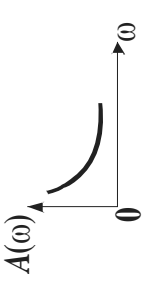
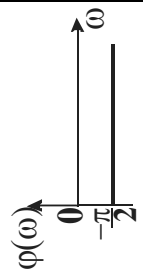

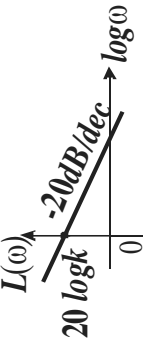
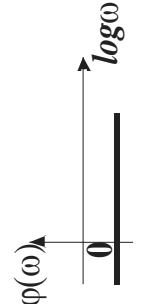
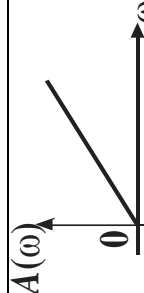
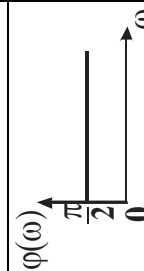
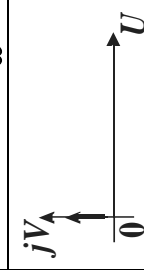
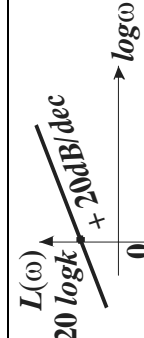
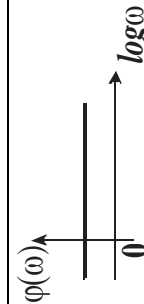
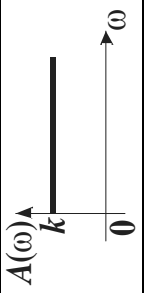
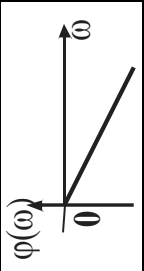
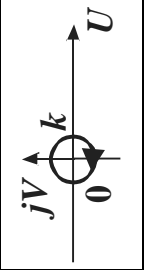
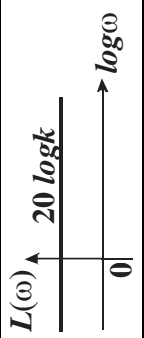
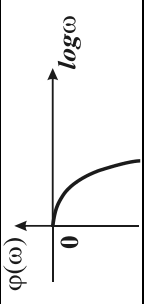
Appendix C. Continuation

Number	Name of the link	Input-output equation	Transfer function	Step response	Pulse response
5	Forcing	$y(t) = k\left(T \frac{dx}{dt} + x(t)\right)$ $Y(s) = k(Ts + 1)X(s)$	$W(s) = \frac{Y(s)}{X(s)} = k(Ts + 1)$	 $h(t) = k\mathbf{1}(t) + kT\delta'(t)$ $h'(t) = k\delta(t) + kT\delta''(t)$	
6	Inertial (aperiodical) of the first order	$T \frac{dy}{dt} + y(t) = kx(t)$ $(Ts + 1)Y(s) = kX(s)$	$\frac{k}{Ts + 1}$	 $h(t) = k(1 - e^{-t/T})$ $h'(t) = \frac{k}{T}e^{-t/T}$	
7	Proportional-integrating (isodrome)	$\frac{dy}{dt} = kT \frac{dx}{dt} + kx(t)$ $sY(s) = k(Ts + 1)X(s)$	$\frac{k(Ts + 1)}{s}$	 $h(t) = kt + kT\mathbf{1}(t)$ $h'(t) = k\mathbf{1}(t) + kT\delta(t)$	

Appendix C. Continuation

Number	Name of the link	Input-output equation	Transfer function $W(s) = \frac{Y(s)}{X(s)}$	Step response $h(t)$	Pulse response $h'(t) = \omega(t)$
8	Aperiodical of the second order	$T_1 T_2 \frac{d^2 y}{dt^2} + (T_1 + T_2) \frac{dy}{dt} + y = kx$ $(T_1 s + 1)(T_2 s + 1)Y(s) = kX(s)$	$\frac{k}{(T_1 s + 1)(T_2 s + 1)}$	 $h(t) = \frac{k}{T_1 - T_2} \cdot (T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}} + 1)$	 $h'(t) = \frac{k}{T_1 - T_2} \cdot (e^{-\frac{t}{T_1}} - e^{-\frac{t}{T_2}})$
9	Oscillatory	$T^2 \frac{d^2 y}{dt^2} + 2\xi T \frac{dy}{dt} + y = kx$ $(T^2 s^2 + 2\xi T s + 1)Y(s) = kX(s)$ $0 < \xi < 1$	$\frac{k}{T^2 s^2 + 2\xi T s + 1}$	 $h(t) = k - C e^{-\xi \frac{t}{T}} \sin(\omega t + \varphi)$	 $h'(t) = \frac{k e^{-\xi \frac{t}{T}}}{T \sqrt{1 - \xi^2}} \sin \sqrt{1 - \xi^2} \frac{t}{T}$
10	Conservative	$T^2 \frac{d^2 y}{dt^2} + y = kx$ $(T^2 s^2 + 1)Y(s) = kX(s)$	$\frac{k}{T^2 s^2 + 1}$	 $h(t) = k(1 - \cos \frac{t}{T})$	 $h'(t) = \frac{k}{T} \cdot \sin \frac{t}{T}$

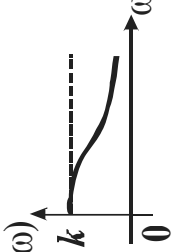
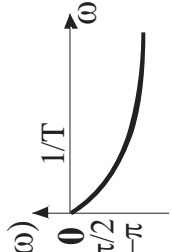
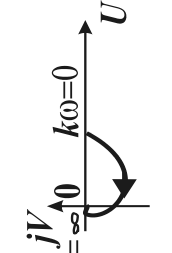
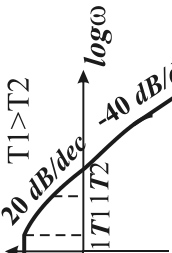
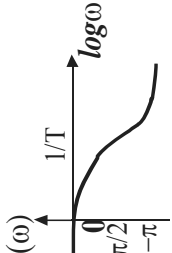
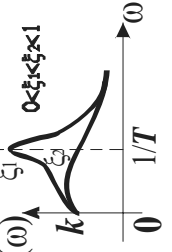
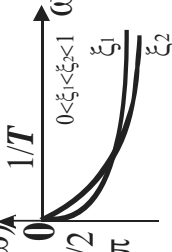
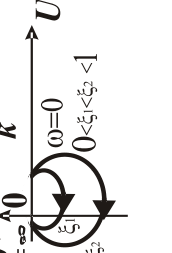
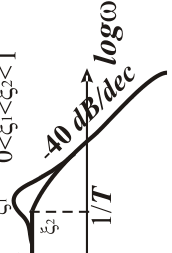
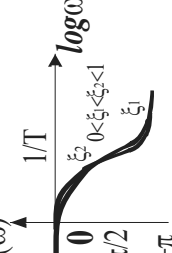
Appendix C. Continuation

Number	Name of the link	Amplitude-frequency characteristic $A(\omega)$	Phase-frequency characteristic $\varphi(\omega)$	Nyquist diagram $W(j\omega)$	Magnitude characteristic $L(\omega) = 20 \log A(\omega)$	Logarithmic phase-frequency characteristic $\varphi(\omega)$
1	Amplifying					
		$A(\omega) = k$	$\varphi(\omega) = 0$	$W(j\omega) = k$	$L(\omega) = 20 \log k$	$\varphi(\omega) = 0$
2	Integrating					
		$A(\omega) = \frac{k}{\omega}$	$\varphi(\omega) = -\frac{\pi}{2}$	$W(j\omega) = -j\frac{k}{\omega}$	$L(\omega) = 20 \log k - 20 \log \omega$	$\varphi(\omega) = -\frac{\pi}{2}$
3	Differentiating					
		$A(\omega) = k\omega$	$\varphi(\omega) = \frac{\pi}{2}$	$W(j\omega) = j\frac{k}{\omega}$	$L(\omega) = 20 \log k + 20 \log \omega$	$\varphi(\omega) = \frac{\pi}{2}$
4	Delaying					
		$A(\omega) = k$	$\varphi(\omega) = -\tau\omega$	$W(j\omega) = ke^{-j\tau\omega} = k\cos(\tau\omega) - jk\sin(\tau\omega)$	$L(\omega) = 20 \log k$	$\varphi(\omega) = -\tau\omega$

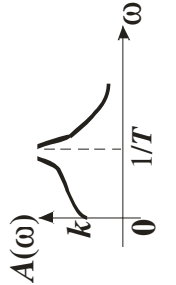
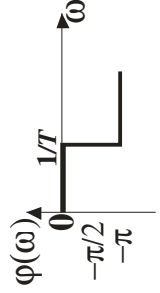
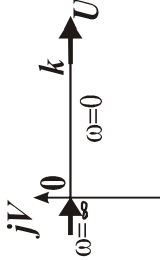
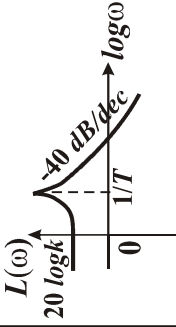
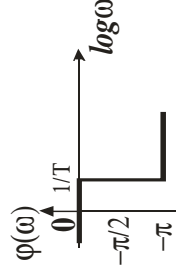
Appendix C. Continuation

Number	Name of the link	Amplitude-frequency characteristic $A(\omega)$	Phase-frequency characteristic $\varphi(\omega)$	Nyquist diagram $W(j\omega)$	Magnitude characteristic $L(\omega) = 20 \log A(\omega)$	Logarithmic phase-frequency characteristic $\varphi(\omega)$
5	Forcing					
6	Inertial (aperiodical) of the first order	$A(\omega) = k \sqrt{1 + T^2 \omega^2}$	$\varphi(\omega) = \arctan T \omega$	$W(j\omega) = k + jkT\omega$	$L(\omega) = 20 \log k + 20 \log \sqrt{1 + T^2 \omega^2}$	$\varphi(\omega) = \arctan T \omega$
6	Inertial (aperiodical) of the first order					
7	Proportional-integrating (isodromic)	$A(\omega) = \frac{k \sqrt{1 + T^2 \omega^2}}{\omega}$	$\varphi(\omega) = -\arctan T \omega$	$W(j\omega) = \frac{k}{1 + jT\omega}$	$L(\omega) = 20 \log k - 20 \log \sqrt{1 + T^2 \omega^2}$	$\varphi(\omega) = -\arctan T \omega$
7	Proportional-integrating (isodromic)					
			$\varphi(\omega) = -\frac{\pi}{2} + \arctan T \omega$	$W(j\omega) = kT - j \frac{k}{\omega}$	$L(\omega) = 20 \log k + 20 \log \frac{\sqrt{1 + T^2 \omega^2}}{\omega}$	$\varphi(\omega) = -\frac{\pi}{2} + \arctan T \omega$

Appendix C. Continuation

№	Name of the link	Amplitude-frequency characteristic $A(\omega)$	Phase-frequency characteristic $\varphi(\omega)$	Nyquist diagram $W(j\omega)$	Magnitude characteristic $L(\omega) = 20 \log A(\omega)$	Logarithmic phase-frequency characteristic $\varphi(\omega)$
8	Aperiodical of the second order	 $A(\omega) = \frac{k}{\sqrt{(1-T_1 T_2 \omega^2)^2 + (T_1 + T_2)^2 \omega^2}}$	 $\varphi(\omega) = -\arctg \left(\frac{(T_1 + T_2) \omega}{1 - T_1 T_2 \omega^2} \right)$	 $W(j\omega) = \frac{k}{(1 - T_1 T_2 \omega^2)^2 + (T_1 + T_2)^2 \omega^2} \times [(1 - T_1 T_2 \omega^2) - j(T_1 + T_2) \omega]$	 $L(\omega) = 20 \log k - 10 \log \left[(1 - T_1 T_2 \omega^2)^2 + (T_1 + T_2)^2 \omega^2 \right]$	 $\varphi(\omega) = -\arctg \left(\frac{(T_1 + T_2) \omega}{1 - T_1 T_2 \omega^2} \right)$
9	Oscillatory	 $A(\omega) = \frac{k}{\sqrt{(1 - T_1 T_2 \omega^2)^2 + (2\xi T \omega)^2}}$	 $\varphi(\omega) = -\arctg \left(\frac{2\xi T \omega}{1 - T^2 \omega^2} \right)$	 $W(j\omega) = \frac{k}{(1 - T^2 \omega^2)^2 + (2\xi T \omega)^2} \times [(1 - T^2 \omega^2) - j2\xi T \omega]$	 $L(\omega) = 20 \log k - 10 \log \left[(1 - T^2 \omega^2)^2 + (2\xi T \omega)^2 \right]$	 $\varphi(\omega) = -\arctg \left(\frac{2\xi T \omega}{1 - T^2 \omega^2} \right)$

Appendix C. Ending

Number	Name of the link	Amplitude-frequency characteristic $A(\omega)$	Phase-frequency characteristic $\varphi(\omega)$	Nyquist diagram $W(j\omega)$	Magnitude characteristic $L(\omega) = 20 \log A(\omega)$	Logarithmic phase-frequency characteristic $\varphi(\omega)$
10	Conservative					
		$A(\omega) = \frac{k}{1 - T^2 \omega^2}$	$\varphi(\omega) = -\text{arctg} 0$	$W(j\omega) = \frac{k}{1 - T^2 \omega^2}$	$L(\omega) = 20 \log \frac{k}{1 - T^2 \omega^2}$	$\varphi(\omega) = -\text{arctg} 0$

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Технический редактор Л.А. Кузьменко
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<http://www.khai.edu>

Издательский центр «ХАИ»

61070, Харьков-70, ул. Чкалова, 17

izdat@khai.edu