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PHYSICS FOR ENGINEERS

Part IV
OSCILLATIONS AND WAVES

2011

MINISTRY OF EDUCATION AND SCIENCE, YOUTH AND SPORT OF UKRAINE

ZHUKOVSKY NATIONAL AEROSPACE UNIVERSITY
“KHARKIV AVIATION INSTITUTE”

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UDK 534 (075.8)
O-83

Навчальний посібник містить основи теорії механічних коливань, механічних біжучих і стоячих хвиль, електромагнітних коливань і електромагнітних хвиль. Розглянуто відповідні диференціальні рівняння та проаналізовано їх розв’язання. Наведено приклади розв'язання задач, а також завдання для самостійного розв’язання.

Для студентів напрямів навчання «Авіація і космонавтика», «Прикладна механіка» та інших, які вивчають фізику англійською мовою. Може бути також використано студентами, що навчаються за спеціальністю «Прикладна лінгвістика». Особливий інтерес становить для іноземних студентів і аспірантів, які навчаються в Україні.

Reviewed by: Doctor of Technical Sciences, Professor E. Strelnikova,
Doctor of Technical Sciences, Professor Ju. Batygin,
Doctor of Technical Sciences, Professor O. Chugay

O-83 Physics for engineers. Part 4. Oscillations and waves: Textbook / O. Naumenko, V. Demenko, V. Pavlenko, V. Sikulskiy. – Kharkov: Zhukovsky National Aerospace University “Kharkov Aviation Institute”, 2011. – 264 p.

ISBN 978-966-662-238-2

The Textbook contains the basis of the theory of mechanical oscillations, mechanical travelling and standing waves, electromagnetic oscillations and electromagnetic waves.

Corresponding differential equations are considered and their solutions are analyzed. The Textbook includes the questions, problems, exercises related to theoretical analysis. Important examples are solved, and a lot of exercises are placed to work without assistance.

The Textbook is intended for practical study of physics by the students of higher technical universities, as well as by those students who are trained in “Applied Linguistics”. It may be also be useful for those students who are preparing for their practical training at universities of Europe and USA, as, well, as for the foreign citizens who are trained in Ukraine.

UDK 534 (075.8)

Illustrations 121. Tables 5. Bibliographical references: 21 titles

ISBN 978-966-662-238-2

Chapter 1

Mechanical Oscillations

In this chapter we consider a very special kind of motion which occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position. If this force is always directed toward the equilibrium position, repetitive back-and-forth motion occurs about this position. Such motion is called *oscillation,* or *vibration* (the terms are completely equivalent).

You are most likely familiar with several examples of periodic motion, such as the oscillations of a block attached to a spring, the swinging of a child on a playground swing, the motion of a pendulum, and the vibrations of a stringed musical instrument. In addition to these everyday examples, numerous other systems exhibit periodic motion. For example, the molecules in a solid oscillate about their equilibrium positions; electromagnetic waves, such as light waves, radar and radio waves, are characterized by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage, current, and electrical charge vary periodically with time.

Thus a study of periodic motion gives us an important foundation for further study in many different areas of physics.

1.1 Basic Concepts

One of the simplest systems that can undergo periodic motion is a block of mass m attached to a spring, as shown in Figure 1.1. The body is attached to one end of a spring, and the other end of the spring is held stationary.

Let x be the *displacement* of the body from its equilibrium position. When $x = 0$, the spring is neither stretched nor compressed. When the body is displaced to the right, x is positive and the spring stretches. The force F that the spring exerts on the body is toward the left (negative x-direction), toward the equilibrium position, and F is negative. When the body is displaced to the left, x is negative and the spring is compressed. The force on the body is toward the right (positive x-direction), again toward equilibrium, and F is positive. Thus the sign of F is always opposite to the sign of x itself. We call such a force a *restoring force.*

We know that for some springs the force is *directly proportional* to the deformation, at least for small deformations. This proportionality is called Hook's law. Thus in Figure 1.1, if the spring obeys Hook's law, we may represent the relationship of F to x as

$$F = -kx, \tag{1.1}$$

where k is the *force constant* for the spring. This relation is valid for both positive and negative x; in both cases F and x have opposite signs.

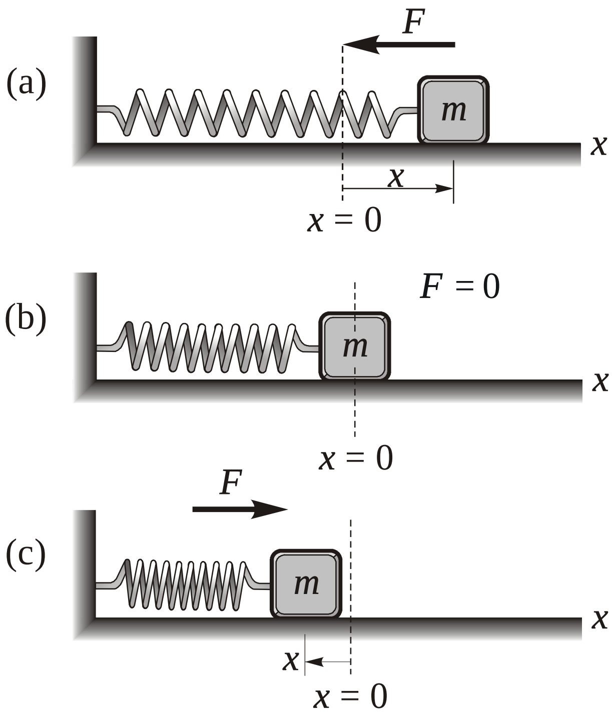


Figure 1.1 A block attached to a spring moving on a frictionless surface. (a) When the block is displaced to the right of equilibrium ($x > 0$), the force exerted by the spring acts to the left; (b) When the block is at its equilibrium position ($x = 0$), the force exerted by the spring is zero; (c) When the block is displaced to the left of equilibrium ($x < 0$), the force exerted by the spring acts to the right

Suppose we displace the body a distance A the right and release it, with no initial velocity. The spring exerts a force toward the equilibrium position, and the body accelerates in this direction. The acceleration is not constant, because the force decreases as the body approaches the equilibrium position.

When the body reaches $x = 0$, the force and acceleration have decreased to zero, but the velocity that the body has acquired, causes it to "overshoot" the equilibrium position and continue to move to the left. The force then reverses direction, and the body's speed starts to decrease. The body comes to rest at some point to the left of O and starts back toward the equilibrium position.

The motion is confined to a range $x = \pm A$ on both sides of the equilibrium position, and each complete back-and-forth trip takes the same amount of time. If there were no loss of mechanical energy due to friction, the motion would continue forever. This specific motion, under the influence of a restoring force proportional to displacement and without any friction, is called *simple harmonic motion*, abbreviated *SHM*.

Simple harmonic motion is the simplest of all periodic motions to analyze. In more complex examples the force may depend on displacement in a more complicated way; but only when it is *directly proportional to displacement* do we use the expression "simple harmonic motion." However, many more complex periodic motions are approximately simple harmonic, if the displacements are small enough. Thus, simple harmonic motion is a model that serves for an approximate representation of many periodic motions.

An experimental arrangement that exhibits simple harmonic motion is illustrated in Figure 1.2. A mass oscillating vertically on a spring has a pen attached to it. While the mass is oscillating, a sheet of paper is moved perpendicular to the direction of the spring motion, and the pen traces out a wavelike pattern.

In general, a particle moving along the x axis exhibits simple harmonic motion when x, the particle's displacement from equilibrium, varies in time according to the relationship

$$x = A\cos(wt + f), \qquad (1.2)$$

where A, w, and f are constants. To give physical significance to these constants, we have labeled a plot of x as a function of t in Figure 1.3a. This is just the pattern that is observed with the experimental apparatus shown in Figure 1.2. The *amplitude* A of the motion is the maximum displacement of the particle in either the positive or negative x direction. The constant w is called the *angular frequency* of the motion and has units of radians per second. The constant angle f, called the *phase constant* (or *phase angle*), is determined by the initial displacement and velocity of the particle. If the particle is at its maximum position $x = A$ at $t = 0$, then $f = 0$ and the curve of x versus t is as shown in Figure 1.3b. If the particle is at some other position at $t = 0$, the constants f and A tell us what the position was at time $t = 0$. The quantity $(wt + f)$ is called the *phase* of the motion. The phase constant f is important when we compare the motion of two or more oscillating objects. Imagine two identical pendulum bobs swinging side by side in simple harmonic motion, with one having been released later than the other. The pendulum bobs have different phase constants.

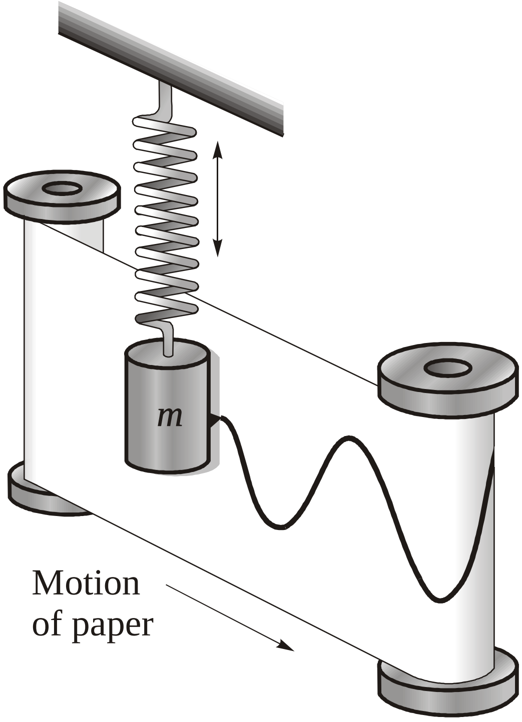


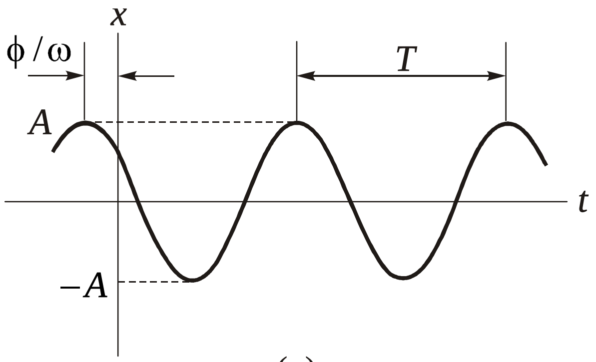
Figure 1.2 An experimental apparatus for demonstrating simple harmonic motion. A pen attached to the oscillating mass traces out a wavelike pattern on the moving chart paper

Note from Eq. (1.2) that the trigonometric function x is periodic and repeats itself every time wt increases by $2p$ rad. The *period* T of the motion is the time it takes for the particle to go through one full cycle. We say that the particle has made one oscillation. This definition of T tells us that the value of x at time t equals the value of x at time $t + T$. We can show that $T = 2p / w$ by using the preceding observation that the phase $(wt + f)$ increases by $2p$ rad in a time T:

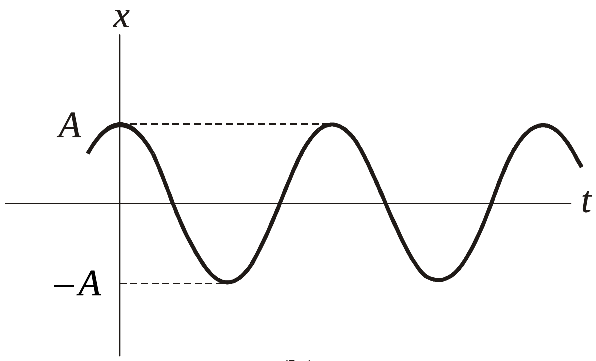
$$wt + f + 2p = w(t + T) + f .$$

Hence, $wT = 2p$, or

$$T = \frac{2p}{w}. \qquad (1.3)$$



(a)



(b)

Figure 1.3 (a) An $x-t$ curve for a particle undergoing simple harmonic motion. The amplitude of the motion is A, the period is T and the phase constant is ϕ; (b) The $x-t$ curve in the special case in which $x = A$ at $t = 0$ and hence $\phi = 0$

The inverse of the period is called the *frequency* f of the motion. *Frequency* represents the number of oscillations that the particle makes per unit time:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{1.4}$$

The units of f are cycles per second s^{-1}, or *hertz* (Hz).

Rearranging Eq. (1.4), we obtain the angular frequency:

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{1.5}$$

We can obtain the *linear velocity* of a particle undergoing simple harmonic motion by differentiating Eq. (1.2) with respect to time:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi). \tag{1.6}$$

The *acceleration* of the particle is

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi). \tag{1.7}$$

Because $x = A\cos(\omega t + \phi)$, we can express Eq. (1.7) in the form

$$a = -\omega^2 x. \tag{1.8}$$

From Eq. (1.6) we see that, because the sine function oscillates between ±1, the extreme values of v are $\pm\omega A$. Because the cosine function also oscillates between ±1, Eq. (1.7) tells us that the extreme values of a are $\pm\omega^2 A$. Therefore, the magnitude of maximum speed and that of the maximum acceleration of a particle moving in simple harmonic motion are

$$v_{\max} = \omega A, \tag{1.9}$$

$$a_{\max} = \omega^2 A. \tag{1.10}$$

Figure 1.4a represents the displacement versus time for an arbitrary value of the phase constant. The velocity and acceleration curves are illustrated in Figures 1.4b and c. These curves show that the phase of the velocity differs from the phase of the displacement by $\pi/2$ rad, or 90°. That is, when x is a maximum or a minimum, the velocity is zero. Likewise, when x is zero, the speed is maximum.

Furthermore, note that the phase of the acceleration differs from the phase of the displacement by π rad, or 180°. That is, when x is a maximum, a is a maximum in the opposite direction.

Summarizing, the following properties of a particle moving in simple harmonic motion are important:

1. The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the *necessary and sufficient condition* for simple harmonic motion.

2. The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase, as shown in Figure 1.4.

3. The frequency and the period of the motion are independent of the amplitude.

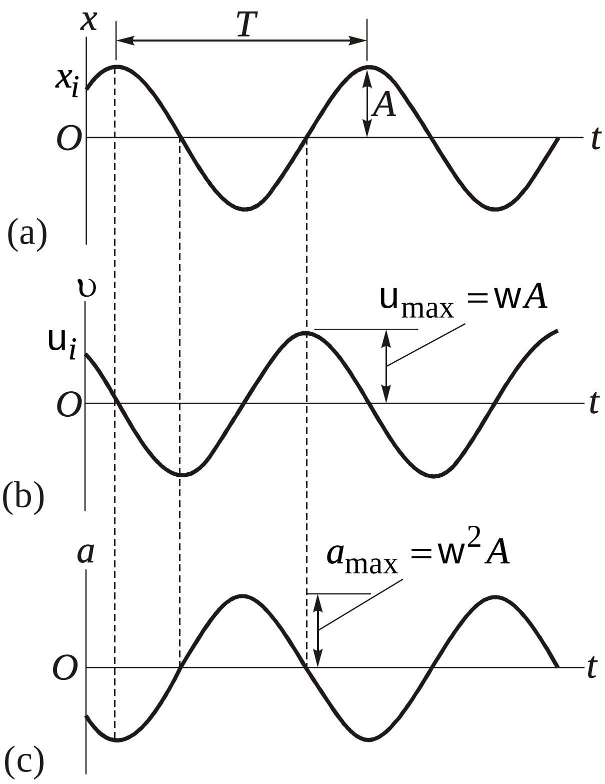


Figure 1.4 Graphical representation of simple harmonic motion. (a) Displacement versus time; (b) Velocity versus time; (c) Acceleration versus time. Note that at any specified time the velocity is 90° out of phase with the displacement and the acceleration is 180° out of phase with the displacement

Example 1.1

An object oscillates with SHM along the x axis. Its displacement from the origin varies with time according to the equation

$$x = 4\cos\left(\pi t + \frac{\pi}{4}\right),$$

where x is in metres, t is in seconds and the angles in the parentheses are in radians.

a) Determine the amplitude, frequency and period of the motion.

Solution. By comparing this equation with Eq. (1.2), ($x = A\cos(\omega t + \phi)$), the general equation for SHM, we see that $A = 4$ m and $\omega = \pi$ rad/s. Therefore,

$$f = \omega/2\pi = \pi/2\pi = 0.5 \text{ Hz}.$$

and $T = 1/f = 2$ s.

b) Calculate the velocity and acceleration of the object at any time t.

Solution. $$v = \frac{dx}{dt} = -(4\pi \text{ m/s})\sin\left(\pi t + \frac{\pi}{4}\right),$$

$$a = \frac{dv}{dt} = -(4\pi^2 \text{ m/s}^2)\cos\left(\pi t + \frac{\pi}{4}\right).$$

c) Determine the position, velocity and acceleration of an object at $t = 1$ s.

Solution. Noting that the angles in the trigonometric functions are the same, we obtain at $t = 1$:

$$x = (4)\cos\left(\pi + \frac{\pi}{4}\right) = (4)\cos\left(\frac{5\pi}{4}\right) = (4)(-0.707) = -2.83 \text{ m},$$

$$v = -(4\pi)\sin\left(\frac{5\pi}{4}\right) = -(4\pi)(-0.707) = 8.89 \text{ m/s},$$

$$a = -(4\pi^2)\cos\left(\frac{5\pi}{4}\right) = -(4\pi^2)(-0.707) = 27.9 \text{ m/s}^2.$$

d) Determine the maximum speed and maximum acceleration of the object.

Solution. In the general expressions for v and a found in part (b), we use the fact that the maximum values of the sine and cosine functions are unity. Therefore, v varies between $\pm 4\pi$ m/s, and a varies between $\pm 4\pi^2$ m/s. Thus,

$$v_{max} = \pm 4\pi \text{ m/s} = 12.6 \text{ m/s},$$

$$a_{max} = \pm 4\pi^2 \text{ m/s}^2 = 39.5 \text{ m/s}^2.$$

We obtain the same result using $v_{max} = \omega A$ and $a_{max} = \omega^2 A$, where $A = 4$ m and $\omega = \pi$ rad/s.

e) Find the displacement of the object between $t = 0$ and $t = 1$ s.

Solution. The x coordinate at $t = 0$ is

$$x = (4)\cos\left(0 + \frac{\pi}{4}\right) = (4)(-0.707) = 2.83 \text{ m}.$$

In part (c), we found that the x coordinate at $t = 1$ s is -2.83 m, therefore, the displacement between $t = 0$ and $t = 1$ s is

$$\Delta x = x_f - x_i = -2.83 \text{m} - 2.83 \text{m} = -5.66 \text{ m}.$$

Because the body's velocity changes its sign during the first second, the magnitude of Δx is not the same as the distance covered in the first second. (By the time the first second is over, the object has been through the point $x = -2.83$ m once, traveled to $x = -2.83$ m and come back to $x = -2.83$ m.)

Exercises

1.1. Which of the following is a necessary and sufficient condition for SHM?

a) constant period;

b) constant acceleration;

c) proportionality between acceleration and displacement from equilibrium position;

d) proportionality between restoring force and displacement from equilibrium position.

1.2. For a particle executing SHM which of the following statements is valid:

a) the total energy of the particle always remains the same;

b) the restoring force is maximum at extreme positions;

c) the restoring force is always directed towards a fixed point;

d) the velocity of the particle is maximum at the center of motion of the particle.

1.3. A vibrating object goes through five complete oscillations in 1 s. Find the angular frequency and the period of the motion. (Ans. $w = 31.4$ rad/s, $T = 0.2$ s.)

1.4. In Figure 1.1 the mass is displaced 0.12 m from its equilibrium position and released with no initial velocity. After 2 s its displacement is found to be 0.12 m on the opposite side and it has passed the equilibrium position once during this interval. Find the amplitude, the period, the frequency and the angular frequency.

1.5. Which of the following relationships between the acceleration a and the displacement x of a particle involve SHM: $a = 0.5x$, $a = 400x^2$, $a = -20x$, $a = -3x^2$?

1.6. Given $x = (2\,\text{m})\cos(5t)$ for SHM and needing to find the velocity at $t = 2$ s, should you substitute for t and then differentiate with respect to t or vice versa?

1.7. An object of mass 0.01 kg moves with SHM of amplitude 0.24 m and period 4 s. The coordinate is +0.24 m when $t = 0$. Compute

a) the position of the object when $t = 0.5$ s.

b) the magnitude and direction of the force acting on the object when $t = 0.5$ s.

c) the minimum time required for the object to move from its initial position to the point where $x = -0.12$ m.

d) the velocity of the object when $x = -0.12$ m.

1.8. An object is vibrating with SHM of amplitude 15 cm and frequency 4 Hz. Compute

a) the maximum values of the acceleration and velocity. (Ans. $94.7\,\text{m/s}^2$, 3.77 m/s);

b) the acceleration and velocity when the coordinate is 9 cm (Ans. –56.8 m/s^2, ± 3.02 m/s);

c) the time required to move from the equilibrium position to a point 12 cm distant from it. (Ans. 0.0369 s).

1.2 Block-Spring System

Let us return to the block-spring system (Figure 1.1). Again, we assume that the surface is frictionless; hence, when the block is displaced from equilibrium, the only force acting on it is the restoring force of the spring. Applying Newton's second law to the motion of the block, together with the Hook's law, we obtain:

$$F = -kx,$$
$$F = ma,$$
$$a = -\frac{k}{m}x. \qquad (1.11)$$

As we saw in Eq. (1.11), when the block is displaced a distance x from equilibrium, it experiences an acceleration $a = -(k/m)x$. If the block is displaced a maximum distance $x = A$ at some initial time and then released from rest, its initial acceleration at this instant is $-kA/m$ (extreme negative value). When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum. Then the block continues to travel to the left of equilibrium and finally reaches $x = -A$, at this time its acceleration is kA/m (maximum positive) and its speed is again zero. Thus, we see that the block oscillates between the turning points $x = \pm A$.

Recall that $a = dv/dt = d^2x/dt^2$ and so we can express Eq. (1.11) as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x,$$

$$\text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

If we denote the ratio k/m with the symbol w^2,

$$w^2 = \frac{k}{m}, \qquad (1.12)$$

this equation becomes

$$\frac{d^2x}{dt^2} + w^2 x = 0. \qquad (1.13)$$

Now we require a solution to Eq. (1.13), that is, a function $x(t)$ that satisfies this second-order differential equation. From the theory of differential equations it is known that this solution has form:

$$x = A\sin wt, \qquad (1.14a)$$
$$\text{or } x = A\cos wt, \qquad (1.14b)$$
$$\text{or } x = A\cos(wt + f). \qquad (1.14c)$$

To see this explicitly, assume that $x = A\cos(\omega t + \phi)$. Then

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi),$$

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 A\cos(\omega t + \phi).$$

Comparing this expressions for x and d^2x/dt^2, we see that $d^2x/dt^2 = -\omega^2 x$, and Eq. (1.13) is satisfied. We conclude that whenever *the force acting on a particle is linearly proportional to the displacement from some equilibrium position and in the opposite direction, the particle moves in simple harmonic motion.*

It can be shown in the same manner that function (1.14a) and (1.14b) are the solutions for the differential equation (1.13). We see that the essential difference among them is the position of the body at the instant of time we choose to call $t = 0$.

Equations (1.14) will form the basis of our further description of simple harmonic motion. For given values of A and ω, they differ in the position of the particle at time $t = 0$, that is, in the particular point in the cycle at which $t = 0$. If the body is given an initial displacement A at time $t = 0$ and released with no initial velocity, the motion is described by Eq. (1.14b). If the body is given an initial velocity v_0 at the equilibrium position ($x = 0$) at time $t = 0$, the appropriate equation is Eq. (1.14a). In that case, v_0 and A are related by

$$v_0 = \omega A,$$

since the velocity at $x = 0$ is the maximum velocity.

When the body is given *both* an initial displacement x_0 and an initial velocity v_0 at time $t = 0$, it is better to use Eq. (1.14c).

Suppose that at $t = 0$ the initial position of an oscillator is $x = x_0$ and its initial speed is $v = v_0$. Under these conditions, Eqs. (1.2) and (1.6) give

$$x_0 = A\cos\phi, \qquad (1.15)$$

and

$$v_0 = -\omega A\sin\phi. \qquad (1.16)$$

Dividing Eq. (1.16) by Eq. (1.15) eliminates A, giving $v_0/x_0 = -\omega\tan\phi$, or

$$\tan\phi = -\frac{v_0}{\omega x_0}. \qquad (1.17)$$

Furthermore, if we square Eq. (1.15) and (1.16), divide the velocity equation by ω^2, and then add terms, we obtain

$$x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = A^2(\cos^2\phi + \sin^2\phi),$$

or, using the identity $\cos^2\phi + \sin^2\phi = 1$,

$$x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = A^2, \tag{1.18}$$

which can be solved for A:

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}\ . \tag{1.19}$$

The amplitude is not equal to the initial displacement. This is reasonable: if at time $t = 0$ the particle has both an initial displacement x_0 in the positive direction and also a positive velocity v_0 in that direction, then it will *move farther* in that direction before returning; hence, A must be greater than x_0.

Recall that the period of any simple harmonic oscillator is $T = 2\pi/\omega$ and that the frequency is the inverse of the period. We know that $\omega = \sqrt{k/m}$, so we can express the period and frequency of the block-spring system as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}, \quad \text{and} \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{1.20}$$

That is, *the frequency and period depend only on the mass of the block and on the force constant of the spring.* Furthermore, the frequency and period are independent of the amplitude of the motion. As we might expect, the frequency is greater for a stiffer spring (the stiffer the spring, the greater the value of k) and decreases with increasing mass.

Example 1.2

A spring is mounted as in Figure 1.1. By attaching a spring balance to the free end and pulling sideways, we determine that the force is proportional to the displacement and that a force of 4 N causes a displacement of 0.02 m. We attach a 2-kg body to the end, pull it aside a distance of 0.04 m, and release it.

a) Find the force constant of the spring.

Solution. $k = \dfrac{F}{x} = \dfrac{4}{0.02} = 200$ N/m.

b) Find the period and frequency of vibration.

Solution. The period is:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{200}} = \frac{\pi}{5}\,\text{s} = 0.628\ \text{s}.$$

The frequency: $f = \dfrac{1}{T} = \dfrac{5}{\pi} = 1.59\,\text{Hz}$.

The angular frequency: $\omega = 2\pi f = 10$ rad/s.

c) Compute the maximum velocity attained by the vibrating body.

Solution. The maximum velocity occurs at the equilibrium position, where $x = 0$. For any x, from Eq. (1.18)

$$v = \pm\omega\sqrt{A^2 - x^2},$$

so when $x = 0$,

$$v = v_{\max} = \pm\omega A = (10)(0.04) = \pm 0.4 \text{ m/s}.$$

The same result we obtain, if we use Eq. (1.9).

d) Compute the maximum acceleration.

Solution. From Eq. (1.8),

$$a = -\frac{k}{m}x = -\omega^2 x.$$

The maximum acceleration occurs at the ends of the path, where $x = \pm A$. Therefore,

$$a_{\max} = \pm(10)^2(0.04) = \pm 0.4\,\text{m/s}^2.$$

e) Compute the velocity and acceleration of the body when it has moved halfway to the center from its initial position.

Solution. At this point,

$$x = \frac{A}{2} = 0.02 \text{ m},$$

$$v = -(10)\sqrt{(0.04)^2 - (0.02)^2} = -\left(\frac{2\sqrt{3}}{10}\right) = -0.346 \text{ m/s},$$

$$a = -\omega^2 x = -(10)^2(0.02) = -2.0\,\text{m/s}^2.$$

f) How much time is required for the body to move halfway to the center from the initial position?

Solution. The position at any time is given by $x = A\cos\omega t$. From this,

$$A/2 = A\cos(10\,t),$$

$$\cos(10t) = \frac{1}{2},$$

$$(10t) = \arccos\frac{1}{2} = \frac{\pi}{3},$$

$$t = \frac{\pi}{30} \text{ s}.$$

Example 1.3

Consider a mass m on a frictionless table connected to fixed points A and B (Figure 1.5) by two springs of equal natural length, negligible mass and spring constants k_1 and k_2 respectively. The mass is displaced horizontally and then released. What is the period of oscillation?

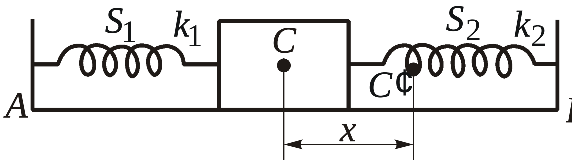


Figure 1.5 Mass m is connected by two springs

Solution. Let the mass be displaced horizontally to the position C' such that $CC' = x$. The spring S_1 is stretched through a distance x and exerts a force $-k_1 x$ along CA. Similarly, the string S_2 is compressed through a distance x and exerts a force $-k_2 x$ along BC. If d^2x/dt^2 be the acceleration of the block, then from the Newton's second law:

$$m\frac{d^2x}{dt^2} = -k_1 x - k_2 x = -(k_1 + k_2)x\,,$$

or

$$\frac{d^2x}{dt^2} + \frac{(k_1 + k_2)}{m}x = 0\,.$$

This equation represents SHM. Hence, the angular frequency and period are, corespondingly:

$$\omega = \sqrt{\frac{k_1 + k_2}{m}} \text{ and } T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}\,.$$

Exercises

1.9. A block of mass $m = 680$ g is fastened to a spring whose spring constant $k = 65$ N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

a) What are the angular frequency, frequency and the period of the resulting motion? (Ans. $\omega = 9.8$ rad/s, $f = 1.6$ Hz, $T = 640$ ms.)

b) What is the amplitude of oscillation? (Ans. $A = 11$ cm.)

c) What is the maximum speed v_{max} of the oscillating block, and where is the block when it occurs? (Ans. $v_{max} = 1.1$ m/s.)

d) What is the magnitude a_{max} of the maximum acceleration of the block? (Ans. $a_{max} = 11 \text{m/s}^2$.)

e) What is the phase constant? (Ans. $\varphi = 0$ rad.)

f) What is the displacement $x(t)$ for the block-spring system? (Ans. $x(t) = 0.11\cos(9.8t)$).

1.10. At $t = 0$, the displacement of the block is -9.50 cm. The block's velocity $v(0)$ is then -0.920 m/s and its acceleration $a(0) = 47.0 \text{m/s}^2$.

a) What is the angular frequency of this system? (Ans. $\omega = 23.5$ rad/s.)

b) What is the phase constant φ and amplitude A? (Ans. $\varphi = 155°$ and $A = 9.4$ cm.)

1.11. A harmonic oscillator has a mass of 0.5 kg and a spring of unknown force constant. It is found to have a period of 0.20 s. Find the force constant of the spring.

1.12. A block of unknown mass is attached to a string of force constant 200 N/m. It is found to vibrate with a frequency of 3.0 Hz. Find the period, the angular frequency and the mass. (Ans. $T = 0.333\text{s}$, $\omega = 18.8$ rad/s, $m = 0.563$ kg.)

1.3 Block-Spring System in Vertical Plane

Suppose we turn the system of Figure 1.1 by 90°, so the mass hangs vertically from the spring (see Figure 1.6a). The motion does not change in any essential way. In Figure 1.6b, a body of mass m hangs in equilibrium from a spring with force constant k. In this position the spring is stretched an amount Δl just great enough so that the spring's upward vertical force $k\Delta l$ on the body balances its weight mg. In this case, $k\Delta l = mg$.

When the body is at a distance x above its equilibrium position, as in Figure 1.6c, the extension of the spring is $\Delta l - x$. The upward force it exerts on the body is then $k(\Delta l - x)$, and the resultant force F on the body is

$$F = k(\Delta l - x) - mg = -kx,$$

that is, a net downward force of magnitude kx. Similarly, when the body is *below* the equilibrium position, there is a net upward force proportional to x. Therefore, the equation of motion is the same as Eq. (1.13). Hence, if the body is set in vertical motion, it oscillates with SHM, with the same angular frequency as though it were horizontal, i.e., the motion is described by Eq. (1.13) and its angular frequency is

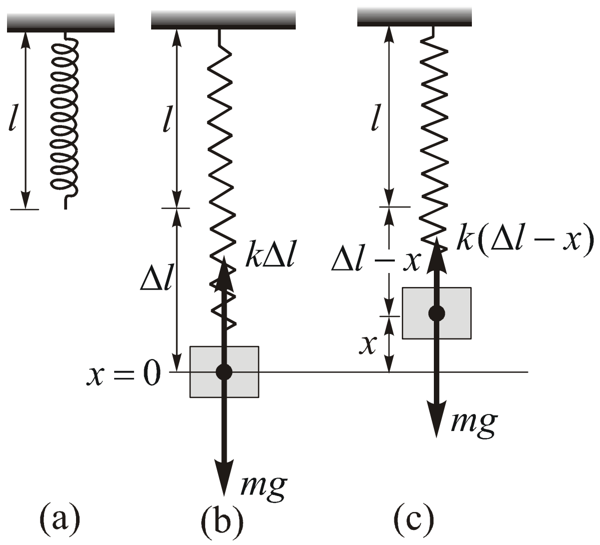


Figure 1.6 The restoring force on a body suspended by a spring is proportional to the coordinate measured from the equilibrium position

$$\omega = \sqrt{k/m}.$$

Example 1.4

Two springs A and B each of length l, have a force constants k_1 and k_2. Find the force constant k of the spring system, if they are connected (a) in parallel, (b) in series in vertical plane.

Solution.

Let body of mass m be suspended from the combination of the springs.

(a) Springs are connected in parallel.

Force $F = mg$, acted on the spring's system can be resolved into two components, $F = F_1 + F_2$. One of them, F_1, is applied to the spring A, the other – to the spring B. Elongations of the both spring x are the same.

The tension in A is $F_1 = k_1 x$ and tension in B is $F_2 = k_2 x$. Hence for the system of spring we can write

$$F = kx$$

where k the spring constant of the combination. We can rewrite this equation as

$$F_1 + F_2 = kx, \text{ ors}$$

$$k_1 x + k_2 x = kx, \text{ or}$$

$$k = k_1 + k_2 .$$

(b) Springs are connected in series,

The elongations in springs A and B, produced by force F, will be $x_1 = F / k_1$ and $x_2 = F / k_2$, respectively. The total elongation is

$$x = x_1 + x_2 = \left(\frac{F}{k_1} + \frac{F}{k_2}\right) = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = F\left(\frac{k_2 + k_1}{k_1 k_2}\right).$$

And, finally, the force constant of the system is

$$k = \frac{F}{x} = \frac{F}{F\left(\frac{k_2 + k_1}{k_1 k_2}\right)} = \frac{k_1 k_2}{k_2 + k_1}.$$

Example 1.5

A body of mass 5 kg is suspended by a string, which stretches 0.1 m when the body is attached. The body is then displaced downward an additional 0.05 m and released. Find the amplitude, the period and the frequency of the resulting SHM.

Solution. Since the initial position is 0.05 m from equilibrium and there is no initial velocity, $A = 0.05$ m. To find the period we first find the force constant of the string. The string is stretched 0.1 m by a force of $(5\,\text{kg})(9.8\,\text{m/s}^2)$, so

$$k = \frac{mg}{\Delta l} = \frac{(5)(9.8)}{0.1\text{m}} = 490 \text{ N/m}.$$

The period T is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5\,\text{kg}}{490\,\text{N/m}}} = 0.635 \text{ s}.$$

The frequency of SHM:

$$f = \frac{1}{T} = 1.57 \text{ Hz}.$$

Example 1.6

The vertical motion of a huge piston in a machine is approximately simple harmonic with a frequency $f = 0.5$ Hz. A block of 10 kg is placed on the piston. What is the maximum amplitude of the piston's SHM, for the block and piston to remain together?

Solution. Here, $f = 0.5$ Hz, $g = 9.8$ m/s^2. When displacement is x, the acceleration of SHM is given by

$$a = w^2 x = (2pf)^2 x = 4p^2 f^2 x .$$

The acceleration will be maximum at the extreme position $x = A$, i.e.

$$a_{\max} = 4p^2 f^2 A .$$

The block will remain in contact with the piston, if $a_{\max}$ does not exceed the acceleration due to gravity, i.e. $a_{\max}$ is the most equal to g. i.e.

$$4p^2 f^2 A = g ,$$

or $A = \dfrac{g}{4p^2 f^2} = \dfrac{9.8 \text{ m/s}^2}{4p^2 (0.5)^2} = 0.993$ m.

Exercises

1.13. Four passengers whose combined mass is 300 kg are observed to compress the automobile by 5 cm when they enter the automobile. If the total load supported by the string is 900 kg, find the period of vibration of the loaded automobile.

1.14. Choose the answer: Two bodies M and N of equal masses are suspended from two separate massless spring with spring constants k_1 and k_2 respectively. If the two bodies oscillate vertically such that their maximum velocities are equal, the ratio of the amplitude of M to that of N is

k_1 / k_2, k_2 / k_1, $\sqrt{k_1 / k_2}$, $\sqrt{k_2 / k_1}$. (Recall that $v_{\max} = wA$.)

1.15. The period of a mass suspended by a spring (force constant k) is T. If the spring is cut in three equal pieces, what will be the force constant of each part and what will be the period? (Ans. $3k$, $T\sqrt{3}$.)

1.16. A block of mass 2 kg is suspended from a spring of negligible mass and is found to stretch the string 0.20 m.

a) What is the force constant of the spring? (Ans. 98.0 N/m.)

b) What is the period of oscillation of the block if pulled down and released? (Ans. 0.898 s.)

c) What would be the period of the block of mass 4 kg hanging from the same spring? (Ans. 1.27 s.)

1.17. The scale of a spring balance reading from zero to 180 N is 9 cm long. A fish suspended from the balance is observed to oscillate vertically at 1.5 Hz. What is the mass of the fish? Neglect the mass of the spring.

1.18. A block of mass 5 kg hangs from a spring and oscillates with a period of 0.5 s. How much will the spring shorten when the block is removed? (Ans. 0.0620 m.)

1.4 Simple Pendulum

A simple pendulum is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Figure 1.7. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle θ is small (less than about 10°) the motion is that of a simple harmonic oscillator.

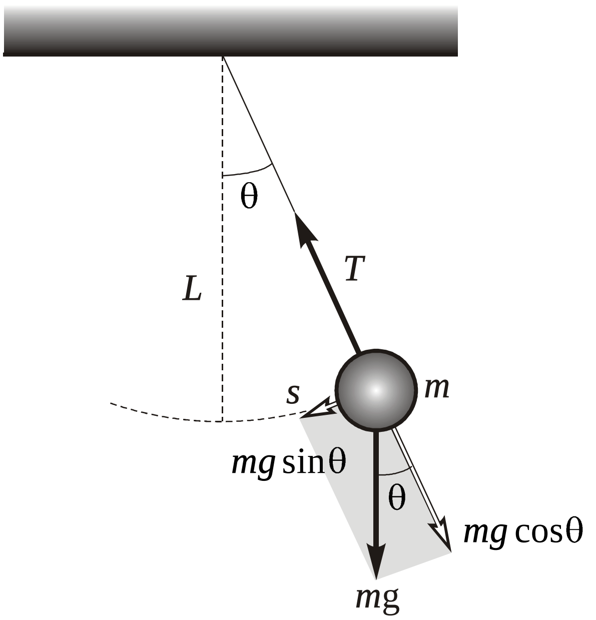


Figure 1.7 When θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\theta = 0$. The restoring force is $mg\sin\theta$, the component of the gravitational force tangent to the arc

The forces acting on the bob are the force T exerted by the string and the gravitational force mg. The tangential component of the gravitational force $mg\sin\theta$ always acts towards $\theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force, and we can apply Newton's second law for motion in the tangential direction:

$$\sum F_t = -mg\sin\theta = m\frac{d^2s}{dt^2},$$

where s is the bob's displacement measured along the arc. The minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because $s = L\theta$ and L is constant, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta .$$

The right side is proportional to $\sin\theta$ rather than to θ; hence, with $\sin\theta$ present, we would not expect simple harmonic motion because this expression is not of the form of Eq. (1.13). However, if we assume that θ is small, we can use the approximation $\sin\theta \approx \theta$; thus, the equation of motion for the simple pendulum becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta . \qquad (1.21)$$

Now we have an expression of the same form as Eq. (1.13), and we conclude that the motion for small amplitudes of oscillation is simple harmonic motion, therefore, θ can be written as

$$\theta = \theta_{max} \cos(\omega t + \phi), \tag{1.22}$$

where θ_{max} is the *maximum angular displacement* and the angular frequency ω is

$$\omega = \sqrt{\frac{g}{L}}.$$

The period of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}. \tag{1.23}$$

In other words, *the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.* Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and are at the same location (so that g is the same), oscillate with the same period.

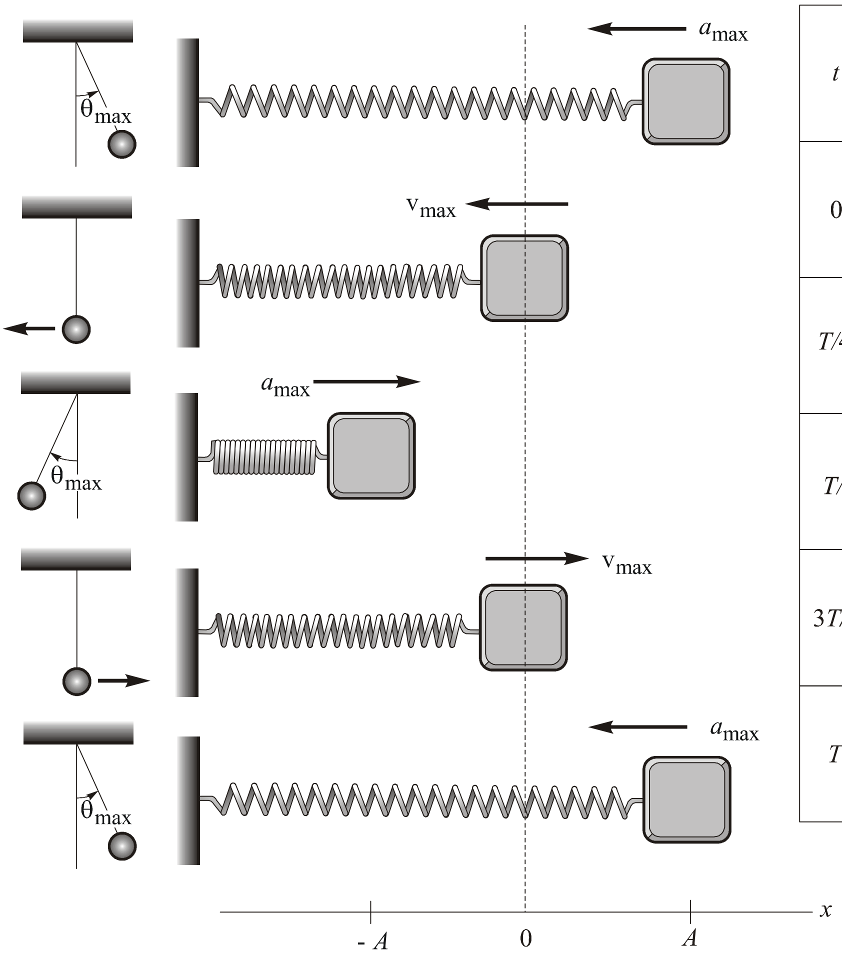
We emphasize again that the motion of a pendulum is only approximately SHM; when the amplitude is not small, the departures from SHM can be substantial. But how small is this "small"? The period can be expressed by an infinite series; when the maximum angular displacement is θ_{max}, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\theta}{2} + \frac{1^2 \times 3^2}{2^2 \times 4^2}\sin^4\frac{\theta}{2} + \ldots\right).$$

We can compute the period to any desired degree of precision by taking enough terms in series. When $\theta_{max} = 15°$, the true period differs from that given by the approximate Eq. (1.23) by less than 0.5%.

The analogy between the motion of a simple pendulum and that of a block-spring system is illustrated in Figure 1.8.

The simple pendulum of certain length can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and of other valuable underground resources.



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Figure 1.8 Simple harmonic motion for a block-spring system and its relationship to the motion of a simple pendulum. The parameters in the table refer to the block-spring system, assuming that $x = A$ at $t = 0$; thus, $x = A\cos wt$

Simple pendulum was first used by the French physicist Jean Foucault to verify the Earth's rotation experimentally. As the pendulum swings, the vertical plane in which it oscillates appears to rotate as the bob successively knocks over the indicators arranged in a circle on the floor. In reality, the plane of oscillation is fixed in space, and the Earth rotating beneath the swinging pendulum moves the indicators into the position to be knocked down, one after the other.

Example 1.7

A simple pendulum of length L and mass m is suspended in a car that is traveling with a constant speed v around a circle of radius r. If the pendulum undergoes small oscillations about its equilibrium position, what will be its frequency of oscillation?

Solution. Here the car is an accelerated frame of reference. A fictitious force mv^2 / r is to be introduced as a centrifugal force (Figure 1.9).

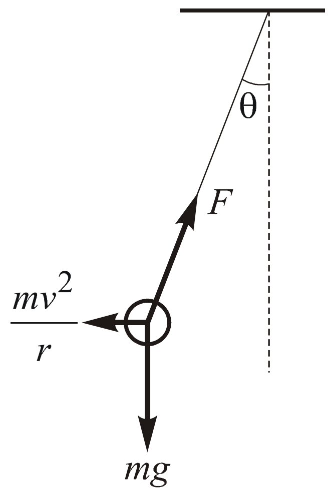


Figure 1.9 A simple pendulum in a car traveling around a circle

From Figure 1.9

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = \frac{mv^2}{r},$$

Hence $F = m\sqrt{\left(g^2 + \frac{v^4}{r^2}\right)}$.

When the pendulum is slightly displaced such that it makes an angle $(\theta + d\theta)$ with the vertical, then there will be a restoring force $F\sin d\theta \approx F d\theta = Fx/L$, where $x = L d\theta$ is linear displacement. Restoring force per unit displacement is F/L.

The period is:

$$T = 2\pi\sqrt{\frac{m}{F/L}} = 2\pi\sqrt{\frac{mL}{m\sqrt{g^2 + \frac{v^4}{r^2}}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2 + \frac{v^4}{r^2}}}}.$$

The frequency of oscillation:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\sqrt{g^2 + \frac{v^4}{r^2}}}{L}}.$$

Exercises

1.19. A simple pendulum 4 m long swings with amplitude of 0.2 m.

a) Compute the linear velocity v of the pendulum at its lowest point.

b) Compute its linear acceleration a at the end of its path.

1.20. Find the length of a simple pendulum whose period is exactly 1 s at a point where $g = 9.8\,\text{m/s}^2$ (Ans. 0.248 m.)

1.21. A simple pendulum has a period of 2.0 s on the Earth. What is its period on the Moon, where $g = 1.7\,\text{m/s}^2$?

1.22. When period of a simple pendulum is doubled

a) its length is doubled;

b) the mass of the bob is doubled;

c) its length is made four times;

d) the mass of the bob and the length of the pendulum are doubled?

1.23. The period of oscillation of a simple pendulum with length L at the Earth surface is T. Its period inside a mine is

a) greater than T,
b) less than T,
c) equals to T,
d) can not be computed.

1.24. What is the length of a simple pendulum that marks seconds by completing a full swing from left to right and then back again every 2.0 s?

1.25. A performer seated on a trapeze is swinging back and forth with a period of 8.85 s. If he stands up, so that the center of mass of the trapeze+performer system rises by 35.0 cm, what will be the new period of oscillation of the system? Treat trapeze + performer as a simple pendulum.

1.26. When a small sphere is hung from the end of an elastic string, the length of the simple pendulum obtained is 40 cm. The time for 20 small oscillations of this pendulum is 26 s. The bob is then changed to a sphere of the same size but different mass. The new time for 20 oscillations is 26.4 s. Calculate the ratio of the masses. (Ans. $\frac{m_1}{m_2} = \frac{1.98}{3.29} = 0.602$.)

1.5 Physical Pendulum

Suppose you balance a small wheel so that it is supported by your extended index finger. When you give the wheel a small displacement (with your other hand) and then release it, it oscillates. If a *hanging object oscillates about a fixed axis that does not pass through its center of mass and the object cannot be approximated as a point mass*, we cannot treat the system as a simple pendulum. In this case the system is called a *physical pendulum*.

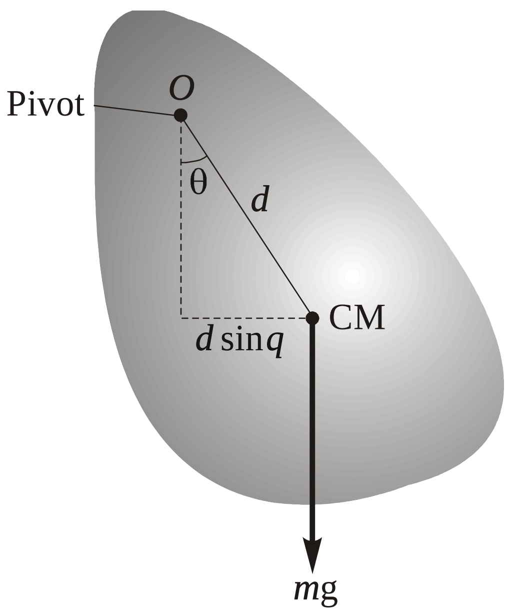


Figure 1.10 A physical pendulum

Consider a rigid body pivoted at a point O that is a distance d from the center of mass (Figure 1.10). The force of gravity provides a torque about an axis through O, and the magnitude of that torque is $mgd \sin q$, where d is as shown in Figure 1.10. Using the law of motion $t = Ia$, where I is the moment of inertia about the axis passig through O, $d^2 q / dt^2$ is the angular acceleration, we obtain

$$- mgd \sin q = I \frac{d^2 q}{dt^2}.$$

The minus sign indicates that the torque about O tends to decrease q.

That is, the force of gravity produces a restoring torque. Because this equation gives us the angular acceleration $d^2\theta / dt^2$ of the pivoted body, we can consider it the equation of motion for the system. If we again assume that θ is small, the approximation $\sin\theta \approx \theta$ is valid, and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta . \tag{1.24}$$

Because this equation is of the same form as Eq. (1.13), the motion is simple harmonic motion. That is, the solution of Eq. (1.24) is $\theta = \theta_{\max}\cos(\omega t + \varphi)$, where $\theta_{\max}$ is the maximum angular displacement and

$$\omega = \sqrt{\frac{mgd}{I}} . \tag{1.25}$$

The period of oscillations is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} . \tag{1.26}$$

We can use this result to measure the moment of inertia of a flat rigid body. If the location of the center of mass – and hence the value of d – is known, the moment of inertia can be obtained by measuring the period. Finally, note that Eq. (1.26) reduces to the period of a simple pendulum when $I = md^2$, that is, when all the mass is concentrated at the center of mass.

Sometimes the quantity

$$L_e = \frac{I}{md} , \tag{1.27}$$

which is called the ***effective length*** of physical pendulum, is used. As it is clear from Eq. (1.27), the effective length depends on moment of inertia, that is, distribution of mass over the pendulum and its shape. Substituting the expression for L_e into the Eq. (1.26), we obtain

$$T = 2\pi\sqrt{\frac{L_e}{g}} ,$$

that is, the same expression as for the simple pendulum. Therefore, the effective length of physical pendulum is the length of simple pendulum which has the same period of oscillations as the given physical one.

Example 1.8

How can the period of a physical pendulum be used to determine its moment of inertia?

Solution. Eq. (1.26) may be solved for the moment of inertia I, giving

$$I = \frac{T^2 mgh}{4\pi^2} .$$

The quantities on the right of the equation can all be measured directly. Hence the moment of inertia of a body of any complex shape may be found by suspending the body as a physical pendulum and measuring its period of oscillation. We can find the center of gravity by balancing. Since T, m, g and h are known, we can compute I.

Example 1.9

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of oscillations if the amplitude of the motion is small.

Solution. Moment of inertia of a uniform rod about an axis through one end is $\frac{1}{3}ML^2$. The distance d from the pivot to the center of mass is $L/2$. Substituting these quantities into Eq.(1.26) gives

$$T = 2p\sqrt{\frac{\frac{1}{3}ML^2}{mg(L/2)}} = 2p\sqrt{\frac{2L}{3g}}.$$

Exercises

1.27. A thin uniform rod of length L and mass m is pivoted about a perpendicular axis through the rod at a distance $L/4$ from one end.

a) Find the moment of inertia about this axis. (Ans. $7mL^2/48$.)

b) Find the period of oscillation of the rod. (Ans. $2p\sqrt{7L/12g}$.)

1.28. A monkey wrench is pivoted at one end and allowed to swing as a physical pendulum. The period is 0.9 s, the pivot is 0.20 m from the center of gravity.

a) What is the ratio of moment of inertia to mass for the wrench, about an axis through the pivot?

b) If the wrench was initially displaced 0.1 rad from its equilibrium position, what is the angular velocity of the wrench as it passes through the equilibrium position?

1.29. A meter stick swings about a pivot at one end at distance h from its center of mass. What its period of oscillation T? (Ans. 1.64 s.)

1.6 Torsional Pendulum

In the Figure 1.11, there is shown an angular version of simple harmonic oscillator, the element of springless or elasticity is associated with twisting of suspention wire rather than the extension and compression. The divice is called a *torsional pendulum*, with torsion referring to the twisting. Torsional pendulum is a rigid body suspended by a wire attached at the top to a fixed support. When the body is twisted through an angle q, the twisted wire exerts a restoring torque on the body . The torque is proportional to the angular displacement. That is,

$$t = -kq,$$

where k (kappa) is called the *torsion constant* of the support wire. The value of k can be obtained by applying a known torque to twist the wire through a measurable angle q.

Applying Newton's second law for rotational motion, we find

$$t = -kq = I\frac{d^2q}{dt^2},$$

$$\frac{d^2q}{dt^2} = -\frac{k}{I}q. \qquad (1.28)$$

Again, this is the equation of motion for a simple harmonic oscillator, with $w = \sqrt{k/I}$ and a period

$$T = 2p\sqrt{\frac{I}{k}} \qquad (1.29)$$

There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded.

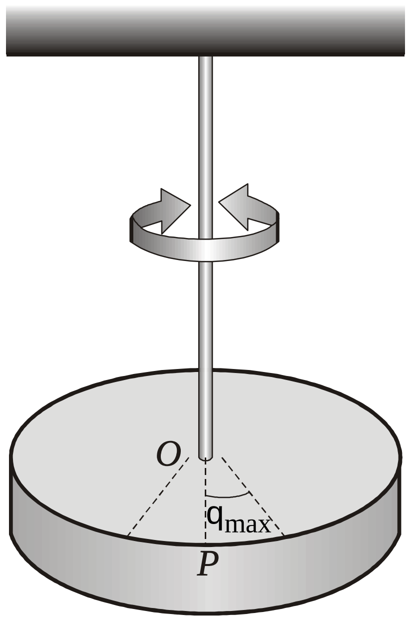


Figure 1.11 A torsional pendulum of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line OP with an amplitude q_{max}

Example 1.10

A watch has a balanced wheel which performs angular simple harmonic motion of period 0.5 s and maximum angular displacement of p radian. What is the maximum angular velocity of the wheel?

Solution. From $T = \frac{1}{f}$, the angular frequency is $2pf = \frac{2p}{T} = \frac{2p}{0.5} = 4p$ and the maximum angular velocity $w_{max} = 2pfq = 4p(p) = 39.5$ rad/s.

Example 1.11

A thin rod of length L and mass $m = 135$ g is suspended from a long wire at its midpoint. Its period T of angular SHM is measured to be 2.53 s. An irregularly shaped object is then hung from the same wire, and its period $T_o = 4.76$ s. What is the moment of inertia of object about its suspension axis?

Solution. The moment of inertia of either the rod or an object is related to the measured period by Eq. (1.29). We know that moment of inertia of a thin rod about a perpendicular axis passing through its midpoint is $I = \frac{1}{12} mL^2$. Thus, we have

$$I = \frac{1}{12} mL^2 = \frac{1}{12}(0.135)(0.124)^2 = 1.73 \times 10^{-4} \text{kg} \cdot \text{m}^2.$$

The period of rod is $T = 2\pi\sqrt{\frac{I_r}{k}}$, and the period of the object is $T_o = 2\pi\sqrt{\frac{I_o}{k}}$. The constant k, which is the property of the wire, is the same for both bodies. Lets square each of equations, divide the second by the first one and solve the resulting equation for I_o. The result is

$$I_o = I_r \frac{T_o^2}{T_r^2} = (1.73 \times 10^{-4} \text{kg} \cdot \text{m}^2) \frac{(4.76 \text{s})^2}{(2.53 \text{s})^2} = 6.12 \times 10^{-4} \text{kg} \cdot \text{m}^2.$$

Exercises

1.30. The balance wheel of a watch vibrates with the angular amplitude of π rad and a period of 0.5 s.

a) Find its maximum velocity.

b) Find its angular velocity when the displacement is the one-half magnitude.

c) Find its angular acceleration when the displacement is $45°$.

1.31. An alarm clock ticks four times each second, each tick representing half a period. The balance wheel consists of a thin rim of radius 1.5 cm, connected to the balance staff by thin spokes of negligible mass. The total mass of the balance wheel is 0.8 g.

a) What is the moment of inertia of the balance wheel? (Ans. 1.80×10^{-7} $\text{kg} \cdot \text{m}^2$.)

b) What is the torque constant of the hairspring? (Ans. 2.84×10^{-5} $\text{N} \cdot \text{m/rad}^2$.)

1.32. A torsional pendulum is formed by attaching a wire to the center of a meter stick with a mass of 2 kg. If the resulting period is 3 min, what is the torsion constant for the wire?

1.33. A clock balance wheel has a period of oscillation of 0.25 s. The wheel is constructed so that 20 g of mass is concentrated around a rim of radius 0.5 cm. What are (a) the wheel's moment of inertia and (b) the torsion constant of the attached spring?

1.7 Energy of Simple Harmonic Oscillator

Let us examine the mechanical energy of a block-spring system illustrated in Figure 1.12. Because the surface is frictionless, we expect the total mechanical energy to be constant. We can apply Eq.(1.6) to express the kinetic energy as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mw^2A^2\sin^2(wt + f). \quad (1.30)$$

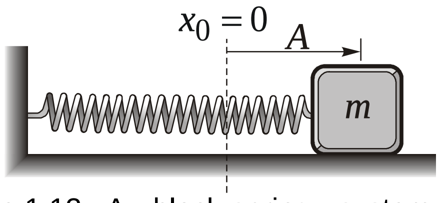


Figure 1.12 A block-spring system that starts from rest at $x = A$. In this case $f = 0$ and, thus, $x = A\cos wt$

As the elastic restoring force is a conservative force, we can represent the work done by this force in terms of potential energy. The elastic potential energy stored in the spring of any elongation x is given by $\frac{1}{2}kx^2$. Using Eq. (1.2), we obtain

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(wt + f). \quad (1.31)$$

We see that K and U are *always* positive quantities. Because $w^2 = k/m$, we can express the total mechanical energy of a simple harmonic oscillator as

$$E = K + U = \frac{1}{2}kA^2\left[\sin^2(wt + f) + \cos^2(wt + f)\right].$$

From the identity $\sin^2(wt + f) + \cos^2(wt + f) = 1$, we see that the quantity in square brackets is a unity. Therefore, this equation reduces to

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \quad (1.32)$$

The total mechanical energy is equal to the maximum potential energy stored in the spring when $x = \pm A$ because $v = 0$ at these points and thus there is no kinetic energy. At the equilibrium position, where $U = 0$ because $x = 0$, the total energy, all in the form of kinetic energy, is again $E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}kA^2$. That is,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = const. \quad (1.33)$$

That is, *the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.* Note that U is small when K is large and vice versa, because the sum must be constant. Plots of the kinetic and potential energies versus time is shown in Figure 1.13, where we have taken $f = 0$. As already mentioned, both K and U are always positive, and at all times their sum is a constant equal to $\frac{1}{2}kA^2$, the total energy of the system. Energy is continuously transformed between potential energy stored in the spring and kinetic energy of the block.

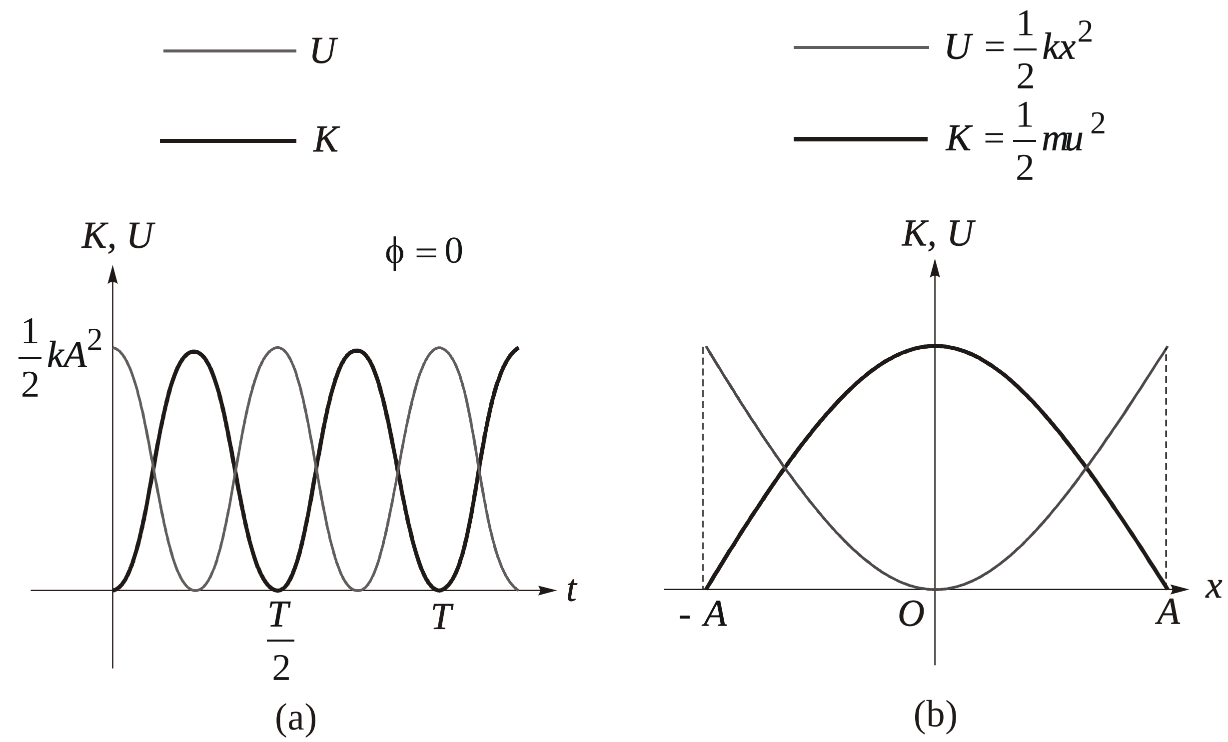


Figure 1.13 (a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with $f = 0$. (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In plot (b), note that $K + U = const$

We can often see another representation of Eq. (1.32), shown by the graph in Figure 1.14, where energy is plotted vertically and the coordinate x horizontally. The curve represents the potential energy, $U = \frac{1}{2}kx^2$. As we can see, the curve is a parabola. The horizontal line at height E represents the constant total energy of the body. We see that the body's motion is restricted to values of x lying between the points where the horizontal line intersects the parabola. If x were outside this range, the potential energy would exceed the total energy that is impossible.

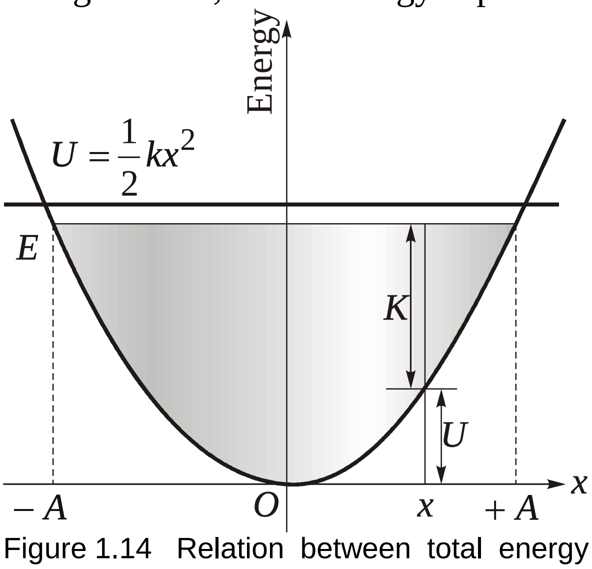


Figure 1.14 Relation between total energy E, potential energy U and kinetic energy K for a body oscillating with SHM

If we draw a vertical line at any value of x within the permitted range, the length of the segment between the x-axis and the parabola represents the potential energy U at that value of x. The length of the segment between the parabola and the horizontal line at height E represents the corresponding kinetic energy K. At the endpoints, the energy is all potential, and at the middle points it is all kinetic. The speed has its maximum value $v_{\max}$ at the midpoint:

$$\frac{1}{2}mv^2{}_{\max} = E, \quad v_{\max} = \sqrt{\frac{2E}{m}}.$$

Let's return to the Figure 1.8 which illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block-spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure.

Finally, we can use the principle of energy conservation to obtain the velocity of an arbitrary displacement by expressing the total energy at some arbitrary position x as

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$$

Then velocity is

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)} = \pm w\sqrt{\left(A^2 - x^2\right)}. \qquad (1.34)$$

When we check this equation to see whether it agrees with known cases, we find that it substantiates the fact that the speed is a maximum at $x = 0$ and is zero at the turning points $x = \pm A$.

Example 1.12

The system in Fig. 1.1 is given an initial displacement of 0.05 m, and an initial velocity of 2 m/s. Find the amplitude, the phase angle and the total energy of the motion and write an equation for the position as a function of time.

Solution

From Eq. (1.19),

$$A = \sqrt{x_0^2 + (v_0 / w)^2} = \sqrt{(0.05\,\text{m})^2 + (2\,(\text{m/s})/10\,\text{s}^{-1})^2} = 0.206\ \text{m}.$$

From Eq. (1.17),

$$q_0 = \arctan\frac{-v_0}{wx_0} = \arctan\frac{-2\,\text{m/s}}{(10\,\text{s}^{-1})(0.05\,\text{m})} = -76.0^\circ = -1.33\ \text{rad}.$$

From Eq. (1.32) and the following discussion,

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(200\,\text{N/m})(0.206\,\text{m})^2 = 4.26\ \text{J}.$$

Alternatively, from the initial conditions,

$$E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = \frac{1}{2}(2\,\text{kg})(2\,\text{m/s}) + \frac{1}{2}(200\,\text{N/m})(0.05\,\text{m})^2 = 4.24\ \text{J}.$$

The x according to Eq. (1.2) is given by

$$x = (0.206\,\text{m})\cos[(10\text{s}^{-1})t - 1.33\,\text{rad}].$$

Example 1.13

A 0.5 kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal frictionless track.

a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

Solution

Using Eq. (1.32), we obtain

$$E = K + U = \frac{1}{2}kA^2 = \frac{1}{2}(20\,\text{N/m})(3\times 10^{-2}\,\text{m}) = 9\times 10^{-3}\ \text{J}.$$

When the cube is at $x = 0$, we know that $U = 0$ and $E = \frac{1}{2}mv_{\max}^2$; therefore,

$$E = \frac{1}{2}mv_{\max}^2 = 9\times 10^{-3}\ \text{J}.$$

b) What is the velocity of the cube when the displacement is zero?

Solution

We can apply Eq. (1.19) directly

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\sqrt{\frac{20\,\text{N/m}}{0.5\,\text{kg}}[(0.03\,\text{m})^2 - (0.02\,\text{m})^2]} = \pm 0.141\ \text{m/s}.$$

In the expression, the positive and negative signs indicate that the cube could be moving either to the right or to the left at this instant.

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Solution

Using the result of (b), we find that

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(0.5\,\text{kg})(0.141\,\text{m/s})^2 = 5\times 10^{-3}\,\text{J},$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(20\,\text{N/m})(0.02\,\text{m})^2 = 4\times 10^{-3}\,\text{J}.$$

Note, that $K + U = E$.

Exercises

1.34. An object of mass 4 kg is attached to a string of force constant $k = 100$ N/m. The object is given an initial velocity of $v_0 = 12$ m/s and an initial displacement of $x_0 = 0$. Find the amplitude, the phase angle, and the total energy of the motion and write the equation for the position as a function of time.

1.35. A block of mass 4 kg is attached to a coil spring and oscillates vertically in SHM. The amplitude is 0.5 m, and at the highest point of the motion, the spring has its natural unstretched length. Calculate the elastic potential energy of the spring, the kinetic energy of the body, its gravitational potential energy relative to the lowest point of the motion and the sum of these three energies, when the body is:

a) at the lowest point, (Ans. 39.2 J, 0, 0, 39.2 J.)

b) at is equilibrium position, (Ans. 9.8 J, 9.8 J, 9.8 J, 39.2 J.)

c) at its highest point. (Ans. 0, 0, 39.2 J, 39.2 J.)

1.36. A 200-g mass is attached to a spring and undergoes simple harmonic motion with a period of 0.250 s. If the total energy of the system is 2 J, find (a) the force constant of the spring and (b) the amplitude of the motion.

1.37. An automobile having a mass of 1 000 kg is driven into a brick wall in a safety test. The bumper behaves as a spring of constant 5×10^6 N/m and compresses 3.16 cm as the car is brought to rest. What was the speed of the car before the impact, assuming that no energy is lost during the impact with the wall?

1.38. A mass-spring system oscillates with amplitude of 3.5 cm. If the spring constant is 250 N/m and the mass is 0.5 kg, determine (a) the mechanical energy of the system, (b) the maximum speed of the mass, and (c) the maximum acceleration.

1.39. A 50-g mass connected to a spring with a force constant of 35 N/m oscillates on a horizontal, frictionless surface with an amplitude of 4 cm. Find (a) the total energy of the system and (b) the speed of the mass when the displacement is 1 cm. Find (c) the kinetic energy and (d) the potential energy when the displacement is 3.00 cm.

1.40. A 2.00-kg mass is attached to a spring and placed on a horizontal, smooth surface. A horizontal force of 20 N is required to hold the mass at rest when it is pulled 0.2 m from its equilibrium position (the origin of the x axis). The mass is now released from rest with an initial displacement of $x_1 = 0.2$ m, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the mass. Where does this maximum speed occur? (d) Find the maximum acceleration of the mass. Where does it occur? (e) Find the total energy of the oscillating system. Find (f) the speed and (g) the acceleration when the displacement equals one third of the maximum value.

1.41. A 1.5-kg block at rest on a tabletop is attached to a horizontal spring having a force constant of 19.6 N/m. The spring is initially unstretched. A constant 20-N horizontal force is applied to the object causing the spring to stretch. (a) Determine the speed of the block after it has moved 0.3 m from equilibrium, assuming that the surface between the block and the tabletop is frictionless. (b) Answer part (a) for a coefficient of kinetic friction of 0.200 between the block and the tabletop.

1.42. The amplitude of a system moving in simple harmonic motion is doubled. Determine the change in (a) the total energy, (b) the maximum speed, (c) the maximum acceleration, and (d) the period.

1.8 Circle of Reference

We can gain additional insight into simple harmonic motion through a geometric representation called the *circle of reference*. This representation makes use of a close relationship between SHM and uniform circular motion, which we studied earlier. The basic idea is shown in Figure 1.15. Point Q moves counterclockwise around a circle with a radius A that is equal to the amplitude of the actual simple harmonic motion, with the constant angular velocity ω (measured in rad/s). Thus, ω is the rate of change of the angle θ; $\omega = d\theta/dt$.

The vector from O to Q is the position vector of point Q relative to O. This vector has the constant magnitude A and at time t is at an angle θ, measured counterclockwise from the positive x-axis. As Q moves, this vector rotates counterclockwise with the constant angular velocity $\omega = d\theta/dt$. The horizontal component of this vector represents the actual motion of the body under study. Such a rotating vector is called a *phasor*. This representation is also useful in many other areas of physics where we encounter quantities that vary sinusoidally with time, including *ac*-circuit analysis and interference phenomena in optics.

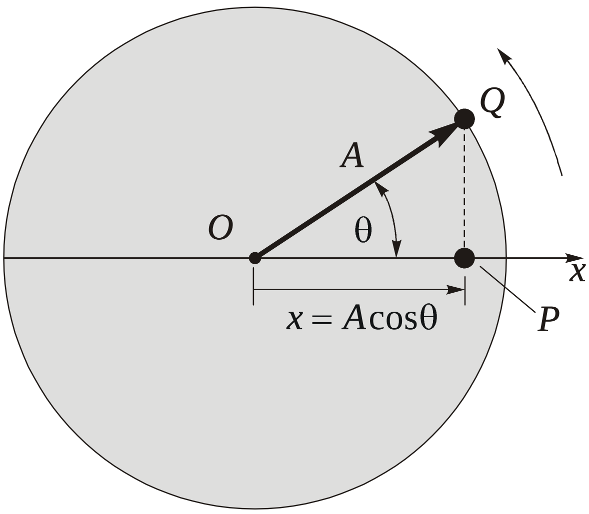


Figure 1.15 Coordinates of a body in SHM

In Figure 1.15, point P lies on the horizontal diameter of the circle, directly below Q. We call Q the *reference point,* the circle – the *reference circle,* and P – the *projection* of Q onto the diameter. The location of P can be treated as a *shadow* of Q on the x-axis, cast by a light beam perpendicular to the x-axis. As Q revolves, P moves back and forth along the diameter, staying always directly below (or above) Q. Now we'll show that the motion of P is *simple harmonic motion.*

The displacement of P from the origin O at any time t is the distance OP, or x. From Figure 1.15, we see that

$$x = A\cos\theta .$$

If point Q is at the extreme right end of the diameter at time $t = 0$, then $\theta = 0$ when $t = 0$, and the time variation of θ is given by

$$\theta = \omega t .$$

Hence,

$$x = A\cos\omega t\,. \qquad (1.35)$$

Now ω, the angular velocity of Q in radians per second, is related to f, the number of complete revolutions of Q per second, by

$$\omega = 2\pi f\,,$$

since there are 2π radians in one complete revolution. Furthermore, the point P makes one complete back-and-forth vibration for each revolution of Q. Hence f is also the number of vibrations per second, or *the frequency* of vibration of point P. Thus Eq. (1.35) may also be written as

$$x = A\cos 2\pi f\,.$$

We can find the instantaneous velocity of P with the aid of Figure 1.16. The reference point Q moves with a tangential velocity given by

$$v_t = \omega A = 2\pi f A\,.$$

Since point P is always directly below or above the reference point, the velocity of P at each instant must equal the x-component of the velocity of Q. That is, from Figure 1.16

$$v = -\omega A\sin\omega t \qquad (1.36)$$

The minus sign is needed because the velocity is directed toward the left. When Q is below the horizontal diameter, the velocity of P is toward the right; but since $\sin\theta$ is negative at such points, the minus sign is still needed. Eq. (1.36) gives the velocity of point P at any time.

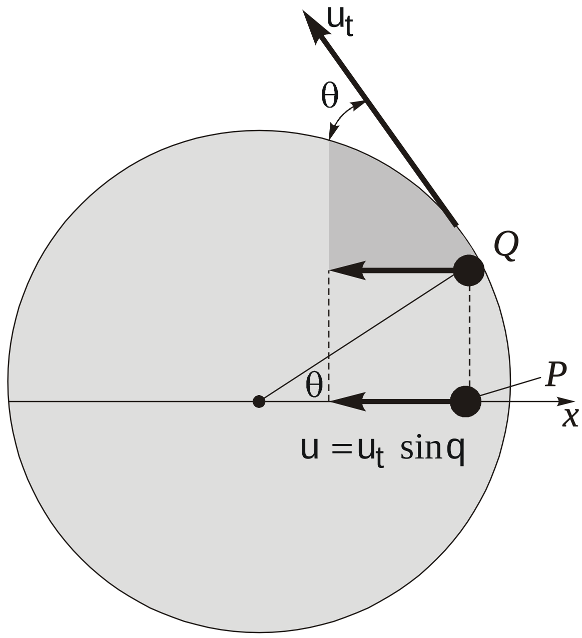


Figure 1.16 Velocity in SHM

We can also find the acceleration of the point P by making use again of the fact that since P is always directly below or above Q, its acceleration must equal the x-component of the acceleration of Q. As point Q moves in a circular path with the constant angular velocity ω, at every instant it has an acceleration toward the center given by

$$a_{\wedge} = -\omega^2 x\,.$$

From Figure 1.17, the x-component of this acceleration is

$$a_x = -a_{\wedge}\cos\theta\,,$$

$$a_x = -\omega^2 A\cos\omega t\,. \qquad (1.37)$$

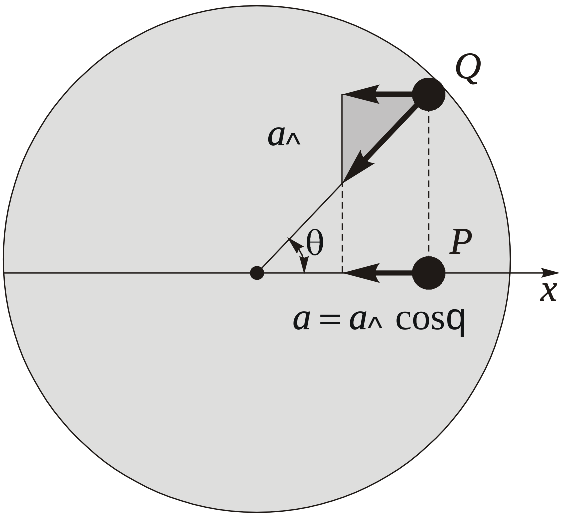


Figure 1.17 Acceleration in SHM

The minus sign is needed because the acceleration is directed toward the left. When Q is to the left from the center, the acceleration of P is directed toward the right; but since $\cos q$ is negative at such points, the minus sign is still required. Eq. (1.37) gives the acceleration of P at any time.

Now comes the final step in showing that the motion of P is simple harmonic. We combine Eqs. (1.35) and (1.37), obtaining

$$a = -w^2 x . \qquad (1.38)$$

As w is constant, the acceleration a at each instant equals a negative constant times the displacement x at this instant. But this is the essential feature of simple harmonic motion: Force and acceleration are proportional to the displacement from equilibrium. Hence, the motion of P is indeed simple harmonic.

In order to make Eqs. (1.12) and (1.38) agree precisely, we must choose an angular velocity w for the reference point Q such that $w^2 = k/m$. Thus, the *angular velocity* of point P is identical to the *angular frequency* of the motion defined by Eq. (1.12).

Throughout this discussion we have assumed that the initial position of the particle (at time $t = 0$) is its maximum positive displacement A, but this is not an essential restriction. Different initial positions of the particle correspond to different initial positions of the reference point Q. For example, if at time $t = 0$ the phasor OQ makes an angle q_0 with the positive x-axis, then the angle q at time t is given not by $q = wt$ as before but by $q = q_0 + wt$.

The only change in the discussion is to replace wt in Eqs. (1.35), (1.36), and (1.37) by $(wt + q_0)$. These equations become then

$$x = A\cos(wt + q_0),$$

$$v = -wA\sin(wt + q_0),$$

$$a_x = -w^2 A\cos(wt + q_0) = -w^2 x .$$

The initial position x_0 and initial velocity v_0 (at time $t = 0$) are then given by

$$x = A\cos q_0 \quad \text{and} \quad v_0 = -wA\sin q_0 .,$$

correspondingly.

Example 1.14

A particle rotates counterclockwise in a circle of radius 3.0 m with a constant angular speed of 8.0 rad/s. At $t = 0$, the particle has an x coordinate is equal to 2.0 m and is moving to the right.

a) Determine the x-coordinate as a function of time.

Solution

Because the amplitude of the particle's motion equals the radius of the circle and $\omega = 8$ rad/s, we have

$$x = A\cos(\omega t + \varphi) = 3\cos(8t + \varphi).$$

We can evaluate φ by using the initial condition that $x = 2$ m at $t = 0$:

$$2 = 3\cos(0 + \varphi),$$

$$\varphi = \cos^{-1}\left(\frac{2}{3}\right).$$

If we were to take our answer as $\varphi = 48.2°$, then the coordinate $x = 3\cos(8t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\varphi = -48.2° = 0.41$ rad. The x coordinate as a function of time is then

$$x = 3\cos(8t - 0.841).$$

Note that φ in the cosine function must be in radians.

b) Find the x components of the particle's velocity and acceleration at any time t.

Solution.

$$v_x = \frac{dx}{dt} = (-3)(8)\sin(8t - 0.841) = -24\sin(8t - 0.841)\,\text{m/s},$$

$$a_x = \frac{dv_x}{dt} = (-24)(8)\cos(8t - 0.841) = -192\cos(8t - 0.841)\,\text{m/s}^2.$$

From these results, we conclude that $v_{max} = 24$ m/s and that $a_{max} = 192\,\text{m/s}^2$. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.

Exercises

1.43. An object is undergoing SHM with period $T = 0.4$ s. Use the circle of reference to calculate the time it takes the object to go from $x = 0$ to $x = A/4$.

1.44. An object is undergoing SHM with period $(\pi/2)$ s and amplitude $A = 0.2$ m. At $t = 0$ the object is at $x = 0$. How far is the object from the equilibrium position when $t = \pi/10$ (Ans. 0.19 m.)

1.45. The motion of the piston in a car is almost simple harmonic with amplitude of 40 mm and the frequency of 120Hz. Calculate (a) the maximum acceleration and (b) the maximum speed of the piston. (Ans. (a) $2.27´\ 10^4$ m/s^2, (b) 30.2 m/s.)

1.46. A vertical rod is fixed to the rim of a horizontal turntable of diameter 4 cm. A horizontal beam of light casts a shadow of the rod on a screen. (a) The turntable rotates at a uniform angular velocity w. Show that the motion of the shadow of the rod on the screen is simple harmonic. (b) If the turntable rotates at 0.5 revolutions per second, what is the maximum speed of the shadow of the vertical rod and at which point does it occur? (Ans. (a) 6.28 cm/s, (b) At the midpoint)

1.47. While riding behind a car that is traveling at 3.00 m/s, you notice that one of the car tires has a small hemispherical boss on its rim. (a) Explain why the boss, from your viewpoint behind the car, executes simple harmonic motion, (b) If the radius of the car tire is 0.3 m, what is the boss' period of oscillations?

1.48. Consider the simplified single-piston engine.If the wheel rotates with the constant angular speed explain why the piston rod oscillates in simple harmonic motion.

1.9 Phasor Addition of Oscillations

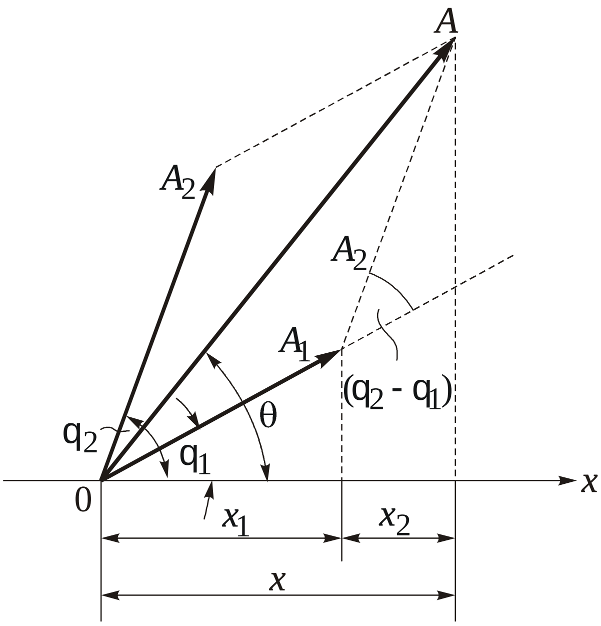


Figure 1.18 Phasor addition of the oscillations $x_1 = A_1 \sin(wt + q_1)$ and $x_2 = A_2 \sin(wt + q_2)$. The resultant oscillation x has amplitude A and phase costant q

There are a lot of problems concerning superposition of several oscillations. Unfortunately, analytical procedure becomes cumbersome when we must add them. Because we are interested in combining a large number of oscillations, we now describe a graphical procedure for this purpose.

Let us consider the addition of two oscillations of the same direction and equal frequencies. The resulting displacement x of the oscillating body is the sum of displacements x_1 and x_2:

$$x_1 = A_1 \cos(wt + q_1),$$

$$x_2 = A_2 \cos(wt + q_2).$$

These oscillations can be represented graphically by phasors of magnitudes A_1 and A_2 rotating about the origin counterclockwise with an angular frequency w, as shown in Figure 1.18.

We can obtain the resultant oscillation, which is the sum of x_1 and x_2, graphically by redrawing the phasors as shown in Figure 1.18, where the tail of the second phasor is placed at the tip of the first one. According to vector addition, the resultant phasor A runs from the tail of the first phasor to the tip of the second one. Furthermore, A rotates about the origin simultaneously with the initial phasors with the same angular frequency ω. The projection of A along the horizontal axis equals the sum of the projections of the two phasors:

$$x = x_1 + x_2 .$$

From Figure 1.18, it is clear that the amplitude of the resulting vector, can be found from cosine theorem as

$$A^2 = A_1^2 + A_2^2 - 2A_1A_2\cos[\pi - (\theta_2 - \theta_1)] = A_1^2 + A_2^2 + 2A_1A_2\cos(\theta_2 - \theta_1), \quad (1.39)$$

and the initial phase of the resultant oscillation is

$$\tan\theta = \frac{A_1\sin\theta_1 + A_2\sin\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2} . \quad (1.40)$$

Example 1.15

Two oscillations of the same period T have amplitudes $A_1 = 4$ mm and $A_1 = 3$ mm, and their phase constants are 0 and $\pi/3$ rad, respectively. What are the amplitude A and phase constant θ of the resultant oscillation? Write the equation of the resultant oscillation.

Solution. The oscillations can be represented by phasors rotating about an origin at the same angular speed $\omega = \frac{2\pi}{T}$. The phase constant is by $\pi/3$ greater for the second oscillation than for first one, that is why phasor 1 must lag phasor 2 by $\pi/3$ rad in their counterclockwise rotation, as shown in Figure 1.19a. The resultant oscillation can be represented by a phasor that is the vector sum of phasors 1 and 2.

To simplify the vector summation, we draw phasors 1 and 2 in Figure 1.19a at the instant when phasor 1 lies along the horizontal axis. Then we drew lagging phasor 1 at positive angle $\pi/3$ rad. In Figure 1.19b, we shift phasor 2 so that its tail is at the head of phasor 1. We can draw the phasor A of the resultant oscillation from the tail of phasor 1 to the head of phasor 2. The phase constant θ is the angle it makes with phasor 1.

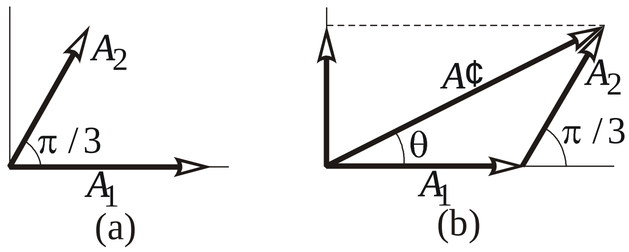


Figure 1.19 (a) Two phasors of magnitudes A_1 and A_2 and with phase difference $\pi/3$; (b) Vector addition of these phasors at any instant during their rotation gives the magnitude A of the phasor for the resultant oscillation

To find values of A and θ, we can add phasors 1 and 2 by components. For the horizontal components we have

$$A_x = A_1\cos 0 + A_2\cos\pi/3 = 4 + 3\cos\pi/3 = 5.5 \text{ mm}.$$

For vertical components we get

$$A_y = A_1 \sin 0 + A_2 \sin \pi/3 = 4 + 3\sin \pi/3 = 2.6 \text{ mm.}$$

Thus, the resultant oscillation has amplitude of:

$$A = \sqrt{(5.5)^2 + (2.6)^2} = 6.1 \text{ mm}$$

and phase constant θ of

$$\theta = \arctan\frac{2.6\,\text{mm}}{5.5\,\text{mm}} = 0.44 \text{ rad.}$$

From Figure 1.19b, phase constant θ is a ***positive*** angle relative to phasor 1, Thus, the resultant oscillation leads oscillation 1 in their travel by phase constant $+0.44$ rad. From Eq. (1.39), we can write the resultant oscillation as

$$A(t) = (6.1\text{mm})\sin(\omega t + 0.44).$$

1.10 Addition of Mutually Perpendicular Oscillations

Assume that a particle can be set into oscillations both along x and y axes. When both types of oscillations are generated, particle moves, in general, along a curved trajectory and form of the trajectory depends on the phase difference of the two oscillations.

Let's chose the initial moment of time in such a way that the initial phase constant of the first oscillation is zero. Then equations of oscillations can be written as

$$x = A\cos\omega t, \tag{1.41}$$

$$y = B\cos(\omega t + \theta), \tag{1.42}$$

where θ is the phase difference between oscillations.

Expressions (1.41) and (1.42) represent the equation of trajectory in the parametric form. To obtain an equation of trajectory as an equation of a curved line, we have to eliminate the parameter t from them. From the first equation, it follows that

$$\cos\omega t = \frac{x}{A}. \tag{1.43}$$

Therefore,

$$\sin\omega t = \pm\sqrt{1 - \cos^2\omega t} = \pm\sqrt{1 - \frac{x^2}{A^2}}. \tag{1.44}$$

Now, using trigonometric identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

we represent $\cos(\omega t + \varphi)$ in Eq. (1.42) as

$$\cos(\omega t + \theta) = \frac{x}{A}\cos\theta \pm \sin\theta\sqrt{1 - \frac{x^2}{A^2}};$$

and rewrite Eq. (1.42)

$$\frac{y}{B}=\frac{x}{A}\cos\theta \pm \sin\theta\sqrt{1-\frac{x^2}{A^2}}\,.$$

After several transformations, the latest equation can be represented as

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-\frac{2xy}{AB}\cos\theta=\sin^2\theta \qquad (1.45)$$

Eq. (1.45) is the equation of ellipse with semi–axis which are turned relatively coordinate axes x and y. Orientation of the ellipse and magnitude of its semi–axes depend on the amplitudes A and B and phase difference θ in a rather complicated way.

Special cases

1. The phase difference θ equals zero. In this case, Eq. (1.45) takes the form

$$\left(\frac{x}{A}-\frac{y}{B}\right)^2=0\,,$$

and it reduces to the equation of straight line:

$$y=\frac{B}{A}x\,. \qquad (1.46)$$

The resulting motion is the harmonic oscillation along this straight line with the frequency ω and amplitude $\sqrt{A^2+B^2}$ (Figure 1.20).

2. Phase difference is $\theta=\pm\pi$. Eq. (1.45) has the form:

$$\left(\frac{x}{A}+\frac{y}{B}\right)^2=0\,,$$

and the resulting motion is SHM along the straight line (Figure 1.21)

$$y=-\frac{B}{A}x\,. \qquad (1.47)$$

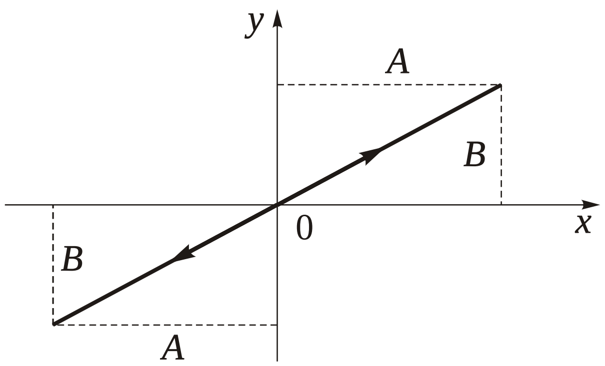


Figure 1.20 The phase difference $\theta=0$

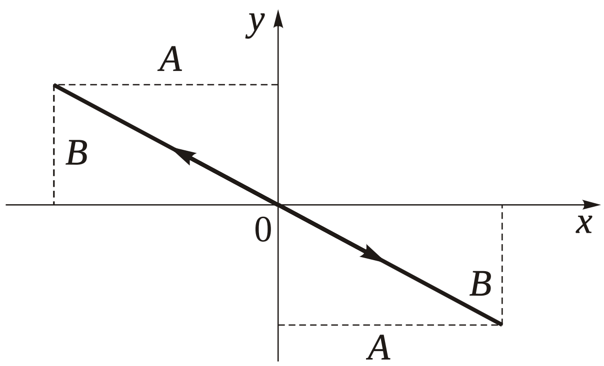


Figure 1.21 Phase difference $\theta=\pm\pi$

3. At $\theta = \pm\dfrac{\pi}{2}$, Eq. (1.45) transforms into

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, \qquad (1.48)$$

that is, into the equation of ellipse with a semi–axis oriented along coordinate axes x and y correspondingly which are equal to the amplitudes A and B. When amplitudes A and B are equal to each other, the ellipse reduces into circumference.

Two cases $\theta = +\pi/2$ and $\theta = -\pi/2$ differ in the direction of the motion along the ellipse or circumference. If $\theta = +\pi/2$, Eqs. (1.41) and (1.42) can be written as

$$x = A\cos\omega t \quad \text{and} \quad y = -B\sin\omega t .$$

At $t = 0$, the particle is in the point 1 (Figure 1.22). When some time elapses, the coordinate x decreases whereas the coordinate y becomes negative. Therefore, the body moves along the clockwise direction.

When $\theta = -\pi/2$, the equations of oscillations have the form

$$x = A\cos\omega t \quad \text{and} \quad y = B\sin\omega t$$

and the motion occurs counterclockwise.

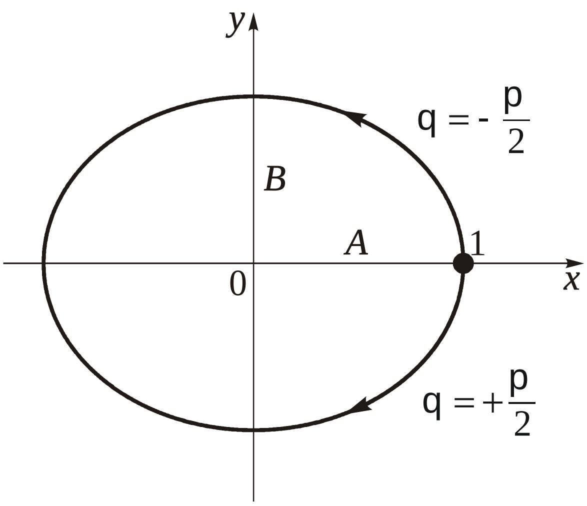


Figure 1.22 Phase difference is $\theta = \pm\dfrac{\pi}{2}$. Two cases $\theta = +\pi/2$ and $\theta = -\pi/2$ differ in the direction of the motion along the ellipse or circumference

It follows from the discussion above, that the uniform motion along the circle of radius R with the angular velocity ω can be represented as a sum of two mutually perpendicular oscillations:

$$x = R\cos\omega t, \quad y = \pm R\sin\omega t .$$

In the expression for y the plus sign corresponds to the motion in the anticlockwise direction, whereas the minus sign corresponds to the clockwise direction.

When the frequencies of mutually perpendicular oscillations differ in very small amount $\Delta\omega$, they can be treated as oscillations of equal frequency but with a slowly varying phase difference. Indeed, their equations can be represented as

$$x = A\cos\omega t, \quad y = B\cos[\omega t + (\Delta\omega t + \theta)],$$

and the expression $(\Delta\omega t + \theta)$ can be treated as the phase difference which slowly varies according to the linear law. The resulting motion occurs along the slowly

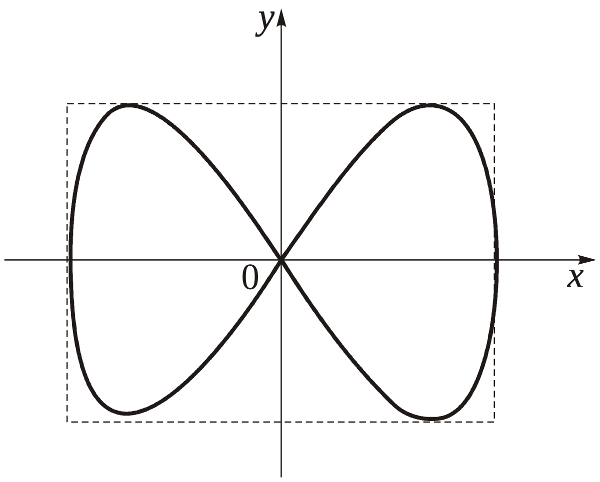


Figure 1.23 The ratio of frequencies is 1:2 and phase difference is $\pi/2$

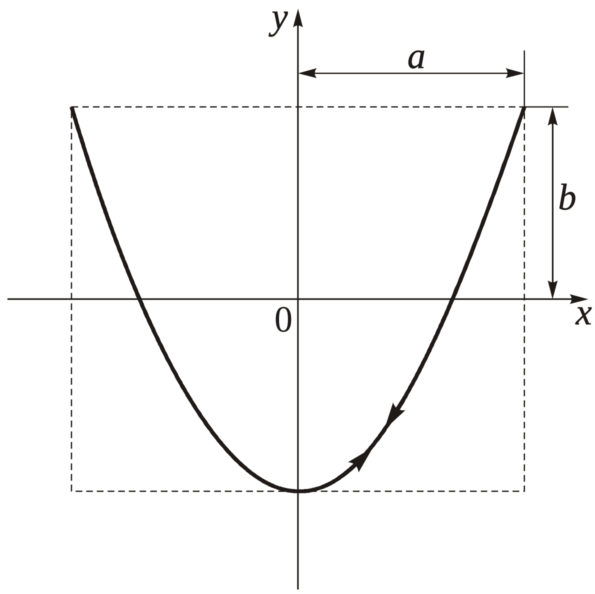


Figure 1.24 The ratio of frequencies is 1:2 and phase difference is 0

varying trajectory which consequently takes forms, inherent to change of a phase difference from $-\pi$ to π.

When frequencies of oscillations are not the same, the trajectory of the resulting oscillation has the form of rather complex curves called the *Lissagy figures*. When the ratio of frequencies is 1:2 and the phase difference is $\pi/2$, the resulting trajectory is a curve represented in Figure 1.23. In this case, the equations of oscillation have the form:

$$x = A\cos\omega t\,,\ y = B\cos\left(2\omega t + \frac{\pi}{2}\right).$$

During the time interval, when the particle displaces from one extreme position to the other along the x–axis, it has time to reach the extreme position and return back into the initial point along y–axis.

When the ratio of frequencies is 1:2 and the phase difference is 0, the trajectory transforms into an open curve in Figure 1.24 along which particle moves back and forth.

When the ratio of frequencies of oscillations approaches to the unity, the Lissagy figures become more and more complex. As an example, the trajectory corresponding to the ratio 3:4 and phase difference $\pi/2$ is shown in Figure 1.25.

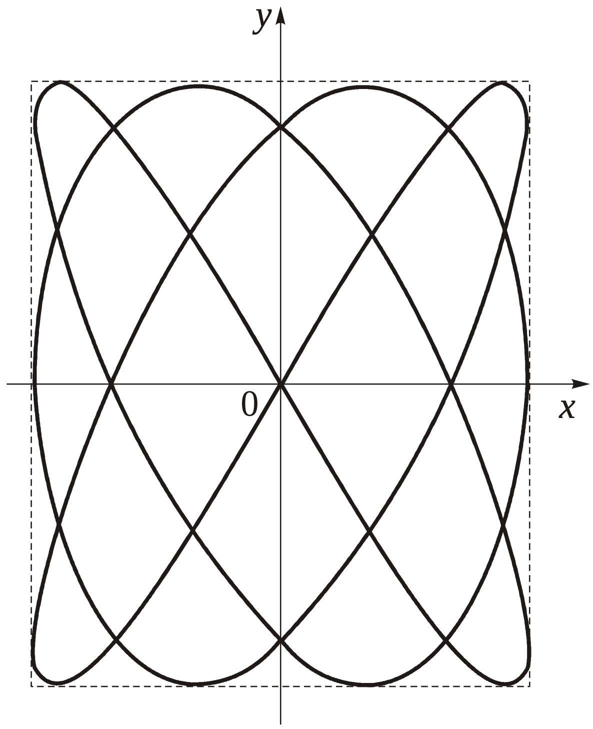


Figure 1.25 The ratio of frequencies is 3:4 and phase difference is $\pi/2$

Exercisers

1.49. Determine the amplitude of the resultant oscillation when two oscillations having the same frequency and same direction are combined if their amplitudes are 3.0 cm and 4.0 cm, and they have phase constants of 0 and $p/2$ rad respectively.

1.50. Two oscillations of the same period with amplitudes of 5.0 and 7.0 mm produce a resultant oscillation with an amplitude of 9.0 mm. The phase constant of the 5.0 mm oscillation is 0. What is the phase constant of the 7.0 mm oscillation?

1.51. Three oscillations of the same frequency and direction have amplitudes A, $A/2$ and $A/3$ and their phase constants are 0, $p/2$, and p respectively. What are (a) the amplitude and (b) the phase constant of the resultant oscillation?

1.11 Damped Oscillations

In the idealized oscillating systems we have discussed so far, there is no friction. Thus, the systems are *conservative;* the total mechanical energy is constant, and a system once set into motion continues oscillating forever with no decrease in amplitude.

Real systems always have some friction, however, and oscillations do die out with time unless some means is provided for replacing the mechanical energy lost to friction. A pendulum clock continues to run because the potential energy stored in the spring is used to replace the mechanical energy lost due to friction in the pendulum and the gears. But when the spring "runs down", and no more energy is available, the pendulum swings decrease in amplitude and stop.

The decrease in amplitude caused by dissipative forces is called *damping,* and the corresponding motion is called *damped oscillation.* The simplest case to analyze in detail is that of a frictional damping force directly proportional to the velocity of the oscillating body. This behavior occurs in systems involving viscous fluid flow, such as sliding between oil-lubricated surfaces, shock absorbers, and many other systems of practical importance. Then we have an additional damping force on the body due to friction, $F_d = -bv$, where $v = dx/dt$ is the velocity and b is a *damping constant* that describes the strength of the damping force. In SI system, b has the unit of kilogram per second. The minus sine indicates that F_d opposes the motion.

The total force on the body is then

$$F = -kx - bv$$

and the Newton's-second-law formulation becomes

$$-kx - bv = ma, \tag{1.49}$$

or

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

or $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0. \tag{1.50}$$

Finding solutions to the Eq. (1.50) is a straightforward problem in differential equations, but we will not go into the details here. If the damping force is relatively small and the body is given an initial displacement A_0, the motion is described by

$$x = A_0 e^{-(b/2m)t} \cos \omega' t, \tag{1.51}$$

where the frequency of oscillation ω' is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{1.52}$$

This motion differs from that of the undamped case in two ways. First, the amplitude

$$A(t) = A_0 e^{-(b/2m)t} \tag{1.53}$$

is not constant but decreases with time because of the exponential factor $e^{-(b/2m)t}$. The larger the value of b, the more quickly the amplitude decreases. For an undampted oscillator, the mechanical energy is constant and is given by $E = \frac{1}{2}kA^2$. If the oscillator is damped, the mechanical energy is not constant but decreases exponentially with time.

Second, the angular frequency of oscillation is no longer equal to $\sqrt{k/m}$ but is somewhat smaller. Figure 1.26 shows graphs of Eq. (1.51) for two different values of the constant b. If $b = 0$ (there is no damping), then Eq. (1.52) reduces to Eq. (1.12) $(\omega = \sqrt{k/m})$ for the angular frequency of an undamped oscillations, and Eq. (1.51) reduces to Eq. (1.2) for the displacement x of an undamped oscillations.

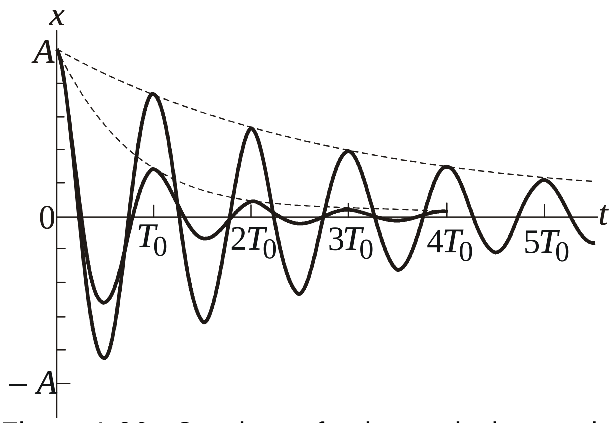


Figure 1.26 Graphs of damped harmonic motion. The period when there is no damping ($b = 0$) is T_0. The grey curve shows the motion when $b = 0.1\sqrt{km}$, and the black curve is for $b = 0.4\sqrt{km}$. The broken lines show the exponential factor $A_0 e^{-(b/2m)t}$ for each case. The amplitude decreases more rapidly for the larger value of b. Close inspection of the points where the curves cross the t-axis also shows that the period increases slightly with the increasing b. The critical-damping condition is $b = 2\sqrt{km}$

It is convenient to express the angular frequency of a damped oscillation in the form of

$$\omega' = \sqrt{\omega^2 - \frac{b^2}{4m^2}},$$

where $\omega = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the *natural frequency* of the system. When the magnitude of the maximum retarding force $R_{max} = bv_{max} < kA$, the system is said to be *underdamped*. As the value of R approaches kA, the amplitudes of the oscillations decrease more and more rapidly.

Note that in Eq. (1.52) the frequency becomes zero when b becomes so large that

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0 \quad \text{or} \quad b = \sqrt{4km}. \qquad (1.54)$$

When b exceeds this value, the system no longer oscillates, when it is displaced and released, but returns to its equilibrium position without oscillation. If Eq. (1.54) is satisfied, the condition is called *critical damping*. If the medium is so viscous that the retarding force is greater than the restoring force – that is, if $bv_{max} > kA$ and $b/2m > \omega$ – the system is *over-damped.* Again, the displaced system, when free to move, does not oscillate but simply returns to its equilibrium position. As the damping increases, the time it takes the system to approach equilibrium also increases. The nonoscillating motion that occurs when b is even larger corresponds to *overdamping*. In these cases, the solutions of Eq. (1.50) are the decreasing exponential functions without any sinusoidal factors. When b is smaller than the critical value, the situation corresponds to underdamping**.**

In all cases, both overdamping and underdamping, the mechanical energy of the system continuously decreases, approaching zero. The lost of mechanical energy dissipates into internal energy in the retarding medium.

The ratio

$$\frac{A(t)}{A(t+T)} = \frac{A_0 e^{-bt}}{A_0 e^{-(b+T)}} = e^{bT} \qquad (1.55)$$

is called the ***damping decrement*** and its natural logarithm

$$l = \ln \frac{A(t)}{A(t+T)} = bT \qquad (1.56)$$

is called the ***logarithmic damping decrement***. In both expressions we denote $b = b/2m$.

To characterize the oscillating system we as ordinarily use the logarithmic damping decrement l . Let the amplitude of oscillation decreases e times during time t . Then,

$$\frac{A(t)}{A(t+T)} = \frac{A_0 e^{-bt}}{A_0 e^{-b(t+t)}} = \frac{e^{-bt}}{e^{-bt} \times e^{-bt}} = e^{bt} = e.$$

Hence, $bt = 1$. As the logarithmic damping decrement $l = bT$, then $b = l/T$ and, on the other hand, $b = 1/t$. Therefore, $l = T/t$, but during time t the system fulfills $N_e = t/T$ oscillation. Hence, the logarithmic decrement of damping is an inverse of the number of oscillations during which amplitude decreases by a factor of e.

Another often used characteristic is a so-called Q-factor:

$$Q = \frac{p}{l} = pN_e. \qquad (1.57)$$

According to its definition, the Q-factor is proportional to the number of oscillation N_e during which the amplitude decreases by a factor e.

The suspension system of an automobile is a familiar example of damped oscillations. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it does not continue bouncing forever. For optimal passenger comfort, the system should be critically damped or, perhaps, slightly underdamped. As the shocks get old and wear off, the value of b decreases and the bouncing persists longer. Not only is this nauseating, but it is bad for the steering because the front wheels have less positive contact with the ground. Thus, damping is an advantage for this system. Conversely, in a system such as a clock or an electrical oscillating system of the type found in radio transmitters, it is usually desirable to minimize damping.

Example 1.16

A damped oscillator consists of $m = 250$ g, $k == 85$ N/m, and the damping constant $b = 70$ g/s.

a) What is the period of motion?

Solution.

When $b << \sqrt{km} = 4.6$ kg/s, the period is approximately that of undamped oscillations:

$$T = 2p\sqrt{\frac{m}{k}} = 2p\sqrt{\frac{0.25\,\text{kg}}{85\,\text{N/m}}} = 0.34\,\text{s}.$$

b) How long does it take the amplitude of the damped oscillations to drop to half its initial value?

Solution. At time t, the amplitude in Eq. (1.53) is $A_0 e^{-bt/2m}$. It has value A_0 at $t = 0$. Thus, we must find the value of t for which $\frac{A_0}{A_0 e^{-bt/2m}} = 2$. Canceling A_0 and taking the natural logarithm of the remaining equation we get $\ln\frac{1}{2}$ on the right side and

$$\ln e^{-bt/2m} = -bt/2m$$

on the left side. Thus,

$$t = \frac{-2m\ln\frac{1}{2}}{b} = \frac{-2(0.25\,\text{kg})(\ln\frac{1}{2})}{0.070\ \text{kg/s}}.$$

Because $T = 0.34$ s, this is about 15 periods of oscillation.

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

Solution. The mechanical energy at time t is $\frac{1}{2}kx_m^2 e^{-bt/m}$. It has the value $\frac{1}{2}kx_m^2$ at $t = 0$. Thus, we must find the value of t for which

$$\frac{1}{2}kx_m^2 e^{-bt/m} = \frac{1}{2}\left(\frac{1}{2}kx_m^2\right).$$

It is clear that

$$e^{-bt/m} = \frac{1}{2}, \text{ or } -bt/m\ln e = \ln\frac{1}{2}, \text{ or}$$

$$t = -\frac{m}{b}\ln\frac{1}{2} = -\frac{0.25}{0.07}\ln\frac{1}{2} = 2.5\ \text{s}.$$

Exercises

1.52. A mass of 0.4 kg is moving on the end of a spring of force constant $k = 300\,\text{N/m}$ and is acted on by a damping force $F = -bv$. (a) If the constant b has the value 5 kg/s, what is the frequency of oscillation of the mass? (Ans. 4.24 Hz.); (b) For what value of the constant b will the motion be critically damped? (Ans. 21.9 kg/s.)

1.53. A mass of 0.2 kg is attached to the end of a spring of force constant $k = 250\,\text{N/m}$ moves with an initial displacement of 0.3 m. There is a damping force $F = -bv$ acting on the mass. It is observed that the amplitude of the motion has decreased to 0.1 m within 5 s. Calculate the magnitude of the damping constant.

1.54. The amplitude of oscillation of a simple pendulum decreases with time. How does the total energy of the pendulum vary with time?

A. It decreases in a steady rate.

B. It decreases exponentially.

C. It remains constant.

D. It oscillates with the same frequency as the pendulum.

1.55. With the aid of suitable graphs, explain what is meant by

(a) free oscillations,

(b) underdamped oscillations,

(c) critically damped oscillations,

(d) overdamped oscillations.

1.56. Show that the time rate of the change of the mechanical energy for a damped, undriven oscillator is given by $dE/dt = -bv^2$ and, hence, is always negative. (Hint: Differentiate the expression for the mechanical energy of an oscillator.)

1.57. A pendulum with the length of 1 m is released from the initial angle of 15º. After 1 000 s, its amplitude is reduced by friction to 5.5°. What is the value of $b/2m$?

1.58. Show that Eq. (1.51) is a solution to Eq. (1.50) provided that $b^2 < 4mk$.

1.59. (a) Describe the energy transformation in a complete cycle for a simple pendulum under free oscillation. Sketch suitable graphs to support your answer; (b) Explain why the amplitude of a damped oscillation decreases.

1.60. A car with bad shock absorbers bounces up and down with a period of 1.5 s after hitting a bump. The car has a mass of 1 500 kg and is supported by four springs of equal force constant k. Determine the value of k.

1.61. A large passenger with a mass of 150 kg sits in the middle of the car described in Exercise 1.60. What is the new period of oscillation?

1.12 Forced Oscillations

There are many real situations where we would like to maintain oscillations of constant amplitude in a damped oscillating system. A familiar example is a child sitting on a swing. We set the system into motion by pulling the child back from the straight-down equilibrium position and releasing it. If that is all we do, the system oscillates with the decreasing amplitude and eventually comes to rest. But by giving the system a little push once each cycle, we can maintain a nearly constant amplitude. More generally, we can maintain a constant-amplitude oscillation in a damped harmonic oscillator by applying an oscillating force, that is, a force that varies with time in a periodic or cyclic way. We call this additional force a *driving force.*

Furthermore, the frequency of the force variation need not to be the same as the natural oscillation frequency of the system. If we apply a periodically varying driving force to the mass of the harmonic oscillator, the mass undergoes a periodic motion ***with the same frequency as that of the driving force.*** We call this motion a *forced oscillation,* or *a driven oscillation;* it is different from the motion that occurs when the system is simply set into motion and then left alone to oscillate with a natural frequency determined by m, k, and b.

When the driving force varies according to the harmonic law, the differential equation of oscillations has the form

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = f_0 \cos\omega_d t\,. \qquad (1.58)$$

Here β is the damping coefficient, ω – is a natural frequency, $f_0 = F_0/m$ (F_0 is the amplitude of the driving force), ω_d is a frequency of a driving force.

Eq. (1.58) has the nonzero right part. According to the theory of differential equations, the solution of nonhomogeneous equation equals the sum of a general solution of a corresponding homogeneous equation and a specific solution of a given nonhomogeneous equation. The general solution we have already got. It has the form

$$x = A_0 e^{-\beta t}\cos(\omega' t + \varphi),$$

where $\omega' = \sqrt{\omega^2 - \beta^2}$, A_0 and φ are the arbitrary constants.

The specific solution can be found using phasors. Suppose that the specific solution has the form:

$$x = A\cos(\omega_d t - \varphi) \qquad (1.59)$$

Then,

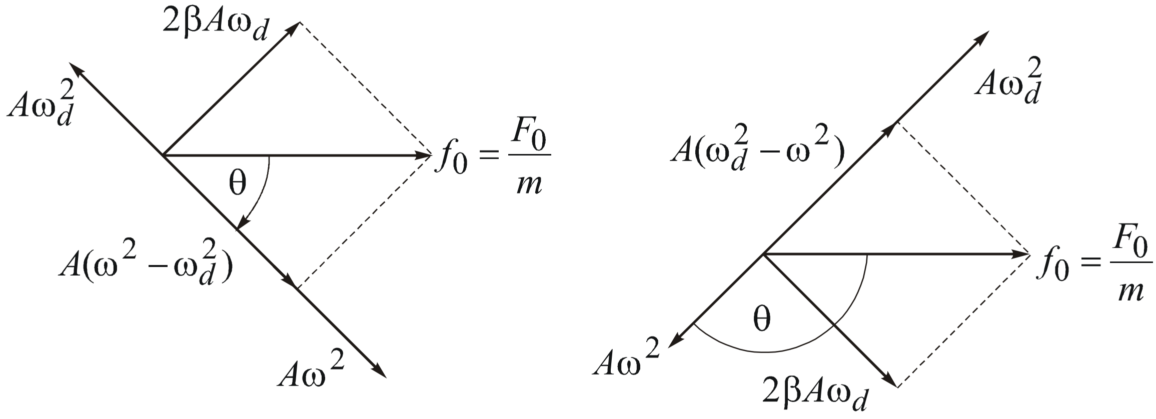
$$\frac{dx}{dt} = -\omega_d A\sin(\omega_d t - \varphi) = \omega_d A\cos(\omega_d t - \varphi + \frac{\pi}{2}) \qquad (1.60)$$

$$\frac{d^2x}{dt^2} = -\omega_d{}^2 A\cos(\omega_d t - \beta) = \omega_d{}^2 A\cos(\omega_d t - \varphi + \pi) \qquad (1.61)$$

Substituting Eqs. (1.61) and (1.60) into Eq. (1.58), we obtain:

$$\omega_d{}^2 A\cos(\omega_d t - \varphi + \pi) + 2\beta\omega_d A\cos(\omega_d t - \varphi + \pi/2) +$$
$$+\omega_d{}^2 A\cos(\omega_d t - \varphi + \pi) = f_0\cos\omega_d t\,. \qquad (1.62)$$

From (1.62) it follows that constants A and φ must have such values that harmonic function $f_0\cos\omega_d t$ is equal to the sum of three harmonic functions of left part in equation (1.58). If we represent the function $\omega_d{}^2 A\cos(\omega_d t - \varphi)$ with phasor of length $\omega_d{}^2 A$ directed to the right, then the function $2\beta\omega_d A\cos(\omega_d t - \varphi + \pi/2)$ will be represented by the phasor $2\beta\omega_d A$ placed at angle $\pi/2$ counterclockwise with respect to the phasor $\omega^2 A$. (Figure 1.27)



a *b*

Figure 1.27 Phasor diagram of forced oscillations for (a) $\omega_d < \omega_0$ and (b) $\omega_d > \omega_0$

The function $\omega_d{}^2 A\cos(\omega_d t - \theta + \pi)$ is represented by the vector $\omega_d{}^2 A$, placed at the angle π relative to the vector $\omega^2 A$. To satisfy Eq. (1.58) the sum of these three phasors must coincide with the phasor $f_0 \cos\omega_d t$. It is clear from Figure 1.27, that such a situation can be valid only at amplitude A satisfying the following equation:

$$(\omega^2 - \omega_d{}^2)A^2 + 4\beta^2\omega_d{}^2 A^2 = f_0^2 \tag{1.63}$$

The amplitude A is the solution of Eq. (1.63):

$$A = \frac{F_0 / m}{\sqrt{(\omega^2 - \omega_d{}^2)^2 + 4\beta^2\omega_d{}^2}} \tag{1.64}$$

Figure 1.27a describes the situation when $\omega_d < \omega$, and Figure 1.27b corresponds the case $\omega_d > \omega_0$.

When the frequency of the driving force is *equal* to the natural frequency of the system, we would expect the amplitude of the resulting oscillation to be larger than when the two are extremely different, and this expectation is borne out by more detailed analysis and experiment. The easiest case to analyze is that of a *sinusoidally* varying force, say $F = F_{\max}\sin\omega_d t$ where ω_d is not necessarily equal to the natural frequency ω of the system. If we vary the frequency ω_d of the driving force, the amplitude of the resulting forced oscillation varies in a rather specific way, as shown in Figure 1.28. When there is very little damping, the amplitude goes through a sharp peak as the driving frequency passes through the natural oscillation frequency. For increased damping, the peak becomes broader and smaller in height and shifts toward lower frequencies.

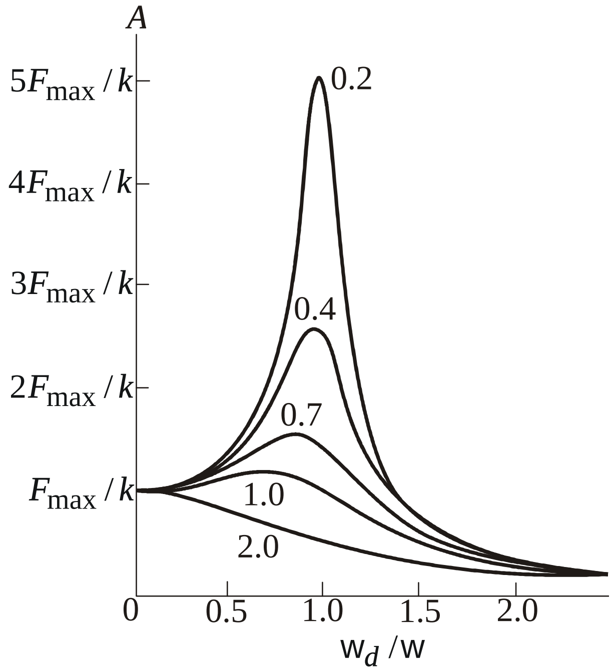


Figure 1.28 Graph of the amplitude A of forced oscillation of a damped harmonic oscillator, as a function of the frequency w_d of the driving force, plotted on the horizontal axis as the ratio of w_d to the angular frequency $w = \sqrt{k/m}$ of an undamped oscillator. The highest curve has $b = 0.2\sqrt{km}$, the next has $b = 0.4\sqrt{km}$, and so on. As b increases, the peak becomes broader and less sharp and shifts toward lower frequencies. When b is as large as $2\sqrt{km}$, the peak disappears completely

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called ***resonance.*** Physics is full of examples of resonance; oscillations of a child on a swing is one of them. A vibrating rattle in a car that occurs only at a certain engine speed is another familiar example. You have probably heard of the dangers of a band marching across a bridge; if the frequency of their steps is close to a natural frequency of the bridge, dangerously large oscillations can build up. A tuned circuit in a radio or television receiver responds strongly to waves having frequencies near to its resonant frequency, and this is used to select a particular station and reject the others.

Example 1.17

A car is driven at a constant speed over a road on which the surface height varies sinusoidally. The shock absorber which normally damps vertical oscillation is not working.

a) Explain why at a critical speed of the car, the amplitude of vertical oscillation of the car becomes very large.

Solution.

When the car moves over a road with sinusoid-like surface, it is forced into vertical oscillation. The frequency of the forced oscillations is $f = \frac{v}{l}$, where v is the speed of the car. When the speed v increases, the frequency of the driving force increases as well. At the critical speed, the frequency of the driving force is the same as the natural frequency of the car suspension system. Resonance occurs, and the amplitude of vertical oscillations of the car is maximum.

b) In terms of the quantities listed below, deduce a formula for

i) the natural frequency of vertical oscillation of the car.

Solution.

i) The suspension system of the car is represented by the spring system shown in Figure 1.29.

When the car is stationary (Figure 1.29b), $R = Mg$ and $R = kx$ (using Hook's law), where k is the force constant of spring. x_0 is the equilibrium compression of the spring.

Hence, $Mg = kx_0$. (i)

Figure 1.29c shows the displacement x of the car, when it is oscillating vertically. Using $F = Ma$, second Newton's law we write as

$Mg - R_1 = Ma$, $Mg - k(x_0 + x) = Ma$,

$-kx = Ma$.

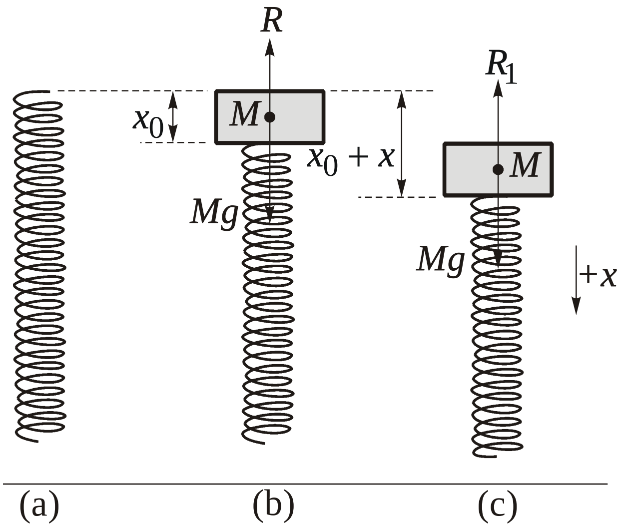


Figure 1.29 The suspension system of the car

Hence, the acceleration $a = -\dfrac{k}{M}x = -\omega^2 x$,

$$\omega = \sqrt{\frac{k}{M}},\ f_0 = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M}},\ mg = ks,\ k = \frac{mg}{s}.$$

and the natural frequency, $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{M}} = \dfrac{1}{2\pi}\sqrt{\dfrac{mg}{Ms}}$.

ii) the critical speed of the car is when the amplitude of vertical oscillations is maximum. Calculate this speed from the data provided: mass of the car and the passengers, $M = 2000$ kg, vertical rise of the car when the passengers get out, is $s = 0.1$ m, mass of the passengers, $m = 500$ kg and the wavelength of the road surface corrugations is $\lambda = 20$ m.

Solution,

If v_0 is the critical speed of the car when resonance occurs, then $v_0 = f\lambda$ and the driving frequency $f = \dfrac{v_0}{\lambda}$. When the resonance occurs,

driving frequency = natural frequency, i.e.

$$\frac{v_0}{\lambda} = \frac{1}{2\pi}\sqrt{\frac{mg}{Ms}},\quad v_0 = \frac{\lambda}{2\pi}\sqrt{\frac{mg}{Ms}} = \frac{20}{2\pi}\sqrt{\frac{5\times 10^2 \times 9.8}{2\times 10^3 \times 1\times 10^{-1}}} = 15.8 \text{ m/s}.$$

(c) Discuss the behavior of the car when the shock absorber mechanism is working properly. Give suitable sketch graphs.

Solution.

When the suspension of the system is working properly, the vertical oscillations are slightly below the critical dumping. After going over each hump, the car does not perform under damped oscillations. (Figure 1.30), but quickly returns to the equilibrium position. (Figure 1.31)

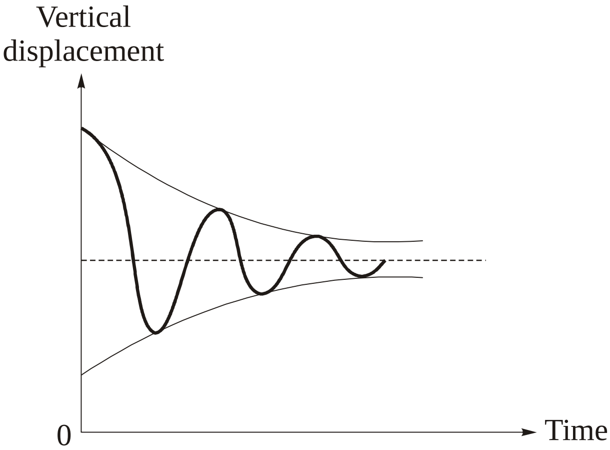


Figure 1.30 Suspension system faulty

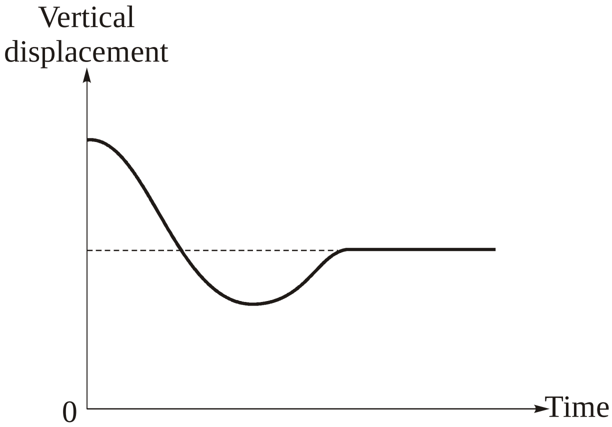


Figure 1.31 Suspension system functions properly

Exercises

1.62. Describe the energy transformation in a complete cycle for a simple pendulum. Sketch on the same axes to show how the amplitude of forced oscillations varies with the frequency of the driving force for

a) underdamping,

b) moderate damping,

c) overdamping.

Mark the resonance frequency f_0 on the graph,. What is the effect of damping on the resonance frequency?

1.63. (a) Explain the terms *forced oscillation* and *resonance*, (b) State the condition for resonance to occur.

1.64. (a) What is meant by the natural frequency of an oscillatory system?

(b) The suspension system of a car consists of a spring under compression and a shock absorber which damps the vertical oscillations of the car. Sketch graphs to illustrate how the vertical height of the car above the road varies with time after the car has just passed over a bump if the shock absorber is (i) not functioning, (ii) functioning normally

(c) When the driver of mass 80 kg steps into the car of mass 920 kg, the vertical height of the car above the road decreases by 20 cm. (i) Explain why when the car is driven over a series of equally spaced bumps, the amplitude of the vertical oscillations of the car becomes large for one particular speed. (ii) Calculate the speed of the car if the separation between successive humps is 15 m.

1.65. A baby rejoices in the day by crowing and jumping up and down in her crib. Its mass is 12.5 kg, and the crib mattress can be modeled as a light spring with the force constant of 4.30 kN/m. (a) The baby soon learns to bounce with maximum amplitude and minimum effort by bending its knees at what frequency? (b) It learns to use the mattress as a trampoline – losing contact with it for part of each cycle – when her amplitude exceeds what value?

1.66. A 2.00-kg mass attached to a spring is driven by an external force $F = (3)\cos(2pt)$. If the force constant of the spring is 20.0 N/m, determine (a) the period and (b) the amplitude of the motion. (*Hint:* Assume that there is no damping – that is, that $b = 0$ and use Eq. (1.64).)

1.67. A weight of 40 N is suspended from a spring that has a force constant of 200 N/m. The system is undamped and is subjected to a harmonic force with a frequency of 10 Hz, which results in the forced-motion amplitude of 2 cm. Determine the maximum value of the force.

1.68. Damping is negligible for a 0.150-kg mass hanging from light 6.30–N/m spring. The system is driven by a force oscillating with amplitude of 1.7 N. At what frequency will the force make the mass vibrate with amplitude of 0.44 m?

Summary

A motion that repeats itself over and over again after a regular time interval is called a *periodic motion*. A motion that repeats itself over and over again about its mean position, such that it remains confined within the well defined limits (called extreme positions) on either side of the mean position is called *oscillation* or *vibrational motion*.

One *oscillation* (vibration) is the to and fro motion of a particle between any two consecutive passages in the same direction.

Characteristics of SHM:

The *displacement* of a particle, executing SHM, at any time is defined as the distance of the particle from the mean position at that time.

$$x = A\cos(wt + q),$$

where x is called displacement of SHM and q is the phase constant of SHM.

The amplitude of a SHM is defined as the maximum displacement on either side of the mean position.

$$\textit{Velocity } v = \frac{dx}{dt} = - Aw\sin(wt + q) = w\sqrt{A^2 - x^2} = \frac{2p}{T}\sqrt{A^2 - x^2}.$$

The velocity at the mean position is $v_{\max} = Aw$ (maximum). The velocity at the extreme position is $v = 0$.

$$\textit{Acceleration } a = \frac{d^2x}{dt^2} = - w^2 x = - w^2 A\cos(wt + q).$$

The negative sign shows that the acceleration of a SHM is always directed towards the mean position of the SHM Acceleration at the mean position $a = 0$. The acceleration at the extreme position is $a_{\max} = -\omega^2 A$.

Time period (T) is the time taken by the particle executing SHM. to complete one oscillation.

Frequency (f) is the number of oscillations completed per second by the particle executing SHM

$$f = \frac{1}{T}.$$

The unit of frequency is s^{-1} or cycle per second (c.p.s.) or hertz (Hz). For high frequencies, the units such as kilohertz (kHz) or megahertz (MHz) are used.

Block–spring system. When a mass m is attached to a massless spring and pulled aside or downwards, it executes SHM. If x is extension in the spring on attaching the mass m and k is its force constant, then time period of SHM executed the spring

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

A *simple pendulum* is a point mass suspended by a weightless, inextensible string of length L from a rigid support about which it can oscillate freely. When the mass is displaced from its mean position, it executes SHM.

Time period, $T = 2\pi\sqrt{\frac{L}{g}}$.

Period does not depend on mass.

A *physical pendulum* is a body suspended from an axis of rotation a distance d from its center of gravity. If the moment of inertia about the axis of rotation is I, the period is given by

$$T = 2\pi\sqrt{\frac{I}{mgd}}.$$

Conservation of energy leads to the following relation among the position and velocity at any time and the amplitude and total energy:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}kx^2\,\frac{1}{2}mv^2 = const.$$

If the body is given an initial displacement A and released with no initial velocity, the position is given as a function of time by

$$x = A\cos\omega t.$$

If it is given an initial velocity v_0 and no initial displacement, the position is given as a function of time by

$$x = A\sin\omega t,$$

with A given by $A = v_0/\omega$. If the body is given an initial displacement x_0 and an initial velocity v_0, the position is given by

$$x = A\cos(\omega t + \theta)$$

with A and θ given by

$$A^2 = x^2 + \frac{v_0^2}{\omega^2} \quad \text{and} \quad \theta = \arctan\left(-\frac{v_0}{\omega x}\right).$$

The *circle-of-reference* construction uses a rotating vector, called a *phasor*, which have a length equal to the amplitude of the oscillation and angle θ with x axis equal to the phase constant of oscillations. Its projection on the horizontal axis represents the actual motion of the body. When we add several oscillations, the resultant oscillation can be determined as sum of phasors.

When a damping force $F = -bv$ proportional to velocity is added to a simple harmonic oscillator, the motion is described as a *damped oscillation*:

$$x = A_0 e^{-(b/2m)t} \cos \omega t,$$

provided that $b^2 < 4km$. This condition is called *underdamping*. When $b^2 = 4km$, the system is *critically damped* and no longer oscillates. When b is still larger, the system is *overdamped*.

When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called a *forced oscillation*. Its amplitude reaches a peak at driving frequencies close to the natural oscillation frequency of the system. This behavior is called *resonance*.

Key Terms

periodic motion – периодическое движение
oscillation – колебание
simple harmonic motion (SHM) – гармоническое колебание
cycle – цикл
period – период
frequency – частота
amplitude – амплитуда
angular frequency – круговая частота
phase angle – угол сдвига фаз, фазовый угол
simple pendulum – математический маятник
physical pendulum – физический маятник
torsional pendulum – крутильный маятник
damping – затухание
damped oscillation – затухающие колебания
critical damping – критическое затухание
overdamping – апериодическое затухание, сильное затухание
underdamping – слабое затухание
driving force – вынуждающая сила
forced oscillation – вынужденные колебания
resonance – резонанс

Mechanical Waves

Particle and wave are the two important physical concepts, in the sense that we are able to associate almost every branch of the subject with one of them. However, these two concepts are completely different. The concept *particle* suggests a tiny concentration of a matter capable of transmitting energy. The concept wave suggests just the opposite - namely, a broad distribution of energy filling the space which it passes.

The world is full of waves. Sound, light, ocean waves, earthquakes, radio and television transmissions are all wave phenomena. *A wave is any disturbance from an equilibrium position that travels or propagates with time from one area to another.* There are three main types of waves:

Mechanical waves. These waves are most familiar because we deal with them almost constantly; common examples include water waves, sound waves and seismic waves. They are governed by Newton's laws and they can exist only within a material medium, such as water, air or rock.

Electromagnetic waves. These waves are less familiar but we use them constantly; common examples include visible and ultraviolet light, radio and television waves, X-rays. These waves require no material medium to exist. For example, light waves from stars travel through the space to reach us.

Matter waves. Although these waves are commonly used in modern technology, this type is probably very unfamiliar. These waves are associated with electrons, protons and even atoms and molecules. As we commonly think of these things as constituting matter, such waves are called matter waves.

Chapter 2.1

Traveling Mechanical Waves

In this chapter we study only mechanical waves. Mechanical waves always travel within some material substance called the *medium* for the wave. When we observe what we call a water wave, what we see is a rearrangement of the water surface. Without water, there would be no wave. It’s important to emphase that the medium itself does not travel through space; its individual particles undergo back-and-force motions around their equilibrium positions.

The mechanical waves require (1) a source of disturbance, (2) a medium that can be disturbed, and (3) some kind of physical connection through which the adjacent portions of the medium can influence each other.

All traveling waves carry energy. The amount of energy transmitted through the medium and the mechanism responsible for this transport of energy differ from case to case. For instance, the power of ocean waves during a storm is much greater than the power of sound waves generated by a single human voice.

Some waves are periodic: in these the particles in the medium undergo periodic motions during wave propagation. If the periodic motion is sinusoidal, the result is a sinusoidal wave, a type of periodic wave of special importance.

2.1.1 Basic Characteristics of Wave Motion

Imagine you are floating in a boat in a large lake. You move slowly up and down as waves move past you. As you look out over the lake, you may be able to see the individual waves approaching. The point at which the displacement of the water from its normal level is the highest is called the *crest* of the wave. The point at which the displacement of the water from its normal level is the lowest is called the *trough* of the wave. The distance between two adjacent crests is called the ***wavelength*** l (Greek letter lambda). More generally, the ***wavelength is a minimum distance between any two adjacent identical points (such as crests) of the wave,*** as shown in Figure 2.1.1.

If you count the number of seconds between the arrivals of two adjacent identical points of wave, you are measuring the *period* T of the wave. In general, we define the *period* of oscillation T of a wave to be the time any element of medium takes to move through one full oscillation.

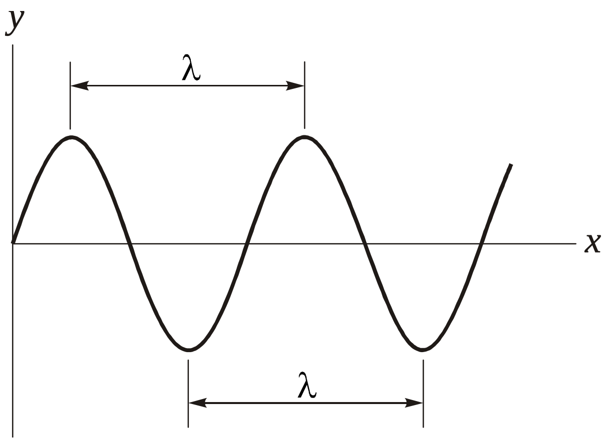


Figure 2.1.1 The wavelength l of a wave is the distance between two adjacent identical points

The inverse of the period, which is called the *frequency* f and measured in Hz in SI, is often used. In general, the frequency of a periodic wave is *the number of crests (or troughs, or any other point on the wave) that passes a given point in a unit time interval.* Since the waveform, traveling with constant speed v, advances a distance of one wavelength in a time interval of one period, it follows that

$$v = \frac{l}{T} = l f \ . \qquad (2.1.1)$$

The maximum displacement of a particle of the medium from its equilibrium level is called the *amplitude* A of the wave. For our water wave, it represents the highest distance of a water molecule above the undisturbed surface of the water as the wave passes by.

Waves travel with a specific speed, and this speed depends on the properties of the medium being disturbed. For instance, sound waves travel through room temperature air with a speed of about 343 m/s, whereas they travel through most solids with a speed much greater than that 343 m/s.

Example 2.1.1

What is the wavelength of a sound wave having a frequency of 262 Hz (the approximate frequency of the note "middle C" on the piano)? The speed of sound in air at 20°C is 344 m/s.

Solution.

From Eq. (2.1.1),

$$\lambda = \frac{v}{f} = \frac{344\ \text{m/s}}{262\ \text{s}^{-1}} = 1.31\ \text{m}.$$

For comparison, the "high C" sung by coloratura sopranos is two octaves above the middle C. The corresponding frequency is four times as large, $f = 4(262\,\text{Hz}) = 1048\,\text{Hz}$, and the wavelength is one-fourth as large, $\lambda = (1.31\text{m})/4 = 0.328\ \text{m}$.

Exercises

2.1.1. If you shake one end of a taut rope periodically, three times each second, what is the period of the sinusoidal waves set up in the rope?

2.1.2. Which property is common to all types of mechanical waves?

2.1.3. The speed of sound waves in the air depends on temperature, but the speed of light does not. Why?

2.1.4. The speed of sound in air is 343 m/s at 20°C. What is the wavelength of a sound wave of frequency 32 Hz, the lowest pedal note of the medium-sized pipe organs? What is the frequency of a wave having a wavelength of 1.22 m, corresponding approximately to the note D above the middle C of the piano?

2.1.5. Provided the amplitude is sufficiently large, the human ear can respond to longitudinal waves over the range of frequencies from about 20 Hz to about 20,000 Hz. Compute the wavelengths corresponding to these frequencies

a) for waves in air ($v = 343$ m/s);

b) for waves in water ($v = 1480$ m/s).

2.1.6. What is the wavelength of the wave in Figure 2.1.2, where each segment of the wave has length d ?

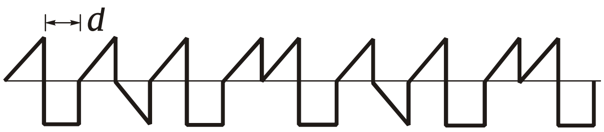
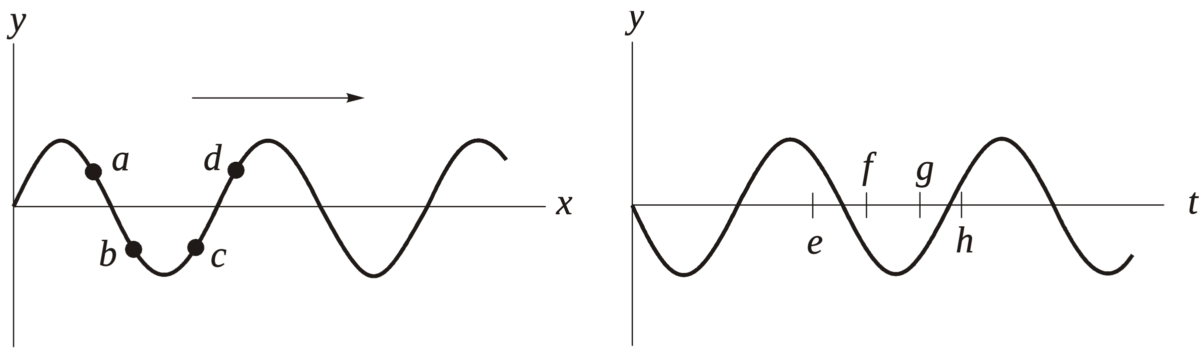


Figure 2.1.2 Wavelength is the minimum distance between any two identical points (such as the crests)

2.1.7. Figure 2.1.3a gives a snapshot of a wave traveling in the direction of positive x along a string under tension. Four string elements are indicated by the lettered points. For each of these elements, determine whether, at the instant of the snapshot the element is moving upward or downward or is momentarily at rest? *(Hint:* Imagine the wave as it moves through the four string elements.). Figure 2.1.3b gives the displacement of a string element located at $x = 0$ as a function of time. At the lettered times, is the element moving upward or downward or is momentarily at rest?



(a) (b)

Figure 2.1.3 (a) Wave traveling in the direction of positive x; (b) Displacement of a string element as a function of time

2.1.8. For a certain transverse wave, the distance between two successive crests is 1.20 m, and eight crests pass a given point along the direction of travel every12.0 s. Calculate the wave speed.

2.1.9. A sinusoidal wave is traveling along a rope. The oscillator that generates it completes 40.0 vibrations in 30.0 s. The given maximum travels 425 cm along the rope in 10.0 s. What is the wavelength?

2.1.10. For a certain transverse wave, the distance between two successive crests is 1.20 m, and eight crests pass a given point along the direction of travel every 12.0 s. Calculate the wave speed.

2.1.2 Transverse and Longitudinal Waves

Let's tie one end of a long flexible rope to a stationary object and held the other end, stretching the rope tight. Then we give this end some transverse (sideways) motion. If we give a single "flip", the result is a single *wave pulse* which travels down the length of the string (Figure 2.1.4).

This pulse and its motion can occur because the string is under tension. When you pull your end of the string upward, it begins to pull upward on the adjacent section of the string via tension between two sections. As the adjacent section moves upward, it begins to pull the next section upward and so on. Meanwhile, you have pulled down on your end of the string. As each section moves upward, in turn, it begins to be pulled back downward by neighboring sections that are already on the way down. The net result is that a distortion in the string shape (the pulse) moves along the string at some velocity $\vec{v}$.

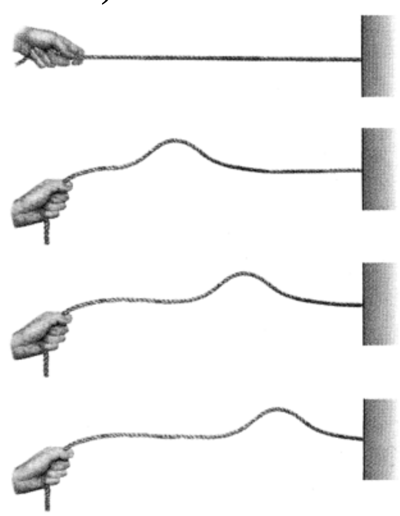


Figure 2.1.4 A wave pulse traveling down a stretched rope. The shape of the pulse is approximately unchanged as it travels along the rope

This type of disturbance is called a ***traveling wave***, and Figure 2.1.5 represents four consecutive "snapshots" of the creation and propagation of the traveling wave pulse. The rope is the medium through which the wave travels. Such a single pulse, in contrast to a train of pulses, has no frequency, no period, and no wavelength. However, the pulse does have definite amplitude and definite speed. As we shall see later, the properties of this particular medium that determine the speed of the wave are the tension in the rope and it's mass per unit length. The shape of the wave pulse changes very little as it travels along the rope. (Strictly speaking, the pulse changes its shape and gradually spreads out during the motion. This effect is called *dispersion* and is common to mechanical waves, as well as to electromagnetic waves.) As the wave pulse travels, each small segment of the rope, as it is disturbed, moves in a direction perpendicular to the wave motion. Note that no part of the rope ever moves in the direction of the wave.

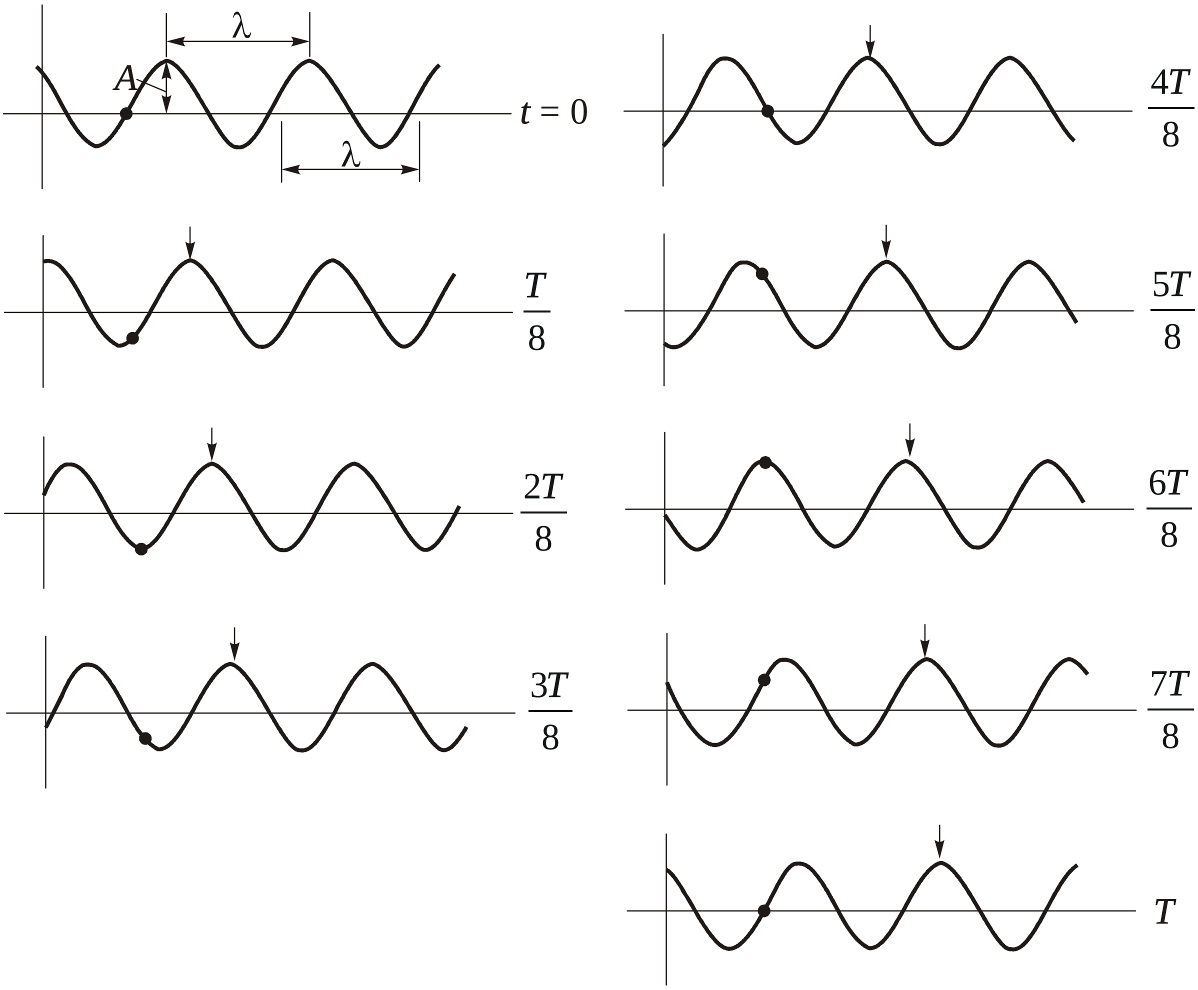


Figure 2.1.5 A sinusoidal transverse wave traveling toward the right. The shape of the string is shown at intervals of one-eighth of a period

A more interesting situation develops when we give the free end of the rope a repetitive or *periodic* motion. In particular, suppose we move it back and forth with *simple harmonic motion* of amplitude A, frequency f, and period T, where, as usual, $f = 1/T$ (Figure 2.1.5). A continuous succession of transverse sinusoidal waves then advances along the string. The shape of a portion of the string near the end, at intervals of 1/8 period, is shown in Figure 2.1.5 for a total time of one period. The waveform advances steadily toward the right, as indicated by the short arrow pointing to one particular wave crest, while any one point on the string (black dot) oscillates back and forth about its equilibrium position with simple harmonic motion. We distinguish between the motion of a *waveform,* which moves with the constant speed *along* the string, and the motion *of a particle of the string,* which moves with simple harmonic motion *transverse* to the string. A traveling wave that causes the particles of the disturbed medium to move perpendicular to the wave motion is called *a transverse wave.*

Compare this with the another type of wave – one moving down a long, stretched spring, as shown in Figure 2.1.6. The left end of the spring is pushed briefly to the right and then pulled briefly to the left. This movement creates a sudden compression of a region of the coils. The compressed region travels along the spring (to the right in Figure 2.1.6). The compressed region is followed by a region where the coils are extended. Notice that the direction of the displacement of the coils is *parallel* to the direction of propagation of the compressed region. A traveling wave that causes the particles of the medium to move parallel to the direction of wave motion is called *a longitudinal wave.*

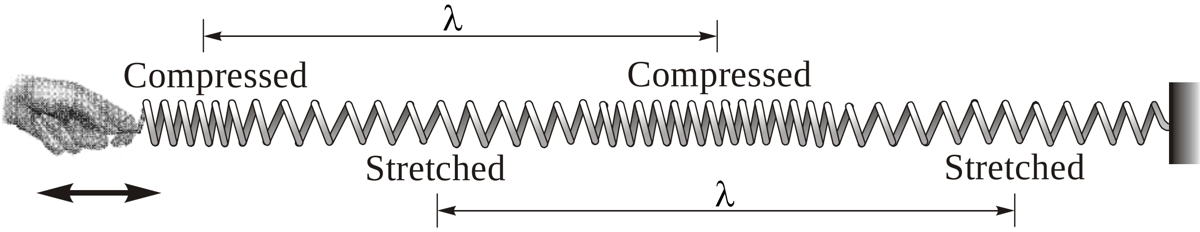


Figure 2.1.6 A longitudinal wave travels along a stretched spring. The displacement of the coils is in the direction of the wave motion. Each compressed region is followed by a stretched region

Sound waves, which we shall further discuss, are another example of longitudinal waves. The disturbance in a sound wave is a series of high-pressure (condensation) and low-pressure (rarefaction) regions that travel through air or any other material medium.

In nature some waves exhibit a combination of transverse and longitudinal displacements. Surface water waves are a good example. When a water wave travels on the surface of deep water, the water molecules at the surface move in nearly circular path, as shown in Figure 2.1.7.

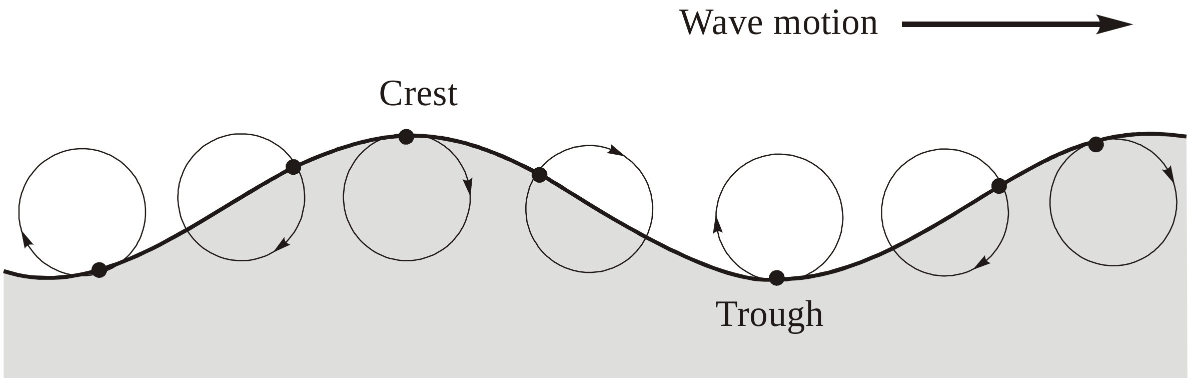


Figure 2.1.7 The motion of water molecules on the surface of deep water in which a wave is propagating is a combination of transverse and longitudinal displacement

Note that the disturbance has both transverse and longitudinal components.

An important property of transverse waves is *polarization.* When we produce a transverse wave on a string, we can choose between moving the end up and down or sideways; in either case, the wave displacements are perpendicular, or *transverse,* to the length of the string. If the end moves up and down, the motion of the entire string is confined to a vertical plane; if the end moves sideways, the wave moves in a horizontal plane. In either case, the wave is said to be *linearly polarized* because the individual particles move back and forth in straight lines perpendicular to the string.

Longitudinal waves, unlike transverse waves, *do not* exhibit polarization. This concept has no meaning for a longitudinal wave.

Figures 2.1.4 and 2.1.5 show the oscillations of particles those equilibrium positions are located along x-axis. In reality, not only particles of x-axis oscillate but neighboring particles too. Propagating from the origin of oscillation, wave process involves new and new regions of space. The locus of points oscillating with the same phase is called a *wave surface*. Wave surface can be constructed through any point of space which is involved in wave process. The boundary wave surface which separates the space, already involved in wave process, from the space which the wave process hasn't reach yet is called *wave front*. The wave surfaces are constructed through equilibrium positions of particles, oscillating with the same phase. The wave front moves in the direction of propagation of wave all the time. Therefore, there are a lot of wave surfaces, whereas there is only one wave front.

Wave surfaces can have different forms. In simplest cases, they have a form of a plane or a sphere. The waves in such cases are called *plane* or *spherical* ones, correspondingly. In a plane wave, wave surfaces represent a set of planes, parallel to each other (Figure 2.1.8); in a spherical one, they are concentric spheres.

Spherical waves are represented with a series of circular arcs concentric with the source, as shown in Figure 2.1.9. Each arc represents a wave surface. The distance between adjacent wave surfaces equals the wavelength l . The radial lines pointing outwards from the source are rays.

Now consider a small portion of a wave front far from the source, as shown in Figure 2.1.10. In this case, the rays passing through the wave front are nearly parallel to one another, and the wave front is very close to being planar. Therefore, at distances from the source that are great compared with the wavelength, we can approximate a wave front with a plane. Any small portion of a spherical wave far from its source can be considered a plane wave.

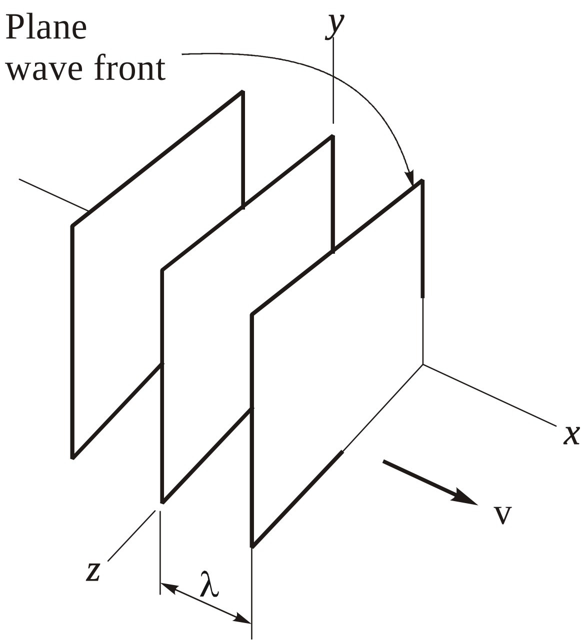


Figure 2.1.8 A plane wave moving in the positive x direction with a speed v. The wave front is plane

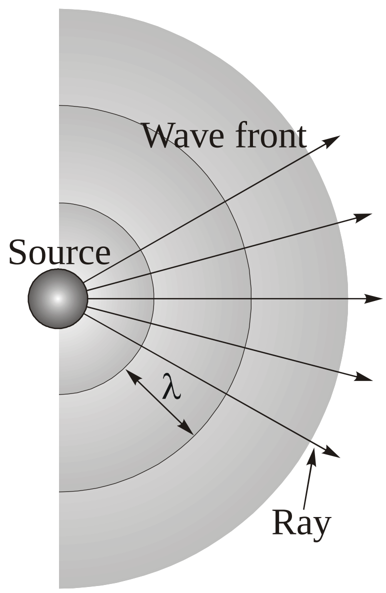


Figure 2.1.9 A spherical wave propagating radially outward from an oscillating spherical body. The intensity of the spherical wave varies as $1/r^2$

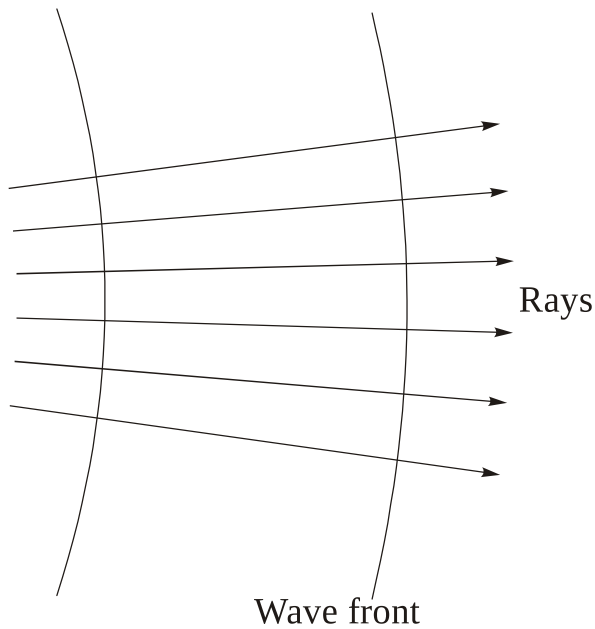


Figure 2.1.10 Spherical waves emitted by a point source. The circular arcs represent the spherical wave fronts that are concentric with the source

Exercises

2.1.11. What is the difference between longitudinal and transverse waves?

2.1.12. Why is a wave pulse traveling on a string considered a transverse wave?

2.1.13. Is it possible to have a longitudinal wave on a stretched string?

2.1.14. Is it possible to have a transverse wave on a steel rod?

2.1.15. A solid can transport both longitudinal wave and transverse waves, but a fluid can transport only longitudinal waves. Why?

2.1.16. What is the distance between compression and its nearest rarefaction in a longitudinal wave?

2.1.17. State definition of wavelength and frequancy of a wave.

2.1.18. What phenomenon is exhibited by transverse waves but not by longitudinal waves?

2.1.3 Wave Functions of Plane and Spherical Waves

To make the analysis of wave motion complete, we need a mathematical language that provides a detailed description of the motion of the medium during wave propagation. A central element of this language is the concept of *wave function* which is a function that describes the position of an arbitrary particle in the medium as a function χ of its coordinates x, y, z and time t:

$$\chi = \chi(x, y, z, t).$$

This function must be periodic both with respect to time t and coordinates x, y, z. In this discussion, we'll concentrate primarily on sinusoidal waves in which each particle of the medium undergoes simple harmonic motion about its equilibrium position.

As an example, we look first at waves on a stretched string. If we ignore the sag of the string due to gravity, the equilibrium position of the string is along a straight line. We take this to be the x-axis of a coordinate system. Waves on a string are *transverse:* During wave motion, a particle with equilibrium position x is displaced some distance y to the direction perpendicular to the x-axis. The value of y depends on the particle (that is, on x) as well as on the time t, when we look at it. Thus, y is a *function of* x and t; $y = f(x,t)$. If we know this function for a particular wave motion, we can use it to predict the position of any particle at any time. From this we can determine the velocity and acceleration of any particle, the shape of the string, its slope at any point, and anything else related to the position and motion of the string at any time.

Thus, the wave function $y = f(x,t)$, once it is known, contains a complete description of the motion. Let us now consider wave functions for sinusoidal waves. Suppose a wave travels from left to right along the string (the direction of increasing x). We can compare the motion of any particle of the string with the motion of a second particle to the right of the first one. We find that the second particle moves in the same way as the first one, but after a time lag that is proportional to the distance between the particles. If one end of a stretched string oscillates with simple harmonic motion, all other points on it oscillate with simple harmonic motion of the same amplitude and frequency. The phase of the motion is, however, different for different points. This means that the cyclic motions of different points are out of step with each other by various fractions of a cycle. For example, if, at the same time, one point has its maximum positive displacement, and another has its maximum negative displacement, the two are a half-cycle out of phase. A phase angle of π (180°) corresponds to 1/2 cycle, $\pi/2$ to 1/4 cycle, and so on.

Suppose the displacement of a particle at the left end (at $x=0$), where the motion originates, is given by

$$y = A\sin\omega t. \qquad (2.1.2)$$

The time required for the wave disturbance to travel from $x=0$ to some point x to the right of the origin is given by x/v, where v is the wave speed. The motion of the point x at time t is the same as the motion of point $x=0$ at the earlier time $(t - x/v)$. Thus the displacement of the point x at time t is obtained simply by replacing t by $(t - x/v)$, in Eq. (2.1.2) and we have

$$y(x,t) = A\sin\omega\left(t - \frac{x}{v}\right) = A\sin 2\pi f\left(t - \frac{x}{v}\right). \qquad (2.1.3)$$

The notation $y = f(x,t)$ is a reminder that the displacement y is a function of both the location x of the point and the time t. Eq. (2.1.3) can be rewritten in several alternative forms, conveying the same information in different ways. In terms of the period T and wavelength λ, we get:

$$y(x,t) = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right). \qquad (2.1.4)$$

Another convenient form is obtained by defining a quantity k, called the *propagation constant,* or the *wave number*:

$$k = \frac{2\pi}{\lambda}. \qquad (2.1.5)$$

In terms of k and the angular frequency ω, the wavelength-to-frequency relation $v = \lambda f$ becomes

$$\omega = vk \qquad (2.1.6)$$

and we can rewrite Eq. (2.1.4) as

$$y(x,t) = A\sin(\omega t - kx). \qquad (2.1.7)$$

Which of these various forms we use is a matter of convenience for a specific problem; they all say the same.

The *phase* of the wave is the argument ($\omega t - kx$) of the sine in Eq. (2.1.7). As the wave sweeps through a string element at a point at a particular position x, the phase changes linearly with time t. This means that its sine also changes, oscillating between +1 and -1. Its extreme positive value (+1) corresponds to a peak of the wave moving through the element; then, the value of y at position x is A. This extreme negative value (-1) corresponds to a valley of the wave moving through the element; then, the value of y at x is $-A$. Thus, the sine function and the time-dependent phase of a wave correspond to the oscillation of a string element, and the amplitude of the wave determines the extremes of the element displacement.

For any given time t, Eqs.(2.1.3), (2.1.4) or (2.1.7) give the displacement y of a particle from its equilibrium position, as a function of the coordinate x of the particle. If the wave is a transverse one on a string, the equation represents the *shape* of the string at a certain instant, as if we have taken a photograph of the string. Thus, at time $t = 0$

$$y(x,t) = A\sin(-kx) = -A\sin kx = -A\sin 2\pi\frac{x}{\lambda}.$$

This curve is plotted in Figure 2.1.11.

At the same time, at any given *coordinate* x, Eqs. (2.1.3), (2.1.4) or (2.1.7) give the displacement y of the particle at that coordinate, as a function of *time*. That is, it describes the motion of the particle. Thus, at the position $x = 0$,

$$y(x,t) = A\sin\omega t = A\sin 2\pi\frac{t}{T}.$$

This curve is plotted in Figure 2.1.12.

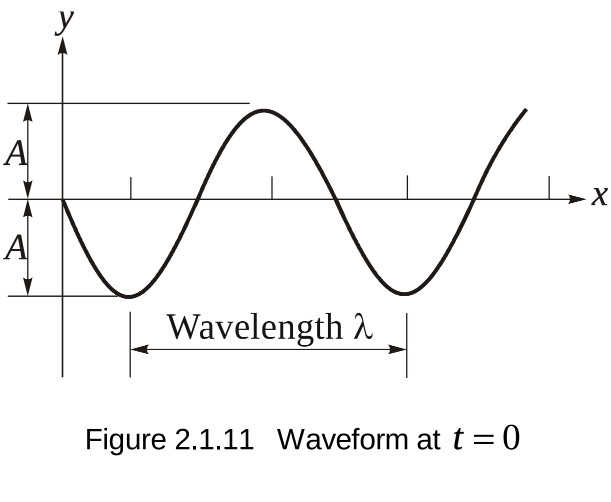


Figure 2.1.11 Waveform at $t = 0$

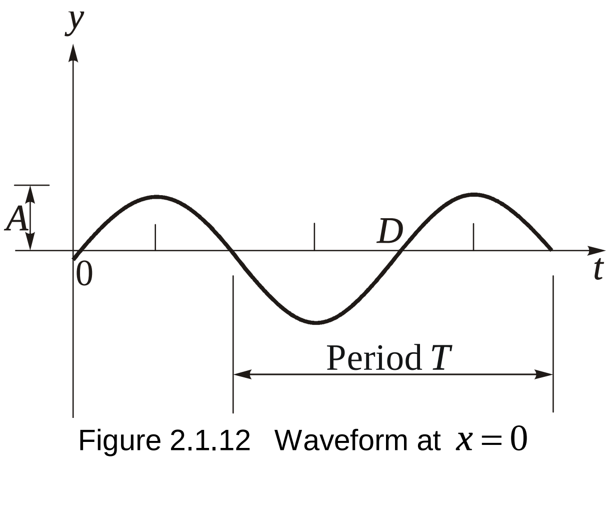


Figure 2.1.12 Waveform at $x = 0$

The above formula may be used to represent a wave traveling in the negative x-direction by making a simple modification. In this case, the displacement of a point x at time t is the same as the motion of point $x = 0$ at the later time $(t+x/v)$. In Eq. (2.1.2) we must therefore replace t by $(t+x/v)$. Thus, for a wave traveling in the negative x-direction,

$$y(x,t)=A\sin 2\pi f\left(t+\frac{x}{T}\right)=A\sin 2\pi\left(\frac{t}{T}+\frac{x}{\lambda}\right)=A\sin(\omega t+kx). \quad (2.1.8)$$

We must be careful to distinguish between the *speed of propagation* v of the waveform and the *particle speed* u of a particle of the medium in which the wave is traveling. The wave speed v is given by

$$v=\lambda f=\frac{\omega}{k}. \quad (2.1.9)$$

For general case of the wave propagating along arbitrary direction, the wave function can be written as follows:

$$\xi(\vec{r},t)=A\sin(\omega t-\vec{k}\vec{r}+\varphi), \quad (2.1.10)$$

where ξ is displacement, φ is an initial phase (or phase constant) and vector $\vec{k}$ is a so-called *wave vector*. The wave vector $\vec{k}$ equals in magnitude to the wave number $k=2\pi/\lambda$ and is directed along the perpendicular to the wave surface.

When we derived the above equations, we have suggested that the amplitude of oscillations does not depend on x. For a plane wave, this assumption is valid if the medium does not absorb energy of the wave. But if the wave propagates in the absorbing medium, the intensity of wave decreases with the distance from the origin, and the wave damps. Experiments show that in homogenous and isotropic medium, such damping occurs according to the exponential law: $A=A_0e^{-\gamma x}$ and the wave function of plane wave becomes:

$$y=A_0e^{-\gamma x}\sin(\omega t-kx). \quad (2.1.11)$$

Here A_0 is the amplitude of wave in plane $x=0$.

Now let's derive expression of the wave function for spherical wave. In isotropic and homogenous medium, wave from point source will be spherical. Let the phase of oscillation of the source be $(\omega t+\varphi)$. Then points at the wave surface of radius r will oscillate with the phase $[\omega(t-r/v)+\varphi]$, because it takes time $\tau=r/v$ for a wave to cover the distance r. In this case, the amplitude of oscillations would not be constant even if there is no absorption: It decreases with distance as $1/r$. Hence, the wave function of the spherical wave has the form:

$$\xi=\frac{A}{r}\cos(\omega t-\vec{k}\vec{r}), \quad (2.1.12)$$

In the above equation, A is the amplitude of the wave at the unit distance from the source. Dimension of A is determined by the product of oscillating quantity and dimension of length. It should be mentioned that Eq. (2.1.12) is valid only for distances r which is much greater than the size of the source.

Although we have introduced the concept of wave function with reference to transverse waves on a string, the concept is valid for longitudinal waves as well. The quantity y still measures the displacement of a particle of the medium from its equilibrium position; the only difference is that in a longitudinal wave this displacement is parallel to the x-axis instead of being perpendicular to it in transverse wave.

Example 2.1.2

A sinusoidal wave traveling in the positive x direction has an amplitude of 15 cm, a wavelength of 40 cm, and a frequency of 8.00 Hz. The vertical displacement of the medium at $t=0$ and $x=0$ is also 15.0 cm, as shown in Figure 2.1.13. (a) Find the angular wave number k, period T, angular frequency ω, and speed v of the wave.

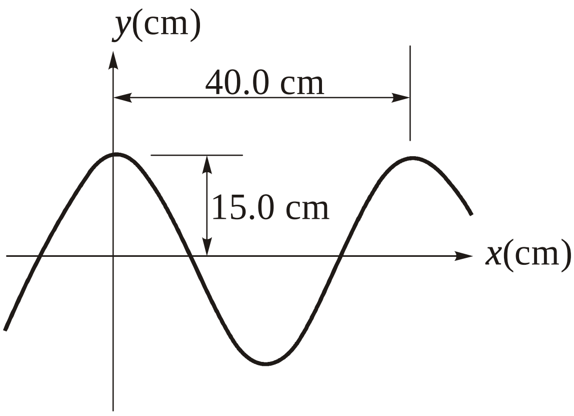


Figure 2.1.13 A sinusoidal wave of wavelength $\lambda = 40$ cm and amplitude $A=15$ cm. The wave function can be written in the form $y = A\cos(kx - \omega t)$

Solution.

a) Using Eqs. (2.1.5) and (2.1.9), we find the following:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{40} = 0.157 \text{ rad/cm},$$

$$\omega = 2\pi f = 2\pi(8) = 50.3 \text{ rad/s},$$

$$T = \frac{1}{f} = \frac{1}{8} = 0.125 \text{ s},$$

$$v = \lambda f = (40)(8) = 320 \text{ cm/s}.$$

b) Determine the phase constant ϕ, and write a general expression for the wave function.

Solution.

As $A=15$ cm and because $y=15$ cm at $x=0$ and $t=0$, substitution into Eq. (2.1.7) gives

$$15 = 15\sin\phi \quad \text{or} \quad \sin\phi = 1.$$

We may take the principal value $\phi = \pi/2$ rad (or 90°). Hence, the wave function is of the form

$$y = A\sin(kx - \omega t + \frac{\pi}{2}) = A\cos(kx - \omega t).$$

By inspection, we can see that the wave function must have this form, noting that the cosine function has the same shape as the sine function displaced by 90º. Substituting the values for A, k, and ω into this expression, we obtain

$$y = (15)\cos(0.157x - 50.3t).$$

Example 2.1.3

A wave traveling along the string is described by

$$y(x,t) = 0.00327\sin(2.72t - 72.1x),$$

in which the numerical constants are in SI units.

a) What is the amplitude of the wave?

Solution.

The equation of this wave is of the same form as Eq. (2.1.7),

$$y = A\sin(\omega t - kx),$$

so we have a sinusoidal wave. By comparing the two equations, we see that the amplitude is

$$A = 0.00327\,\text{m} = 3.27\,\text{mm}.$$

b) What are the wavelength, period, and frequency of this wave?

Solution.

By comparing equations, we see that the angular wave number and angular frequency are

$$k = 72.1 \text{ rad/m} \quad \text{and} \quad \omega = 2.72 \text{ rad/s}.$$

Now we relate the wavelength λ to the wave number k:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi \text{ rad}}{72.1\,\text{rad/m}} = 0.0871\text{m} = 8.71 \text{ cm}.$$

Next, we relate T to ω via equation:

$$T = \frac{2\pi}{\omega} = \frac{2\pi\,\text{rad}}{2.72\,\text{rad/s}} = 2.31 \text{ s}.$$

and the frequency is

$$f = \frac{1}{T} = \frac{1}{2.31\text{s}} = 0.433 \text{ Hz}.$$

c) What is the speed of this wave?

Solution.

The speed of the wave is given by

$$v = \frac{\omega}{k} = \frac{2.72\,\text{rad/s}}{72.1\text{rad/m}} = 0.0377\,\text{m} = 3.77 \text{ cm/s}.$$

Because the phase of wave contains the position variable x, the wave is moving along the x axis. Also, the minus sing in front of the kx term indicates that the wave is moving in the positive direction of the x axis. (Note that the quantities calculated in (b) and (c) are independent of the amplitude of the wave).

d) What is the displacement y at $x = 22.5$ cm and $t = 18.9$ s?

Solution.

The Eq. (2.1.7) gives the displacement as a function of position x and time t. Substituting the given values into the equation yields

$$y = 0.00327\sin(2.72 \times 18.9 - 72.1 \times 0.225) = 1.92 \text{ mm}.$$

Displacement is positive. (Don't forget express the phase in radians before calculating the sine function).

Exercises

2.1.19. In Figure 2.1.14, five points are indicated on a snapshot of a sinusoidal wave. What is the phase difference between point 1 and (a) point 2, (b) point 3, (c) point 4, and (d) point 5? Answer in radians and in terms of the wavelength of the wave. The snapshot shows a point of zero displacement at $x = 0$. In terms of the period T of the wave, when will (e) a peak and (f) the next point of zero displacement reach $x = 0$?

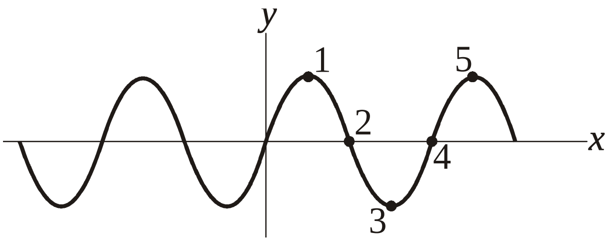


Figure 2.1.14 Sinusoidal wave

2.1.20. A sinusoidal wave train is described by the equation

$$y = (0.25\,\text{m})\sin(0.30x - 40.0t),$$

where x and y are in meters and t is in seconds. Determine for this wave the (a) amplitude, (b) angular frequency, (c) angular wave number, (d) wavelength, (e) wave speed, and (f) direction of motion.

2.1.21. A transverse wave on a string is described by the expression

$$y = (0.12\text{m})\sin(\pi x/8 + 4\pi t).$$

Determine the transverse speed and acceleration of the string at $t = 2.0$ s for the point on the string located at $x = 1.60$ m. (b) What are the wavelength, period, and speed of propagation of this wave?

2.1.22. (a) Write the expression for y as a function of x and t for a sinusoidal wave traveling along a rope in the negative x direction with the following characteristics: $A = 8.0$ cm, $\lambda = 80.0$ cm, $f = 3.0$ Hz and $y(0, t) = 0$ at $t = 0$. (b) Write the expression for y as a function of x and t for the wave in part (a), assuming that $y(x, 0) = 0$ at the point $x = 10.0$ cm.

2.1.23. Show that Eq. (2.1.7) may be written as:

$$y = -A\sin\frac{2\pi}{\lambda}(x - vt).$$

2.1.24. The equation of a certain traveling transverse wave is

$$y = 2\sin 2\pi\left(\frac{t}{0.01} - \frac{x}{30}\right),$$

where x and y are in centimeters and t is in seconds. What are the wave's amplitude, wavelength, frequency, speed of propagation?

2.1.4 Wave Equation

The wave function of any wave is a solution of the differential equation, called *wave equation*. To obtain this equation, we take wave function (2.10) for a plane wave and compare second partial derivative of the displacement ξ with respect to coordinates x, y, z and second partial derivative with respect to time t:

$$\frac{\partial^2 \xi}{\partial x^2} = -k_x{}^2 A\sin(\omega t - \vec{k}\vec{r} + \varphi) = -k_x{}^2 \xi,$$

$$\frac{\partial^2 \xi}{\partial y^2} = -k_y{}^2 A\sin(\omega t - \vec{k}\vec{r} + \varphi) = -k_y{}^2 \xi,$$

$$\frac{\partial^2 \xi}{\partial z^2} = -k_z{}^2 A\sin(\omega t - \vec{k}\vec{r} + \varphi) = -k_z{}^2 \xi,$$

$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 A\sin(\omega t - \vec{k}\vec{r} + \varphi) = -\omega^2 \xi.$$

By summing the derivatives with respect to coordinates x, y, z we get:

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = -(k_x{}^2 + k_y{}^2 + k_z{}^2)\xi = -k^2 \xi.$$

After comparing this equation with the second time derivative and substitute k^2/ω^2 with $1/v^2$, we obtain

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}. \tag{2.1.13}$$

The equation can be written in the form:

$$\Delta \xi = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}, \tag{2.1.14}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the so-called Laplas's operator. This equation is one of the most important equations in all physics. It is called the *wave equation*, and whenever it appears, we can conclude immediately that the disturbance described by the function ξ propagates as a traveling wave with a speed of wave v. In a particular case, when wave propagates along x-axis, Eq. (2.1.14) reduces to:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}. \tag{2.1.15}$$

This expression can be applied to various types of traveling waves. For waves on strings, x represents the vertical displacement of the string. For sound waves quantity x corresponds to displacement of air molecules from equilibrium or variations in either pressure or density of the gas through which the sound waves are propagating. In the case of electromagnetic waves x corresponds to electric or magnetic field components.

Alternative Derivation from the Newton's Second Law. Alternative way of the wave equation derivation is the application of the Newton's second law. Suppose a traveling wave is propagating along a string that is under a tension T. Let us consider one small string segment of length Δx (Figure 2.1.15). Ends of the segment make small angles θ_A and θ_B with the x axis. The net force acting on the segment in the vertical direction is

$$\sum F_y = T\sin\theta_B - T\sin\theta_A = T(\sin\theta_B - \sin\theta_A).$$

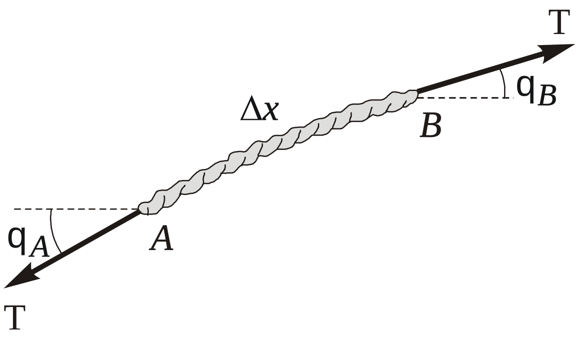


Figure 2.1.15 The segment of a string under tension T. At points A and B the slopes are given by $\tan\theta_A$ and $\tan\theta_B$, respectively

As the angles are small, we can use the small-angle approximation $\sin\theta \approx \tan\theta$ to express the net force as

$$\sum F_y \approx T(\tan\theta_B - \tan\theta_A).$$

However, at A and B the tangents of the angles are defined as the slopes of the string segment at these points. Because the slope of a curve is given by $\partial y/\partial x$, we have

$$\sum F_y \approx T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]. \qquad (2.1.16)$$

We now apply Newton's second law to the segment, with the mass of the segment given by $m = \mu\Delta x$, where μ is mass per unit length.

$$\sum F_y = ma_y = \mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right). \qquad (2.1.17)$$

Combining Eq. (2.1.16) with Eq. (2.1.17), we obtain:

$$\mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) = T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right], \text{ or}$$

$$\frac{\mu}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{(\partial y/\partial x)_B - (\partial y/\partial x)_A}{\Delta x}. \qquad (2.1.18)$$

The right side of this equation can be expressed in other form if we note that the partial derivative of any function is defined as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If we associate $f(x+\Delta x)$ with $(\partial y/\partial x)_B$ and $f(x)$ with $(\partial y/\partial x)_A$, we see that, in the limit $\Delta x \to 0$, Eq.(2.1.18) becomes

$$\frac{\mu}{T}\left(\frac{\partial^2 y}{\partial t^2}\right)=\frac{\partial^2 y}{\partial x^2} \quad \text{or}$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\left(\frac{\partial^2 y}{\partial t^2}\right), \tag{2.1.19}$$

where

$$v^2=\frac{T}{\mu}. \tag{2.1.20}$$

This is the linear wave equation as it applies to waves on a string.

Now we show that the sinusoidal wave function (Eq.2.1.7) represents a solution of the linear wave equation. If we take the sinusoidal wave function to be of the form $y(x,t)=A\sin(\omega t-kx)$, then the appropriate derivatives are

$$\frac{\partial^2 y}{\partial t^2}=-\omega^2 A\sin(\omega t-kx), \tag{2.1.21a}$$

$$\frac{\partial^2 y}{\partial x^2}=-k^2 A\sin(\omega t-kx). \tag{2.1.21b}$$

Substituting these expressions into Eq. (2.1.19), we obtain

$$-\frac{\mu\omega^2}{T}\sin(\omega t-kx)=-k^2\sin(\omega t-kx).$$

This equation must be true for all values of the variables x and t in order for the sinusoidal wave function to be a solution of the wave equation. Both sides of the equation depend on x and t through the same function $\sin(\omega t-kx)$. Because this function divides out, we do indeed have an identity, provided that

$$k^2=\frac{\mu\omega^2}{T}.$$

Using the relationship $v=\omega/k$ in this expression, we see that

$$v^2=\frac{\omega^2}{k^2}=\frac{T}{\mu},$$

and, finally

$$v=\sqrt{\frac{T}{\mu}}. \tag{2.1.22}$$

In this section, we have shown that the sinusoidal wave function is a solution of the linear wave equation (Eq. 2.1.14). Although we do not prove it here, the linear wave equation is satisfied by *any* wave function having the form $x=f(x\pm vt)$. Furthermore, we have seen that the linear wave equation is a direct consequence of the Newton's second law applied to any segment of the string.

Exercises

2.1.25. Show that the wave function $y = e^{b(x-vt)}$ is a solution of the wave equation (Eq. 2.1.13), where b is a constant.

2.1.26. Show that the wave function $y = \ln[b(x-vt)]$ is a solution to Eq. (2.1.13) where b is a constant.

2.1.27. Show that the function $y(x,t) = x^2 + v^2 t^2$ is a solution to the wave equation.

2.1.28. A wave on a string is described by the wave function

$$y = (0.10\,\mathrm{m})\sin(0.50x - 20t)].$$

(a) Show that a particle in the string at $x = 2.0$ m executes simple harmonic motion; (b) Determine the frequency of oscillation of this particular point.

2.1.5 Sound Waves

Sound waves are the most important example of longitudinal waves. Seismic prospecting teams use such waves to probe Earth's crust for oil. Ships carry sound-ranging gear (sonar) to detect underwater obstacles. Submarines use sound waves to stalk other submarines, largely by listening for the characteristic noises produced by the propulsion system. Sound waves can be used to explore the soft tissues of the human body.

Sound waves are divided into three categories that cover different frequency ranges:

1) Audible waves are waves that lie within the range of sensitivity of the human ear. The ear is sensitive to range of sound frequencies from about 20 Hz to about 20,000 Hz. The corresponding wavelength range is from about 17 m, corresponding to a 20-Hz, to about 1.7 cm, corresponding to 20,000 Hz.

2) Infrasonic waves are waves having frequencies below the audible range. Elephants can use infrasonic waves to communicate with each other even when separated by many kilometers.

3) Ultrasonic waves are waves having frequencies above audible range. You may have used “silent” whistle to retrieve your dog. The ultrasonic sound it emits is easily heard by dogs, although humans cannot detect it at all. Dolphins and bats use high-frequency sound waves for navigation. For bat a typical frequency is 100,000 Hz; the corresponding wavelength in air is about 3.5 mm, small enough to permit detection of flying insects useful as food. Ultrasonic waves are also used in medical imaging.

Sound waves can travel through any material with a speed that depends on the properties of the medium. As the waves travel, the particles in the medium vibrate to produce changes in density and pressure along the direction of motion of the wave. These changes result in a series of high-pressure and low-pressure regions. If the source of the sound waves vibrates sinusoidally, the pressure variations are also sinusoidal.

In this discussion, we have ignored the *molecular* nature of a gas and have treated it as a continuous medium. Actually, we know that a gas is composed of molecules in random motion, separated by distances that are large compared with their diameters. The vibrations that constitute a wave in a gas are superposed on the random thermal motion. At atmospheric pressure, a molecule travels an average distance of about 10^{-5} cm between collisions while the displacement amplitude of a faint sound may be only a few ten-thousandths of this amount.

The simplest sound waves are sinusoidal waves with definite frequency, amplitude and wavelength. When such a wave arrives at the ear, the air particles at the eardrum vibrate with definite frequency and amplitude. This vibration may also be described in terms of the variation of air pressure at the same point. The pressure fluctuates above and below atmospheric pressure with a sinusoidal variation having the same frequency as the motions of the air particles.

A sinusoidal sound wave in an elastic medium is described by a wave function of the form:

$$y = A\sin(\omega t - kx),$$

where y is the displacement from equilibrium of a point in the medium, and the amplitude A is, as usual, the maximum displacement from equilibrium. From a practical standpoint, it is nearly always easier to measure the pressure variations in a sound wave than to measure the displacements, so it is worthwhile to develop a relation between the two. Let p be the instantaneous pressure fluctuation at any point; that is, the amount by which the pressure differs from normal atmospheric pressure. If the displacements of two neighboring points x and $x+\Delta x$ are the same, the air between these points is neither compressed nor expanded, there is no volume change, and consequently $p = 0$. Only when y varies from one point to a neighboring one, there is a change of volume and, therefore, of pressure.

The fractional volume change $\Delta V/V$ in a volume element near point x turns out to be given simply by $\partial y/\partial x$, which is the rate of change of y with x as we go from one point to a neighboring point. To see why this is so, consider an imaginary cylinder of air, (as in Figure 2.1.16), with the cross-sectional area A and the axis along the direction of propagation. The grey cylinder shows the undisplaced position, and the dashed lines show the displaced position. When no sound disturbance is present, the cylinder's length is Δx and its volume is $V = A\Delta x$. When a wave is present, the end of the cylinder (initially at x) is displaced a distance $y_1 = y(x,t)$, and the end initially at $x+\Delta x$ is displaced a distance $y_2 = y(x+\Delta x,t)$. The change in volume ΔV of this element is

$$\Delta V = A(y_2 - y_1) = A[y(x+\Delta x,t) - y(x,t)],$$

and in the limit as $\Delta x \to 0$, the fractional change in volume $\Delta V/V$ is given by

$$\frac{\Delta V}{V} = \frac{y(x+\Delta x,t) - y(x,t)}{\Delta x} = \frac{\partial y}{\partial x}.$$

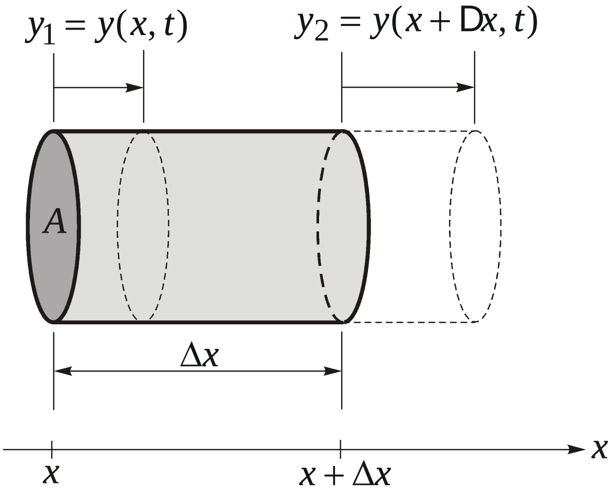


Figure 2.1.16 A cylindrical volume of gas with the cross-sectional area A. The length in the undisplaced position is Δx. During wave propagation along the axis, the left end is displaced to the right a distance y_1, and the right end is displaced a different distance y_2. The resulting change in volume is $A(y_2 - y_1)$

Now from the definition of the bulk modulus B, $p = -B\Delta V/V$, we find that

$$p = -B\left(\frac{\partial y}{\partial x}\right). \qquad (2.1.23)$$

The negative sign arises because, when $\partial y/\partial x$ is positive, the displacement is greater at $x + \Delta x$ than at x corresponding to an increase in volume and a decrease in pressure. For the sinusoidal wave of Eq. (2.1.7), we find

$$p = BkA\cos(\omega t - kx). \qquad (2.1.24)$$

This expression shows that the quantity BkA represents the maximum pressure variation. This maximum is called the ***pressure amplitude*** and is denoted by p_{max}. Thus,

$$p_{max} = BkA. \qquad (2.1.25)$$

The pressure amplitude is directly proportional to the displacement amplitude A, as might be expected, and it also depends on wavelength. Waves of shorter wavelength (larger k) have greater pressure variations for a given amplitude because the maxima and minima are squeezed closer together.

Ultrasound Waves

Ultrasonics cover a frequency range from 20 000 Hz upwards. Compared with sonics (i.e. sound waves we can hear), ultrasonics have shorter wavelengths because their frequencies are higher.

Ultrasonics is widely used. For example, ultrasonics of frequency 40 kHz is used for industrial cleaning. In air where the speed of sound is approximately 340 m/s, the wavelength of these ultrasonic waves is 8.5 mm. By comparison, sound waves of frequency 1000 Hz have in air wavelength of 0.33 m. Equipment to be cleaned by ultrasonics is placed in a tank of water which ultrasonics pass through. The ultrasonic waves pass through the water, agitating and loosening particles of dirt and grease.

Another use of ultrasonics is in medical imaging. For example, they are used in prenatal care to give an image of a baby inside the womb (Figure 2.1.17).

Ultrasonic scans do not harm the baby, and are much safer than X-ray scans in this situation.

Figure 2.1.17 Ultrasonic imaging of a baby in the womb

To produce the ultrasonic waves, an ultrasonic *transducer* is used. (Figure 2.1.18).

It converts electrical energy into ultrasonic energy by applying an alternating voltage across a quartz crystal. Quartz is used because it changes length slightly when a voltage is applied across it. So an alternating voltage makes the quartz vibrate. By making the applied frequency equal to the natural frequency of vibration of the crystal, the vibrations become very large: the crystal resonates. So, the crystal produces ultrasonic waves of frequency equal to its natural frequency. The frequency is in the megahertz (MHz) range, so the ultrasonics passes directly into the body when the transducer is placed on the body surface. Inside the body, tissue boundaries reflect part of the incoming ultrasonic energy, and the transducer can be used to detect the reflected ultrasonics. The transducer converts ultrasonic energy back into the electrical energy, thus enabling an image at an oscilloscope to be built up showing the internal boundaries.

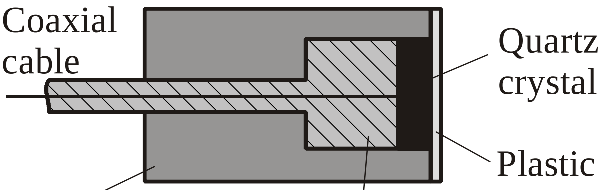
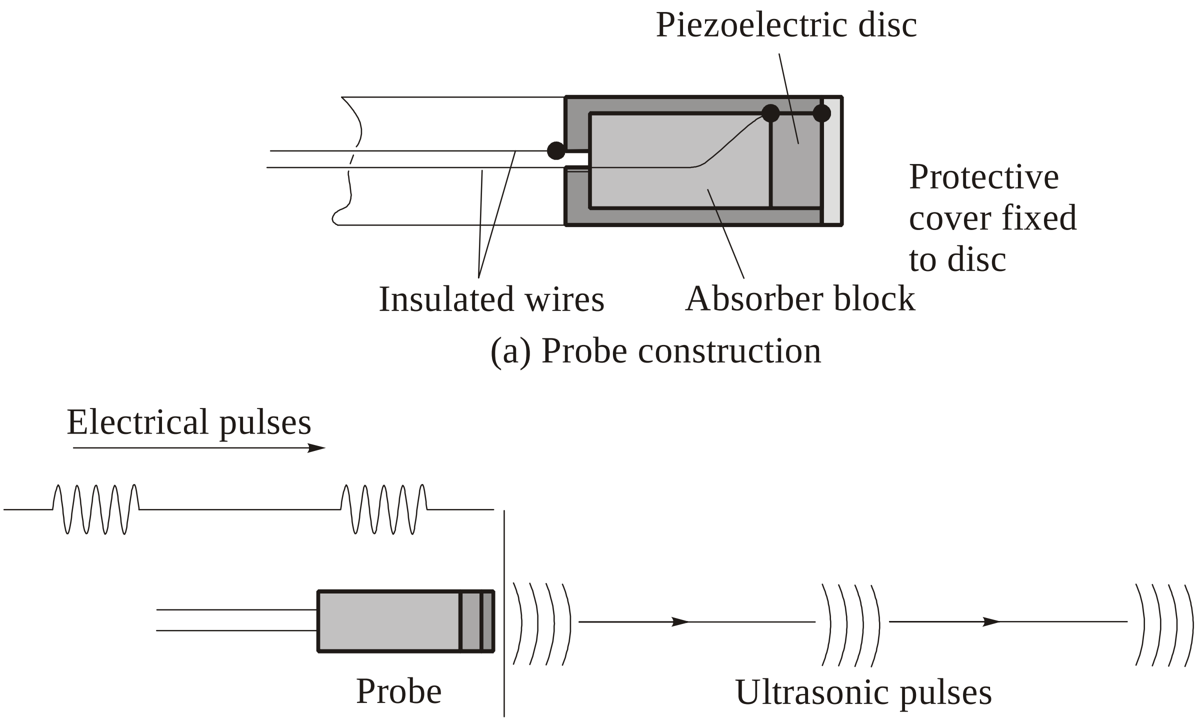


Figure 2.1.18 An ultrasonic transducer

For medical imaging purposes, ultrasonics is used at frequencies between about 1 and 10 MHz. This frequency range represents a compromise between lower frequencies which would diffract and spread out too much and higher frequencies which would be absorbed too easily by tissues. The higher the frequency the smaller the wavelength and, hence, the greater the detail of the image. Since the frequency used depends on the depth and density of the organ to be imaged, low density organs near the surface (e.g. the eye) can be imaged in more detail than higher density internal structures (e.g. a baby in the womb).

Producing ultrasonics. An ultrasonic probe contains a piezoelectric transducer in the shape of disc which vibrates when an alternating voltage is applied across its surfaces (Figure 2.1.19). When the applied frequency is equal to the natural frequency of vibration of the transducer disc, the disc vibrates at resonance and creates sound waves at the same frequency as the alternating voltage in the surrounding medium. The thickness of the disc determines its resonant frequency on the same principle as the resonance in a pipe. An absorber block behind the disc prevents ultrasonic waves created at two surfaces of the disk canceling each other. Normaly, the alternating voltage is supplied in pulses so the disc produces ultrasonic pulses. The backing block is made of epoxy resin which damps the disc vibrations rapidly at the end of each pulse before the next pulse is produced.

An ultrasonic scanning system. The probe is connected to a control system which includes a visual display unit. In operation, the probe is held in contact with the body surface via a gel so that ultrasonic pulses are directed into the body. These pulses reflect at the surface of internal organs and at tissue boundaries and are detected by the transducer, which acts as a receiver when it is not producing pulses.



(a) Probe construction

(b) Ultrasonic pulses

Figure 2.1.19 An ultrasonic probe

The speed of ultrasound in tissue is about 1500 m/s so it takes an ultrasonic pulse less than 1 millisecond to travel across the body and back. In operation, the probe must therefore produce pulses at a rate of no more than one per millisecond to allow received pulses to return before the next pulse is transmitted. Also, the pulses must last no more than a few microseconds to ensure the end of a pulse is clearning the probe before the reflection of the front end returns.

Each boundary in the body is a partial reflector of ultrasonics, so each pulse from the transducer produces a series of reflected pulses which return to the transducer. These reflected pulses are received by the transducer when it is in the "receiver" mode to produce a pulsed signal from the transducer. This signal is amplified and displayed (the A-scan) or used to modulate the brightness of an image built up on a VDU (the B-scan) as the probe is moved across the body surface.

In the A-scan system, the spacing of each reflected pulse at the oscilloscope screen from the transmitted pulse is proportional to the time taken by each ultrasonic pulse to travel from the probe to the reflecting boundary and return back (Figure 2.1.20). The A-scan system is used when precise locations are to be measured.

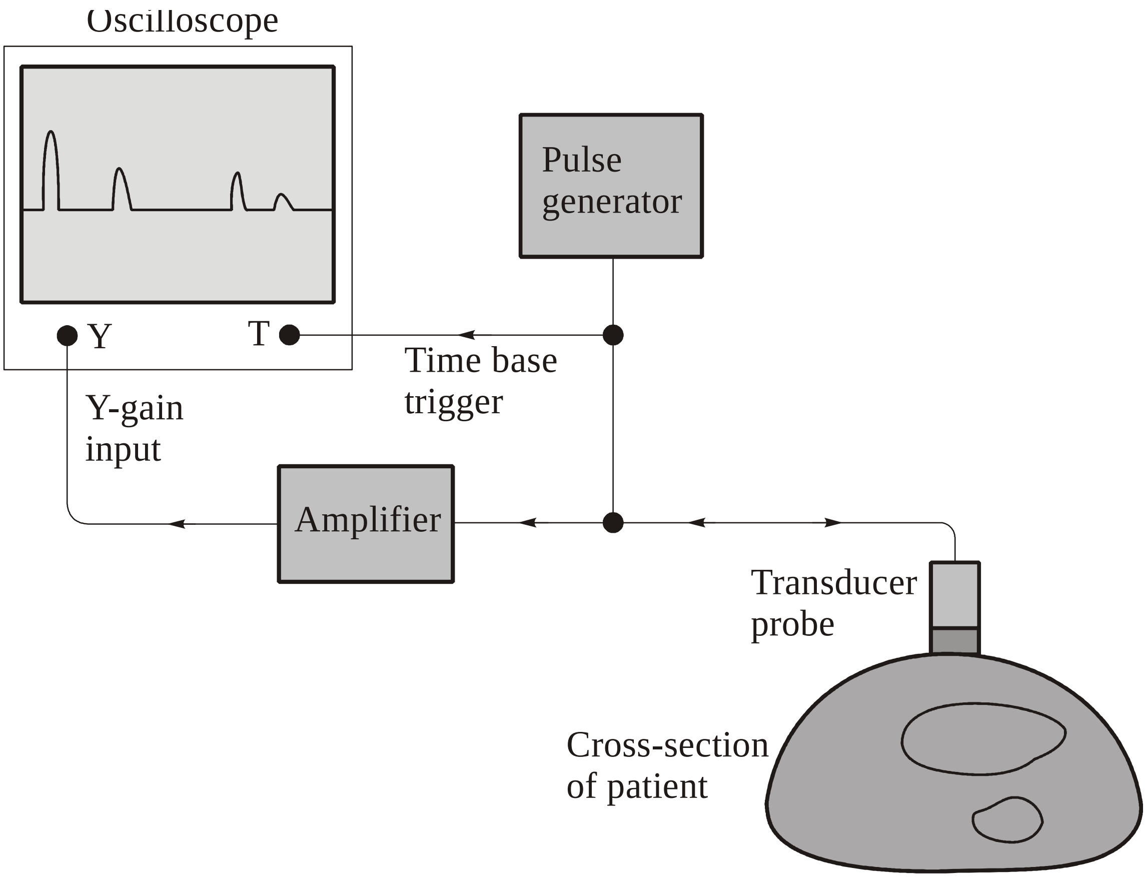


Figure 2.1.20 The A-scan system

In the B-scan system position sensors attached to the probe provide signals to control the direction of the electron beam in the VDU as it moves across the screen. Received pulses control the beam current. The B-scan system therefore gives a two-dimensional image (Figure 2.1.21).

Reflection of ultrasound. The intensity of an ultrasonics beam reflected at a boundary between two substances is given by the equation

$$I = \frac{(r_2 v_2 - r_1 v_1)^2}{(r_2 v_2 - r_1 v_1)^2} I_0,$$

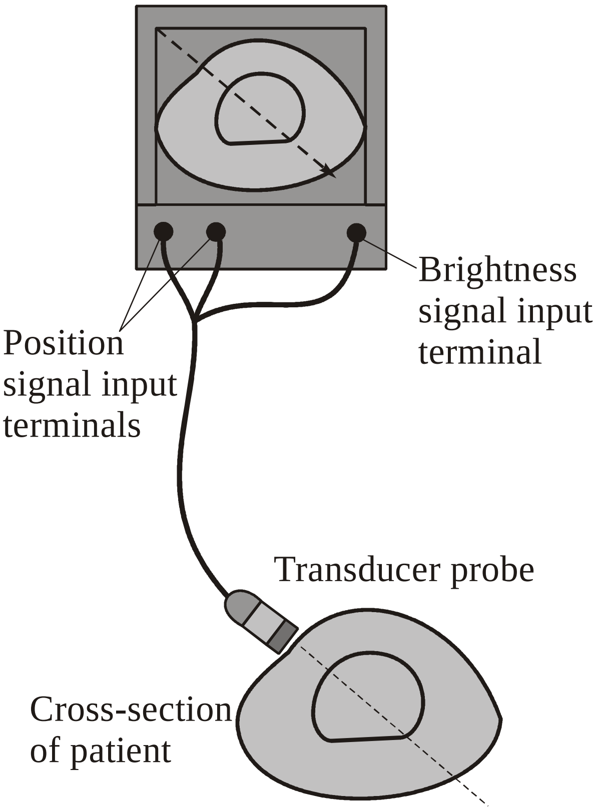


Figure 2.1.21 The B-scan system

where I_0 is the incident intensity, r_1 and r_2 are the densities of the incident substance and the transmitted substance, v_1 and v_2 are the speeds of ultrasonics in the two substances.

The reflection coefficient R of a boundary is defined as I / I_0. The acoustic impedance of a substance is defined as $r v$.

Typical values of densities, speeds, and acoustic impedances, for different types of tissue in the body, are given in Table 2.1.1. These values may be used to calculate the reflection coefficient for different tissue boundaries. Several implications follow from these calculations:

– Almost 100% reflection occurs at an air-skin boundary. This is why the probe is applied to the body via a gel or a water bag so that most of the ultrasound energy enters the body.

– Ultrasonics reflects at the boundaries between different soft tissues in the body. Hence an ultrasonic imaging system can detect and display such boundaries unlike an X-ray imaging system which cannot. Note that the strength of a reflected pulse depends on the distance travelled by the ultrasonic pulse in the body as well as the reflection coefficient.

Table 2.1.1 Typical values for densities, speeds and acoustic impedances for different types of tissue in the body

Substance type	Speed, m s^{-1}	Density, k gm^{-3}	Acoustic impedance, kg m^{-2} s^{-1}
Air	1.2	340	410
Water	1000	1500	1.5´ 10^6
Soft tissue	1050	1550	1.6´ 10^6
Fat	900	1450	1.3´ 10^6
Muscle	1080	1600	1.7´ 10^6
Bone	1900	4000	7.8´ 10^6

Example 2.1.4. Measurement of sound waves show that maximum pressure variations in the loudest sound, that the ear can tolerate without pain are of the order of 30 Pa (above and below atmospheric pressure, which is about 100,000 Pa). Find the corresponding maximum displacement if the frequency is 1000 Hz and $v = 350$ m/s.

Solution.

We have

$$\omega = (2\pi)(1000\,\text{Hz}) = 6283\,\text{s}^{-1} \text{ and}$$

$$k = \frac{\omega}{v} = \frac{6283}{350} = 18\,\text{m}^{-1}.$$

For air the adiabatic bulk modulus is

$$B = \gamma p = (1.4)(1.01 \times 10^5\,\text{Pa}) = 1.42 \times 10^5 \text{ Pa}.$$

From Eq. (2.1.25) we find

$$A = \frac{p_{\max}}{Bk} = \frac{(30\,\text{Pa})}{(1.42 \times 10^5\,\text{Pa})(18\,\text{m}^{-1})} = 0.0118 \text{ mm}.$$

Thus, the displacement amplitude of even the loudest sound is extremely small. The maximum pressure variation in the faintest audible sound of frequency 1000 Hz is only about 3×10^{-5} Pa. The corresponding displacement amplitude is about 6×10^{-3} cm. Thus, the ear is an extremely sensitive organ.

Exercises

Note: The equilibrium density of air is $\rho = 1.29\,\text{kg/m}^3$; the speed of sound in air is $v = 343$ m/s. Pressure variations ΔP are measured relative to atmospheric pressure, 1.013×10^5 Pa.

2.1.29. In air a sound wave has pressure amplitude equal to 4×10^{-3} Pa. Calculate the displacement amplitude of the wave at a frequency of 10.0 kHz.

2.1.30. A sinusoidal sound wave is described by the displacement

$$y(x,t) = (2.0\ \mu\text{m})\cos[(15.7\,\text{m}^{-1})x - (858\,\text{s}^{-1})t].$$

(a) Find the amplitude, wavelength, and speed of this wave; (b) Determine the instantaneous displacement of the molecules at the position $x = 0.05$ m at $t = 3$ ms; (c) Determine the maximum speed of a molecules oscillatory motion.

2.1.31. As a sound wave travels through the air, it produces pressure variations (above and below atmospheric pressure) that are given by $\Delta P = 1.27\sin(\pi x - 340\pi t)$ in SI units. Find (a) the amplitude of the pressure variations; (b) the frequency of the sound wave; (c) its wavelength in air, and (d) its speed.

2.1.32. Write an expression that describes the pressure variation as a function of position and time for a sinusoidal sound wave in air, if $\lambda = 0.100$ m and $\Delta P = 0.20$ Pa.

2.1.33. Write the function that describes the displacement wave corresponding to the pressure wave in Ex. 2.32.

2.1.34. In a traveling sound wave the pressure is given by the equation

$$\Delta p = (1.5\ \mathrm{Pa})\sin\pi[(0.9\,\mathrm{m}^{-1})x - (315\,\mathrm{s}^{-1})t].$$

Find (a) the frequency; (b) the pressure amplitude; (c) the wavelength; and (d) the speed of wave.

2.1.35. Calculate the pressure amplitude of a 2.0-kHz sound wave in air if the displacement amplitude is equal to 2×10^{-8} m.

2.1.36. (1) Calculate the wavelength of ultrasonics of frequency 2 MHz in (a) air, (b) water. The speed of sound in air = 340 m/s; in water = 1500 m/s. (Ans. (a) 0.17 mm; (b) 0.75 mm.)

(2) Calculate the reflection coefficient at (a) an air-tissue boundary, (b) a water-tissue boundary, (c) a boundary between fat and tissue. Use the data from Table 2.1. (Ans. (a) 0.995; (b) 0.04; (c) 0.10.)

(3) The diagram shows an A-scan trace. (a) Explain why there are several pulses on the display after each transmitted pulse. (b) Calculate the distance from the probe to the boundary that caused pulse X on the display. See Table 2.1.1 for the speed of sound in the body. (Ans. 0.18 m)

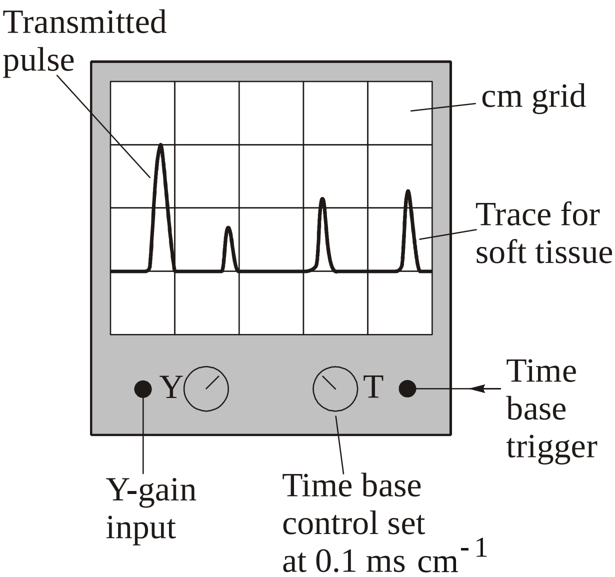


Figure 2.1.22. The diagram shows an A-scan trace

2.1.37. (1) With the aid of a diagram (Figure 2.1.22), describe and explain the construction of an ultrasonic transducer.

(2) (a) With the aid of a diagram, describe an ultrasonic B-scan system. (b) Why is it necessary to use a gel where the ultrasonic probe is applied? (c) What advantage does an ultrasonic scanner have in medicine in comparison with X-ray imaging?

2.1.38. Bats can ascertain distances, directions, and size of the obstacle without any eyes. Explain, why.

2.1.6 Phase Speed and Group Speed of a Traveling Wave

Figure 2.1.23 shows two snapshots of the wave taken a small time interval Δt apart. The wave is traveling in the positive x-direction (to the right in the Figure 2.1.23), the entire wave pattern moves a distance Δx in the direction during the time interval Δt. The ratio $\Delta x/\Delta t$ (or, in the differential limit dx/dt), is the wave speed v. How can we find it’s value?

As the wave in the Figure 2.1.23 moves, each point of the moving wave form (such as the point A marked on a peak) retains its displacement y. (Points on the string do not retain their displacement, but points on the wave form do). If the point A retains its displacements as it moves, the phase which ensures the displacement must be constant:

$$kx - \omega t = const \qquad (2.1.26)$$

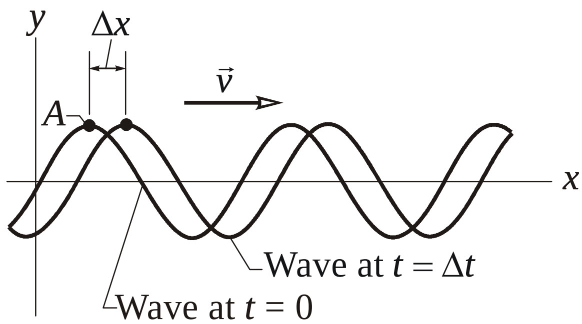


Figure 2.1.23 Two snapshots of the wave at time $t = 0$ and then at $t = \Delta t$. As the wave moves to the right at velocity v, the entire curve shifts a distance Δx during Δt. Point A “rides” with the wave form but the string elements move only up and down

Note that although this argument is constant, both x and t are changing. In fact, as t increases, x must as well, to keep the argument constant. This confirms that the wave pattern is moving in the positive x-direction.

If we take the derivative dx/dt, we obtain the speed with which the given phase propagates. This speed is known as the ***phase speed*** of wave.

$$k\frac{dx}{dt} - \omega = 0, \text{ or } \quad \frac{dx}{dt} = v = \frac{\omega}{k}. \qquad (2.1.27)$$

Using $k = 2\pi/\lambda$, we can rewrite the phase speed of the wave as

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f. \qquad (2.1.28)$$

The equation $v = \lambda/T$ tells us that the phase speed of a wave is one wavelength per period; the wave moves a distance of one wavelength in one period of oscillation.

When we deal with a packet of waves with different wavelength and so-called dispersion medium (that is, medium in which speed of wave propagation depends on its wavelength), it is useful, as Rayleigh showed, to introduce additional speed, a ***group speed***. We shell not discuss the dispersion phenomena in this book; we only mention that the group speed u is defined as

$$u = v - \lambda\frac{dv}{d\lambda}. \qquad (2.1.29)$$

From Eq. (2.1.29), we can see, that group speed is greater than phase speed ($u > v$) if $\frac{dv}{dl} < 0$ or less ($u < v$) if $\frac{dv}{dl} > 0$. Hence, group speed can be smaller or greater than phase speed. When $\frac{dv}{dl} > 0$, i.e. when waves with longer wavelength propagate faster, we speak about *normal* dispersion; when $\frac{dv}{dl} < 0$, i.e. when waves with shorter wavelength propagate faster, dispersion is called *abnormal*. When there is no dispersion, $\frac{dv}{dl} = 0$, all waves of the packet propagate with equal speeds, and then the phase and the group speeds are equal.

Exercises

2.1.39. A wave has an angular frequency of p rad/s and a wavelength of 1.80 m. Calculate (a) the angular wave number and (b) the speed of the wave.

2.1.40. The ocean floor is underlain by a layer of basalt that constitutes the crust, or uppermost layer, of the Earth in that region. Below the crust is found denser peridotite rock, which forms the Earth's mantle. The boundary between these two layers is called the Mohorovicic discontinuity ("Moho" for short). If an explosive charge is set off at the surface of the basalt, it generates a seismic wave that is reflected back out at the Moho. If the speed of the wave in basalt is 6.50 km/s and the two-way travel time is 1.85 s, what is the thickness of this oceanic crust?

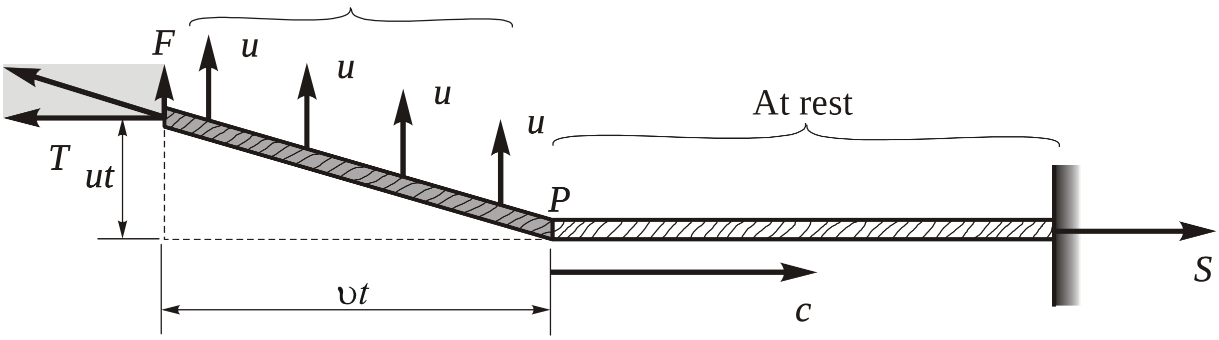
2.1.41. A traveling wave is represented by the equation $y = 0.25\sin(6000t - 20x)$. (a) Calculate the wave frequency, wavelength, and speed of the wave. (b) Write the equation to represent a similar wave of twice the amplitude and frequency and traveling with the same speed but in the opposite direction.

2.1.42. The wave in a ripple tank is represented by the equation $y = 0.6\sin(20t - 4x)$, where x and y are in cm and t in second. Calculate the speed of the wave. Write an equation to represent a wave in the ripple tank which has half the amplitude and twice the frequency but travels with the same speed.

2.1.7 Speed of a Transverse Wave on Strings

How is the speed of propagation v of a transverse wave on a string related to the *mechanical* properties of the system? The relevant physical quantities are the *tension* in the string and it's *mass per unit length.* Intuition suggests that the speed should grow with the increase of tension, and should decrease with the growing mass. We now develop this relationship by two different methods. The first is simple and considers a specific type of waveform; the second is more general but also more formal.

For our first development, we consider a perfectly flexible string, as shown in Figure 2.1.24 having linear mass density (mass per unit length) m and stretched with the tension T. Initially the string is at rest. At time $t = 0$, a constant transverse force $\vec{F}$ is applied at the left end of the string. We might expect that the end would move with the constant acceleration; this would certainly occur if the force were applied to a point mass. But here the effect of the force is to set successively more and more mass in motion. The wave travels with constant speed, so the division point P between moving and nonmoving portions also travels with definite speed. Hence, the total mass in motion is proportional to the time the force actied and, thus, to the impulse of the force. This, in turn, is equal to the total momentum mu of the moving part of the string. The total momentum thus must increase proportionately with time, so the change of momentum must be associated entirely with the increasing amount of mass in motion, not with the increasing velocity of an individual mass element. Force is the rate of change of momentum mu, and mu changes because m changes, not u. Hence, the end of the string moves upward with the constant velocity u.



(a)

Figure 2.1.24 Propagation of transverse disturbance in a string

Figure 2.1.24 shows the shape of the string after the time t has elapsed. All particles of the string to the left of the point P move upward with the speed u, while all particles to the right of the point P are still at rest. The boundary point P between the moving and the stationary portions is traveling to the right with the speed of propagation v. The left end of the string has moved up a distance ut, and the boundary point P has advanced a distance vt along the string.

The tension at the left end of the string is the vector sum of the forces $\vec{T}$ and $\vec{F}$. As no motion occurs in the direction along the length of the string, there is no unbalanced horizontal force, so T, the magnitude of the horizontal component, does not change when the string is displaced. As a result of the increased tension, the string clearly stretched somewhat. It can be shown that, for small displacement, the amount of stretch is approximately proportional to the increase in tension, as we would expect from Hook's law.

We can obtain an expression for the speed v of propagation by applying the impulse-momentum relation to the portion of the string in motion at time t: that is the darkly shaded portion in Figure 2.1.24. We set the transverse impulse ($\textit{transverse forse´ time}$) equal to the change of transverse momentum of the moving portion ($\textit{mass´ transverse velocity}$). The impulse of the transverse force F in time t is Ft. By similar triangles,

$$\frac{F}{T} = \frac{ut}{vt}, \qquad F = T\frac{u}{v}.$$

Hence, transverse impulse $Ft = T\dfrac{u}{v}t$.

The mass of the moving portion of the string is the product of mass per unit length m and the length vt. (In its displaced position, the section of string is stretched, making its mass per unit length somewhat less than m and its length somewhat greater than vt. But the mass of the section is still mvt, the same as in the undisplaced position.) Hence, transverse momentum

$$mu = mvtu.$$

Note again that the momentum increases with time not because the mass moves faster, but because more mass is brought into motion. Nevertheless, the impulse of the force F is still equal to the total change in momentum of the system. Applying this relation, we obtain

$$T\frac{u}{v}t = mvtu,$$

and therefore we obtain the same equation as Eq. (2.1.21)

$$v = \sqrt{\frac{T}{m}}. \qquad \text{(transverse wave)}$$

Hence the speed of propagation of a transverse pulse in a string depends only on the tension (a force) and the mass per unit length. Although this calculation of the wave speed considered only a very special kind of pulse, it can be shown that any shape of wave disturbance can be considered as a series of pulses with different rates of transverse displacement. Thus, although derived for a special case, Eq. (2.1.21) is valid for any transverse wave motion on a string, including, in particular, the sinusoidal and other periodic waves.

Here is an alternative derivation of Eq. (2.1.21). Instead of the sinusoidal wave, let us consider a single symmetrical pulse such as that of Figure 2.1.25, moving from left to right along the string with speed v. For convenience, we choose a reference frame in which the pulse remains stationary; that is, we run along with the pulse, keeping it constantly in view. In this frame, the string appears to move past us, from right to left with speed v.

Consider a small string element of length Δl within the pulse, forming an arc of a circle of radius R and subtending an angle 2θ at the center of that circle. A force $\vec{T}$ with a magnitude equal to the tension in the string pulls tangentially on this element at each end. The horizontal components of these forces cancel, but the vertical components add to form a radial restoring force F:

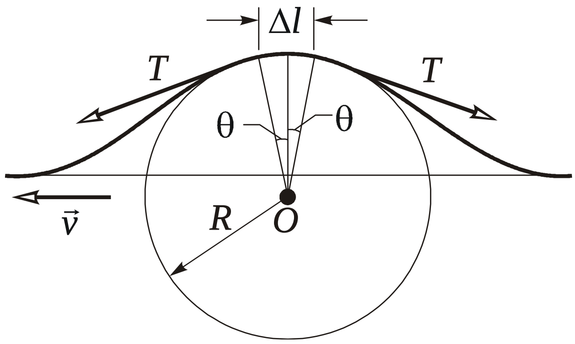


Figure 2.1.25 A symmetric pulse, viewed from a reference frame in which the pulse is stationary and the string appears to move right to left with speed v

$$F = 2(T\sin\theta) \approx T(2\theta) = T\frac{\Delta l}{R}, \qquad (2.1.24)$$

where we approximated $\sin\theta$ as θ for small angles θ in Figure 2.1.25. From that figure, we have also used $2\theta = \Delta l / R$.

The mass of the element is given by

$$\Delta m = \mu \Delta l, \qquad (2.1.25)$$

where μ is the string mass linear density.

At the moment shown in Figure 2.1.25, the string element Δl is moving in an arc of a circle. Thus, it has a centripetal acceleration toward the center of that circle, given by

$$a = \frac{v^2}{R}. \qquad (2.1.26)$$

Eqs. (2.1.24), (2.1.25), and (2.1.26) contains the elements of Newton's second law. Combining them in the form

$$\text{force} = \text{mass} \times \text{acceleration}$$

we get

$$\frac{T\Delta l}{R} = (\mu \Delta l)\frac{v^2}{R}.$$

After solving this equation for the speed v,

$$v = \sqrt{\frac{T}{\mu}}. \qquad (2.1.27)$$

Example 2.1.5

A uniform cord has a mass of 0.30 kg and a length of 6.0 m (Figure 2.1.26). The cord passes over a pulley and supports a 2.0-kg object. Find the speed of a pulse traveling along this cord.

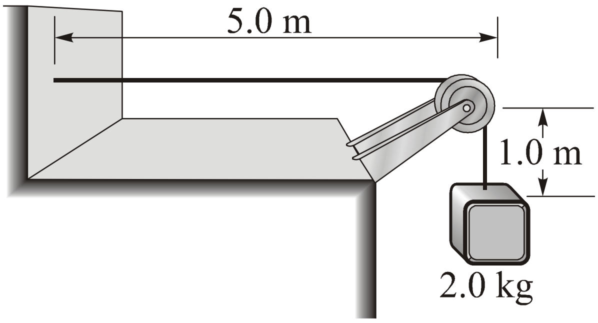


Fig. 2.1.26. The tension T in the cord is maintained by the suspended object. The speed of any wave traveling along the cord is given by $v=\sqrt{T/m}$

Solution.

The tension T in the cord is equal to the weight of the suspended 2.0-kg mass:

$$T = mg = (2.0\,\text{kg})(9.8\,\text{m/s}^2) = 19.6\ \text{N}.$$

(This calculation of the tension neglects the small mass of the cord. Strictly speaking, the cord can never be exactly horizontal, and therefore, the tension is not uniform.) The mass per unit length m, of the cord is

$$m = \frac{m}{l} = \frac{0.30\,\text{kg}}{6.0\,\text{m}} = 0.050\ \text{kg/m}.$$

Therefore, the wave speed is

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{19.6\,\text{N}}{0.050\,\text{kg/m}}} = 19.8\,\text{m/s}\,.$$

Exercises

2.1.43. A transverse traveling wave on a taut wire has an amplitude of 0.20 mm and a frequency of 500 Hz. It travels with a speed of 196 m/s. (a) Write an equation in SI units of the form $y = A\sin(kx - wt)]$ for this wave; (b) The mass per unit length of this wire is 4.10 g/m. Find the tension in the wire.

2.1.44. A phone cord is 4.0 m long and of mass of 0.20 kg. A transverse wave pulse is produced by plucking one end of the taut cord. The pulse makes four trips down along the cord and returnes in 0.80 s. What is the tension in the cord?

2.1.45. Transverse waves with a speed of 50.0 m/s are to be produced in a taut string. A 5.0-m length of string with a total mass of 0.06 kg is used. What is the required tension?

2.1.46. A piano string having a mass per unit length $5.00´\,20^{-3}$ kg/m is under the tension of 1 350 N. Find the speed with which a wave travels on this string.

2.1.47. A sinusoidal wave of wavelength 2 m and amplitude 0.1 m travels on a string with the speed of 1 m/s to the right. Initially, the left end of the string is at the origin. Find (a) the frequency and the angular frequency; (b) the angular wave number; and (c) the wave function for this wave. Determine the equation of motion for (d) the left end of the string and (e) the point on the string at $x = 1.50$ m to the right of the left end. (f) What is the maximum speed of any point on the string?

2.1.48. The linear density of the string is 1.6×10^{-4} kg/m. A tranverse wave on the string is described by the equation

$$y = 0.021\sin[2x + 30t].$$

What is (a) the wave speed and (b) the tention in the string?

2.1.8 Speed of a Longitudinal Wave in Solids

To obtain the expression for speed of longitudinal wave in solids we consider a medium with the plane sinusoidal wave propagating along x-axis. Let's choose the cylindrical volume element with the cross section A and the height Δx in the medium (Figure 2.1.27). The plane wave, as usual, is described by the equation

$$\xi = \xi_{\max}\cos(\omega t - kx + \varphi),$$

where ξ is the displacement of particle.

The graph of this function is represented in Figure 2.1.28. If the base of the cylinder with the coordinate x has displacement ξ in a certain moment of time, then the base with the coordinate $x + \Delta x$ has the displacement $\xi + \Delta\xi$. Hence, the volume becomes deformed – it obtains the elongation $\Delta\xi$. We call the ratio of variation of the displacement $\Delta\xi$ to the initial separation Δx the percent elongation $\Delta\xi / \Delta x$. Quantity $\langle\varepsilon\rangle = \Delta\xi / \Delta x$ determines the average relative elongation of the cylinder. As ξ changes with x nonlinearly, real deformation in the different cross-section of the cylinder would not be the same. To obtain deformation ε in a particular cross-section x, we use approximation $\Delta x \to 0$, i.e. take derivative

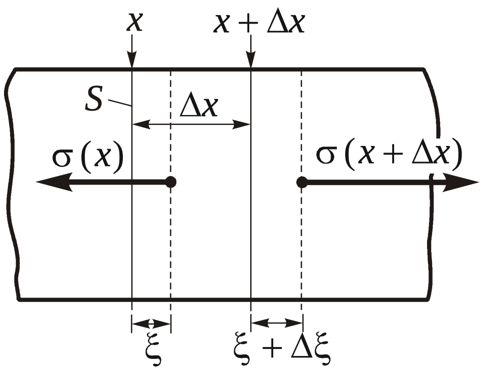


Figure 2.1.27 Relative deformation in different cross–section

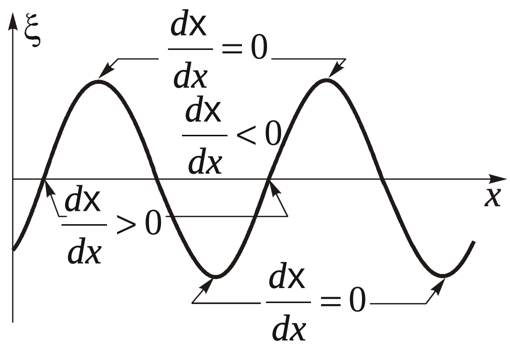


Figure 2.1.28 Deformation of cylinder when longitudinal wave is propagating through it

$$\varepsilon = \frac{\partial\xi}{\partial x}.$$

We use the partial derivative because ξ depends not only on x, but on t as well. When $\varepsilon = \dfrac{\partial\xi}{\partial x} > 0$, the distance between points increases.This situation corresponds to *elongation* of a medium. When $\varepsilon = \frac{\partial \xi}{\partial x} < 0$, the distance between points decreases and such kind of ε describes the *compression* of a medium. As $\xi = \xi_{max}\cos(\omega t - kx + \varphi)$, the deformation is:

$$\varepsilon = \frac{\partial \xi}{\partial x} = \frac{\partial}{\partial x}[\xi_{max}\cos(\omega t - kx + \varphi)] = k\xi_{max}\sin(\omega t - kx + \varphi).$$

It is clear from the above equation, that deformation has its maximum value in the same points where speed $v = \frac{\partial \xi}{\partial t} = \omega\xi_{max}\sin(\omega t - kx + \varphi)$ reaches its maximum, i.e. in points of equilibrium.

To obtain the speed of wave, recall Hook's law for deformation of the elastic medium, according to which the elastic force is proportional to the deformation of medium: $F = \frac{1}{\alpha}\left(\frac{\partial \xi}{\partial t}\right)A$ where α is the elastic coefficient. For cylindrical element this coefficient equals $\alpha = \frac{1}{E}$ where E is the Young's modulus.

Let's return to the cylindrical volume with a longitudinal wave of Figure 2.1.27 and consider forces exerted by the cylinder. The force exerted by the left base of cylinder is $F_1 = AE\left(\frac{\partial \xi}{\partial x}\right)_x$, and the force at the right base is $F_2 = AE\left(\frac{\partial \xi}{\partial x}\right)_{x+\Delta x}$. As forces F_1 and F_2 are directed in opposite directions, the resulting force is $F = F_1 - F_2$. Now we are ready to write the equation of motion for cylinder. Let's take the small element Δx in the cylinder so small that projections of acceleration for all its points are the same and equal to $\partial^2 \xi / \partial t^2$. The mass of the cylinder we can be express as $\rho A \Delta x$ where ρ is the density of the nondeformed medium, the resulting force is:

$$F_x = AE\left[\left(\frac{\partial \xi}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial \xi}{\partial x}\right)_x\right].$$

As Δx is small, we assume that

$$\left(\frac{\partial \xi}{\partial x}\right)_{x+\Delta x} = \left(\frac{\partial \xi}{\partial x}\right)_x + \left(\frac{\partial^2 \xi}{\partial x^2}\right)_x \Delta x.$$

Hence, for the resultant force, we obtain:

$$F_x = AE\frac{\partial^2 \xi}{\partial x^2}\Delta x.$$

Now we substitute expressions, obtained for acceleration, mass, and force into Newton's second law:

$$\rho A\Delta x\frac{\partial^2\xi}{\partial t^2}=AE\frac{\partial^2\xi}{\partial x^2}\Delta x .$$

Dividing out $A\Delta x$, we obtain the wave equation:

$$\frac{\partial^2\xi}{\partial x^2}=\frac{\rho}{E}\frac{\partial^2\xi}{\partial t^2} . \qquad (2.1.28)$$

Comparing Eqs. (2.1.28) and (2.1.13), we conclude that the phase speed of a longitudinal wave in solids is

$$v=\sqrt{\frac{E}{\rho}} . \qquad (2.1.29)$$

It can be shown in the similar way that the speed of the transverse wave in solids is

$$v=\sqrt{\frac{G}{\rho}} , \qquad (2.1.30)$$

where G is the *modulus of rigidity*.

Typical value for the speed of sound in solids is much greater than the speed of sound in gases, as Table 2.1.2 shows. This difference in speeds makes sense because the molecules of a solid are bound together into a much more rigid structure than those in a gas and hence, respond more rapidly to a disturbance.

Table 2.1.2 Speeds of sound in various media

Medium	Speed of sound, υ (m/s)
Gases	
Hydrogen $(0° C)$	1286
Helium $(0° C)$	972
Air $(20° C)$	343
Air $(0° C)$	331
Oxygen $(0° C)$	317
Liquids at $25° C$	
Glycerol	1984
Sea water	1533
Water	1493
Mercury	1450
Kerosene	1324
Methyl alcohol	1143
Carbon tetrachloride	926

Table 2.1.2 (continued)

Medium	Speed of sound, u (m/s)
Solids	
Diamond	12000
Pyrex glass	5640
Iron	5130
Aluminum	5100
Brass	4700
Copper	3560
Gold	3240
Lucite	2680
Lead	1322
Rubber	1600

In solids, a combination of longitudinal waves can propagate, for example during earthquakes. The three-dimensional waves that travel from the point under the Earth surface at which an earthquake occurs are of both types, transverse and longitudinal. Longitudinal waves are faster of the two, and travel at speeds in the range of 7 to 8 km/s near the surface. These are called P waves with "P" standing for *primary* because they travel faster than the transverse waves and arrive at a seismograph first. The slower transverse waves called S waves (with "S" standing for *secondary),* travel through the Earth at 4 to 5 km/s near the surface. By recording the time interval between the arrivals of these two sets of waves at a seismograph, the distance from the seismograph to the point of origin of the waves can be determined. A single measurement establishes an imaginary sphere centred at the seismograph, with the radius of the sphere determined by the difference in arrival times of the P and S waves. The origin of the waves is located somewhere on that sphere. Imaginary spheres from three or more monitoring stations located far apart intersect at one region of the Earth, and this region is where the earthquake occurred.

Example 2.1.6

If a solid bar is struck at one end with a hammer, a longitudinal pulse propagates down the bar with the speed $v = \sqrt{E/\rho}$ where E is the Young's modulus for the material, $E = 7\times 10^{10}$ N/m^2 for aluminum. Find the speed of sound in an aluminum bar.

Solution.

Density of aluminum $\rho = 2.7\times 10^{3}$ kg/m^3. Therefore,

$$v = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{7\times 10^{10}}{2.7\times 10^{3}}} = 5.1 \text{ km/s}.$$

Exercises

2.1.49. A wire of density 9 gm/cm^3 is stretched between two damps 1 m apart while subjected to an extension of 0.05 cm. What is the lowest frequency of vibrations in the wire? Assume Young's modulus $E = 9\times 10^{10}$ N/m^2. (Ans. 35.3 Hz.)

2.1.50. Earthquakes generate sound waves inside Earth. Unlike a gas, Earth can experience both transverse (S) and longitudinal (P) sound waves. Typically, the speed of S waves is about 4.5 km/s, and that of P waves 8.0 km/s. A seismograph records P and S waves from an earthquake. The first P waves arrive 3.0 min before the first S waves (Figure 2.29). Assuming the waves travel in a straight line, how far away does the earthquake occur?

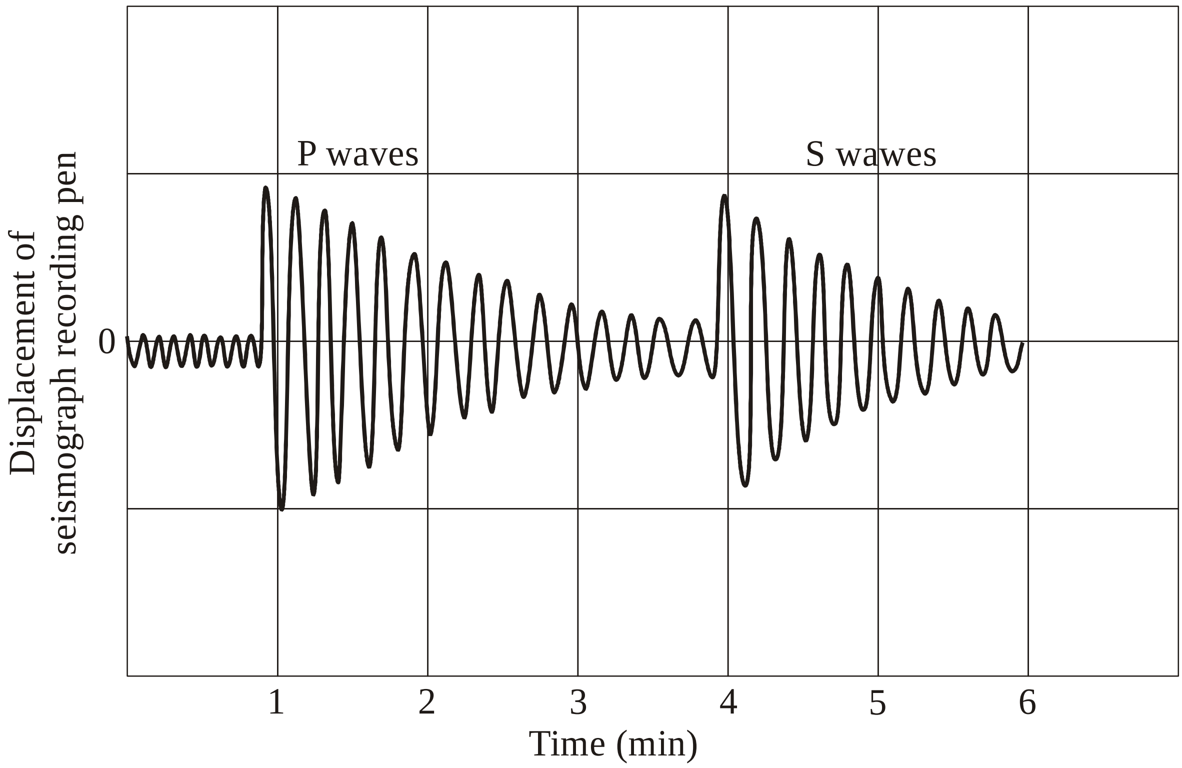


Figure 2.1.29 The first P waves arrive 3.0 min before the first S waves

2.1.51. The tensile stress in a thick copper bar is 99.5% of its elastic breaking point of 13×10^{10} N/m^2. A 500-Hz sound wave is transmitted through the material. (a) What displacement amplitude will cause the bar to break? (b) What is the maximum speed of the particles at this moment?

2.1.52. The speed of sound in a certain metal is v. One end of a long pipe of this metal of length L is struck a hard blow. A listener at the other end hears two sounds, one from the wave that travels along the pipe and the other from the wave that travels through the air. (a) If v_a is the speed of sound in air, what time interval t elapses between the arrivals of the two sounds? (b) Suppose that $t = 1$ s and the metal is steel. Find the length L.

2.1.53. A steel pipe 100 m long is struck at one end. A person at the other end hears two sounds as a result of two longitudinal waves, one traveling in the metal pipe and the other traveling in the air. What is the time interval between the two sounds? Take Young's modulus of steel to be 2×10^{11} Pa, the density of steel to be 7800 kg/m^3, and the speed of sound in air to be 345 m/s.

2.1.9 Speed of a Longitudinal Wave in Fluids

Propagation speed of longitudinal as well as transverse waves is determined by the mechanical properties of the medium, and we can derive relation for speed of longitudinal waves, analogous to Eq. (2.1.21) for transverse waves on a string. Here is an example of such a derivation for longitudinal waves in a liquid in a tube.

Figure 2.1.30 shows a fluid with the density r in a tube with the cross-sectional area A. In the equilibrium state, the fluid is under a uniform pressure p. In Figure 2.1.30a, the fluid is at rest. At time $t = 0$, we start the piston at the left end moving toward the right with the constant speed u. This initiates a wave motion that travels to the right along the length of the tube in which successive sections of fluid begin to move and become compressed at successively later times.

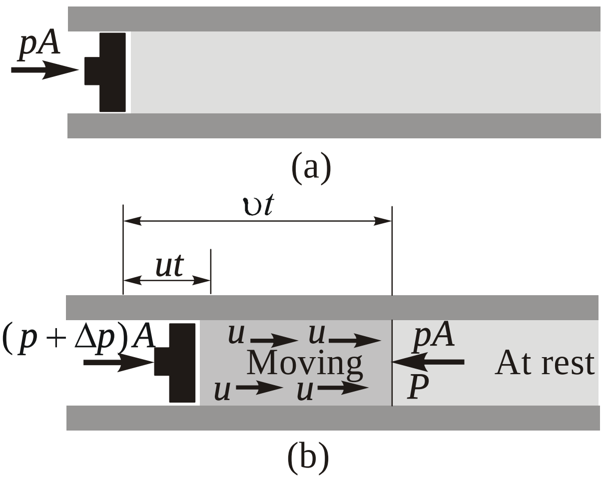


Figure 2.1.30 Propagation of a longitudinal disturbance in a fluid confined in a tube

Figure 2.1.30b shows the fluid after a time t has elapsed. All portions of fluid to the left of the point P are moving with the speed u, and all portions to the right of P are still at rest. The boundary between the moving and stationary portions travels to the right with the speed equal to the speed of propagation v. At time t, the piston has moved a distance ut, and the boundary has advanced a distance vt. As with a transverse disturbance in a string, we can compute the speed of propagation from the impulse-momentum theorem.

The quantity of fluid set in motion at time t is the amount that originally occupied a volume of length vt and of cross-sectional area A. The mass of this fluid is therefore, $pvtA$, and the longitudinal momentum it has acquired is

$$\text{Longitudinal momentum} = pvtAu.$$

Next we compute the increase of pressure, Δp, in the moving fluid. The original volume of the moving fluid, Avt, has decreased by the amount Aut. From the definition of bulk modulus B,

$$B = \frac{\text{Change in pressure}}{\text{Fractional change in volume}} = \frac{\Delta p}{Aut / Avt}.$$

Therefore,

$$\Delta p = B\frac{u}{v}.$$

In the moving fluid, the pressure is $p + \Delta p$, and the force exerted on it by the piston is $(p + \Delta p)A$. The net force on the moving fluid (see Figure 2.1.30) is $A\Delta p$, and the longitudinal impulse is

$$\text{Longitudinal impulse} = \Delta p A t = B\frac{u}{v}At.$$

Applying the impulse-momentum theorem, we find

$$B\frac{u}{v}At = \rho vtAu.$$

Hence,

$$v = \sqrt{\frac{B}{\rho}} \qquad \text{(longitudinal wave)}. \qquad (2.1.31)$$

Therefore, the speed of propagation of a longitudinal pulse in a fluid depends only on the bulk modulus and the density of the medium. The form of this relation is similar to that of Eq. (2.1.21); in both cases, the numerator is a quantity characterizing the strength of the restoring force, and the denominator is a quantity describing the inertial properties of the medium.

Example 2.1.6

(a) Find the speed of sound in water which has a bulk modulus of 2.1×10^9 N/m^2 and a density of 10^3 kg/m^3.

Solution.

Using Eq. (2.1.31), we find that

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.1 \times 10^9}{10^3}} = 1.4 \text{ km/s}.$$

In general, sound waves travel slowlier in liquids than in solids because liquids are more compressible than solids.

(b) Dolphins use sound waves to locate food. Experiments have shown that a dolphin can detect a 7.5-cm target 110 m away, even in murky water. For a bit of dinner at that distance, how much time passes between the moment the dolphin emits a sound pulse and the moment the dolphin hears its reflection and, thereby, detects the distant target?

Solution.

The total distance covered by the sound wave as it travels from dolphin to the target and return is $2\times 110 = 220$ m. From Eq. (2.1.2), we have

$$\Delta t = \frac{\Delta x}{v} = \frac{220}{1400} = 0.16 \text{ s.}$$

Exercises

2.1.54. Find the speed of sound in mercury, which has a bulk modulus of approximately 2.8×10^{10} N/m^2 and a density of 13 600 kg/m^3.

2.1.55. Diagnostic ultrasound of frequency 4.5 MHz is used to examine tumors in soft tissue. (a) What is the wavelength in air of such a sound wave?; (b) If the speed of sound in tissue is 1500 m/s, what is the wavelength of this wave in tissue?

2.1.56. Provided the amplitude is sufficiently great, the human ear can respond to longitudinal waves over a range of frequencies from about 20 Hz to about 20,000 Hz. Compute the wavelengths corresponding to these frequencies

a) for waves in air ($v = 345$ m/s);

b) for waves in water ($v = 1480$ m/s).

2.1.57. When sound travels from air into water, does the frequency of the wave change? The wavelength? The speed?

2.1.10 Speed of a Sound Wave in Gases

In Section 2.1.9, we have derived the expression for calculating the speed of sound in a fluid in a pipe, in terms of its density ρ and bulk modulus B. We have learned that when a gas is compressed adiabatically, its temperature rises; when it expands adiabatically, its temperature drops. Does this also happen when a wave travels through a gas, or does enough heat conduction occur between the adjacent layers of gas to maintain a nearly constant temperature throughout? This is a crucial question because it determines what we use for the bulk modulus B in Eq. (2.1.27). The bulk modulus is defined in general as:

$$B = -V\frac{dp}{dV}.$$

If the temperature is constant, then according to Boyle's law, the product pV is constant, and we can use it to evaluate dp/dV. But if the process is adiabatic, then pV^{γ} is constant, and we get a different result for B.

Experiments show that for ordinary sound frequencies, say 20 to 20000 Hz, the thermal conductivity of gases is so small that the propagation of sound is, in fact, very nearly ***adiabatic***. Thus we must use the adiabatic bulk modulus B_{ad} derived from the assumption

$$pV^{\gamma} = const. \qquad (2.1.32)$$

We take the derivative of Eq. (2.1.32) with respect to V

$$\frac{dp}{dV}V^{\gamma} + \gamma p V^{\gamma-1} = 0 .$$

Dividing out it by $V^{\gamma-1}$ and rearranging, we find that the adiabatic bulk modulus for an ideal gas is simply

$$-V\frac{dp}{dV} = B_{ad} = \gamma p . \qquad (2.1.33)$$

For an isothermal process, however, $pV = const$, and the isothermal bulk modulus is

$$B_{ist} = p .$$

In each case, the bulk modulus (characterizing the material's resistance to compression) is proportional to the pressure, but the adiabatic modulus is *larger* than the isothermal by a factor γ.

Combining Eqs. (2.1.31) and (2.1.33), we obtain

$$v = \sqrt{\frac{\gamma p}{\rho}} . \quad \text{(ideal gas)} \qquad (2.1.34)$$

an alternative form can be obtained by using the relation

$$\frac{p}{\rho} = \frac{RT}{M} ,$$

where R is the gas constant, M is the molar mass, and T is the absolute temperature. Therefore,

$$v = \sqrt{\frac{\gamma RT}{M}} . \qquad (2.1.35)$$

For a given gas, γ, R, and M are constants, so the speed of propagation is proportional to the square root of the absolute temperature.

The speed of sound also depends on the temperature of the medium. For sound traveling through air, the relationship between wave speed and medium temperature is

$$v = (331\text{m/s})\sqrt{1 + \frac{t°C}{273°C}} , \qquad (2.1.36)$$

where 331 m/s is the speed of sound in air at 0ºC, and T_C is the temperature in Celsius degrees. Using this equation, we can find that at 20°C, the speed of sound in air is approximately 343 m/s.

This information provides a convenient way to estimate the distance to a thunderstorm. During a lightning flash, the temperature of a long channel of air rises rapidly as the bolt passes through it. This temperature increase causes the air in the channel to expand rapidly, and this expansion creates a sound wave. The channel produces sound throughout, its entire length at essentially the same instant. If the orientation of the channel is such that all of its parts are approximately the same distance from you, sounds from the different parts reach you at the same time, and you hear a short, intense thunderclap. However, if the distances between your ear and different portions of the channel vary, sounds from different portions arrive at your ears at different times. If the channel were a straight line, the resulting sound would be a steady roar but the zigzag shape of the path produces variations in loudness.

Example 2.1.7

Compute the speed of longitudinal waves in air at an absolute temperature of 300 K.

Solution

The mean molecular mass of air is

$$28.8\,\text{g/mol} = 28.8\times 10^{-3}\,\text{kg/mol}.$$

Also, $\gamma = 1.4$ for air, and $R = 8.314\,\text{J/mol}\cdot\text{K}$. At $T = 300$ K, we obtain

$$v = \sqrt{\frac{(1.4)(8.314\ \text{J/mol}\cdot\text{K})(T = 300\,\text{K})}{(28.8\times 10^{-3}\text{kg/mol})}} = 348\ \text{m/s}.$$

This result agrees with the measured speed at this temperature to within 0.3%.

Exercises

2.1.58. What is the difference between the speeds of longitudinal waves in air at – 3°C and at 57°C?

2.1.59. Use the definition $B = -V(dp/dV)$ and the relation between p and V for an adiabatic process to derive Eq. (2.35).

2.1.60. If the propagation of sound waves in gases were characterized by isothermal rather than adiabatic expansions and compressions, and assuming that the gas behaves as an ideal gas, show that the speed of sound would be given by $\sqrt{RT/\mu}$. What is the speed of sound in air at 27°C in this case?

2.1.61. At a temperature of 27°C, what is the speed of longitudinal waves in argon? hydrogen? Compare your answers to (a) and (b) with the speed in air at the same temperature.

2.1.62. What is the difference between the speeds of longitudinal waves in air at – 3°C and at 57°C?

2.1.11 Energy of Wave Motion

Every wave motion has the energy associated with it. To initiate a wave motion, we exert a force on a portion of the wave medium; the point where the force is applied moves, so we perform work on the system. A wave can transport energy from one region of space to another. For example, transmission of energy by electromagnetic waves is familiar, and the destructive power of ocean surf is a convincing demonstration of energy transported by water waves.

As an example of energy in wave motion, consider the plane longitudinal wave which propagates through a medium:

$$\xi = A\cos(\omega t - kx + \varphi)\,. \qquad (2.1.37)$$

Let's take the small volume ΔV in a medium so small that speeds of propagation and deformations of all its particles are the same and equal to $\partial\xi/\partial t$ and $\partial\xi/\partial x$, correspondingly.

The volume ΔV has the kinetic energy

$$\Delta W_k = \frac{\rho}{2}\left(\frac{\partial\xi}{\partial t}\right)^2 \Delta V\,, \qquad (2.1.38)$$

where $\rho\Delta V$ is volume mass, $\partial\xi/\partial t$ – its speed.

The volume ΔV has the elastic potential energy as well:

$$\Delta W_p = \frac{E\varepsilon^2}{2}\Delta V = \frac{E}{2}\left(\frac{\partial\xi}{\partial x}\right)^2 \Delta V\,, \qquad (2.1.39)$$

where $\varepsilon = \partial\xi/\partial x$ is the elongation, E is the Young's modulus. As $v = \sqrt{E/\rho}$, then $E = \rho v^2$, and the expression for potential energy can be rewritten in the form:

$$\Delta W_p = \frac{\rho v^2}{2}\left(\frac{\partial\xi}{\partial x}\right)^2 \Delta V\,. \qquad (2.1.40)$$

The total energy of the wave is the sum of potential (2.1.40) and kinetic (2.1.38) energies:

$$\Delta W = \Delta W_k + \Delta W_p = \frac{\rho}{2}\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + v^2\left(\frac{\partial\xi}{\partial x}\right)^2\right]\Delta V\,.$$

If we divide the total energy by the volume ΔV, we obtain the energy density $w = \Delta W/\Delta V$:

$$w = \frac{\rho}{2}\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + v^2\left(\frac{\partial\xi}{\partial x}\right)^2\right]. \qquad (2.1.41)$$

To obtain the speed and elongation we take the partial derivative $\partial\xi/\partial t$ and $\partial\xi/\partial x$ of Eq. (2.1.37):

$$\frac{\partial\xi}{\partial t} = -A\omega\sin(\omega t - kx + \varphi)\,, \qquad (2.1.42)$$

$$\frac{\partial\xi}{\partial x} = -Ak\sin(\omega t - kx + \varphi)\,. \qquad (2.1.43)$$

After substituting Eqs. (2.1.42) and (2.1.43) into Eq. (2.1.41) and recalling that $k^2v^2 = \omega^2$, we obtain

$$w = \rho A^2\omega^2 \sin^2(\omega t - kx + \varphi). \tag{2.1.44}$$

It follows from (2.1.44) that the energy density differs from point to point for any instant of time. At a fixed point, energy density w varies according to squared sine function.

To obtain the average energy density $\langle w\rangle$, we have take the average value of the square of a sine function, which is ½. Hence, average energy density at any point of medium is:

$$\langle w\rangle = \frac{1}{2}\rho A^2\omega^2. \tag{2.1.45}$$

Eq. (2.1.44) shows that the energy density depends on the density ρ of the material; the amplitude A, and the frequency ω. The dependence of the average energy density of a wave on the square of its amplitude and also on the square of its angular frequency is a general result, true for waves of all types.

As we have seen medium with the wave in it has additional energy. This energy is transported by the wave from the source of oscillation to different points of space. Quantity of energy transported by the wave through a surface per unit time is called *energy current* Φ.

$$\Phi = \frac{dW}{dt}. \tag{2.1.46}$$

In SI system energy current is measured in J/s, or W.

Energy current has different value for different points of medium. To characterize the energy current at certain point of space, another physical quantity, called *energy current density*, is introduced. Energy current density is a vector quantity. The direction of this vector is the same as the direction of travelling wave, and it's magnitude equals to the energy current Φ through the unit surface $A_\perp$, perpendicular to the direction of the energy current. Let energy ΔW be transported through the area $A_\perp$ at time Δt. The energy current density j is

$$j = \frac{\Delta\Phi}{\Delta A_\perp} = \frac{\Delta W}{\Delta A_\perp \Delta t}. \tag{2.1.47}$$

Imagine the small volume ΔV with area the $A_\perp$ and the height $v\Delta t$, where v is the propagating speed of the wave. The energy ΔW, transported through the volume, is then:

$$\Delta W = w\Delta A_\perp v\Delta t.$$

Substituting this expression into Eq. (2.1.47) we obtain expression for energy current density:

$$j = wv, \tag{2.1.48}$$

or, in a vector form:

$$\vec{j} = w\vec{v}. \tag{2.1.49}$$

The vector defined by Eq. (2.1.49), is named Umov's vector, in honour of Russian physicist N.A. Umov who introduced it.

It should be mentioned, that the intensity I of wave is average in time energy current density. The *intensity* I of a traveling wave is defined as the average time rate at which energy is transported by a wave per unit area across a surface perpendicular to the direction of propagation. Briefly, the intensity is the average *power* transported per unit area.

$$I = \frac{W}{S_{\perp} t} = \frac{P}{S_{\perp}}. \qquad (2.1.50)$$

It is interesting to note that the rate of energy transfer is proportional to the *square* of the amplitude and is also proportional to the square of the frequency.

For a particular case of transverse wave on the string, instantaneous rate of the energy transmission along the string can be expressed as follows:

$$P = \sqrt{\mu T}\,\omega^2 A^2 \cos^2(\omega t - kx).$$

Average power will be

$$P_{av} = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2.$$

Analogous relationship can be worked out for longitudinal waves. It can be shown that the intensity – that is, average power per unit cross-sectional area for fluids in a pipe – is given by

$$I = \frac{1}{2}\sqrt{\rho B}\,\omega^2 A^2,$$

and for a solid rod

$$I = \frac{1}{2}\sqrt{\rho E}\,\omega^2 A^2.$$

Again the power is proportional to A^2 and ω^2.

Exercises

2.1.63. A string of mass 4 g and length 2 m is stretched with a tension of 30 N. Waves of frequency $f = 60$ Hz and amplitude 8 cm are traveling along the string. (a) Calculate the average power carried by these waves. (b) What happens to the average power if the amplitude of the waves is doubled?

2.1.64. Show that Eq. $P_{av} = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2$ can also be written as $P_{av} = \frac{1}{2} T k \omega A^2$, where k is the wave number of the wave.

2.1.65. A string along which waves can travel is 2.70 m long and has a mass of 260 g. The tension in the string is 36.0 N. What must be the frequency of travelling waves (of amplitude 7.70 mm) be for the average power to become 85.0 W?

2.1.66. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar that moves up and down through the distance of 1.0 cm. The motion is continuous and is repeated regularly 120 times per second. The string has linear density of 120 g/m and is kept under the tension of 90.0 N. Find the maximum value of (a) the transverse speed u and (b) the transverse component of the tension T_t. (Hint: This component is $T\sin\theta$, where θ is the angle the string makes with the horizontal. You will need to relate angle θ to dy/dx). (c) Show that two maximum values calculated above occur at the same phase values for the wave. What is the transverse displacement y of the string at these phases? (d) What is the maximum rate of energy transfer along the string? (e) What is the transverse displacement y when the maximum transfer occurs? (f) What is the minimum rate of energy transfer along the string? (g) What is the transverse displacement y when this minimum transfer occurs?

2.1.67. A taut rope has a mass of 0.18 kg and a length of 3.6 m. What power must be supplied to the rope to generate sinusoidal waves having an amplitude of 0.1 m and a wavelength of 0.5 cm and traveling with a speed of 30 m/s?

2.1.68. Transverse waves are being generated on a rope under constant tension. By what factor is the required power increased or decreased if (a) the length of the rope is doubled and the angular frequency remains constant; (b) the amplitude is doubled and the angular frequency is halved; (c) both the wavelength and the amplitude are doubled; and (d) both the length of the rope and the wavelength are halved?

2.1.69. Sinusoidal waves 5 cm in amplitude are to be transmitted along a string that has a linear mass density of 4×10^{-2} kg/m. If the source can deliver the maximum power of 300 W and the string is under the tension of 100 N, what is the highest vibrational frequency at which the source can operate?

2.1.70. It is found that a 6.0-m segment of a long string contains four complete waves and has a mass of 180 g. The string is vibrating sinusoidally with the frequency of 50 Hz and the peak-to-valley displacement of 15 cm. (The "peak-to-valley" distance is the vertical distance from the farthest positive displacement to the fathest negative displacement.) (a) Write the function that describes this wave traveling in the positive x direction. (b) Determine the power supplied to the string.

2.1.71. A sinusoidal wave on a string is described by the equation

$$y = (0.15)\sin(0.8x - 50t),$$

where x and y are in meters and t is in seconds. If the mass per unit length of this string is 12.0 g/m, determine (a) the speed of the wave; (b) the wavelength; (c) the frequency; and (d) the power transmitted to the wave.

2.1.72. A two-dimensional water wave spreads in circular wave fronts. Show that the amplitude A at a distance r from the initial disturbance is proportional to $1/r$ (Hint: Consider the energy carried by one outward-moving ripple.)

2.1.12 Intensity of Spherical and Plane Waves

If a spherical body oscillates so that its radius varies sinusoidallv with time, a spherical wave is produced. The wave moves outward from the source at a constant speed if the medium is uniform.

As of this uniformity, we conclude that the energy in a spherical wave propagates equally in all directions. That is, no one direction is preferred over any other. If P_{av} is the average power emitted by the source, then this power at any distance r from the source must be distributed over a spherical surface of area $4pr^2$. Hence, the wave intensity at a distance r from the source is

$$I = \frac{P_{av}}{A} = \frac{P_{av}}{4pr^2}. \qquad (2.1.51)$$

Because P_{av} is the same for any spherical surface centered at the source, we see that the intensities at distances r_1 and r_2 are

$$I_1 = \frac{P_{av}}{4pr_1^2} \quad \text{and} \quad I_2 = \frac{P_{av}}{4pr_2^2}.$$

Therefore, the ratio of intensities on these two spherical surfaces is

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}.$$

This inverse-square law states that the intensity decreases in proportion to the square of the distance from the source. Intensity is proportional to $A^2{}_{max}$. Thus, we conclude that the displacement amplitude A_{max} of a spherical wave must vary as $1/r$. Therefore, we can write the wave function y (Greek letter psi) for an outgoing spherical wave in the form

$$y(r,t) = \frac{A_0}{r}\sin(wt - \vec{k}\vec{r} + f),$$

where A_0, the displacement amplitude at the unit distance from the source, it is a constant parameter characterizing the given wave.

2.1.13 Intensity of Periodic Sound Waves

We know that the power developed by a force equals the product of force and velocity. Hence, the power *per unit area* in a sound wave equals the product of the excess pressure (force per unit area), given by Eq. (2.1.24), and the *particle* velocity v, obtained by taking the time derivative of wave function $\xi = A\sin(\omega t - kx)$. We find

$$v = \omega A\cos(\omega t - kx)\,,$$

hence,

$$pv = \omega BkA^2\cos^2(\omega t - kx)\,.$$

By definition, the intensity is, the average value of this quantity. The average value of the function $\cos^2\alpha = 1/2$, so we find

$$I = \frac{1}{2}\omega BkA^2\,.$$

By using the relations $\omega = vk$ and $v^2 = B/\rho$, we can transform it into the form

$$I = \frac{1}{2}\sqrt{\rho B}\,\omega^2 A^2\,.$$

It is usually more convenient to express I in terms of the pressure amplitude p_{max}. Using Eq. (2.1.25) and the relation $\omega = vk$, we find

$$I = \frac{\omega p_{max}^2}{2Bk} = \frac{v p_{max}^2}{2B}\,.$$

By using the wave speed relation $v^2 = B/\rho$, we can also write this in the alternative forms

$$I = \frac{p_{max}^2}{2\rho v} = \frac{p_{max}^2}{2\sqrt{\rho B}}\,. \qquad (2.1.52)$$

The intensity of a sound wave of the largest amplitude tolerable to the human ear (about $p_{max} = 30$ Pa) is

$$I = \frac{(30\,\mathrm{Pa})^2}{2(1.22\,\mathrm{kg/m^3})(343\,\mathrm{m/s})} = 1.07\,\mathrm{J/s\cdot m^2} = 1.07\ \mathrm{W/m^2} = 1.07\times 10^{-4}\,\mathrm{W/cm^2}\,.$$

The unit 1 $\mathrm{W/cm^2}$ is a mixed one, neither cgs nor SI. We mention it here because it is unfortunately in general use among acousticians.

The pressure amplitude of the *faintest* sound wave that can be heard is about 3×10^{-5} Pa, and the corresponding intensity is about 10^{-12} $\mathrm{W/m^2}$.

The *total* power carried across a surface by a sound wave equals the product of the intensity at the surface and the surface area if the intensity over the surface is uniform. The average total sound power emitted by a person speaking in a conversational tone is about 10^{-5} W, while a loud shout corresponds to about 3×10^{-2} W.

Because of the extremely large range of intensities over which the ear is sensitive, a logarithmic rather than an arithmetic intensity scale is convenient. The intensity b of the sound wave is defined by the equation:

$$b = 10 \log \frac{I}{I_0}, \tag{2.1.53}$$

where I_0 is an arbitrary reference intensity, taken as 10^{-12} W/m^2. This value corresponds roughly to the faintest sound that can be heard. Intensity levels are expressed in decibels, abbreviated dB. A decibel is 1/10 of a bel, a unit named after Alexander Graham Bell. The bel is inconveniently large for most purposes, and the decibel is the usual unit of sound intensity level.

If the intensity of a sound wave equals I_0 or 10^{-12} W/m^2, its intensity level is 0 dB. The maximum intensity that the ear can tolerate without pain is about 1 W/m^2, which corresponds to an intensity level of 120 dB. Table 2.1.3 gives the intensity levels in decibels of several familiar noises.

Table 2.1.3 Noise levels due to various sources

Source or Description of Noise	Noise level, dB	Intensity, $W \cdot m^{-2}$
Threshold of pain	120	1
Riveter	95	3.2×10^{-3}
Elevated train	90	10^{-3}
Busy street traffic	70	10^{-5}
Ordinary conversation	65	3.2×10^{-6}
Quiet automobile	50	10^{-7}
Quiet radio in home	40	10^{-8}
Average whisper	20	10^{-10}
Rustle of leaves	10	10^{-11}
Threshold of hearing	0	10^{-12}

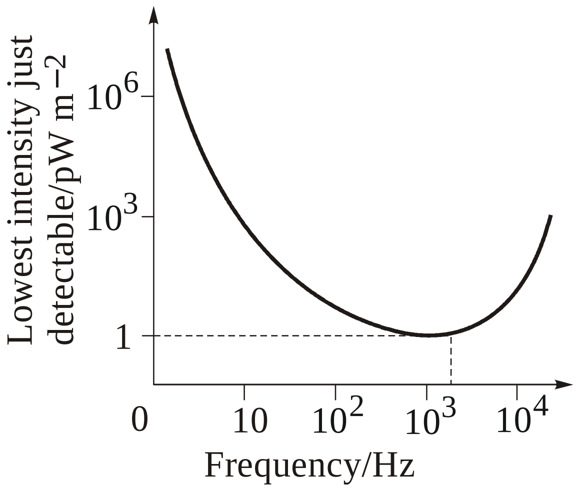


Figure 2.1.31 Sensitivity of the ear varies with frequency

Within the range of audibility, the sensitivity of an ear varies with frequency. At any frequency the threshold of audibility is the minimum intensity of sound at that frequency that can be detected. An ear is most sensitive at about 3000 Hz (Figure 2.1.31). For a young adult with normal hearing, the threshold of audibility at 1000 Hz is about 0 dB; at 200 and 15,000 Hz, it is about 20 dB; and at 50 and 18,000 Hz, it is about 50 dB Thus the ear sensitivity drops at low and high ends of the frequency scale. Frequencies above 20,000 Hz (20 kHz) are not audible to humans at any intensity, and such frequencies are referred to as ultrasonic.

Example 2.1.8

Two identical machines are positioned at the same distance from a worker. The intensity of sound delivered by each machine at the location of the worker is 2.0×10^{-7} W/m. Find the sound level heard by the worker (a) when one machine is operating.

Solution.

a) The sound level at the location of the worker with one machine operating is calculated from Eq. (2.1.53):

$$\beta_1 = 10\log\left(\frac{2.0\times 10^{-7}\,\mathrm{W/m^2}}{1.0\times 10^{-12}\,\mathrm{W/m^2}}\right) = 10\log(2.0\times 10^{3}) = 53\ \mathrm{dB}.$$

b) When both machines are operating.

Solution.

When both machines are operating, the intensity is doubled to $4.0\times 10^{-7}\ \mathrm{W/m^2}$; therefore, the sound level now is

$$\beta_2 = 10\log\left(\frac{4.0\times 10^{-7}\,\mathrm{W/m^2}}{1.0\times 10^{-12}\,\mathrm{W/m^2}}\right) = 10\log(4.0\times 10^{3}) = 56.$$

From these results, we see that when the intensity is doubled, the sound level increases by 3 dB only.

Example 2.1.9

The faintest sounds the human ear can detect at a frequency of 1000 Hz correspond to an intensity of about 10^{-12} W/m^2 - the so–called *threshold of hearing*, and the loudest sounds the ear can tolerate at this frequency correspond to an intensity of about 1 W/m^2 – the *threshold of pain.* Determine the pressure amplitude and displacement amplitude associated with these two limits.

Solution.

First, consider the faintest sounds. Using Eq. (2.1.52) and taking $v = 343$ m/s as the speed of sound waves in air and $\rho = 1.2$ kg/m^3 as the density of air, we obtain

$$\Delta p_{max} = \sqrt{2\rho I} = \sqrt{2(1.2)(343)(10^{-12})} = 2.87\times 10^{-5}\ \text{N/m}^2 .$$

As atmospheric pressure is about 10^5 N/m^2, this result tells us that the ear can discern pressure fluctuations as small as 3 parts in 10^{10}.

We can calculate the corresponding displacement amplitude by using Eqs. (2.1.25), (2.1.6) and (2.1.36):

$$y_{max} = \frac{\Delta P_{max}}{\rho v \omega} = \frac{2.87\times 10^{-5}}{(1.2\,\text{kg/m}^3)(343\,\text{m/s})(2\pi\times 10^3\,\text{Hz})} = 1.11\times 10^{-11}\text{m}.$$

This is a remarkably small number! If we compare this result for y_{max} with the diameter of a molecule (about 10^{-10} m), we see that the ear is an extremely sensitive detector of sound waves.

In a similar manner, it can be found that the loudest sounds the human ear can tolerate correspond to a pressure amplitude of 28.7 N/m^2 and a displacement amplitude equal to 1.11×10^{-5}m.

Example 2.1.10

An electric spark jumps along a straight line of length $L = 10$ m (Figure 2.1.32), emitting a pulse of sound that travels radially outward from the spark. (The spark is said to be a line source of sound.) The power of the emission is $P_S = 1.6\times 10^4$ W.

a) What is the intensity I of the sound when it reaches a distance $r = 12$ m from the spark?

Solution.

Let us centre an imaginary cylinder of radius $r = 12$ m and length $L = 10$ m (open at both ends) on the spark, as shown in Figure 2.1.32. First, the intensity I at the cylindrical surface is the ratio P/A of the time rate P at which sound energy passes through the surface to the surface area A. Second, the principle of conservation of energy applies to the sound energy. This means that the rate P at which energy is transferred through the cylinder must equal the rate P_S at which energy is emitted by the source. Putting these facts together and noting that the area of the cylindrical surface is $A = 2\pi rL$, we get

$$I = \frac{P}{A} = \frac{P_S}{2\pi rL}.$$

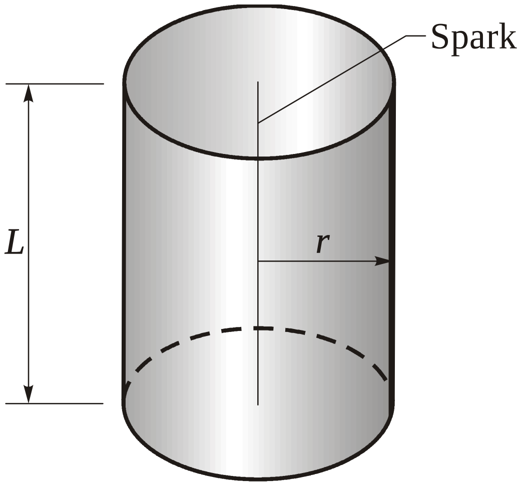


Figure 2.1.32 A spark along a straight line of length L emits sound waves radially outward. The waves pass through an imaginary cylinder of radius r and length L that is centered on the spark

This tells us that the intensity of the sound from a line source decreases with distance r (and not with the square of distance r as for a point source). Substituting the given data, we find

$$I = \frac{1.6 \times 10^4 \text{ W}}{2\pi(12\,\text{m})(10\,\text{m})} = 21.2 \text{ W/m}^2.$$

b) At what time rate P_d is sound energy intercepted by an acoustic detector of area $A_d = 2$ cm^2, aimed at the spark and located a distance $r = 12$ m from the spark?

Solution.

We know that the intensity of sound at the detector is the ratio of the energy transfer rate P_d there to the detector's area A_d:

$$I = \frac{P_d}{A_d}.$$

We can imagine that the detector lies on the cylindrical surface of (a). Then the sound intensity at the detector is the intensity I (= 21.2 W/m^2) at the cylindrical surface. Solving above equation for P_d gives us

$$P_d = I \cdot A_d = (21.2)(2 \times 10^{-4}) = 4.2 \text{ mW}.$$

Exercises

2.1.73. Of the following sounds, which is most likely to have a sound level of 60 dB: – rock concert, turning of a page in this text, normal conversation, or a cheering crowd at a football game?

2.1.74. Estimate the decibel level of each sound in the previous question.

2.1.75. Calculate the sound level, in decibels, of a sound wave that has an intensity of 4 μW/m^2.

2.1.76. A vacuum cleaner has a measured sound level of 70 dB. (a) What is the intensity of this sound in watts per square meter? (b) What is the pressure amplitude of the sound?

2.1.77. The intensity of a sound wave of frequency 1.0 kHz at a fixed distance from a speaker is 0.6 W/m^2. (a) Determine the intensity if the frequency is increased to 2.50 kHz while the constant displacement amplitude is maintained, (b) Calculate the intensity if the frequency is reduced to 0.50 kHz and the displacement amplitude is doubled.

2.1.78. The intensity of a sound wave at a fixed distance from a speaker at a frequency f is I. (a) Determine the intensity if the frequency is increased to f' while the constant displacement amplitude is maintained, (b) Calculate the intensity if the frequency is reduced to $f/2$ and the displacement amplitude is doubled.

2.1.79. A family ice show is held in an enclosed arena. Skaters perform to music with a sound level of 80 dB. This is too loud for your baby who consequently yells at a level of 75.0 dB. (a) What total sound intensity engulfs you? (b) What is the combined sound level?

2.1.80. A violin plays a melody line and is then joined by nine other violins, all playing at the same intensity as the first violin, in a repeat of the same melody (a) When all of the violins are playing together, by how many decibels does the sound level increase? (b) If ten more violins join in, how much has the sound level increased over that for the single violin?

2.1.14 Doppler Effect

Ambulance car is parked by the side of the highway, sounding its 1000 Hz siren. It you also park by the highway, you will hear the same frequency. However, if there is relative motion between you and the ambulance car, either toward or away from each other, you will hear a different frequency. For example, if you are driving *toward* the ambulance car at 120 km/h, you will hear a *higher* frequency (1096 Hz, an increase of 96 Hz). If you are driving *away from* the ambulance car at that same speed, you will hear a *lower* frequency (904 Hz, a decrease of 96 Hz).

Those motion-related frequency changes are examples of the *Doppler effect*. The effect was proposed (although not fully worked out) in 1842 by Austrian physicist Johann Christian Doppler. It was tested experimentally in 1845 by Buys Ballot in Holland, using a locomotive drawing an open car with several trumpeters. The Doppler effect holds not only for sound waves but also for electromagnetic waves.

To see what causes this apparent frequency change, imagine you are in a boat that is lying at anchor on a gentle sea where the waves have a period of $T = 3$ s. This means that every 3 s a crest hits your boat. If you set your watch to $t = 0$ just as one crest hits, the watch reads 3.0 s when the next crest hits, 6.0 s when the third crest hits, and so on. From these observations, you conclude that the wave frequency is $f = 1/T = 1/3$ Hz. Now suppose you start your motor and head directly into the oncoming waves. Again you set your watch to $t = 0$ as a crest hits the front of your boat. Now, however, because you are moving toward the next wave crest as it moves toward you, it hits you less than 3 s after the first hit. In other words, the period you observe is shorter than the 3-s period you observed when you were stationary. As $f = 1/T$, you observe a higher wave frequency than when you were at rest.

If you turn around and move in the same direction as the waves, you observe the opposite effect. You set your watch to $t = 0$ as a crest hits the back of the boat. Because you are now moving away from the next crest, more than 3 s has elapsed on your watch by the time, that crest catches you. Thus, you observe a lower frequency than when you were at rest.

These effects occur because the relative speed between your boat and the waves depends on the direction of travel and on the speed of your boat. When you are moving toward the waves, this relative speed is higher than that of the wave speed which leads to the observation of an increased frequency. When you turn around and move away from waves, the relative speed is lower, as is the observed frequency of the water waves.

Let us now examine an analogous situation with sound waves, in which the water waves become sound waves, the water becomes the air, and the person on the boat becomes an observer listening to the sound.

As a reference frame we shall take the air through which these waves travel. This means that we shall measure the speeds of source *S* of sound waves and detector *D* of these waves relative to air. We shall assume that *S* and *D* move either directly toward or directly away from each other, at speeds smaller than the speed of sound.

If either the detector or the source is moving, or both are moving, the emitted frequency f and the detected frequency f' are related by

$$f' = f\frac{v \pm v_D}{v \pm v_S} \qquad \text{general Doppler effect,} \qquad (2.1.54)$$

where v is the speed of sound through the air, v_D is the detector's speed relative to the air and v_S is the source's speed relative to the air. The choice of plus or minus signs is set by the rule: When detector or source moves towards the other, the sign on its speed must give an upward shift in frequency. When the detector or the source moves away from the other, the sign on its speed must give a downward shift in frequency.

In short, *toward* means *shift up,* and *away* means *shift down.*

Here are some examples of the rule. If the detector moves toward the source, use the plus sign in the numerator of Eq. (2.1.54) to get a shift up in the frequency. If it moves away, use the minus sign in the numerator to get a shift down. If it is stationary, substitute 0 for v_D. If the source moves toward the detector, use the minus sign in the denominator of Eq. (2.1.54) to get a shift up in the frequency. If it moves away, use the plus sign in the denominator to get a shift down. If the source is stationary, substitute 0 for v_S.

Next, we derive equations for the Doppler effect for the following two specific situations and then derive Eq. (2.1.54) for a general situation.

When the detector moves relative to the air and the source is stationary relative to the air, the motion changes the frequency at which the detector intercept wave surfaces and thus the detected frequency of the sound wave.

When the source moves relative to the air and the detector is stationary relative to the air, the motion changes the wavelength of the sound wave and, thus, the detected frequency (recall that frequency is related to wavelength).

Detector Moving; Source Stationary. In Figure 2.1.33, a detector D is moving at speed v_D toward a stationary source S that emits spherical wavefronts, of wavelength l and frequency f, moving at the speed v of sound in air. The wave surfaces are drawn one wavelength apart. Frequency detected by the detector D is the rate at which D intercepts wave surfaces (or individual wavelengths). If D were stationary, the rate would be f but since D is moving into the wave surfaces, the rate of interception is greater, and thus the detected frequency f' is greater than f.

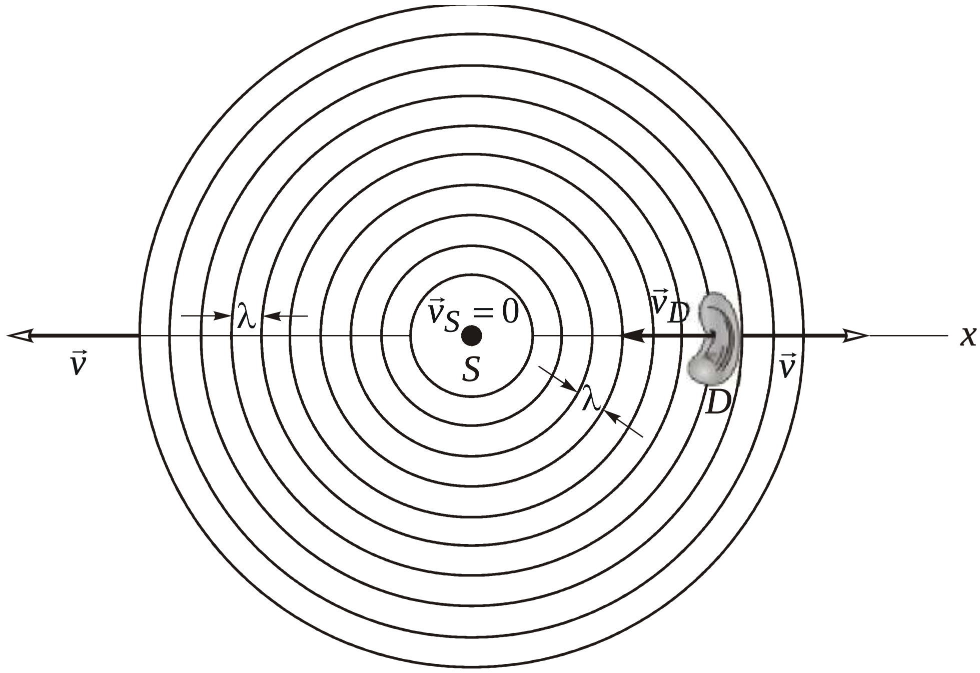


Figure 2.1.33 A stationary source of sound S emits spherical wavefronts, (shown one wavelength apart), that expand outward at speed v. A sound detector D moves with the velocity v_D toward the source. The detector senses a higher frequency because of its motion

Let us for the moment consider the situation in which D is stationary (Figure 2.1.34). In time t, the wave front moves to the right a distance vt. The number of wavelengths in that distance vt is the number of wavelengths intercepted by D in time t, and that number is vt/l. The rate at which D intercepts wavelengths - that is the frequency f detected by D, is

$$f = \frac{vt/l}{t} = \frac{v}{l}. \tag{2.1.55}$$

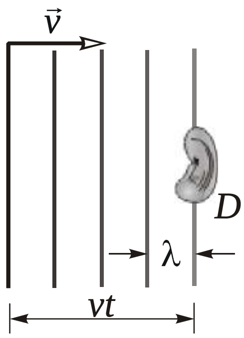


Figure 2.1.34 When *D* is stationary, there is no Doppler effect – the frequency detected by *D* is the frequency emitted by *S*

In this situation, with *D* stationary, there is no Doppler effect: The frequency detected by *D* is the frequency emitted by *S*.

Now let us again consider the situation in which *D* moves opposite the wave surfaces (Figure 2.1.35). In time t, the wavefront moves to the right a distance vt as previously, but now *D* moves to the left a distance $v_D t$. Thus, in this time t the distance moved by the wavefronts relative to *D* is $vt + v_D t$. The number of wavelengths in this relative distance $vt + v_D t$ is the number of wavelengths intercepted by *D* in time t and is $(vt + v_D t)/\lambda$. In this situation, The rate at which *D* intercepts wavelengths is the frequency f' given by

$$f' = \frac{(vt + v_D t)/\lambda}{t} = \frac{v + v_D}{\lambda}. \quad (2.1.56)$$

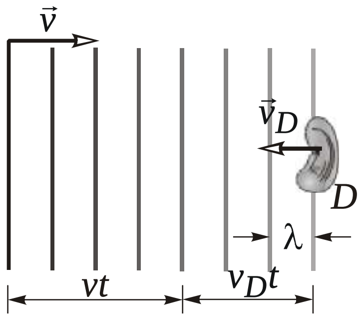


Figure 2.1.35 In time t, the distance moved by the wavefronts relative to *D* is $vt + v_D t$

As $\lambda = v/f$, then Eq. (2.1.56) becomes

$$f' = \frac{v + v_D}{v/f} = f\,\frac{v + v_D}{v}. \quad (2.1.57)$$

Note that in Eq. (2.1.57), f' must be greater than f unless $v_D = 0$ (the detector is stationary).

Similarly, we can find the frequency detected by *D* if *D* moves away from the source. In this situation, the wavefronts move a distance $vt - v_D t$ relative to *D* in time t and f' is given by

$$f' = f\,\frac{v - v_D}{v}. \quad (2.1.58)$$

In Eq. (2.1.58) f' must be smaller than f unless $v_D = 0$. We can summarize Eqs. (2.1.57) and (2.1.58) with

$$f' = f\,\frac{v \pm v_D}{v} \text{ (detector moving; source stationary).} \quad (2.1.59)$$

Source Moving; Detector Stationary. Let detector *D* be stationary with respect to the air, and the source *S* move toward *D* at speed v_S (Figure 2.1.36). The motion of *S* changes the wavelength of the sound waves it emits and, thus, the frequency detected by *D*.

To see this change let $T=(1/f)$ be the time between the emission of any pair of successive wavefronts W_1 and W_2. During T wavefront W_1 moves a distance vT, and the source moves a distance $v_S T$. At the end of vT wavefront W_2 is emitted. In the direction in which S moves the distance between W_1 and W_2, (which is the wavelength λ' of the waves moving in that direction), is $vT - v_S T$. If D detects those waves it detects frequency f' given by

$$f'=\frac{v}{\lambda'}=\frac{v}{vT-v_S T}=\frac{v}{v/f-v_S/f}=f\frac{v}{v-v_S}. \qquad (2.1.60)$$

Note that f' must be greater than f unless $v_S=0$.

In the direction opposite to that taken by S, the wavelength λ' of the waves is $vT+v_S T$. If D detects those waves, it detects the frequency f' given by

$$f'=f\frac{v}{v+v_S};. \qquad (2.1.61)$$

Now f' must be smaller than f unless $v_S=0$.

We can summarize Eqs. (2.1.60) and (2.1.61) with

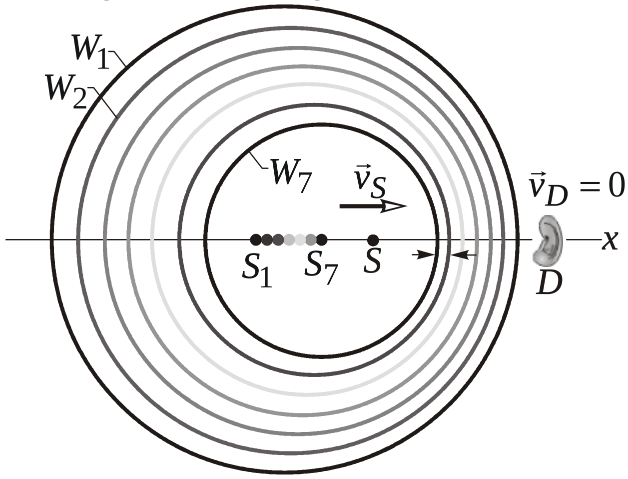


Figure 2.1.36 In the direction in which S moves, the distance between W_1 and W_2, which is the wavelength λ' of the waves moving in that direction, is $vT - v_S T$. The detector senses a higher frequency

$$f'=f\frac{v}{v\pm v_S}; \text{ (source moving; detector stationary)}. \qquad (2.1.62)$$

General Doppler Effect Equation

We can now derive the general Doppler effect equation by replacing f in Eq. (2.1.62) (the frequency of the source) with f' of Eq. (2.1.58) (the frequency associated with motion of the detector). The result is Eq. (2.1.54) for the general Doppler effect:

$$f'=f\frac{v\pm v_D}{v\pm v_S}.$$

The general equation holds not only when both detector and source are moving but also in the two specific situations we just discussed. For the situation in which the detector is moving and the source is stationary, substitution of $v_S=0$ into Eq. (2.1.54) gives us Eq. (2.1.59) which we previously found. For the situation in which the source is moving and the detector is stationary, substitution of $v_D=0$ into Eq. (2.1.54) gives us Eq. (2.1.62) previously found. Thus, Eq. (2.1.54) is the equation to remember.

Although the Doppler effect is most typically experienced with sound waves, it is a phenomenon that is common to all waves. For example, the relative motion of source and observer produces a frequency shift in light waves. The Doppler effect is used in police radar system to measure the speeds of cars. Astronomers use the effect to determine the speeds of stars, galaxies and other celestial object, and even some animals use the effect.

*Bat Navigation***.** Bats navigate and search for prey by emitting, and then detecting reflections of ultrasonic waves. These are sound waves with frequencies greater than can be heard by a human. For example, a horseshoe bat emits ultrasonic waves at 83 kHz, well above the 20 kHz limit of human hearing.

After the sound is emitted through the bat's nostrils, it might reflect (echo) from a moth, and then return to the bat's ears. The motions of the bat and the moth relative to the air cause the frequency heard by the bat to differ by a few kilohertz from the frequency it emitted. The bat automatically translates this difference into a relative speed between itself and the moth, so it can zero on the moth.

Some moths evade capture by flying away from the direction in which they hear ultrasonic waves. That choice of flight path reduces the frequency difference between what the bat emits and what it hears, and then the bat may not notice the echo. Some moths avoid capture by clicking to produce their own ultrasonic waves, thus "jamming" the detection system and confusing the bat. (Surprisingly, moths and bats do all this without first studying physics.)

Doppler ultrasonic systems. Doppler effect is widely used in medicals diagnostic. If a reflecting surface moves in a direction parallel to the direction of the ultrasonic waves, the frequency of the reflected waves is changed by the motion of the reflector.

The shift of frequency of the reflected waves Δf depends on the speed u of the reflector and the direction of the incident beam in accordance with the equation

$$\Delta f = \frac{2uf}{v}\cos\theta,$$

where f is the frequency of the incident waves, v is the wave speed and θ is the angle between the direction of the beam and the direction of motion of the reflecting surface. For example, the frequency shift of an ultrasonic beam of frequency 3 MHz at a speed of 1500 m/s in a substance due to a reflecting boundary moving at a speed of l m/s
towards the source is 4 kHz.

In a Doppler ultrasonic system, the reflected signal is detected by the transducer probe and mixed electronically with a signal at the incident frequency. (Figure 2.1.37)

The resultant waveform is modulated at the Doppler shift frequency which is filtered and measured. The speed of the reflecting surface can then be determined using the equation above. Uses of the system include:

– monitoring the heart beat of a baby in the womb,

– measuring the flow of blood in a blood vessel by measuring the speed of corpuscles in the blood.

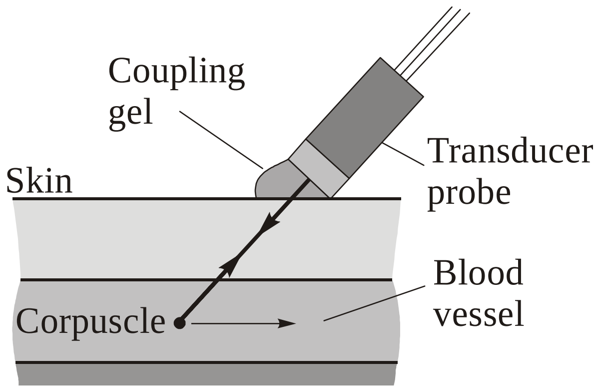


Figure 2.1.37 Transducer probe for measuring the flow of blood

Example 2.1.11

Let $f_S = 300$ Hz and $v = 300$ m/s. The wavelength of the waves emitted by a stationary source is then $v / f_S = 1.0$ m.

a) What are the wavelengths ahead of and behind the moving source in Figure 2.1.38 if its velocity is 30 m/s?

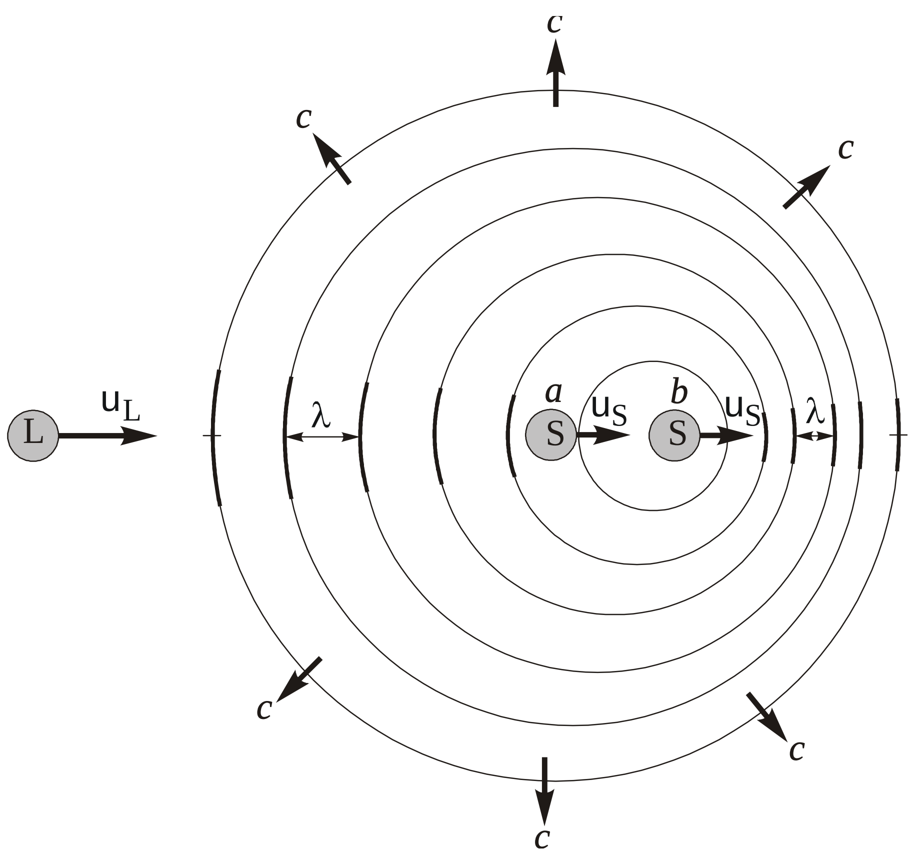


Figure. 2.1.38 Wave surfaces emitted by a moving source are crowded together in front of the source and stretched out behind it

Solution.

In front of the source,

$$\lambda = \frac{v - v_S}{f_S} = \frac{300\,\text{m/s} - 30\text{m/s}}{300\,\text{Hz}} = 0.90\ \text{m}.$$

Behind the source,

$$\lambda = \frac{v + v_S}{f_S} = \frac{300\,\text{m/s} + 30\text{m/s}}{300\,\text{Hz}} = 1.10\ \text{m}.$$

b) If the listener L in Figure 2.1.38 is at rest and the source is moving away from L at 30 m/s, what is the frequency as heard by the listener?

Solution.

Since $v_L = 0$ and $v_S = 30$ m/s, we have

$$f_L = f_S \frac{v}{v + v_S} = 300\,\text{Hz}\,\frac{300\,\text{m/s}}{300\,\text{m/s} + 30\text{m/s}} = 273\ \text{Hz}.$$

c) If the source in Figure 2.1.37 is at rest and the listener is moving toward the left at 30 m/s, what is the frequency heard by the listener?

Solution.

The positive direction (from the listener to the source) is still from left to right, so

$$v_L = -30 \quad \text{and} \quad v_S = 0\ \text{m/s},$$

$$f_L = f_S \frac{v + v_S}{v} = 300\,\text{Hz}\,\frac{300\,\text{m/s} - 30\text{m/s}}{300\,\text{m/s}} = 270\ \text{Hz}.$$

Thus, while the frequency f_L heard by the listener is smaler than the frequency f_S both when the source moves away from the listener and when the listener moves away from the source, the decrease in frequency is not the same for the same speed of recession.

Example 2.12

As an ambulance travels east down a highway at a speed of 33.5 m/s, its siren emits sound at a frequency of 400 Hz. 'What frequency is heard by a person in a car traveling west at 24.6 m/s.

a) as the car approaches the ambulance?

Solution.

We can use Eq. (2.1.54) in both cases, taking the speed of sound in air to be $v = 343$ m/s. As the ambulance and car approach each other, the person in the car hears the frequency

$$f' = \frac{v + v_0}{v - v_S} f = \left(\frac{343\,\text{m/s} + 24.6\,\text{m/s}}{343\text{m/s} - 33.5\,\text{m/s}}\right)(400\,\text{Hz}) = 475\,\text{Hz},$$

b) as the car moves away from the ambulance?

Solution.

As the vehicles recede from each other, the person hears the frequency

$$f' = \frac{v - v_0}{v + v_s} f = \left(\frac{343\,\text{m/s} - 24.6\,\text{m/s}}{343\,\text{m/s} + 33.5\,\text{m/s}}\right)(400\,\text{Hz}) = 338\,\text{Hz}.$$

The change in frequency detected by the person in the car is 475 – 338 = 137 Hz, which is more than 30% of the true frequency.

Example 2.1.13

A rocket moves at a speed of 242 m/s directly toward a stationary target (through the stationary air) while emitting sound waves at frequency $f = 1250$ Hz.

a) What frequency f' is measured by a detector that is attached to the target?

Solution.

We can find f' with Eq. (2.1.54) for the general Doppler effect. Because the sound source (the rocket) moves through the air *toward* the stationary detector on the target, we need to choose the sign on v_s that gives a shift up in the frequency of the sound. Thus, in Eq. (2.1.54), we use the minus sign in the denominator. We then substitute 0 for the detector speed v_d, 242 m/s for the source speed v_s, 343 m/s for the speed of sound v, and 1250 Hz for the emitted frequency.

We find

$$f' = f\frac{v \pm v_d}{v \pm v_s} = (1250)\frac{343 \pm 0}{343 - 242} = 4245\text{ Hz}.$$

which, indeed, is a greater frequency than the emitted frequency.

b) Some of the sound reaching the target reflects back to the rocket as an echo. What frequency f'' does a detector on the rocket detect for the echo?

Solution.

The target is now the source of sound (because it is the source of the echo), and the rocket's detector is now the detector (because it detects the echo). The frequency of the sound emitted by the source (the target) is equal to f', the frequency of the sound the target intercepts and reflects.

We can rewrite Eq. (2.1.54) in terms of the source frequency f' and the detected frequency f'' as

$$f'' = f'\frac{v \pm v_d}{v \pm v_s}. \qquad (2.1.63).$$

A third idea here is that, because the detector (on the rocket) moves through the air toward the stationary source, we need to use the sign on v_d that gives a shift up in the frequency of the sound. Thus, we use the plus sign in the numerator of Eq. (2.1.63). Also, we substitute $v_d = 242$ m/s, $v_s = 0$, $v = 343$ m/s, and $f' = 4245$ Hz. We find

$$f'' = 4245\frac{343 \pm 242}{343 \pm 0} = 7240 \text{ Hz},$$

which, indeed, is greater than the frequency of the sound reflected by the target.

Exercises

2.1.81. Explain what happens to the frequency of your echo as you move in a vehicle toward a canyon wall. What happens to the frequency as you move away from the wall?

2.1.82. If the wavelength of a sound source is reduced by a factor of 2, what happens to its frequency? Its speed?

2.1.83. Trooper *B* is chasing speeder *A* along a straight stretch of road. Both are moving at speed of 160 km/h, The trooper *B*, failing to catch up, sounds his siren again. Take the speed of sound in air to be 343 m/s and the frequency of the source to be 500 Hz. What is the Doppler shift in the frequency heard by the speeder *A*?

2.1.84. At what frequency the 16 000 Hz, whine of the turbines in the jet engines of an aircraft moving with speed 200 m/s is heard by the pilot of a second craft trying to overtake the first at a speed of 250 m/s? (Ans. 17.5 kHz.)

2.1.85. An ambulance with a siren emitting a whine at 1600 Hz overtakes and passes a cyclist pedaling a bike at 2.44 m/s. After being passed, the cyclist hears a frequency of 1590 Hz. How fast is the ambulance moving?

2.1.86. A whistle of frequency 540 Hz moves in a circle of radius 60.0 cm at an angular speed of 15.0 rad/s. What are (a) the lowest and (b) the highest frequencies heard by a listener a long distance away, at rest with respect to the center of the circle? [Ans. (a) 526 Hz, (b) 555 Hz.]

2.1.87. A stationary motion detector sends sound waves of frequency 0.15 MHz toward a truck approaching at a speed of 45.0 m/s. What is the frequency of the waves reflected back to the detector?

2.1.88. A French submarine and a U.S. submarine move toward each other during maneuvers in motionless water in the North Atlantic. The French .sub moves at 50.0 km/h. and the U.S., sub at 70.0 km/h. The French sub sends out a sonar signal (sound wave in water) at 1000 Hz. Sonar waves travel at 5470 km/h. (a) What is the signal frequency as detected by the U.S. sub? (b) What frequency is detected by the French sub in the signal reflected back to it by the U.S. sub? [Ans. (a) 1.02 kHz, (b) 1.04 kHz.]

2.1.89. Explain how the Doppler effect is used with microwaves to determine the speed of an automobile.

2.1.15 Supersonic Speeds. Shock Waves

If a source is moving toward a stationary detector at a speed equal to the speed of sound – that is, if $v_S = v$, the Eqs. (2.1.53) and (2.1.61) predict that the detected frequency f' will be infinitely great. This means that the source is moving so fast that it keeps pace with its own spherical wavefronts, as Figure 2.1.39a suggests. What happens when the speed of the source *exceeds* the speed of sound?

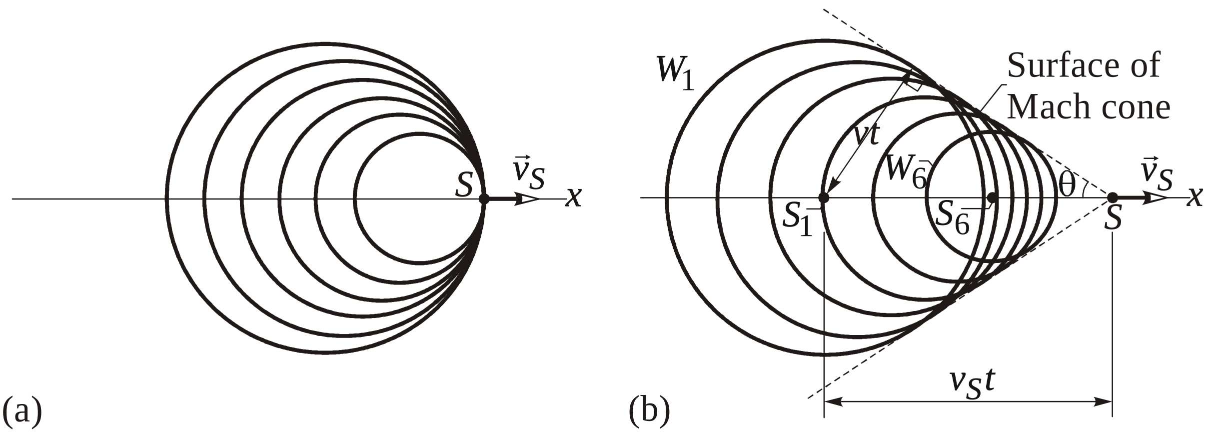


Figure 2.1.39 (a) A source of sound S moves at speed v_S equal to the speed of sound and, thus, as fast as the wave-fronts it generates; (b) A source S moves at speed v_S faster than the speed of sound and, thus, faster than the wavefronts. When the source was at position S_1, it generated wavefront W_1 and at position S_6 it generated W_6. All the spherical wavefronts expand at the speed of sound v and bunch along the surface of a cone called the Mach cone, forming a shock wave. The surface of the cone has the half-angle q and is tangent to all the wavefronts

For such *supersonic* speeds, Eqs. (2.1.53) and (2.1.61) no longer apply.

Figure 2.1.39b depicts the spherical wavefronts originated at various positions of the source. The radius of any wavefront in this figure is vt where v is the speed of sound and t is the time that has elapsed since the source emitted that wavefront. Note that all the wavefronts bunch along a V-shaped envelope in the two-dimensional drawing of Figure 2.1.39b. The wavefronts actually extend in three dimensions, and the bunching actually forms a cone called the *Mach cone.* A *shock wave* is said to exist along the surface of this cone because the bunching of wavefronts causes an abrupt rise and fall of air pressure as the surface passes through any point. From Figure 2.1.39b,we see that the half-angle q of the cone, called the *Mach cone angle,* is given by

$$\sin q = \frac{vt}{v_S t} = \frac{v}{v_S}. \qquad \text{(Mach cone angle).} \quad (2.1.63)$$

The ratio v_S / v is called the *Mach number.* When you hear that a particular plane has flown at Mach 2.3, it means that its speed was 2.3 times the speed of sound in the air through which the plane was flying. The shock wave generated by a supersonic aircraft or projectile produces a burst of sound called a *sonic boom,* in which the air pressure first suddenly increases and then suddenly decreases below normal before returning to normal. Part of the sound that is heard when a rifle is fired is the sonic boom produced by the bullet. A sonic boom can also be heard from a long bullwhip when it is snapped quickly: Near the end the motion of whip, its tip is moving faster than sound and produces a small sonic boom – the *crack* of the whip.

Exercises

2.1.90. A jet plane passes over you at a height of 5000 m and with a speed of Mach 1.5. (a) Find the Mach cone angle; (b) After the jet passes directly overhead, how long does the shock wave reach you? Use 331 m/s for the speed of sound. [Ans. (a) 42º; (b) 11 s]

2.1.91. A plane flies at 1.25 times the speed of sound. Its sonic boom reaches a man on the ground 1.00 min after the plane passes directly overhead. What is the altitude of the plane? Assume the speed of sound to be 330 m/s.

2.1.92. A bullet is fired with a speed of 685 m/s. Find the angle made by the shock cone with the line of motion of the bullet.

2.1.16 Tsunami

A tsunami can be generated by any disturbance that displaces a large water mass from its equilibrium position. Submarine landslides, which often occur during a large earthquake, can also create a tsunami. During a submarine landslide, the equilibrium sea level is altered by sediment moving along the sea floor. Gravitational forces propagate the tsunami given the initial perturbation of the sea level. Similarly, a violent marine volcanic eruption can create an impulsive force that displaces the water column and generates a tsunami. Above water landslides and space born objects can disturb the water from above the surface. The falling debris displaces the water from its equilibrium position and produces a tsunami. Unlike ocean-wide tsunamis caused by some earthquakes, tsunamis generated by non-seismic mechanisms usually dissipate quickly and rarely affect coastlines far from the source area.

Tsunamis are characterized as shallow-water waves. Shallow-water waves are different from wind-generated waves, the waves many of us have observed at the beach. Wind-generated waves usually have period of five to twenty seconds and a wavelength of about 100 to 200 meters. A tsunami can have a period in the range of ten minutes to two hours and a wavelength in excess of 500 km. It is because of their long wavelengths that tsunamis behave as shallow-water waves. A wave is characterized as a shallow-water wave when the ratio between the water depth and its wavelength is very small. The speed of a shallow-water wave is equal to the square root of the product of the acceleration of gravity and the depth of the water. The rate at which a wave loses its energy is inversely related to its wavelength. Since a tsunami has a very large wavelength, it will lose little energy as it propagates. Hence, in the very deep water, a tsunami will cover at high speeds and travel great transoceanic distances with a limited energy loss. They can move from one side of the Pacific Ocean to the other side in less than one day. As a tsunami leaves the deep water of the open sea and propagates into the more shallow waters near the coast, it undergoes a transformation. Since the speed of the tsunami is related to the water depth, as the depth of the water decreases, the speed of the tsunami diminishes. The change of total energy of the tsunami remains constant. Therefore, the speed of the tsunami decreases as it enters shallower water, and the height of the wave grows. Because of this "shoaling" effect, a tsunami that was imperceptible in deep water may grow to be many meters in height.

When a tsunami finally reaches the shore, it may appear as a rapidly rising or falling tide, a series of breaking waves, or even a bore. Reefs, bays, entrances to rivers, undersea features, and the slope of the beach all help to modify the tsunami as it approaches the shore. Tsunamis rarely become great, towering breaking waves. Sometimes the tsunami may break far offshore. Or it may form into a bore: a step-like wave with a steep breaking front. A bore can happen if the tsunami moves from deep water into a shallow bay or river. The water level on shore can rise many meters. In extreme cases, water level can rise to more than 15 m for tsunamis of distant origin and over 30 m for tsunami generated near the epicenter of the earthquake. The first wave may not be the largest in the series of waves. One coastal area may see no damaging wave activity while in another area destructive waves can be large and violent. The flooding of an area can extend inland by 300 m or more, covering large expanses of land with water and debris. Flooding tsunami waves tend to carry loose objects and people out to sea when they retreat.

A tsunami generally consists of a series of waves, often referred to as the tsunami wave train. The amount of time between successive waves, known as the wave period, is usually a few minutes; in some instances, waves are over an hour apart. Many people have lost their lives after returning home in between the waves of a tsunami, thinking that the waves had stopped coming.

Because tsunami can strike at any time, being adequately prepared and knowing what to do beforehand can save your life. Hawaii State and County Civil Defense agencies provide maps of evacuation zones and information on how to be prepared for this type of natural disaster in the front pages of the telephone book.

If you are at the beach and you feel an earthquake or observe a rapid withdrawal of the sea, head for higher ground immediately. When a tsunami warning has been issued, do not attempt to use the telephone or head to low-lying areas to view the oncoming waves. Remember, tsunamis travel at very fast speeds across the ocean; therefore, once a warning has been issued you should evacuate immediately.

Summary

A wave is any disturbance from an equilibrium condition that propagates from one region to another. A mechanical wave always travels within some material called the medium. In a periodic wave the motion of each point of the medium is cyclic or periodic; if the motion is sinusoidal, the wave is called a sinusoidal wave. The frequency f of a periodic wave is the number of repetitions per unit time, and the period T is the time for one cycle. The wavelength λ is the distance between two adjacent identical points of the wave. The speed of propagation v is the speed with which the wave disturbance travels. For any periodic wave, these quantities relate as

$$v = \lambda f .$$

A wave function describes the displacements of individual particles in the medium. It is a function of the coordinate x and time t. The wave function for a sinusoidal wave traveling in the $+x$-direction can be written as

$$y(x,t) = A\sin\omega\left(t - \frac{x}{v}\right) = A\sin 2\pi f\left(t - \frac{x}{v}\right) = A\sin(\omega t - kx) = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right).$$

In all forms, A is the amplitude which is the maximum displacement of a particle from its equilibrium position.

The wave function must obey a partial differential equation called the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} .$$

The speed of a transverse wave on a string having tension T and mass per unit length μ is given by

$$v = \sqrt{\frac{T}{\mu}} .$$

Transverse waves have the property of polarization; longitudinal waves do not.

The speed of a longitudinal wave in a fluid having bulk modulus B and density ρ is given by

$$v = \sqrt{\frac{B}{\rho}} .$$

The speed of a longitudinal wave in a solid rod having Young's modulus E and density ρ is given by

$$v = \sqrt{\frac{E}{\rho}}.$$

The speed of sound in an ideal gas is given by

$$v = \sqrt{\gamma \frac{RT}{\mu}} \text{ (ideal gas).}$$

Wave motion conveys energy from one region to another. For a transverse wave on a stretched string, a portion of string exerts a transverse force on an adjacent portion while it undergoes a displacement; hence, one section does work on another, and energy is transferred along the length of the string

For sinusoidal sound waves, the variation in the displacement is given by

$$\xi(x,t) = A\cos(\omega t - kx),$$

and the variation in pressure from the equilibrium value is

$$\Delta p = p_{\max}\sin(\omega t - kx),$$

where $\Delta p_{\max}$ is the pressure amplitude. The pressure wave is 90º out of phase with the displacement wave. The relationship between $\xi_{\max}$ and $\Delta p_{\max}$ given by

$$\Delta p_{\max} = \rho v \omega \xi_{\max}.$$

The intensity of a periodic sound wave, which is the power per unit area, is

$$I = \frac{1}{2}\rho v \omega^2 A^2.$$

The sound level of a sound wave, in decibels, is given by

$$\beta = 10\log\left(\frac{I}{I_0}\right).$$

The constant I_0 is a reference intensity, usually taken to be at the threshold of hearing (1×10^{-12} W/m^2), and I is the intensity of the sound wave in watts per square meter.

The intensity of a spherical wave produced by a point source is proportional to the average power emitted and inversely proportional to the square of the distance from the source:

$$I = \frac{P_{av}}{4\pi r^2}.$$

The change in frequency heard by an observer whenever there is relative motion between a source of sound waves and the observer is called the Doppler effect. The observed frequency is

$$f' = \left(\frac{v \pm v_D}{v \pm v_S}\right).$$

The upper signs ($+v_D$ and $-v_S$) are used with motion of one toward the other, and the lower signs ($-v_D$ and $+v_S$) are used with motion of one away from the other. You can also use this formula when v_S or v_D is zero.

Key Terms

Wave – волна
mechanical waves – механическая волна
medium – среда
periodic wave – периодическая волна
sinusoidal wave – гармоническая волна, синусоидальная волна
transverse wave – поперечная волна
longitudinal wave – продольная волна
wave speed – скорость волны
wavelength – длина волны
wave function – волновая функция
wave number – волновое число
wave equation – волновое уравнение
sound – звук
pressure amplitude – амплитуда давления
intensity – интенсивность
intensity level – уровень интенсивности
threshold of audibility – порог слышимости, слуховой порог
ultrasonic – сверхзвуковой, ультразвуковой
Doppler effect – эффект Допплера

Chapter 2.2

Standing Waves

In previous chapter we studied the propagation of mechanical waves in media without ends or boundaries; we were not concerned with what happens when a wave arrives at an end or boundary of the medium in which it propagates. But in many wave phenomena such boundaries do play a significant role. A familiar example is the echo that occurs when a sound wave reflects from a rigid wall. Such reflections lead to overlapping, or *superposition,* of two waves – the initial and reflected waves – in the same region of the medium. When there are two or more boundary points or surfaces, repeated reflections can occur. In such cases, it turns out that sinusoidal wave motion is possible only for certain special values of the frequency of the wave, determined by the dimensions and mechanical properties of the medium. These special frequencies and their associated wave patterns are called *normal modes*. Many familiar phenomena are associated with normal modes. This concept will also reappear later in some unexpected places, such as the energy levels of atoms.

2.2.1 Superposition of Waves

Many interesting natural wave phenomena cannot be described by a single moving pulse. Instead, one must analyze complex waves in terms of a combination of many traveling waves. To analyze such wave combinations, we can make use of the *superposition principle*:

If two or more traveling waves are moving through a medium, the resultant function at any point is the algebraic sum of the wave functions of the initial waves.

Mathematically speaking, the principle of superposition states that the wave function describing the resulting motion is obtained by adding two wave functions for two separate waves. This additive property of wave functions depends, in turn, on the form of the wave equation, Eq. (2.1.14), which every physically possible wave function must satisfy. Specifically, the wave equation is *linear*. As a result, if each of any two functions $y_1(x,t)$ and $y_2(x,t)$ satisfies the wave equation separately, their sum $y_1 + y_2$ automatically satisfies the wave equation as well and, hence, is a physically possible motion. In view of this linearity of the wave equation and the corresponding linear-combination property of its solutions, the principle is also called the *principle of linear superposition.*

Waves that obey this principle are called *linear waves* and are generally characterized by small amplitudes. Waves that violate the superposition principle are called *nonlinear waves* and are often characterized by large amplitudes. We will deal only with linear waves.

One consequence of the superposition principle is that *two traveling waves can pass through each other without being destroyed or even altered.* For instance, when sound waves from two sources move through air, they pass one through each other. The resulting sound that hears at a given point is the resultant of the two disturbances.

Figure 2.2.1 is a pictorial representation of superposition. Wave function for a pulse moving to the right is y_1, and wave function for a pulse moving to the left is y_2. The pulses have the same speed but different shapes. Each pulse is assumed to be symmetric, and the displacement of the medium is in the positive y direction for both pulses. When the waves begin to overlap (Figure 2.2.1b), the wave function for the resulting complex wave is given by $y_1 + y_2$.

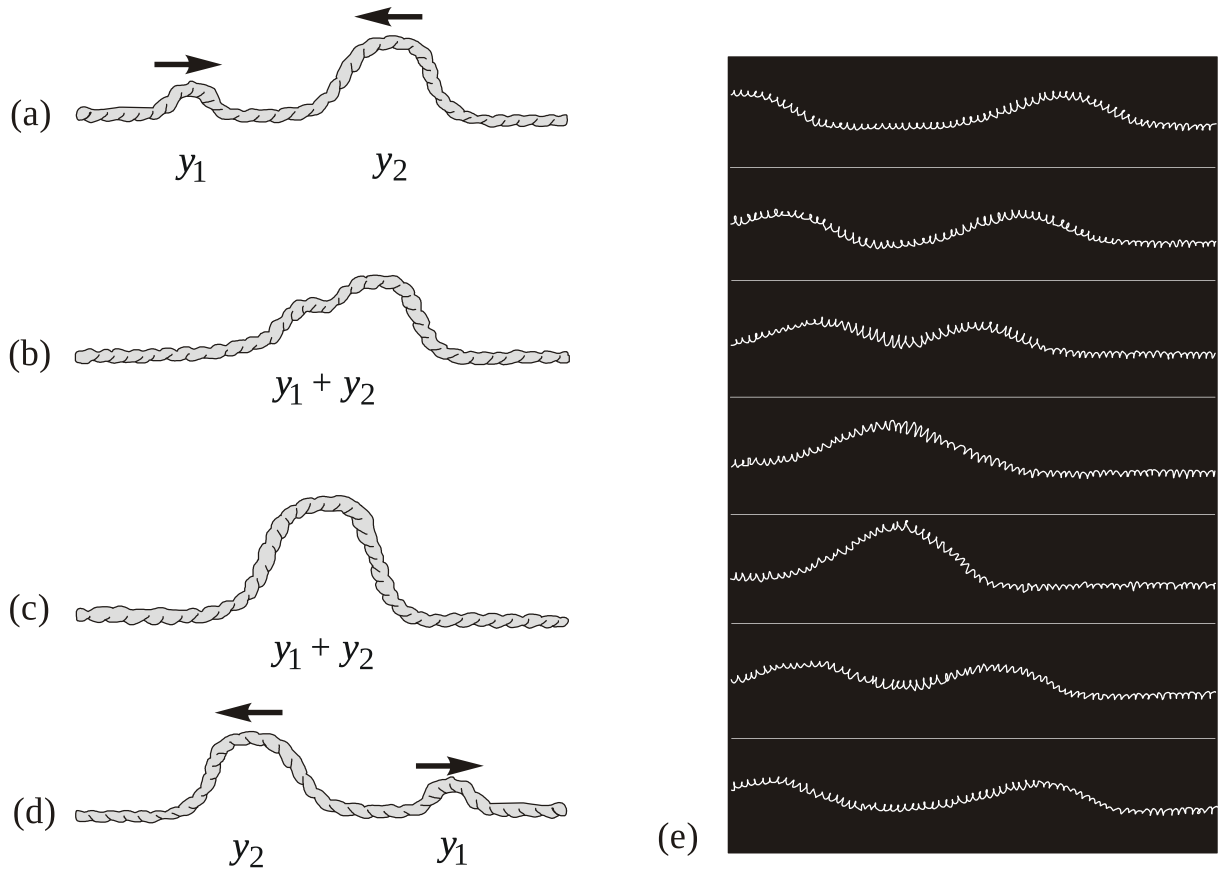


Figure 2.2.1 (a - d) Two wave pulses traveling on a stretched string in opposite directions pass through each other. When the pulses overlap, as shown in (b) and (c), the net displacement of the string equals the sum of the displacements produced by each pulse; (d) The two pulses separate and continue moving in their initial directions

When the crests of the pulses coincide (Figure 2.2.1c), the resulting wave given by $y_1 + y_2$ is symmetric. The two pulses finally separate and continue moving in their original directions (Figure 2.2.1d). Note that the pulse shapes remain unchanged, as if the two pulses had never met.

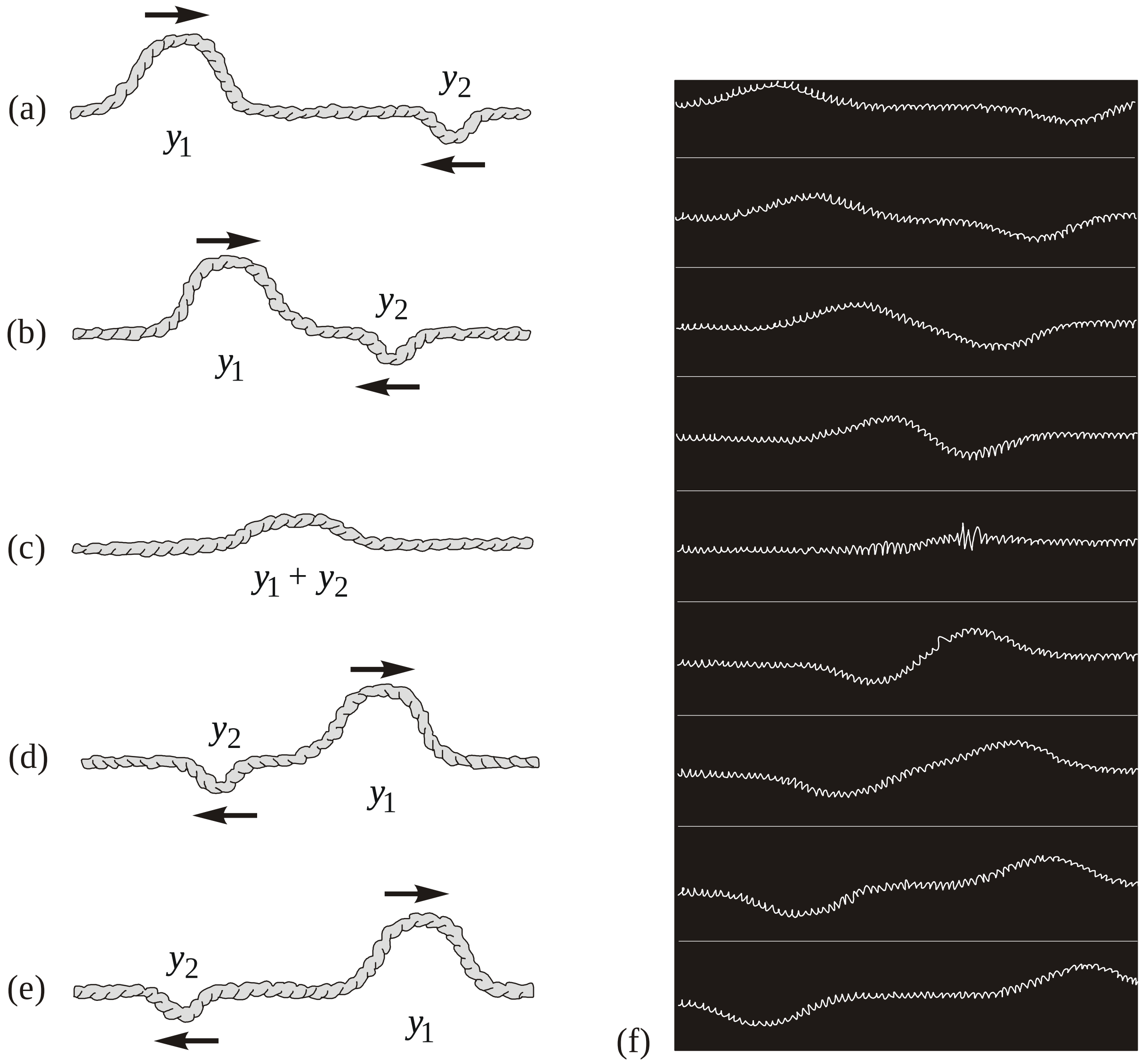


Figure 2.2.2 (a-e) Two wave pulses traveling in opposite directions and having displacements that are inverted relative to each other. When the two overlap in (c), their displacements partially cancel each other;(d, e) two pulses pass through each other; (f) photograph of two pulses traveling in opposite directions

Now consider two pulses traveling in opposite directions on a taut string where one pulse is inverted relative to the other, as illustrated in Figure 2.2.2. In this case, when the pulses begin to overlap, the resultant wave is given by $y_1 + y_2$, but the values of the function y_2 are negative. Again, the two pulses pass through each other; however, the displacements caused by the two pulses are in opposite directions.

2.2.2 Reflection and Transmission

We have discussed traveling waves moving through a uniform medium. We now consider how a traveling wave is affected when it encounters a change in the medium. For example, consider a pulse traveling on a string that is rigidly attached to a support at one end (Figure 2.2.3). When the pulse reaches the support, a severe change in the medium occurs – the string ends. The result of this change is that the wave undergoes *reflection* – that is, the pulse moves back along the string in the opposite direction.

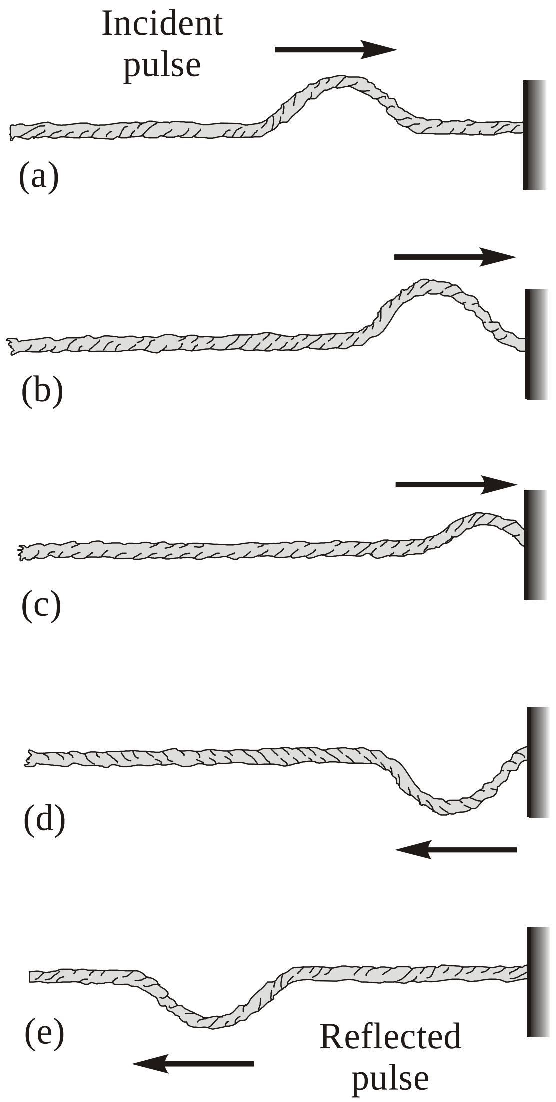


Figure 2.2.3 The reflection of a traveling wave pulse at the fixed end of a stretched string. The reflected pulse is inverted, but its shape is unchanged

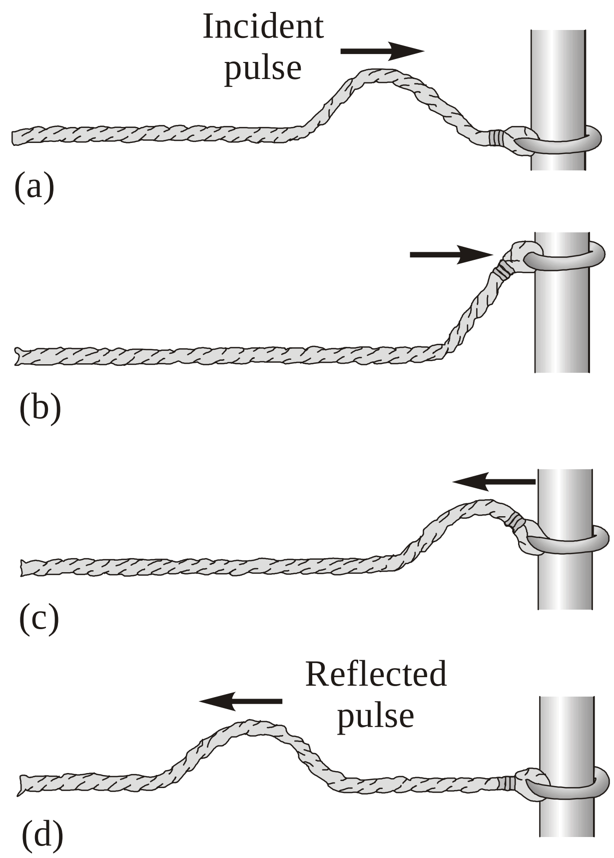


Figure 2.2.4 The reflection of a traveling wave pulse at the free end of a stretched string. The reflected pulse is not inverted

Note that the reflected pulse is inverted. This inversion can be explained as follows: If the end is fastened to a rigid support, it must remain at rest. The arriving pulse exerts a force on the support; the reaction to this force (by Newton's third law), exerted by the support on the string, ‘kicks back’ on the string and sets up an inverted reflected pulse or wave traveling in the reverse direction.

The opposite extreme (to a rigidly fixed end) is the one that is perfectly free to move in the direction, transverse to the length of the string. For example, the string might be tied to a light ring that slides on a smooth rod perpendicular to the length of the string (Figure 2.2.4). Again, the pulse is reflected, but this time it is not inverted. When it reaches the post, the pulse exerts a force on the free end of the string, causing the ring to accelerate upward. The ring overshoots the height of the incoming pulse, and then the downward component of the tension force pulls the ring back down. This movement of the ring produces a reflected pulse that is not inverted and that has the same amplitude as the incoming pulse.

Finally, we may have a situation in which the boundary is intermediate between these two extremes. In this case, part of the incident pulse is reflected and part undergoes *transmission* – that is, some of the pulse passes through the boundary. For instance, suppose a light string is attached to a heavier string, as shown in Figure 2.2.5. When a pulse traveling on the light string reaches the boundary between the two, part of the pulse is reflected and inverted and part is transmitted to the heavier string. The reflected pulse is inverted for the same reasons described earlier in the case of the string rigidly attached to a support.

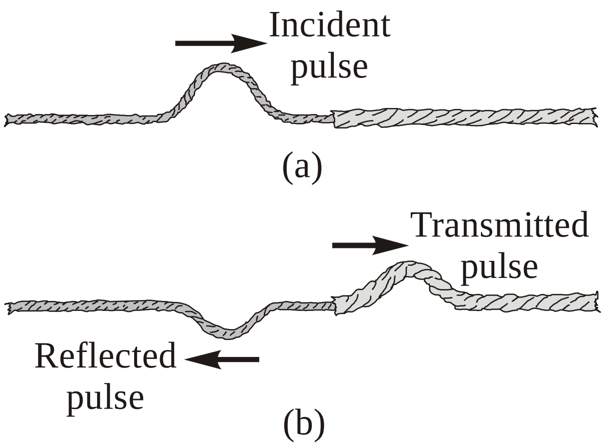


Figure 2.2.5 (a) A pulse traveling to the right on a light string attached to a heavier string; (b) Part of the incident pulse is reflected (and inverted), and part is transmitted to the heavier string

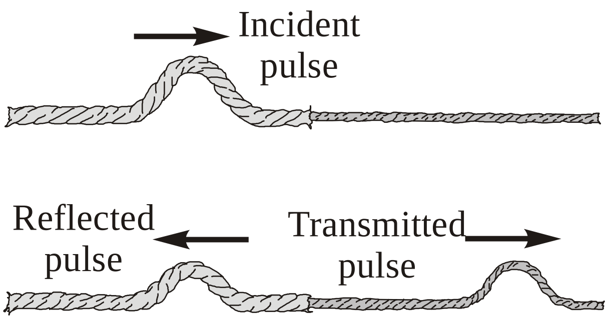


Figure 2.2.6 (a) A pulse traveling to the right on a heavy string attached to a lighter string; (b) The incident pulse is partially reflected and partially transmitted, and the reflected pulse is not inverted

Note that the reflected pulse has a smaller amplitude than the incident pulse. We know that the energy carried by a wave proportional to its amplitude. Thus, according to the principle of the conservation of energy, when the pulse breaks up into a reflected pulse and a transmitted pulse at the boundary, the sum of the energies of these two pulses must equal the energy of the incident pulse. Because the reflected pulse contains only part of the energy of the incident pulse, its amplitude must be smaller.

When a pulse traveling on a heavy string strikes the boundary between the heavy string and a lighter one, as shown in Figure 2.2.6, again part is reflected and part is transmitted. In this case, the reflected pulse is not inverted.

In either case, the relative heights of the reflected and transmitted pulses depend on the relative densities of the two strings. If the strings are identical, there is no discontinuity at the boundary and no reflection takes place.

The condition imposed on the motion of the end of the string, such as attachment to a rigid support or the complete absence of transverse force, are called *boundary conditions*.

According to Eq. (2.1.22) ($v = \sqrt{T/\mu}$), the speed of a wave on a string increases as the mass per unit length of the string decreases. In other words, a pulse travels more slowly on a heavy string than on a light string if both are under the same tension. The following general rules apply to reflected waves:

When a wave pulse travels from medium A to medium B and $v_A > v_B$ (that is, when B is denser than A), the pulse is inverted upon reflection. When a wave pulse travels from medium A to medium B and $v_A < v_B$ (that is, when A is denser than B), the pulse is not inverted upon reflection.

2.2.3 Interference

The principle of superposition is of central importance in all types of wave motion. It applies not only to waves on a string, but also to sound waves, electromagnetic waves (such as light), and all other wave phenomena in which the wave equation is linear. Superposition of two or more waves passing through the same region at the same time is called *interference*.

The superposition principle states that when two or more waves move in the same linear medium, the net displacement of the medium (that is, the resultant wave) at any point equals the algebraic sum of all the displacements caused by the individual waves. Let us apply this principle to two sinusoidal waves traveling in the same direction in a linear medium. If the two waves are traveling to the right and have the same frequency and amplitude but differ in phase, we can express their individual wave functions as

$$y_1 = A\sin(kx - \omega t), \quad \text{and} \quad y_2 = A\sin(kx - \omega t + \phi),$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant. Hence, the resultant wave function y is:

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + A\sin(kx - \omega t + \phi)].$$

To simplify this expression, we use the trigonometric identity

$$\sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right).$$

If we let $\alpha = kx - \omega t$ and $\beta = kx - \omega t + \phi$, we find that the resultant wave function y reduces to

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right). \tag{2.2.1}$$

This result has several important features. The resultant wave function y is also sinusoidal and has the same frequency and wavelength as the individual waves, since the sine function incorporates the same values of k and ω that appear in the original wave functions. The amplitude of the resultant wave is $y = 2A\cos\left(\frac{\phi}{2}\right)$, and its phase is $\phi/2$. If the phase constant ϕ equals 0, then $\cos\phi/2 = \cos 0 = 1$, and the amplitude of the resultant wave is $2A$ – twice the amplitude of either individual wave. In this case, in which $\phi = 0$, the waves are said to be *in phase* and, thus, *interfere constructively*. That is, the crests and troughs of the individual waves y_1 and y_2 occur at the same positions. In general, constructive interference occurs when $\cos\phi/2 = \pm 1$. This is true, for example, when $\phi = 0, 2\pi, 4\pi, ...$ rad – that is, when ϕ is an *even* multiple of π.

When ϕ is equal to π rad or to any *odd* multiple of π, $\cos\phi/2 = \cos\pi/2 = 0$, and the crests of one wave occur at the same positions as the troughs of the second wave. Thus, the resultant wave has *zero* amplitude, and we say that this is *destructive interference*. Finally, when the phase constant has an arbitrary value other than 0 or other than an integer multiple of π rad, the resultant wave has an amplitude whose value is somewhere between 0 and $2A$.

We illustrate this with the aid of sound waves. One simple device for demonstrating interference of sound waves is illustrated in Figure 2.2.7. Sound from a loudspeaker S is sent into a tube at point P where there is a T-shaped junction. Half of the sound power travels in one direction, and half travels in the opposite direction. Thus, the sound waves that reach the receiver R can travel along either of the two paths. The distance along any path from speaker to receiver is called the *path length* r. The lower path length r_1 is fixed, but the upper path length r_2 can be varied by sliding the U-shaped tube. When the *difference in the path lengths*

$$\Delta r = |r_2 - r_1|$$

is either zero or some integer multiple of the wavelength λ (that is, $r = n\lambda$, where $n = 0, 1, 2, ..$), the two waves reaching the receiver at any instant are in phase and interfere constructively. For this case, a

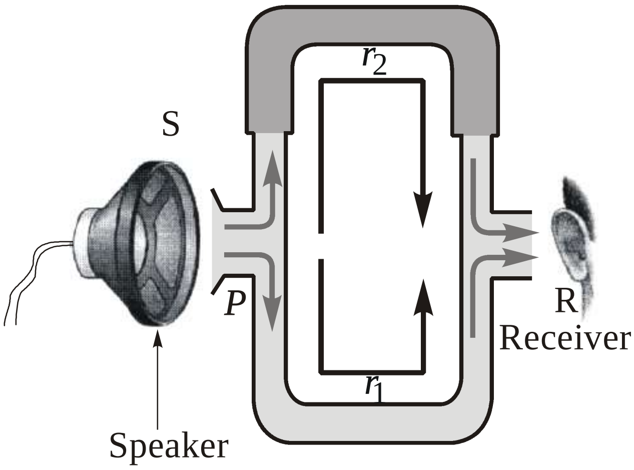


Figure 2.2.7 An acoustical system for demonstrating interference of sound waves. A sound wave from the speaker (S) propagates into the tube and splits into two parts at point P. The two waves, which superimpose at the opposite side, are detected at the receiver (R). The upper path length r_2 can be varied by sliding the upper section

maximum in the sound intensity is detected at the receiver. If the path length r_2 is adjusted such that the path difference $\Delta r = \lambda/2,\ 3\lambda/2,\ ...,\ (2n+1)\frac{\lambda}{2}$ (for $n = 1, 2, .3, ...$), two waves are exactly π rad, or 180°, out of phase at the receiver and, hence, cancel each other. In this case of destructive interference, no sound is detected at the receiver. This simple experiment demonstrates that a phase difference may arise between two waves generated by the same source when they travel along paths of unequal lengths.

It is often useful to express the path difference in terms of the phase angle ϕ between the two waves.

As a *path difference* Δr of one wavelength corresponds to a phase angle of 2π rad, we obtain the ratio $\frac{\phi}{2\pi} = \frac{\Delta r}{\lambda}$, or

$$\Delta r = \frac{\phi}{2\pi}\lambda \ . \tag{2.2.2}$$

Using the notion of path difference, we can express our conditions for constructive and destructive interference in alternative way. If the path difference is any even multiple of $\lambda/2$, then the phase angle $\phi = 2n\pi$, where $n = 0, 1, 2, 3, ...$, and the interference is constructive. For path differences of odd multiples of $\lambda/2$, $\phi = (2n+1)\pi$, where $n = 1, 2, 3, ...$, and the interference is destructive. Thus, we have the conditions

$$\Delta r = 2n\frac{\lambda}{2} \quad \text{for constructive interference} \tag{2.2.3}$$

and

$$\Delta r = (2n+1)\frac{\lambda}{2} \quad \text{for destructive interference.} \tag{2.2.4}$$

Now we understand why the speaker wires in a stereo system should be connected properly. When connected the wrong way – that is, when the positive wire is connected to the negative terminal – the speakers are said to be 'out of phase' because the sound wave coming from one speaker destructively interferes with the wave coming from the other. In this situation, one speaker cone moves outward while the other moves inward. Along a line midway between the two, a rarefaction region from one speaker is superposed on a condensation region from the other speaker. Although the two sounds probably do not completely cancel each other (because the left and right stereo signals are usually not identical), a substantial loss of sound quality still occurs at points along this line.

Example 2.2.1

A pair of speakers placed 3.00 m apart are driven by the same oscillator (Figure 2.2.8). A listener is originally at point O which is located 8.0 m from the center of the line connecting these two speakers. Then the listener walks to point P, which is a perpendicular distance 0.350 m from O, before reaching the *first minimum* in sound intensity. What is the frequency of the oscillator?

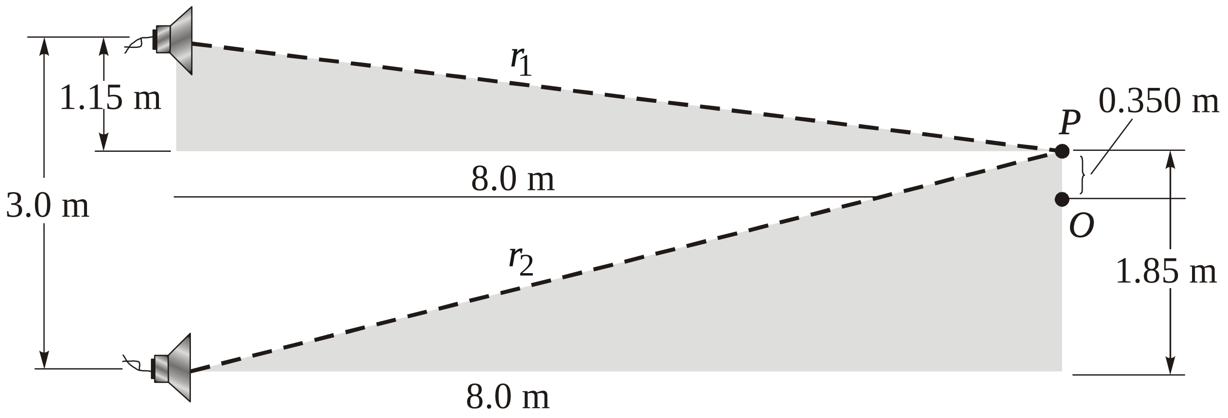


Figure 2.2.8 A pair of speakers placed 3.0 m apart are driven by the same oscillator

Solution.

To find the frequency, we need to know the wavelength of the sound coming from the speakers. With this information, combined with our knowledge of the speed of sound, we can calculate the frequency. We can determine the wavelength from the interference information given. The first minimum occurs when the two waves reaching the listener at point P are 180° out of phase– in other words, when their path difference Δr equals $\lambda/2$. To calculate the path difference, we must first find the path lengths r_1 and r_2

Figure 2.2.8 shows the physical arrangement of the speakers, along with two shaded right triangles that can be drawn on the basis of the lengths described in the problem. From these triangles, we find that the path lengths are

$$r_1 = \sqrt{(8.0\,\text{m})^2 + (1.15\,\text{m})^2} = 8.08\,\text{m}$$

and

$$r_2 = \sqrt{(8.0\,\text{m})^2 + (1.85\,\text{m})^2} = 8.21\,\text{m}\,.$$

Hence, the path difference is $r_2 - r_1 = 0.13\,\text{m}$. As we require that this path difference be equal to $\lambda/2$ for the first minimum, we find that $\lambda = 0.26$ m.

To obtain the oscillator frequency, we use equation $v = \lambda f$ where v is the speed of sound in air, 343 m/s:

Exercises

2.2.1. Two identical traveling waves moving in the same direction are out of phase by $\pi/2$ rad. What is the amplitude of the resultant wave in terms of the common amplitude A of the two combining waves? (Ans. $1.4A$.)

2.2.2. Two sinusoidal waves, identical except for phase, travel in the same direction along a string and interfere to produce a resultant wave given by $y' = (3\text{mm})\sin(20x - 4t + 0.82\,\text{rad})$, with x in meters and t in seconds. What are (a) the wavelength λ of the two waves, (b) the phase difference between them, and (c) their amplitude A?

2.2.3. Two sinusoidal waves are described by the equations

$$y_1 = (5.0\,\text{m})\sin[\pi(4.0x - 1200t)] \text{ and}$$

$$y_2 = (5.0\,\text{m})\sin[\pi(4.0x - 1200t - 0.250)],$$

where x, y are in meters and t is in seconds.

a) What is the amplitude of the resultant wave? (Ans. 9.24 m)

b) What is the frequency of the resultant wave? (Ans. 600 Hz)

2.2.4. A sinusoidal wave is described by the equation

$$y_1 = (0.08\,\text{m})\sin[2\pi(0.10x - 180t)],$$

where y_1 and x are in meters and t is in seconds. Write an expression for a wave that has the same frequency, amplitude, and wavelength as y_1 but which, when added to y_1, gives a resultant with an amplitude of $8\sqrt{3}$ cm.

2.2.5. Two waves are traveling in the same direction along a stretched string. The waves are 90° out of phase. Each wave has amplitude of 4.0 cm. Find the amplitude of the resultant wave. (Ans. 5.66 cm)

2.2.6. Two identical sinusoidal waves with wavelengths of 3.0 m travel in the same direction at a speed of 2.0 m/s. The second wave originates from the same point as the first, but at a later time. Determine the minimum possible time interval between the starting moments of the two waves if the amplitude of the resultant wave is the same as that of each of the two initial waves.

2.2.7. Let $y_1(x,t) = A_1\sin(\omega_1 t - k_1 x)$ and $y_2(x,t) = A_2\sin(\omega_2 t - k_2 x)$ be the solution of the wave equation for the same v. Show that $y(x,t) = y_1(x,t) + y_2(x,t)$ is also a solution of the wave equation.

2.2.4 Standing Waves

We now consider a special example of interference of two identical waves traveling in opposite directions in the same medium, for example, incident and reflected waves. These waves combine in accordance with the superposition principle and so-called *standing waves* are the result of this superposition.

We can analyze such a situation by considering wave functions for two transverse sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium:

$$y_1 = A\sin(kx - \omega t) \qquad \text{and} \qquad y_2 = A\sin(kx + \omega t),$$

where y_1 represents a wave traveling to the right and y_2 represents reflected wave, traveling to the left. By adding these two functions we get the resultant wave function y:

$$y = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx + \omega t).$$

When we use the trigonometric identity:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

letting $\alpha = kx$ and $\beta = \omega t$, this expression reduces to

$$y = (2A\sin kx)\cos\omega t, \tag{2.2.5}$$

which is the wave function of a standing wave. A *standing wave,* such as the one shown in Figure 2.2.9, is an oscillation pattern with a stationary outline that results from the superposition of two identical waves traveling in opposite directions.

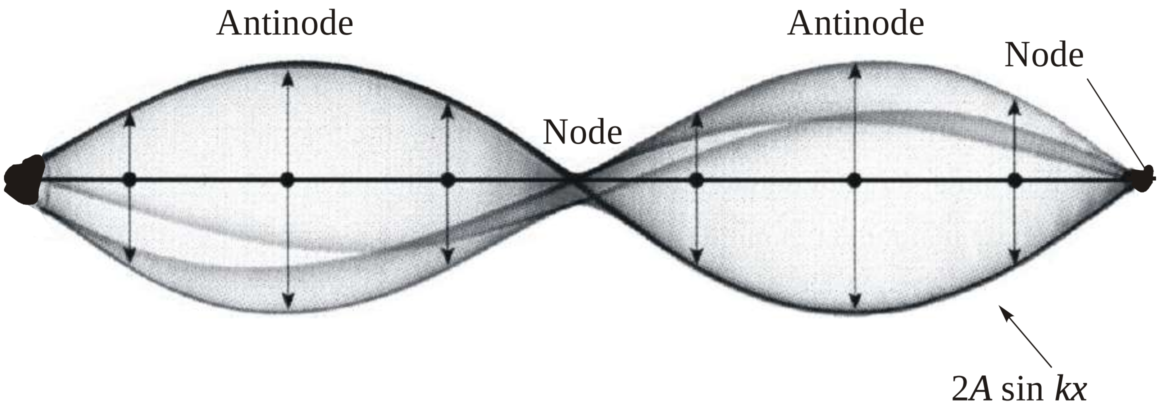


Figure 2.2.9 Standing wave on a string. The time behavior of the vertical displacement from equilibrium of an individual particle of the string is given by $\cos\omega t$. That is, each particle vibrates at an angular frequency ω. The amplitude of the vertical oscillation of any particle on the string depends on the horizontal position of the particle. Each particle vibrates within the confines of the envelope function $2A\sin kx$

Notice that Eq. (2.2.5) does not contain a function of ($kx \pm \omega t$). Thus, it is not an expression for a traveling wave. If we observe a standing wave, we have no sense of motion in the direction of propagation of either of the original waves. If we compare this equation with wave equation, we see that Eq. (2.2.5) describes a special kind of simple harmonic motion. Every particle of the medium oscillates in simple harmonic motion with the same frequency ω (according to the $\cos\omega t$ factor in the equation). However, the amplitude of the simple harmonic motion of a given particle (given by the factor $2A\sin kx$, the coefficient of the cosine function) depends on the location x of the particle in the medium. We need to distinguish carefully between the amplitude A of an individual waves and the amplitude $2A\sin kx$ of the simple harmonic motion of the particles of the medium. A given particle in a standing wave vibrates within the constraints of the envelope function $2A\sin kx$, where x is the particle's position in the medium. This is in contrast to the situation in a traveling sinusoidal wave in which all particles oscillate with the same amplitude and the same frequency and in which the amplitude of the wave is the same as the amplitude of the simple harmonic motion of the particles.

The maximum displacement of a particle of the medium has a minimum value of zero when x satisfies the condition $\sin kx = 0$, that is, when

$$kx = 0, \pi, 2\pi, 3\pi, ...$$

Because $k = 2\pi/\lambda$, these values for kx give

$$x = \frac{\lambda}{2},\ \lambda,\ x = \frac{3\lambda}{2},\ x = \frac{n\lambda}{2}\ \ (n = 0, 1, 2, 3, ...). \qquad (2.2.6)$$

These points of zero displacement are called *nodes.*

The particle with the greatest possible displacement from equilibrium has an amplitude of $2A$, and we define this as the amplitude of the standing wave. The positions in the medium at which this maximum displacement occurs are called ***antinodes.*** The antinodes are located at positions for which the coordinate x satisfies the condition $\sin kx = \pm 1$, that is, when

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$$

Thus, the positions of the antinodes are given by

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots, \frac{n\lambda}{4}, \quad (n = 0, 1, 2, 3). \qquad (2.2.7)$$

In examining Eqs. (2.2.6) and (2.2.7), we note the following important features of the locations of nodes and antinodes:

The distance between adjacent antinodes is equal to $\lambda / 2$.

The distance between adjacent nodes is equal to $\lambda / 2$.

The distance between a node and an adjacent antinode is $\lambda / 4$.

Example 2.2.2

Two waves traveling in opposite directions produce a standing wave. The individual wave functions are

$$y_1 = (4.0\,\text{cm})\sin(3.0x - 2.0t) \text{ and } y_2 = (4.0\,\text{cm})\sin(3.0x + 2.0t),$$

where x and y are measured in centimetres, t in seconds.

a) Find the amplitude of the simple harmonic motion of a particle medium located at $x = 2.3$ cm.

Solution.

According to Eq. (2.2.5) in this problem, we have $A = 4.0$ cm, $k = 3.0$ rad/cm, and $\omega = 2.0$ rad/s. Thus,

$$y = (2A\sin kx)\cos\omega t = [(8.0\,\text{cm})\sin 3.0x]\cos 2.0t.$$

Thus, we obtain the amplitude of the simple harmonic motion of the particle at the position $x = 2.3$ by evaluating the coefficient of the cosine function at this position:

$$y = (8.0\,\text{cm})\sin 3.0x\big|_{x=2.3} = (8.0\,\text{cm})\sin(6.9\,\text{rad}) = 4.6\,\text{cm}.$$

b) Find the positions of the nodes and antinodes.

Solution.

With $k = 2\pi/\lambda = 3.0\,\text{rad/cm}$, we see that $\lambda = 2\pi/3$ cm. Therefore, from Eq. (2.2.6), we find that the nodes are located at

$$x = n\frac{\lambda}{2} = n\left(\frac{\pi}{3}\right)\text{cm} \qquad n = 0, 1, 2, 3, \dots \quad \text{(nodes)}.$$

From Eq. (2.2.7), we find that the antinodes are located at

$$x = n\frac{\lambda}{4} = n\left(\frac{\pi}{6}\right)\text{cm} \qquad n = 1, 3, 5, \dots \quad \text{(antinodes)}.$$

c) What is the amplitude of the simple harmonic motion of the particle located at an antinode?

Solution.

The standing wave is described by Eq. (2.2.5); the maximum displacement of a particle at an antinode is the amplitude of the standing wave which is twice the amplitude of the individual traveling waves:

$$y_{max} = 2A = 2(4.0\,\text{cm}) = 8.0 \text{ cm.}$$

Let us check this result by evaluating the coefficient of our standing-wave function at the positions we found for the antinodes:

$$y = (8.0\,\text{cm})\sin 3.0x\Big|_{x=n(\pi/6)} = (8.0\,\text{cm})\sin\left[3.0n\left(\frac{\pi}{6}\right)\text{rad}\right] = 8.0\,\text{cm}\,.$$

In evaluating this expression, we have used the fact that n is an odd integer; thus, the sine function is equal to unity.

Exercises

2.2.8. Two sinusoidal waves traveling in opposite directions interfere to produce a standing wave described by the equation

$$y = (1.50\,\text{m})\sin(0.40x)\cos(200t)\,,$$

where x is in metres and t is in seconds. Determine the wavelength, frequency, and speed of the interfering waves.

2.2.9. Two waves on a long string are described by the equations

$$y_1 = (0.015\,\text{m})\cos\left(\frac{x}{2} - 40t\right) \quad \text{and} \quad y_2 = (0.015\,\text{m})\cos\left(\frac{x}{2} + 40t\right),$$

where y_1, y_2, and x are in metres and t is in seconds. (a) Determine the position of the nodes of the resulting standing wave. (b) What is the maximum displacement at the position $x = 0.40$ m?

2.2.10. Two speakers are driven by a common oscillator at 800 Hz and face each other at a distance of 1.25 m. Locate the points along a line joining the two speakers where relative minima would be expected. (Use v = 343 m/s.)

2.2.11. Two waves that set up a standing wave on a long string are given by the expressions

$$y_1 = A\sin(kx - \omega t + \phi) \quad \text{and} \quad y_2 = A\sin(kx + \omega t)\,.$$

Show (a) that the addition of the arbitrary phase angle changes only the position of the nodes, and (b) that the distance between the nodes remains constant in time.

2.2.12. Two sinusoidal waves combining in a medium are described by the equations

$$y_1 = (3c\,\text{m})\sin\pi(x + 0.6t) \quad \text{and} \quad y_2 = (3c\,\text{m})\sin\pi(x - 0.6t),$$

where x is in centimetres and t is in seconds. Determine the *maximum* displacement of the medium at (a) $x = 0.25$ cm, (b) $x = 0.5$ cm, and (c) $x = 1.5$ cm. (d) Find the three smallest values of x, corresponding to antinodes.

2.2.13. A standing wave is formed by the interference of two traveling waves, each of which has an amplitude $A = \pi$ cm, angular wave number $k = \pi/2$ cm^{-1}, and angular frequency $\omega = 10\pi$ rad/s. (a) Calculate the distance between the first two antinodes. (b) What is the amplitude of the standing wave at $x = 0.25$ cm?

2.2.14. Verify by direct substitution that the wave function for a standing wave given in equation $y = 2A \sin kx \cos \omega t$ is a solution of the general linear wave equation: $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$.

2.2.5 Standing Waves in a String Fixed at Both Ends

We know about reflection, or echo, of sound waves from rigid walls and the analogous reflection of transverse wave on a string from rigidly held ends. Now suppose we have two parallel walls. If a sharp sound pulse such as a hand clap originates at a point between the walls, the result is a series of regularly spaced echoes caused by the repeated back-and-forth reflection between the walls. In room acoustics, this phenomenon is called "flutter echo".

For transverse waves on a string, the analogous situation occurs if a string of some definite length L (Figure 2.2.10a) is rigidly held at both ends. If a sinusoidal wave is produced on such a string, the wave is reflected and re-reflected. Note that the ends of the string, because they are fixed and must necessarily have zero displacement, are nodes by definition. The string has a number of natural patterns of oscillation, called *normal modes*, each of which has a characteristic frequency that is easily calculated.

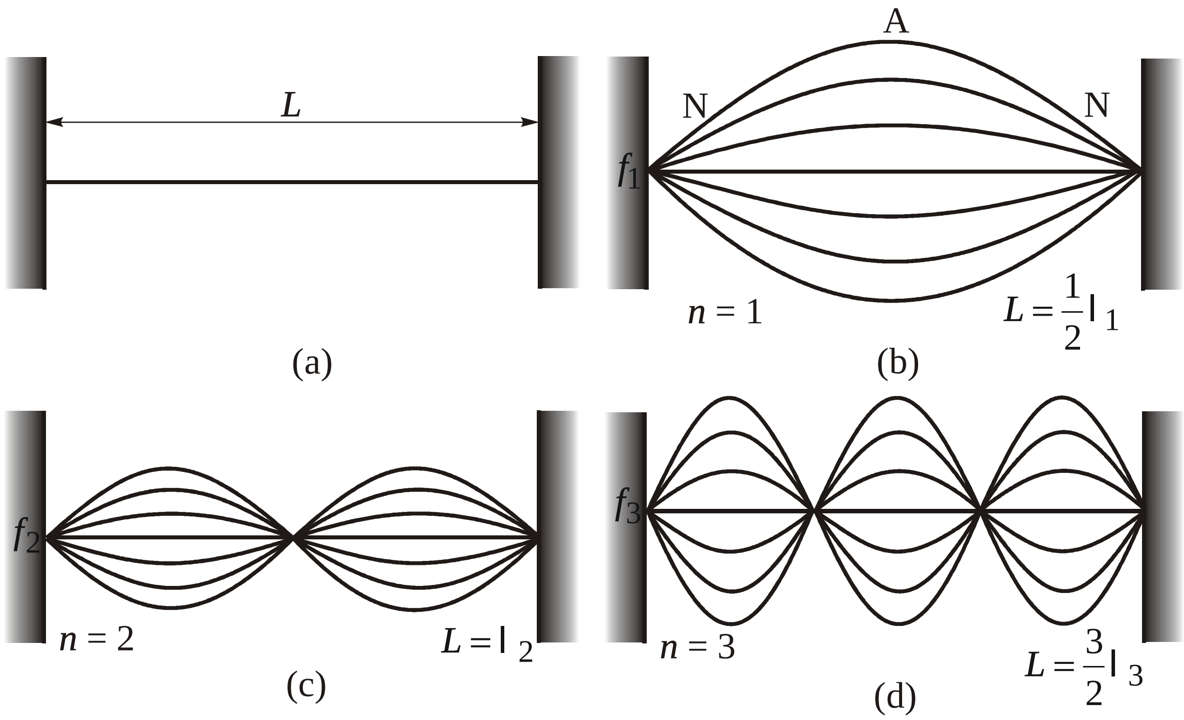


Figure 2.2.10 (a) A string of length L fixed at both ends. The normal modes of vibration form a harmonic series; (b) the fundamental or first harmonic; (c) the second harmonic; (d) the third harmonic

In general, we can describe the normal modes of oscillation for the string by imposing the requirements that the ends must be nodes and that the nodes and antinodes must be separated by one fourth of a wavelength. The first normal mode, shown in Figure 2.2.10b, has nodes at its ends and one antinode in the middle. This is the longest-wavelength mode and is consistent with our requirements. This first normal mode occurs when the wavelength λ_1 is twice the length of the string, that is. $\lambda_1 = 2L$. The next normal mode, of wavelength λ_2 (see Figure 2.2.10c), occurs when the wavelength equals the length of the string, that is, $\lambda_2 = L$. The third normal mode (see Figure 2.2.10d) corresponds to the case when $\lambda_3 = 2L/3$. In general, the wavelengths of various normal modes for a string of length L fixed at both ends is defined as

$$\lambda_n = \frac{2L}{n}, \qquad (n = 1, 2, 3, ...), \tag{2.2.8}$$

where the index n refers to the n*-th* normal mode of oscillation. These are the possible modes of oscillation for the string.

In general, the motion of an oscillating string fixed at both ends is described by the superposition of several normal modes. The exact normal mode depends on how the oscillation is started.

The natural frequencies associated with these modes are obtained from the relationship $f = v/\lambda$, where the wave speed v is the same for all frequencies. Using Eq. (2.2.8), we find that natural frequencies f_n of the normal modes are

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \quad (n = 1, 2, 3, ...). \tag{2.2.9}$$

Because $v = \sqrt{T/\mu}$ where T is the tension in the string and μ is its linear mass density, we can also express the natural frequencies of a taut string as

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \qquad (n = 1, 2, 3, ...). \tag{2.2.10}$$

The lowest frequency f_1, which corresponds to $n = 1$, is called either *fundamental* or *fundamental frequency* and is given by

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}. \tag{2.2.11}$$

Frequencies of the remaining normal modes are integer multiples of the fundamental frequency. Frequencies of normal modes that exhibit an integer–multiple relationship such as this form a *harmonic series*, and the normal modes are called *harmonics*. The fundamental frequency f_1 is the frequency of the first harmonic; the frequency $f_2 = 2f_1$ is the frequency of the second harmonic; and the frequency $f_n = nf_1$ is the frequency of the n-th harmonic. Other oscillating systems, such as a drumhead, exhibit normal modes, but the frequencies are not related as integer multiples of a fundamental. Thus, we do not use the term *harmonic* in association with these types of systems.

This series of frequencies, all integer multiples of the fundamental, is called a *harmonic series*. Musicians sometimes call f_2, f_3, and so on *overtones*; f_2 is the second harmonic, or the the first overtone, f_3 is the third harmonic, or the second overtone, and so on.

In obtaining Eq. (2.2.8), we used a technique based on the separation distance between nodes and antinodes. We can obtain this equation in an alternative manner. These results may also be obtained directly from Eq. (2.2.5):

$$y = (2A\sin kx)\cos \omega t\,.$$

The boundary conditions require that $y_1 + y_2 = 0$ at the ends of the string, that is, at $x = 0$ and $x = L$. Since the sine of zero is zero, the first condition is satisfied automatically. The second condition requires that $\sin kL = 0$, and this is true only when k has certain special values. The sine of an angle is zero only when the angle is zero or an integer multiple of π (180°). Thus, we must have

$$kL = n\pi \qquad (n = 1, 2, 3, \ldots).$$

We do not include the possibility $n = 0$ because that gives $k = 0$, that is, a wave with zero displacement *everywhere* (a possible case, to be sure, but not a very interesting one).

Replacing k above by $2\pi/\lambda$, we obtain

$$\frac{2\pi L}{\lambda} = n\pi \qquad \text{or} \qquad \lambda_n = \frac{2L}{n},$$

in agreement with Eq. (2.2.8).

Each of the frequencies given by Eq. (2.2.9) corresponds to a possible normal mode of motion, that is, a motion in which each particle of the string moves sinusoidally, all with the same frequency. As this analysis shows, there is an infinite number of normal modes, each with its characteristic frequency. This situation is in striking contrast with the simple harmonic oscillator system consisting of a single mass and a spring. The harmonic oscillator has only one normal mode and one characteristic frequency while the vibrating string has an infinite number.

If a string is initially displaced so that its shape is the same as any one of the possible harmonics, it will vibrate, when released, at the frequency of that particular harmonic. But when a piano string is struck or a guitar string is plucked, not only the fundamental but many of the overtones are present in the resulting vibration. This motion is, therefore, a combination, or superposition, of normal modes. Several frequencies and motions are present simultaneously, and the displacement of any point on the string is the sum (or superposition) of displacements associated with the individual modes. Indeed, every possible motion of the string can be represented as some superposition of' normal-mode motions.

The fundamental frequency of a vibrating string is $f_1 = v/2L$ where $v = \sqrt{T/m}$. It follows that

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}.$$

Stringed instruments provide many examples of the implications of this equation. For example, all such instruments are tuned by varying the tension T. An increase of tension increases the frequency or pitch, and vice versa. The inverse dependence of frequency on length L is illustrated by the long strings of the bass section of the piano or the bass viol compared with the shorter strings on the piano treble or the violin. In playing the violin or guitar, the usual means of varying the pitch is to press the strings against the fingerboard with the fingers to change the length of the vibrating portion of the string.

Example 2.2.3

The high E string on a guitar measures 64.0 cm in length and has a fundamental frequency of 330 Hz. By pressing down on it at the first fret, the string is shortened so that it plays an F note that has a frequency of 350 Hz. How far is the fret from the neck end of the string?

Solution.

Eq. (2.2.9) relates the string length to the fundamental frequency. With $n = 1$, we can solve for the speed of the wave on the string,

$$v = \frac{2L}{n} f_n = \frac{2(0.64\,\text{m})}{1}(330\,\text{Hz}) = 422 \text{ m/s}.$$

Because we have not adjusted the tuning peg, the tension in the string, and, hence, the wave speed, remain constant. We can again use Eq. (2.2.9), this time solving for L and substituting the new frequency to find the shortened string length:

$$L = n\frac{v}{2f_n} = (1)\frac{422\,\text{m/s}}{2(350\,\text{Hz})} = 0.603 \text{ m}.$$

The difference between this length and the measured length of 64.0 cm is the distance from the fret to the neck end of the string, or 3.70 cm.

Exercises

2.2.15. A 2.0-m-long wire having a mass of 0.10 kg is fixed at both ends. The tension in the wire is maintained at 20.0 N. What are the frequencies of the first three allowed modes of vibration? If a node is observed at a point 0.40 m from one end, in what mode and with what frequency is it vibrating?

2.2.16. Find the fundamental frequency and the next three frequencies that could cause a standing-wave pattern on a string that is 30.0 m long, has a mass per length of 9×10^{-8} kg/m, and is stretched to a tension of 20.0 N.

2.2.17. A standing wave is established in a 120-cm-long string fixed at both ends. The string vibrates in four segments when driven at 120 Hz. (a) Determine the wavelength. (b) What are the fundamental frequencies of the string?

2.2.18. A string of length L, mass per unit length μ, and tension T is vibrating at its fundamental frequency. Describe the effectt that each of the following conditions has on the fundamental frequency: (a) The length of the string is doubled, but all other factors are held constant; (b) The mass per unit length is doubled, but all other factors are held constant; (c) The tension is doubled, but all other factors are held constant.

2.2.19. A 60.0-cm guitar string under a tension of 50.0 N has a mass per unit length of 0.10 g/cm. What is the highest resonance frequency of the string that can be heard by a person able to hear frequencies of up to 20 000 Hz?

2.2.20. A stretched wire vibrates in its first normal mode at a frequency of 400 Hz. What would be the fundamental frequency if the wire were half as long, its diameter were doubled and its tension were increased four-fold?

2.2.21. A string under tension T_i, oscillates in the third harmonic at a frequency of f_3, and the waves on the string have a wavelength λ_3. If the tension is increased to $T_f = 4T_i$, and the string is again made to oscillate in the third harmonic, what are (a) the frequency of oscillation in terms of f_3 and (b) the wavelength of the waves in terms of λ_3?

2.2.22. A nylon guitar string has a linear density of 7.2 g/m and is under a tension of 150 N. Fixed supports are 90 cm apart. The string is oscillating in the standing wave pattern shown in Figure 2.2.10. Calculate the (a) speed, (b) wavelength and (c) frequency of the traveling waves whose superposition produces this standing wave. [Ans. (a) 140 m/s, (b) 60 cm, $1.4\, y_{max}$)

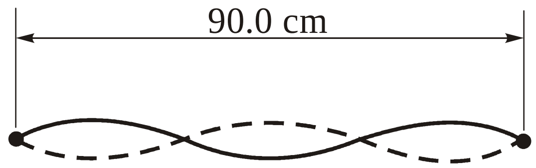


Figure 2.2.10 The string is oscillating in the standing wave pattern

2.2.23. Two sinusoidal waves with identical wavelengths and amplitudes travel in opposite directions along a string with a speed of 10 cm/s. If the time interval between instants when the string is flat is 0.50 s, what is the wavelength of the waves?

2.2.24. A 125 cm length of string has a mass of 2.0 g. It is stretched with a tension of 7.0 N between fixed supports. (a) What is the wave speed for this string? (b) What is the lowest resonant frequency of this string?

2.2.25. What are the three lowest frequencies for standing waves on a wire 10.0 m long having a mass of 100 g, which is stretched under a tension of 250 N?

2.2.26. A string oscillates according to the equation

$$y = (0.5\text{cm})\sin\left(\frac{\pi}{3}x\right)\cos 40\pi t .$$

What are (a) the amplitude and (b) the speed of the two waves (identical except for direction of travel) whose superposition gives this oscillation? (c) What is the distance between nodes? (d) What is the speed of a particle of the string at the position $x = 1.5$ cm when $t = 9/8$ s?

2.2.27. A string 3.0 m long is oscillating as a three-loop standing wave with an amplitude of 1.0 cm. The wave speed is 100 m/s. (a) What is the frequency? (b) Write equations for two waves that, when combined, will result in this standing wave.

2.2.28. In an experiment on standing waves, a string 90 cm long is attached to a prong of an electrically driven tuning fork that oscillates perpendicular to the length of the string at a frequency of 60 Hz. The mass of the string is 0.044 kg. What tension must the string be under (weights are attached to the other end) if it is to oscillate in four loops?

2.2.29. Oscillation of a 600 Hz tuning fork sets up standing waves in a string clamped at both ends. The wave speed for the string is 400 m/s. The standing wave has four loops and an amplitude of 2.0 mm. (a) What is the length of the string? (b) Write an equation for the displacement of the string as a function of position and time.

2.2.30. A rope, under a tension of 200 N and fixed at both ends, oscillates in a second-harmonic standing wave pattern. The displacement of the rope is given by

$$y = (0.1\text{m})\sin\left(\frac{\pi x}{2}\right)\sin 12\pi t ,$$

where $x = 0$ at one end of the rope, x is in metres, and t is in seconds. What are (a) the length of the rope; (b) the speed of the waves on the rope; and (c) the mass of the rope? (d) If the rope oscillates in a third-harmonic standing wave pattern, what is the period of oscillation?

2.2.6 Resonance

We have seen that a system such as a taut string is capable of oscillating in one or more normal modes of oscillation. If a periodic force is applied to such a system, the amplitude of the resulting motion is greater than normal when the frequency of the applied force is equal to or nearly equal to one of the natural frequencies of the system. We have discussed this phenomenon, known as resonance. Although a block-spring system or a simple pendulum has only one natural frequency, standing wave systems can have a whole set of natural frequencies. Because an oscillating system exhibits a large amplitude when driven at any of its natural frequencies, these frequencies are often referred to as *resonance frequencies*.

Figure 2.2.12 shows the response of an oscillating system to various driving frequencies, where one of the resonance frequencies of the system is denoted by f_0. Note that the oscillation amplitude of the system is the greatest when the frequency of the driving force equals the resonance frequency. The maximum amplitude is limited by friction in the system. If a driving force begins to work on an oscillating system initially at rest, the input energy is used both to increase the amplitude of the oscillation and to overcome the frictional force. Once maximum amplitude is reached, the work done by the driving force is used only to overcome friction.

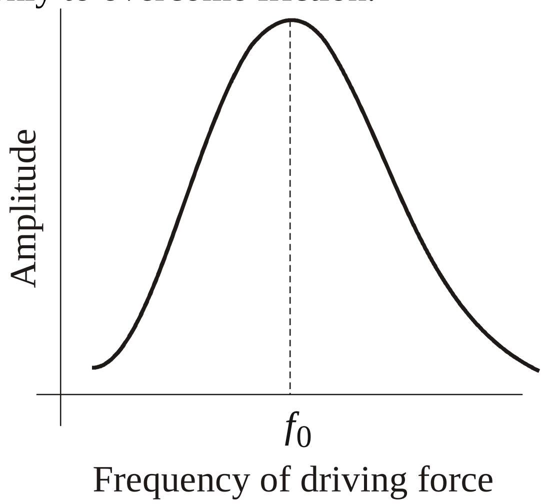


Figure 2.2.12 Graph of the amplitude (response) versus driving frequency for an oscillating system. The amplitude is a maximum at the resonance frequency f_0. Note that the curve is not symmetric

A system is said to be *weakly damped* when the amount of friction to be overcome is small. Such a system has a large amplitude of motion when driven at one of its resonance frequencies, and the oscillations persist for a long time after the driving force is removed. A system in which considerable friction must be overcome is said to be *strongly* damped. For a given driving force applied at a resonance frequency, the maximum amplitude attained by a strongly damped oscillator is smaller than that attained by a comparable weakly damped oscillator. Once the driving force in a strongly damped oscillator is removed, the amplitude decreases rapidly with time.

Exercises

2.2.27. Chains suspending a child swing are 2.0 m long. At what frequency should a big brother's push make the child swing with greatest amplitude?

2.2.28. Standing wave vibrations are set in a crystal goblet with four antinodes equally spaced around the 20.0-cm circumference of its rim. If transverse waves move around the glass at 900 m/s, an opera singer would have to produce a high harmonic with what frequency to shatter the glass with a resonant vibration?

2.2.29. An earthquake can produce a seiche in a lake in which the water sloshes back and forth from end to end with a remarkably large amplitude and long period. Consider a seiche produced in a rectangular farm pond. Suppose that the pond is 9.15 m long and of uniform depth. You measure that a wave pulse produced at one end reaches the other end in 2.50 s. (a) What is the wave speed? (b) To produce the seiche, you suggest that several people stand on the bank at one end and paddle together with snow shovels, moving them in simple harmonic motion. What must be the frequency of this motion?

2.2.30. A 125 cm length of string has a mass of 2.0 g. It is stretched with a tension of 7.0 N between fixed supports. (a) What is the wave speed of the string? (b) What is the lowest resonant frequency of the string?

2.2.31. A string A is stretched between two clamps separated by distance L. A string B with the same linear density and under the same tension as string A, is stretched between two clamps separated by distance $4L$. Consider the first eight harmonics of the string B. Which, if any, has a resonant frequency that matches a resonant frequency of the string A?

2.2.32. A string that is stretched between fixed supports separated by 75.0 cm has resonant frequencies of 420 and 315 Hz with no intermediate resonant frequencies. What are (a) the lowest resonant frequency and (b) the wave speed? [Ans. (a) 105 Hz, (b) 158 m/s.]

2.2.7 Longitudinal Standing Waves

When longitudinal waves propagate in a fluid or gas in a tube of finite length, they are reflected from the ends in the same way that transverse waves on a string are reflected at its ends. The superposition of the waves traveling in opposite directions again forms a standing wave.

When reflection takes place at a *closed* end, the displacement of the particles is always equal to zero. This situation is analogous to a fixed end of a string; in both cases there is no displacement at the end, and the end is a node. For clarity, in the following discussion, we call a closed end of a tube or pipe a displacement node. Furthermore, because the pressure wave is 90° out of phase with the displacement wave, the closed end of an air column corresponds to a pressure antinode (that is, a point of maximum pressure variation).

If the end of the tube is open, the nature of the reflection is more complex and depends on whether the tube is wide or narrow compared with the wavelength. If the tube is narrow compared with the wavelength, which is the case in most musical instruments, the open end is a displacement antinode and a pressure node. (A free end of a stretched string is also a displacement antinode.) Thus longitudinal waves in a column of gas are reflected at the closed and open ends of a tube in the same way that transverse waves in a string are reflected at fixed and free ends, respectively.

You may wonder how a sound wave can reflect from an open end, since there may not appear to be a change in the medium at this point. It is indeed true that the medium through which the sound wave moves is air both inside and outside the pipe. Remember that sound is a pressure wave, however, and a compression region of the sound wave is constrained by the sides of the pipe as long as the region is inside the pipe. As the compression region exits at the open end of the pipe, the constraint is removed and the compressed air is free to expand into the atmosphere. Thus, there is a change in the *character* of the medium between the inside of the pipe and the outside even though there is no change in the *material* of the medium. This change in character is sufficient to allow some reflection.

The first three normal modes of oscillation of a pipe open at both ends are shown in Figure 2.2.13a. When air is directed against an edge at the left, longitudinal standing waves are formed, and the pipe resonates at its natural frequencies. All normal modes are excited simultaneously (although not with the same amplitude). Note that both ends are displacement antinodes (approximately). In the first normal mode, the standing wave extends between two adjacent antinodes which is a distance of half a wavelength. Thus, the wavelength is twice the length of the pipe, and the fundamental frequency is $f_1 = \frac{v}{2L}$. As Figure 2.2.13a shows, the frequencies of the higher harmonics are $2f_1$, $3f_1$. Thus, we can say that *in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.*

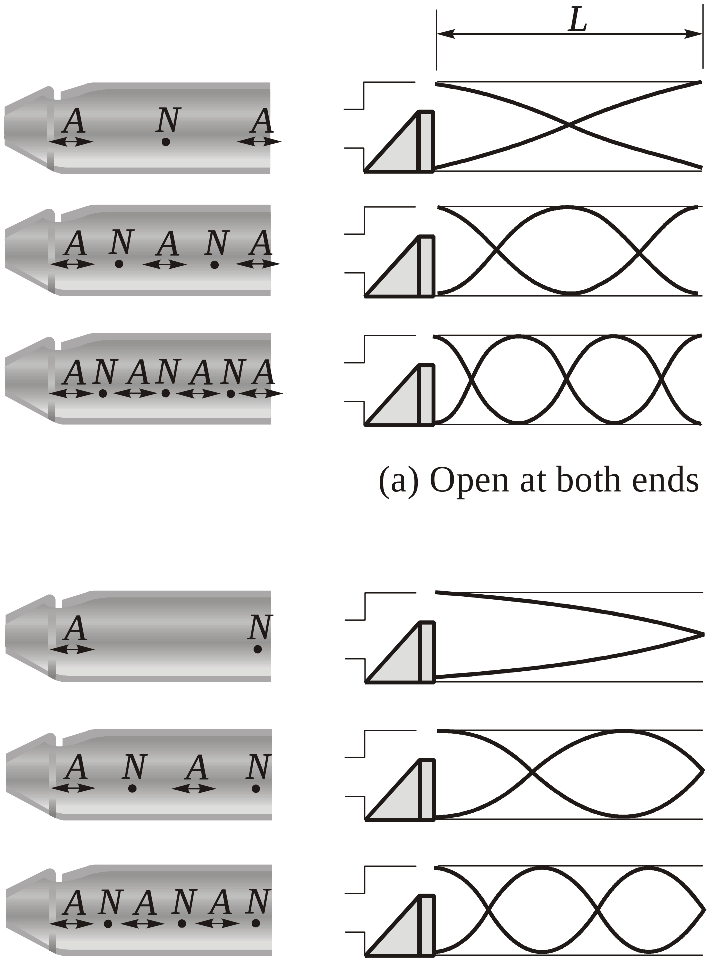
Because all harmonics are present and because the fundamental frequency is given by the same expression as that for a string (see Eq. (2.2.9)), we can express the natural frequencies of oscillation as

$$f_n = n\frac{v}{2L} \quad (n = 1, 2, 3, \ldots). \qquad (2.2.12)$$

Despite the similarity between Eqs. (2.2.9) and (2.2.12), we must remember that in Eq. (2.2.9), v is the speed of waves on the string, whereas in Eq. (2.2.12), v is the speed of sound in air.

If a pipe is closed at one end and open at the other, the closed end is a displacement node (see Figure 2.2.13b). In this case, the standing wave for the fundamental mode extends from an antinode to the adjacent node, which is one fourth of a wavelength. Hence, the wavelength for the first normal mode is $4L$, and the fundamental frequency is $f_1 = v/4L$. As Figure 2.2.13b shows, the higher-frequency waves that satisfy our conditions are those that have a node at the closed end and an antinode at the open end; this means that the higher harmonics have frequencies $3f_1$, $5f_1$.

It is interesting to investigate what happens to the frequencies of instruments based on air columns and strings during a concert as the temperature rises. The sound emitted by a flute, for example, becomes sharp (increases in frequency) as it warms up because the speed of sound increases in the increasingly warmer air inside the flute (consider Eq. (2.2.12)). The sound produced by a violin becomes flat (decreases in frequency) as the strings expand thermally because the expansion causes their tension to decrease.



$\lambda_1 = 2L$, $f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$ First harmonic

$\lambda_2 = L$, $f_2 = \frac{v}{L} = 2f_1$ Second harmonic

$\lambda_3 = \frac{2}{3}L$, $f_3 = \frac{3v}{2L} = 3f_1$ Third harmonic

(a) Open at both ends

$\lambda_1 = 4L$, $f_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$ First harmonic

$\lambda_3 = \frac{4}{3}L$, $f_3 = \frac{3v}{4L} = 3f_1$ Third harmonic

$\lambda_5 = \frac{4}{5}L$, $f_5 = \frac{5v}{4L} = 5f_1$ Fifth harmonic

(b) Closed at one end, open at the other

Figure 2.2.13 Motion of air molecules in standing longitudinal waves in a pipe along with schematic representations of the waves. The graphs represent the displacement amplitudes, not the pressure amplitudes. (a) In a pipe open at both ends, the harmonic series created consists of all integer multiples of the fundamental frequency f_1, $2f_1$, $3f_1$, (b) In a pipe closed at one end and open at the other, the harmonic series created consists of only odd-integer multiples of the fundamental frequency f_1, $3f_1$, $5f_1$

We can demonstrate longitudinal standing waves in a column of gas and measure the wave speed by using the apparatus called Kundt's tube, shown in Figure 2.2.14. A glass tube a metre or so long is closed at one end and has a flexible diaphragm at the other end that can transmit vibrations. We use a sound source which might be a small loudspeaker driven by an audio oscillator and amplifier to vibrate the diaphragm sinusoidally with a variable frequency. A small amount of light powder or cork dust is distributed uniformly along the bottom side of the tube. As we vary the frequency of the sound, we pass through frequencies where the amplitude of the standing waves becomes large enough for the moving gas to sweep the cork dust along the tube at all points where the gas is in motion. The powder, therefore, collects at the displacement nodes, which can be seen and measured easily.

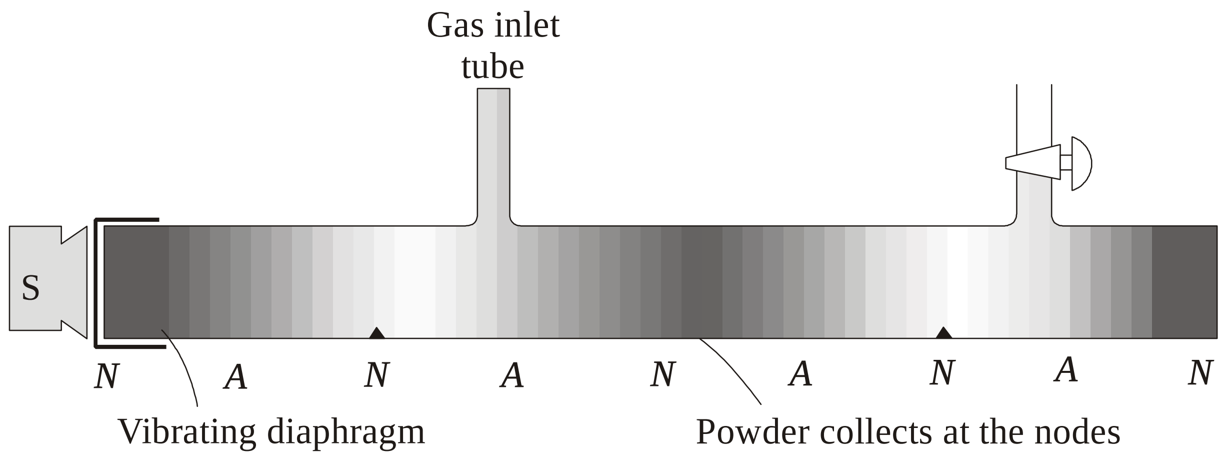


Figure 2.2.14 Kundt's tube for determining the velocity of sound in a gas. The shading represents the density of the gas molecules at an instant when the pressure at the displacement nodes is a maximum or a minimum

In a standing wave, the distance between two adjacent nodes is one-half (not one) wavelength. Thus, we can measure the wavelength by measuring the distances $l/2$ between adjacent clumps of powder. We read the frequency f from the oscillator dial and can calculate the speed v of the waves from the usual relation

$$v = l f \ .$$

At a displacement node, the pressure variations above and below the average have their *maximum* value, while at a displacement antinode the pressure does not vary. To understand this, note that two small masses of gas on opposite sides of a displacement *node* vibrate in *opposite phase.* When the masses of gas approach each other, the gas between them is compressed and the pressure rises; when they recede from each other, the pressure drops. But two small masses of gas on opposite sides of a displacement *antinode* vibrate *in phase* and so cause *no* pressure variations at the antinode.

We can describe this relationship in terms of pressure nodes, which are the points where the pressure does not vary, and pressure antinodes, which are the points where its variation is greatest. *A pressure node is always a displacement antinode, and a pressure antinode is always a displacement node.* An *open* end of a thin tube or pipe is a pressure node because such an end is open to the atmosphere and is, thus, at constant pressure. But for this reason, an open end is always a displacement *antinode.*

Example 2.2.4

A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows.

a) Determine the frequencies of the first three harmonics of the culvert if it is open at both ends. Take $v = 343$ m/s as the speed of sound in air.

Solution.

The frequency of the first harmonic of a pipe open at both ends is

$$f_1 = \frac{v}{2L} = \frac{343\,\text{m/s}}{2(1.23\,\text{m})} = 139 \text{ Hz}.$$

Because both ends are open, all harmonics are present; thus,

$$f_2 = 2f_1 = 278 \text{ Hz} \quad \text{and} \quad f_3 = 3f_1 = 417 \text{ Hz}.$$

b) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

Solution.

The fundamental frequency of a pipe closed at one end

$$f_1 = \frac{v}{4L} = \frac{343\,\text{m/s}}{4(1.23\,\text{m})} = 69.7 \text{ Hz}.$$

In this case, only odd harmonics are present; hence, the next two harmonics have frequencies $f_3 = 3f_1 = 209$ Hz and

$$f_5 = 5f_1 = 549 \text{ Hz}.$$

c) For the culvert open at both ends, how many of the harmonics present fall within the normal human hearing range (20 to 17000 Hz)?

Solution.

Because all harmonics are present, we can express the frequency of the highest harmonic heard as $f_n = nf_1$ where n is the number of harmonics that we can hear. For $f_n = 17000$ Hz, we find that the number of harmonics present in the audible range is

$$n = \frac{17\,000\,\text{Hz}}{139\,\text{Hz}} = 122.$$

Only the first few harmonics are of sufficient amplitude to be heard.

Exercises

2.2.33. Calculate the length of a pipe that has a fundamental frequency of 240 Hz if the pipe is (a) closed at one end and (b) open at both ends.

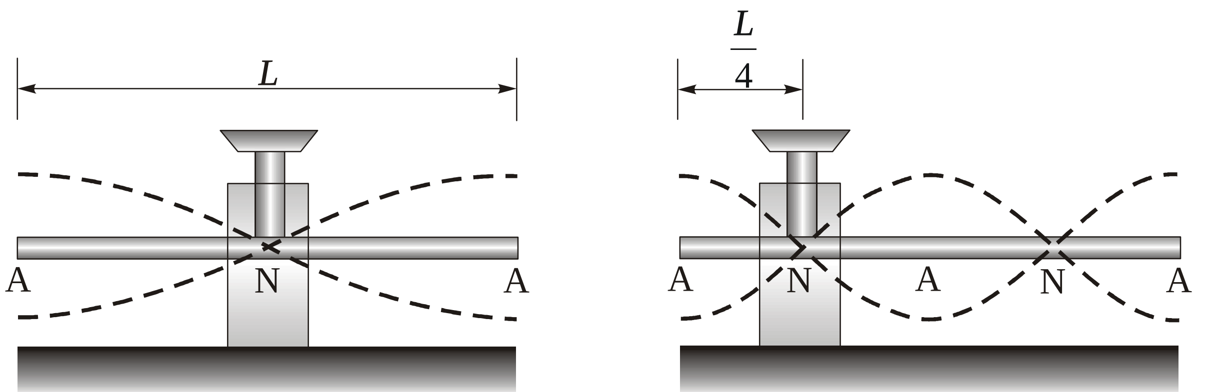
2.2.34. The fundamental frequency of an open organ pipe corresponds to middle C (261.6 Hz on the chromatic musical scale). The third resonance of a closed organ pipe has the same frequency. What are the lengths of the two pipes?

2.2.35. Estimate the length of your ear canal from its opening at the external ear to the eardrum. If you regard the canal as a tube that is open at one end and closed at the other, at approximately what fundamental frequency would you expect your hearing to be most sensitive? Explain why you can hear especially soft sounds just around this frequency.

2.2.36. The longest pipe on an organ that has pedal stops is often 4.88 m. What is the fundamental frequency (at 0°C) if the nondriven end of the pipe is (a) closed and (b) open? (c) What are the frequencies at 20.0°C?

2.2.8 Standing Waves on Rods and Plates

Standing waves can also be set up on rods and plates. A rod clamped in the middle and stroked at one end oscillates, as depicted in Figure 2.2.15a. The oscillations of the particles of the rod are longitudinal, and so the broken lines in Figure 2.2.15b represent *longitudinal displacements* of various parts of the rod. For clarity, we have drawn them in the transverse direction, just as we did for air columns. The midpoint is a displacement node because it is fixed by the clamp, whereas the ends are displacement antinodes because they are free to oscillate. The oscillations in this setup are analogous to those in a pipe open at both ends. In Figure 2.2.15a the broken lines represent the first normal mode for which the wavelength is $2L$ and the frequency is $f = v/2L$ where v is the speed of longitudinal wave in the rod. Other normal modes may be excited by clamping the rod at different points. For example, the second normal mode (Fig. 2.2.15b) is excited by clamping the rod a distance $L/4$ away from one end.



$\lambda_1 = 2L$

$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$

(a)

$\lambda_2 = L$

$f_2 = \frac{v}{L} = 2f_1$

(b)

Figure 2.2.15 Normal-mode longitudinal vibrations of a rod of length L (a) clamped at the middle to produce the first normal mode and (b) clamped at a distance $L/4$ from one end to produce the second normal mode. Note that the dashed lines represent amplitudes parallel to the rod (longitudinal waves)

Two-dimensional oscillations can be set up in a flexible membráne stretched over a circular hoop, such as that in a drumhead. As the membrane is struck at some point, wave pulses that arrive at the fixed boundary are reflected many times. The resulting sound is not harmonic because the oscillating drumhead and the drum's hollow interior together produce a set of standing waves having frequencies that are *not* related by integer multiples. Without this relationship, the sound may be more correctly described as *noise* than as music. This is in contrast to the situation in wind and stringed instruments which produce sounds that we describe as musical.

Some possible normal modes of oscillation for a two-dimensional circular membráne are shown in Figure 2.2.16. The lowest normal mode, which has a frequency f , contains only one nodal curve; this curve runs around the outer edge of the membráne. The other possible normal modes show additional nodal curves that are circles and straight lines across the diameter of the membrane.

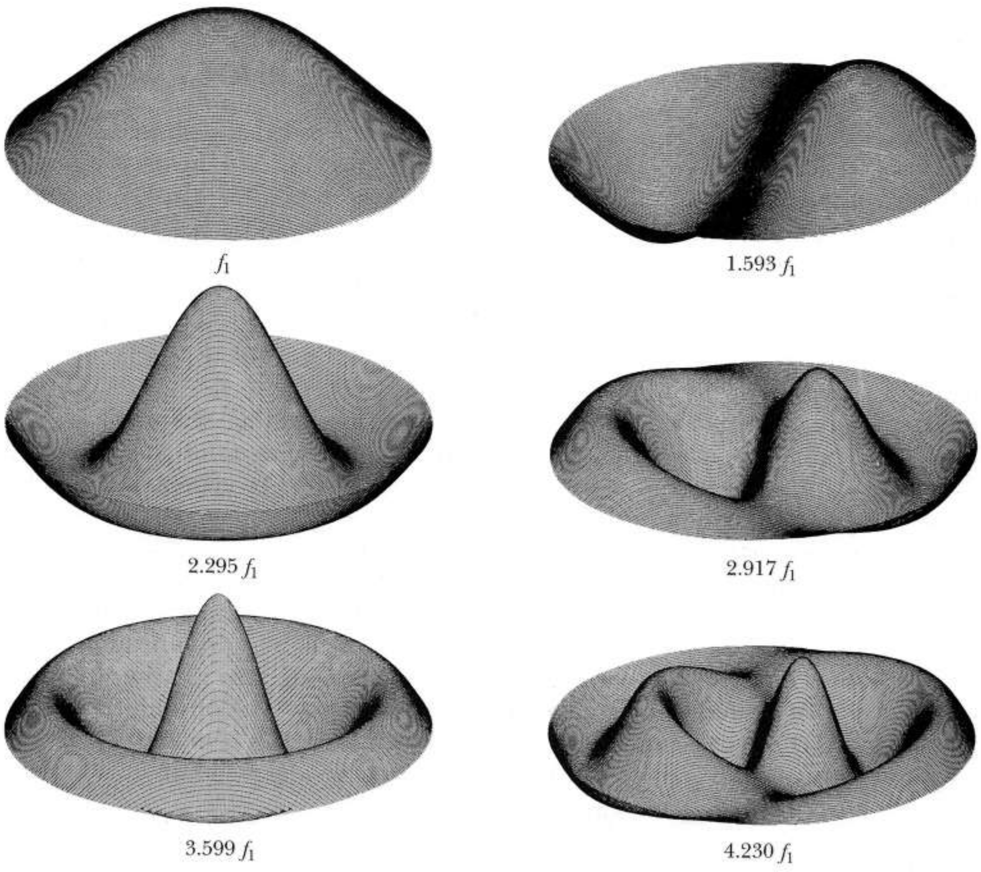


Figure 2.2.16 Representation of some of the normal modes possible in a circular membrane fixed at its perimeter. The frequencies of oscillation do not form a harmonic series

Exercises

2.2.37. An aluminum rod is clamped one quarter of the way along its length and set into longitudinal vibration by a variable-frequency driving source. The lowest frequency that produces resonance is 4400 Hz. The speed of sound in aluminum is 5100 m/s. Determine the length of the rod.

2.2.38. An aluminum rod 1.60 m in length is held at its centre. It is stroked with a rosin-coated cloth to set up a longitudinal vibration. (a) What is the fundamental frequency of the waves established in the rod? (b) What harmonics are set in the rod held in this manner? (c) What would be the fundamental frequency if the rod were made of copper?

2.2.39. A 60.0-cm metal bar that is clamped at one end is struck with a hammer. If the speed of longitudinal (compressional) waves in the bar is 4,500 m/s, what is the lowest frequency with which the struck bar resonates?

2.2.9 Beats: Interference in Time

The interference phenomena with which we have been dealing so far involve the superposition of two or more waves having the same frequency. As the resultant wave depends on the coordinates of the disturbed medium, we refer to the phenomenon as *spatial interference*. Standing waves in strings and pipes are common examples of spatial interference.

We now consider another type of interference, the one that results from the superposition of two waves having slightly different frequencies. In this case, when two waves are observed at the point of superposition, they are periodically in and out of phase. That is, there is a *temporal* (time) alternation between constructive and destructive interference. Thus, we refer to this phenomenon as *interference in time,* or *temporal interference*. For example, if two tuning forks of slightly different frequencies are struck, one hears a sound of periodically varying intensity. This phenomenon is called *beating*:

Beating is the periodic variation in intensity at a given point due to the superposition of two waves having slightly different frequencies.

The number of intensity maxima one hears per second, or the *beat frequency,* equals the difference in frequency between the two sources, as we shall show below. The maximum beat frequency that the human ear can detect is about 20 beats/s. When the beat frequency exceeds this value, the beats blend indistinguishably with the compound sounds producing them.

A piano tuner can use beats to tune a stringed instrument by "beating" a note against a reference tone of known frequency and then adjust the string tension until the frequency of the sound it emits equals the frequency of the reference tone. The tuner does this by tightening or loosening the string until the beats produced by it, and the reference source become too infrequent to notice.

Consider two sound waves of equal amplitude traveling through a medium with slightly different frequencies f_1 and f_2. We use equations similar to Eq. (2.1.7) to represent the wave functions for these two waves at a point that we choose as $x = 0$:

$$y_1 = A\cos \omega_1 t = A\cos 2\pi f_1 t,$$
$$y_2 = A\cos \omega_2 t = A\cos 2\pi f_2 t.$$

Using the superposition principle, we find that the resultant wave function at this point is

$$y = y_1 + y_2 = A(\cos 2\pi f_1 t + \cos 2\pi f_2 t.$$

The trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$$

allows us to write this expression in the form

$$y = 2A\left[\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t. \qquad (2.2.13)$$

Graphs of the individual waves and the resultant wave are shown in Figure 2.2.17. From the factors in Eq. (2.2.13), we see that the resultant sound for a listener standing at any given point has an effective frequency equal to the average frequency $(f_1 + f_2)/2$ and an amplitude given by the expression in the square brackets:

$$A_{result} = 2A\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t. \qquad (2.2.14)$$

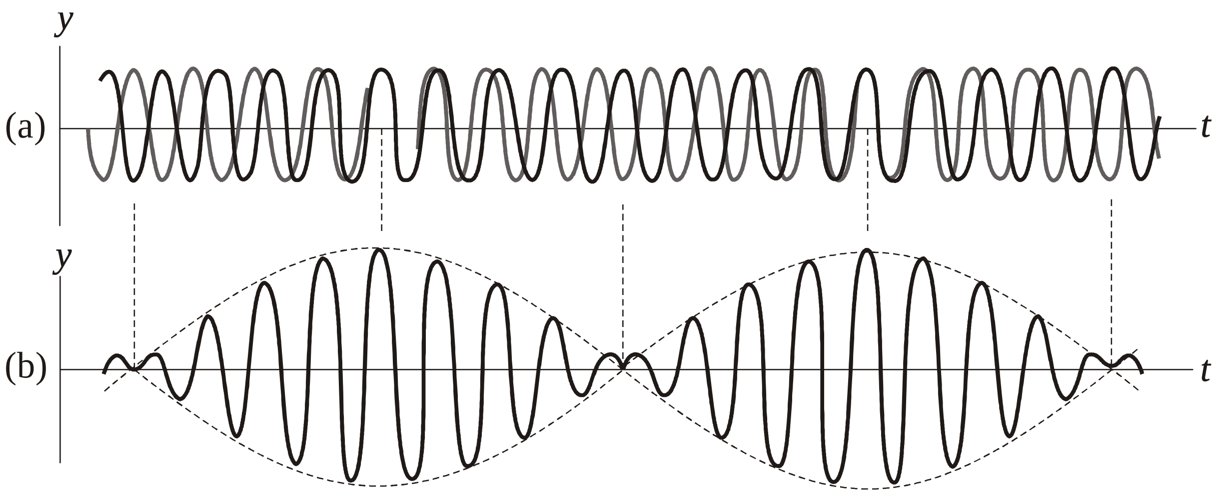


Figure 2.2.17 Beats are formed by the combination of two waves of slightly different frequencies. (a) The individual waves. (b) The combined wave has an amplitude (broken line) that oscillates in time

That is, the amplitude and therefore the intensity of the resultant sound vary in time. The broken line in Figure 2.2.17b is a graphical representation of Eq. (2.2.14) and is a sine wave varying with frequency $(f_1 - f_2)/2$.

Note that a maximum in the amplitude of the resultant sound wave is detected whenever

$$\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pm 1 .$$

This means there are *two* maxima in each period of the resultant wave. Because the amplitude varies with frequency as $(f_1 - f_2)/2$, the number of beats per second, or the beat frequency f_b, is twice this value. That is,

$$f_b = \left| f_1 - f_2 \right| \qquad (2.2.15)$$

For instance, if one tuning fork vibrates at 438 Hz and a second one vibrates at 442 Hz, the resultant sound wave of the combination has a frequency of 440 Hz (the musical note A) and a beat frequency of 4 Hz. A listener would hear a 440-Hz sound wave go through an intensity maximum four times every second.

Example 2.2.5

You wish to tune the note A_3 on a piano to its proper frequency of 220 Hz. You have available a tuning fork of frequency is 440 Hz. How should you proceed?

Solution.

Two frequencies are too far apart to produce beats and the piano string will oscillate not only in its fundamental mode (at 220 Hz when tuned) but also in its second harmonic mode (at 440 Hz when in tune). Thus, with the string somewhat out of tune, the frequency of its second harmonic will beat against the 440 Hz of the tuning fork. To tune the string, you can listen for those beats and then either tighten or loosen the string to decrease the beat frequency until the beating disappears.

Exercises

2.2.40. In certain ranges of a piano keyboard, more than one string is tuned to the same note to provide extra loudness. For example, the note at 110 Hz has two strings that vibrate at this frequency. If one string slips from its normal tension of 600 N to 540 N, what beat frequency is heard when the hammer strikes the two strings simultaneously?

2.2.41. While attempting to tune the note C at 523 Hz, a piano tuner hears 2 beats/s between a reference oscillator and the string. (a) What are the possible frequencies of the string? (b) When she tightens the string slightly, she hears 3 beats/s. What is the frequency of the string now? (c) By what percentage should the piano tuner now change the tension in the string to bring it into tune?

2.2.42. A student holds a tuning fork oscillating at 256 Hz. He walks toward a wall at a constant speed of 1.33 m/s. (a) What beat frequency does he observe between the tuning fork and its echo? (b) How fast must he walk away from the wall to observe a beat frequency of 5.00 Hz?

2.2.10 Energy in a Standing Wave

It is instructive to describe the energy associated with the particles of a medium in which a standing wave exists. Consider a standing wave formed on a taut string fixed at both ends, as shown in Figure 2.2.18. Except for the nodes, which are always stationary, all points on the string oscillate vertically with the same frequency but with different amplitudes of simple harmonic motion. Figure 2.2.18 represents snapshots of a standing wave at various times over one half of a period.

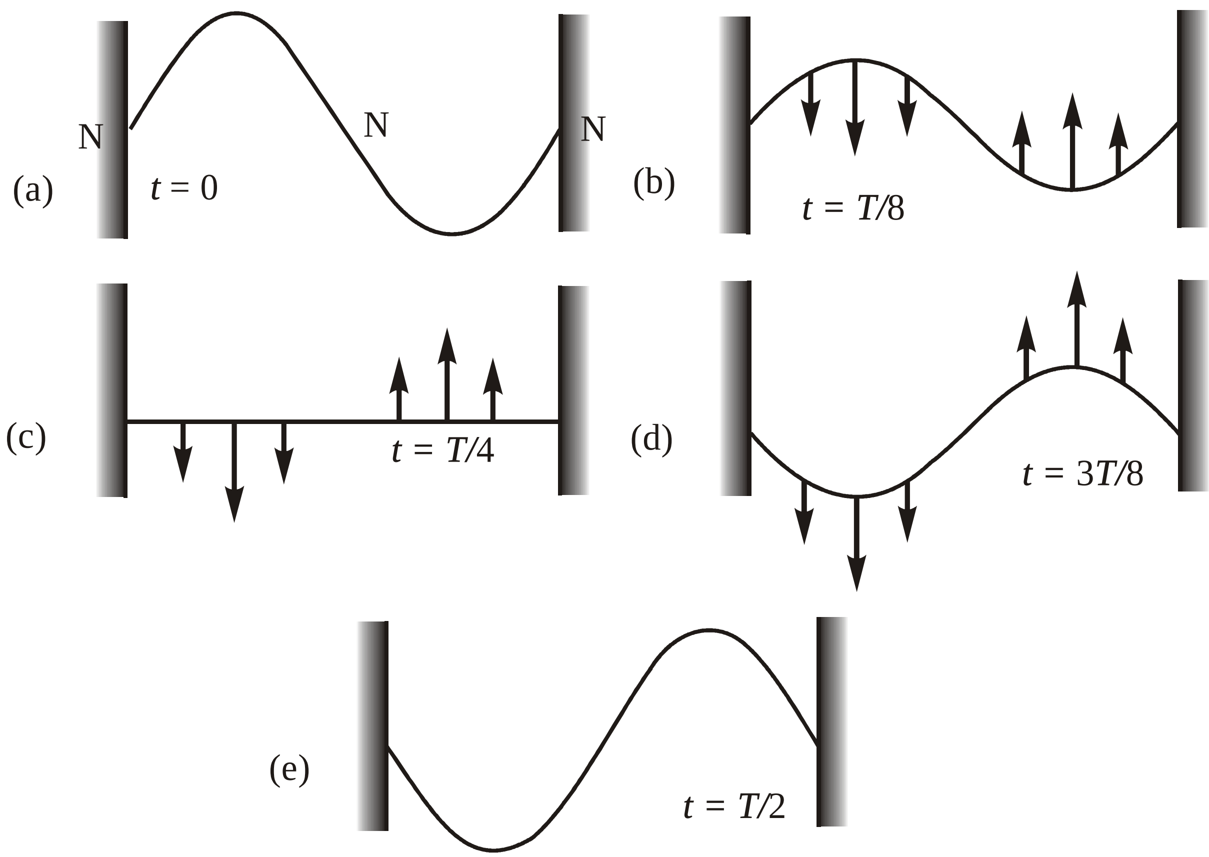


Figure 2.2.18 A standing-wave pattern in a taut string. The five "snapshots" were taken at half-cycle intervals. (a) At $t = 0$, the string is momentarily at rest; thus, $K = 0$, and all the energy is potential energy U associated with the vertical displacements of the string particles. (b) At $t = T/8$, the string is in motion, as indicated by the arrows, and the energy is half kinetic and half potential. (c) At $t = T/4$, the string is moving but horizontal (undeformed); thus, $U = 0$, and all the energy is kinetic. (d) The motion continues as indicated. (e) At $t = T/2$, the string is again momentarily at rest, but the crests and troughs of (a) are reversed. The cycle continues until ultimately, when a time interval equal to T has passed, the configuration shown in (a) is repeated

In a traveling wave energy is transferred along with the wave. We can imagine this transfer to be due to work done by one segment of the string on the next segment. As one segment moves upward, it exerts a force on the next segment, moving it through a displacement – that is, work is done. A particle of the string at a node, however, experiences no displacement. Thus, it cannot do work on the neighboring segment. As a result, no energy is transmitted along the string across a node, and energy does not propagate in a standing wave. For this reason, standing waves are often called *stationary* waves.

The energy of the oscillating string continuously alternates between elastic potential energy, when the string is momentarily stationary (see Figure 2.2.18a), and kinetic energy, when the string is horizontal and the particles have their maximum speed (see Figure 2.2.18c). At intermediate times (see Figure 2.2.18b and d), the string particles have both potential energy and kinetic energy

Exercises

2.2.43. Show that the maximum kinetic energy in each loop of a standing wave produced by two travelling waves of identical amplitudes is $2\pi^2\mu y_{\max}^2 f v$.

2.2.44. Two pulses travel along a string in opposite directions. The wave speed v is 2.0 m/s and the pulses are 6.0 cm apart at $t = 0$. In what form (or type) is the energy of the pulses at $t = 15$ ms?

Summary

A wave that reaches a boundary of the medium in which it propagates is reflected. At any point where the initial and reflected waves overlap, the total wave displacement is the sum of the displacements of the individual waves; this statement is the principle of superposition.

When two traveling waves having equal amplitudes and constant phase difference superimpose, the resultant wave has an amplitude that depends on the phase angle ϕ between the two waves. Constructive interference occurs when the two waves are in phase, corresponding to $\phi = 0, 2\pi, 4\pi, ..$ rad. Destructive interference occurs when two waves are I80º out of phase, corresponding to $\phi = \pi, 3\pi, 5\pi, ...$ rad.

Standing waves are formed from the superposition of two sinusoidal waves having the same frequency and amplitude but traveling in opposite directions. The resultant standing wave is described by the wave function

$$y = (2A\sin kx)\cos\omega t\,.$$

Hence, the amplitude of the standing wave is $2A$, and the amplitude of the simple harmonic motion of any particle of the medium varies according to its position as $2A\sin kx$. The points of zero amplitude (nodes) occur at $x = n\lambda/2$ $(n = 0, 1, 2, 3, ..)$ The maximum amplitude points (called antinodes) occur at $x = n\lambda/4$, $(n = 1, 3, 5, ..)$. Adjacent antinodes are separated by a distance $\lambda/2$. Adjacent nodes are also separated by a distance $\lambda/2$.

When a wave is reflected from a fixed or free end of a stretched string, the incident and reflected waves combine to form a standing wave which does not appear to travel in either direction. Its pattern contains nodes and antinodes; adjacent nodes are spaced a distance $\lambda/2$ apart, as are adjacent antinodes.

When both ends of a string of length L are held, standing waves can occur only when L is an integer multiple of $l/2$; the corresponding possible frequencies are given by

$$f_n = n\frac{v}{2L} \qquad (n = 1, 2, 3,..).$$

Each frequency, with its associated vibration pattern is called a normal mode. The lowest frequency f_1 is called the fundamental frequency. In terms of the mechanical properties T and m of the string, the fundamental frequency is given by

$$f1 = \frac{1}{2L}\sqrt{\frac{T}{m}}, \qquad (n = 1, 2, 3,..).$$

Standing waves also occur in wave motion in pipes or tubes. A closed end is a displacement node and a pressure antinode; an open end is a displacement antinode and a pressure node.

For a pipe open at both ends, the normal-mode frequencies are given by

$$f_n = n\frac{v}{2L} \qquad (n = 1, 2, 3,..).$$

For a pipe open at one end and closed at the other, the normal-mode frequencies are

$$f_n = n\frac{v}{4L} \qquad (n = 1, 2, 3,..).$$

When two or more waves overlap in the same region of space, the resulting effects are called interference. The resulting amplitude can be either larger or smaller than the amplitude of each individual wave, depending on whether the waves are in phase or out of phase. If waves are in phase, the result is called reinforcement or constructive interference; if they are out of phase, it is called cancellation, or destructive interference.

When a periodically varying force is applied to a system having normal modes of vibration, the system vibrates with the same frequency as that of the force; this is called a forced oscillation. If the force frequency is equal or close to one of the normal-mode frequencies, the amplitude of the resulting forced oscillation can become very large; this phenomenon is called resonance.

Key Terms

boundary conditions – граничные условия
principle of superposition – принцип суперпозиции
interference - интерференция
node - узел
antinode - пучность
standing wave – стоячая волна
fundamental frequency – основная частота

Chapter 3

Electromagnetic Oscillations

We have studied the behavior of an RC circuit and that of an RL circuit. In both cases, the behavior is characterized by an exponential approach to some steady-state value. In this chapter, we will see how the electric charge q varies with time in a circuit made of an inductor L, a capacitor C, and a resistor R. From another point of view, we shall discuss how energy shuttles back and forth between the magnetic field of the inductor and the electric field of the capacitor, while it is being gradually dissipated as thermal energy in the resistor. In the absence of energy losses, the charges on the capacitor surge back and forth indefinitely. This process is called an *electromagnetic oscillation*.

We have discussed mechanical oscillations before. We saw how displacement x varies with time in a mechanical oscillating system made of a block of mass m, a spring of spring constant k, and a viscous or frictional element such as oil. We also saw how energy shuttled back and forth between the kinetic energy of the oscillating mass and the potential energy of the spring, gradually dissipated as thermal energy.

The parallel between mechanical and electromagnetic oscillations is exact, and the controlling differential equations are identical. Thus, there is no new mathematics to be learned; we can simply change the symbols and give our full attention to the physics of the process.

3.1 Oscillations in LC Circuit

We now examine the two-element circuit combination LC (Figure 3.1). We'll see that in this case, the charge, current, and potential difference do not decay exponentially with time but vary sinusoidally (with the period T and the angular frequency w). Such a circuit is called LC oscillator.

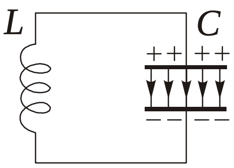


Figure 3.1 Resistanceless LC–circuit

From the energy standpoint (in the absence of energy losses), the oscillations of an electrical circuit consist of a transfer of energy back and forth from the electric field of the capacitor to the magnetic field of the inductor. The total energy associated with the circuit remains constant. This is analogous to the transfer of energy in an oscillating mechanical system from kinetic to potential, and vice versa, with the total energy remaining constant.

If we charge the capacitor C in LC circuit and connect it to inductor L, electromagnetic oscillations occur in the circuit. Succeeding stages of the oscillations in a simple LC circuit are shown in Figure 3.2. The energy stored in the electric field of the capacitor at any time is

$$U_E = \frac{q^2}{2C}, \tag{3.1}$$

where q is the charge on the capacitor at that time.

The energy stored in the magnetic field of the inductor at any time is

$$U_B = \frac{Li^2}{2}, \tag{3.2}$$

where i is the current through the inductor at that time.

From now in this chapter, we shall use small letters for representing *instantaneous values* of the electrical quantities of a sinusoidally oscillating circuit and capital letters for the *amplitudes* of those quantities. Assume that initially the charge q on the capacitor is at its maximum value Q and that the current i through the inductor is zero, state of the circuit is shown in Figure 3.2a. The bar graphs for energy indicate that at this instant, with zero current through the inductor and maximum charge on the capacitor, the energy U_B of the magnetic field is zero and the energy U_E of the electric field is a maximum.

The capacitor now starts to discharge through the inductor, positive charge carriers moving counterclockwise, as shown in Figure 3.2b. This means that a current i given by dq/dt is established. As the capacitor charge decreases, the energy stored in the electric field within the capacitor also decreases. This energy is transferred to the magnetic field that appears around the inductor because of the current i that is building up there. Thus, the electric field decreases and the magnetic field builds up as energy is transferred from the electric field to the magnetic field.

The capacitor eventually loses all its charge (Figure 3.2c) as well as its electric field and the energy stored in that field. The energy has been fully transferred to the magnetic field of the inductor. The magnetic field is at its maximum magnitude, and the current through the inductor is at its maximum value I.

Although the charge on the capacitor is now zero, the counterclockwise current must continue because the inductor does not allow it to change suddenly to zero. The current goes on to transfer positive charge from the top plate to the bottom plate through the circuit (Figure 3.2d). Energy now flows from the inductor back to the capacitor as the electric field within the capacitor builds up again. The current gradually decreases during this energy transfer. When the energy has been transferred completely back to the capacitor (Figure 3.2e), the current has become zero. The situation of Figure 3.2e is similar to the initial one, except that the capacitor is now charged oppositely. The capacitor starts to discharge again but now with a clockwise current (Figure 3.2f). Now the process repeats in opposite direction: We see that the clockwise current builds to a maximum (Figure 3.2g) and then decreases (Figure 3.2h), until the circuit eventually returns to its initial state (Figure 3.2a). The process repeats at some frequency f and, thus, at an angular frequency $w = 2pf$. In the ideal LC circuit with no resistance, there are no energy transfers other than that between the electric field of the capacitor and the magnetic field of the inductor. Owing to the conservation of energy, the oscillations continue indefinitely. It is worth to be mentioned that oscillations need not begin with the energy all in the electric field; the initial situation could be any other stage of the oscillation.

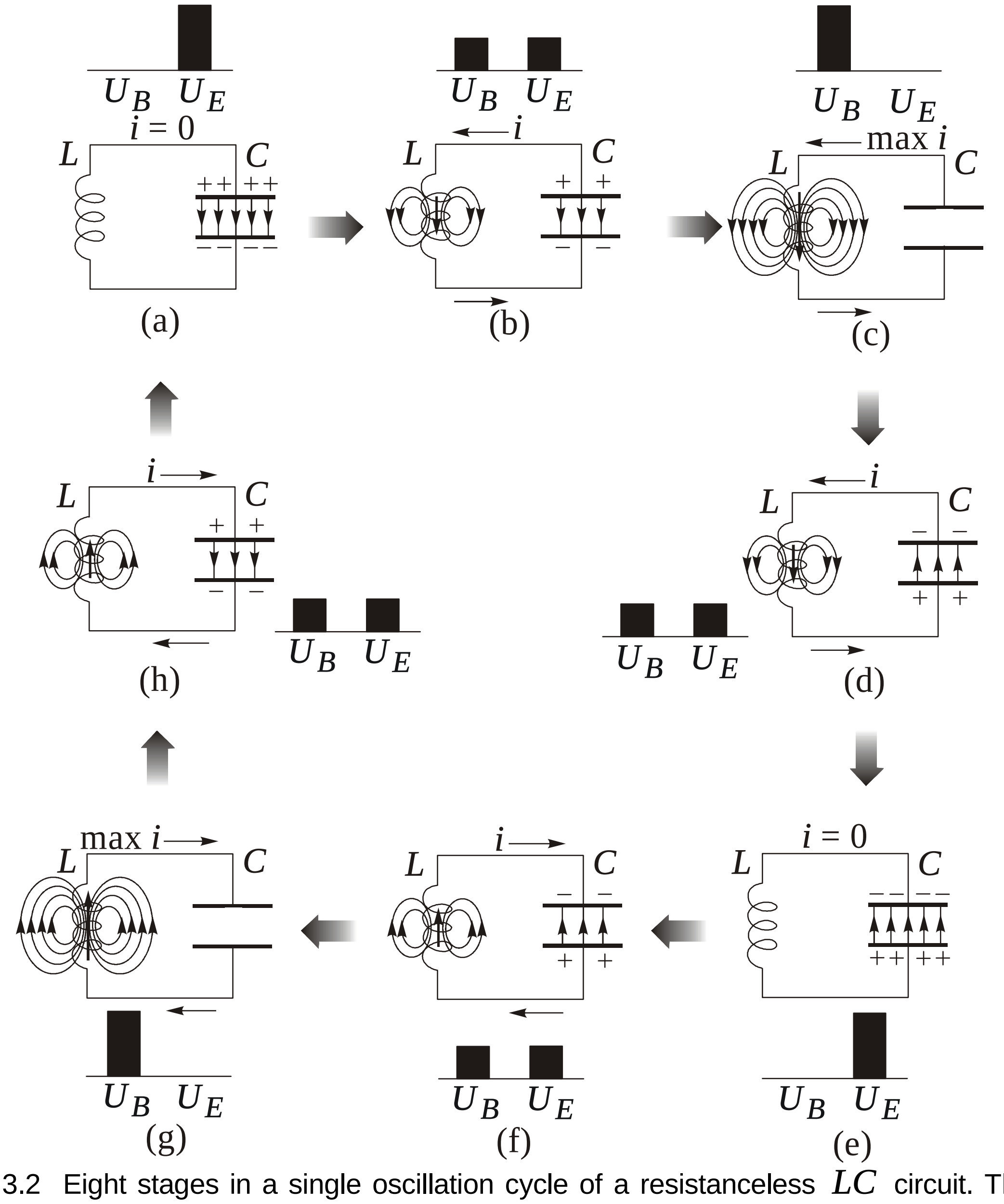


Figure 3.2 Eight stages in a single oscillation cycle of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electric energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (a) Capacitor with the maximum charge, no current. (b) Capacitor discharging, current increasing; (c) Capacitor fully discharged, current maximum. (d) Capacitor charging but with polarity opzposite that in (a), current decreasing. (e) Capacitor with maximum charge. having polarity opposite that in (a), no current; (f) Capacitor discharging, current increasing with direction opposite that in (b); (g) Capacitor fully discharged, current maximum. (h) Capacitor charging, current decreasing

Exercises

3.1. Why does an LC-circuit produce oscillations? (Ans. When a charged capacitor in an LC-circuit discharges through the inductor, the electric energy stored between the plates of capacitor appears as the magnetic energy inside the inductor. When the capacitor is discharged, the magnetic field linked with the inductor starts collapsing. Due to this, the induced EMF is produced in the inductor, the capacitor starts charging; and ultimately, the magnetic energy appears as the electric energy across the capacitor. This process repeats again and again, giving rise to LC-oscillations.)

3.2 Mathematical Description and Comparison with Mechanical Oscillations

Let us look a little closer at the analogy between the oscillating LC system and an oscillating block-spring system. Two kinds of energy are involved in the block-spring system. One is potential energy of the compressed or extended spring; the other is kinetic energy of the moving block.

Table 3.1 Analogies between electrical and mechanical systems

Mass on a Spring	Circuit Containing Inductance and Capacitance
Kinetic energy = $\frac{1}{2}mv^2$	Magnetic energy = $\frac{1}{2}Li^2$
Potential energy = $\frac{1}{2}kx^2$	Electrical energy = $\frac{q^2}{2C}$
$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$	$\frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C}$
$v = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$	$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$
$v = \frac{dx}{dt}$	$i = \frac{dq}{dt}$
$x = A\cos\sqrt{k/m}\,t = A\cos\omega t$	$q = Q\cos\sqrt{1/LC}\,t = Q\cos\omega t$
$v = -\omega A\sin\omega t = -v_{\max}\sin\omega t$	$i = -\omega Q\sin\omega t = -I\sin\omega t$

By looking across the table, we can see an analogy between the forms of the two pairs of energies – the mechanical energy of the block-spring system and the electromagnetic energies of the LC oscillator. The equations for v and i help us see the details of the analogy. They tell us that q corresponds to x, and i corresponds to v. These correspondences suggest that in the energy expressions, $1/C$ corresponds to k and L corresponds to m. Thus,

$$q \rightarrow x, \qquad \frac{1}{C} \rightarrow k,$$

$$i \rightarrow v, \qquad L \rightarrow m.$$

We have seen that the angular frequency of oscillation of a frictionless block-spring system is

$$\omega = \sqrt{\frac{k}{m}}. \qquad \text{(block-spring system)} \qquad (3.3)$$

The correspondences listed above suggest that to find the angular frequency of oscillations for a resistanceless LC circuit, k should be replaced by $1/C$ and m by L, yielding

$$\omega = \sqrt{LC}. \qquad (LC \text{ circuit}) \qquad (3.4)$$

We shell derive this result just now. We shell show explicitly that Eq. (3.4) for the angular frequency of LC oscillations is correct. At first we shell obtain differential equation of oscillation and expression for angular frequency ω for block-spring system, then do the same for LC circuit and compare the obtained results.

The Block-Spring Oscillator. We can write. for the total energy U of a block-spring oscillator at any instant

$$U = U_b + U_s = \frac{mv^2}{2} + \frac{kx^2}{2} \qquad (3.5)$$

where U_b and U_s are, respectively, the kinetic energy of the moving block and the potential energy of the stretched or compressed spring. If there is no friction – we assume, the total energy U remains constant with time even though v and x vary. In more formal language, $dU/dt = 0$. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{mv^2}{2} + \frac{kx^2}{2}\right) = mv\frac{dv}{dt} + kx\frac{dx}{dt}. \qquad (3.6)$$

However, $v = \dfrac{dx}{dt}$, and $\dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$. With these substitutions, Eq. (3.6) becomes

$$m\frac{d^2x}{dt^2} + kx = 0, \quad \text{or}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \qquad \text{(block-spring oscillations)} \qquad (3.7)$$

where $\omega^2 = k/m$. Eq. (3.7) is the fundamental ***differential equation*** that governs the frictionless block-spring oscillations. The general solution to Eq. (3.7) – that is, the function $x(t)$ that describes the block-spring oscillations – is

$$x = A\cos(\omega t + \varphi), \qquad \text{(displacement)}. \quad (3.8)$$

in which A is the amplitude of the mechanical oscillations, ω is the angular frequency of the oscillations, and φ is a phase constant.

The LC Oscillator. Now let us analyze the oscillations of a resistanceless LC circuit, proceeding exactly as we just did for the block-spring oscillator. The total energy U present at any instant in an oscillating LC circuit is given by

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C} \quad (3.9)$$

in which U_B is the energy stored in the magnetic field of the inductor and U_E is the energy stored in the electric field of the capacitor. Since we have assumed the circuit resistance to be zero, no energy is transferred into thermal energy and U remains constant with time. In more formal language, dU/dt must be zero. This leads to

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} \quad . \quad (3.10)$$

However, $i = \dfrac{dq}{dt}$ and $\dfrac{di}{dt} = \dfrac{d^2q}{dt^2}$. With these substitution Eq. (3.10) becomes

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0. \qquad (LC \text{ oscillations}) \quad (3.11)$$

This is the ***differential equation*** that describes the oscillations of a resistanceless LC circuit. If we denote $1/CL$ as ω^2, we obtain the same equation as Eq. (3.7), with the only difference that q instead of x.

$$\frac{d^2q}{dt^2} + \omega^2 q = 0. \quad (3.12)$$

We can obtain Eq. (3.11) from other standpoint: According to Kirhgoff's loop rule, the sum of potential differences in a circuit equals the sum of EMF in it. In our case of a resistanless circuit, the capacitor voltage $U_C = q/C$ must be equal to the EMF of the self-induction at the inductor $\varepsilon_S = -Ldi/dt$ at each instant of time. Hence,

$$\frac{q}{C} = -L\frac{di}{dt}. \quad (3.13)$$

As $i = dq/dt$, then $di/dt = d^2q/dt^2$. When we substitute this expression into Eq. (3.13) and rearrange the later, we obtain the same differential equation as Eq. (3.12):

$$\frac{d^2q}{dt^2} + \omega^2 q = 0,$$

which is analogous to Eq. (1.13) for the mechanical harmonic oscillator. The solutions of this differential equation, functions with their second derivative equal to $-1/LC$ times the original function, are

$$q = Q\cos\omega t, \tag{3.13a}$$

$$q = Q\sin\omega t, \tag{3.13b}$$

$$q = Q\cos(\omega t + \phi) \tag{3.13c}$$

where $\omega = 1/\sqrt{LC}$, and Q and ϕ are constants. Just as with the mechanical harmonic oscillator, the choice of one of these functions is determined by the initial conditions. If at time $t = 0$, the capacitor has maximum charge and $i = 0$, as in the discussion above, then we use Eq. (3.14a). If at $t = 0$, $q = 0$ but i is different from zero, we use Eq. (3.14b). And if both q and i are different from zero at time $t = 0$, the more general form, Eq. (3.14c), must be used. The striking parallel between the mechanical and electrical systems shown in Table 3.1 is only one of many such examples in physics. So close is the parallel between electrical and mechanical (and acoustical) systems that it is possible to solve complicated mechanical and acoustical problems by setting up analogous electrical circuits and measuring the currents and voltages that correspond to the mechanical and acoustical quantities to be determined. This is the basic principle of one kind of *analog computer*.

Exercises

3.2. Show that the angular frequency of free oscillations of an LC circuit is equal to $1/\sqrt{LC}$.

3.3. A charged 30 μF capacitor is connected to a 27 mH inductor. What is the angular frequency of free oscillations of the circuit?

3.4. An inductor having $L = 40$ mH is to be combined with a capacitor to make an LC circuit with the natural frequency of 2×10^6 Hz. What value of capacitance should be used?

3.5. The maximum capacitance of a variable air capacitor is 35 pF. What should be the inductance of a coil connected to this capacitor if the natural frequency of the LC circuit is to be 550×10^3 Hz, corresponding to one end of the AM radio broadcast band, when the capacitor is set to its maximum capacitance? The frequency at the other end of the broadcast band is 1550×10^3 Hz. What must the minimum capacitance of the capacitor be if the natural frequency is to be adjustable over the range of the broadcast band?

3.6. Show that differential equation (3.12) is satisfied by the function $q = Q\cos\omega t$, with ω given by $1/\sqrt{LC}$.

3.7. Calculate the wavelength of radio waves radiated out by a circuit consisting of 0.02 µF capacitor and a 8 µH inductor in series. (Ans. $\lambda = 7.54 \times 10^2$ m.)

3.8. Find the natural frequency of a circuit containing inductance of 100 µH and a capacity of 0.01 µF. To which wavelength its response will be maximum? For how long the oscillations will continue? (Ans. 159.15 kHz; 1884.96 m; forever.)

3.9. A coil of inductance of 0.4 mH is connected to a capacitor of capacitance 400 pF. To what wavelength is the circuit tuned? (Ans. 753.77 m.)

3.10. A 20 µF capacitor is charged to 30 V potential. Then the battery is disconnected and a 200 mH coil is connected across it, so that LC-oscillations are set up. Calculate the frequency of oscillations set up and the maximum current in coil. (Ans. 79.6 Hz; 0.3 A.)

3.3 Charge, Voltage, and Current Oscillations

We can write the general solution of Eq. (3.11) as

$$q = Q\cos(\omega t + \varphi) \text{ (charge)} \qquad (3.15)$$

where Q is the amplitude of the charge variations, $(\omega t + \varphi)$ is the phase of oscillations, ω is the angular frequency, and φ is the phase constant.

The phase of oscillation $(\omega t + \varphi)$ determines the state of oscillating system, that is, the charge on the capacitor at any instant of time. For example, at some moment of time t_1 the phase of oscillation is $(\omega t_1 + \varphi) = 2\pi$. It means that at this time the charge at the capacitor is maximum ($\cos 2\pi = 1$). If in time t_2, the phase equals $\pi/3$, then $q_2 = Q/2$, that is, the capacitor is partially discharged and has half of its maximum charge. Phase of oscillation has a lot of information. If we are given the charge q, then we know the degree of charging or discharging of the capacitor but know nothing about the direction of the process. But if we are given the phase of oscillation, i.e. $(\omega t_1 + \varphi) = 7\pi/4$, then we know the charge on the capacitor ($q = \sqrt{2}Q/2$) and direction of the oscillation process – the capacitor is charging. (For a given instant, with time increasing, the phase increases, cosine function and charge increase as well.)

At the capacitor voltage oscillates in the same manner as its charge:

$$v = \frac{q}{C} = \frac{Q\cos(\omega t + \varphi)}{C} = V\cos(\omega t + \varphi) \text{ (voltage)} \qquad (3.16)$$

where the amplitude value of voltage $V = Q/C$.

Taking the first derivative of Eq. (3.15) with respect to time we get the current of the LC oscillator:

$$i = \frac{dq}{qt} = -wQ\sin(wt + f) \quad . \qquad \text{(current)} \quad (3.17)$$

The amplitude I of this sinusoidally varying current is

$$I = wQ, \qquad (3.18)$$

so we can rewrite Eq. (3.17) as

$$i = -I\sin(wt + f). \qquad (3.19)$$

From Eqs. (3.15), (3.16) and (3.19), it is clear that oscillations of charge and voltage lag the oscillation of current by $p/2$, that is, when current reaches its maximum value I, charge and voltage become equal to zero and vice versa.

We can test whether Eq. (3.15) is a solution of Eq. (3.11) by substituting it and its second derivative with respect to time into Eq. (3.11). The first derivative of Eq. (3.15) is Eq. (3.17). The second derivative is

$$\frac{d^2q}{qt^2} = -w^2Q\cos(wt + f).$$

Substituting them for q and d^2q/dt^2 into Eq. (3.11), we obtain

$$-Lw^2Q\cos(wt + j) + \frac{1}{C}Q\cos(wt + f) = 0.$$

The canceling the $Q\cos(wt + f)$ and its rearrangement lead to

$$w = \frac{1}{\sqrt{LC}}.$$

Thus, Eq. (3.15) is indeed a solution of Eq. (3.11) if w has the value $\dfrac{1}{\sqrt{LC}}$. In Eq. (3.15), the phase constant f is determined by the conditions at any certain time, say, $t = 0$. If the conditions yield $f = 0$ at $t = 0$, Eq. (3.15) requires that $q = Q$, and Eq. (3.17) requires that $i = 0$; these are the initial conditions represented by Figure 3.2a.

Example 3.1

The 9-pF capacitor is charged by the voltage of 12 V and then directly connected across the 2.81 mH inductor. (a) Find the frequency of oscillation of the circuit.

Solution.

(a) The frequency f is

$$f = \frac{w}{2p} = \frac{1}{2p\sqrt{LC}} = \frac{1}{2p\sqrt{(2.81\times 10^{-3})(9\times 10^{-12})}} = 1\times 10^{6}\ \text{Hz}.$$

(b) What are the maximum values of charge on the capacitor and current in the inductor?

Solution.

The initial charge on the capacitor equals the maximum charge, and because $C = Q/V$, we have

$$Q = CV = (9 \times 10^{-12})(12) = 1.08 \times 10^{-10}\ \text{C}.$$

The maximum current is related to the maximum charge according to Eq. (3.18):

$$I = \omega Q = 2\pi f Q = (2\pi \times 10^{6})(1.08 \times 10^{-10}) = 6.79 \times 10^{-4}\ \text{A}.$$

(c) Determine the charge and current as functions of time.

Solution.

According to Eqs. (3.15) and (3.19)

$$q = Q\cos\omega t = (1.08 \times 10^{-10})\cos[(2\pi \times 10^{6})t],$$

$$i = (-6.79 \times 10^{-4})\sin[(2\pi \times 10^{6})t].$$

Exercises

3.11. A 1.00-µF capacitor is charged by a 40 V power supply. Then fully-charged capacitor is discharged through a 10 mH inductor. Find the maximum current in the resulting oscillations.

3.12. An LC circuit consists of a 20.0-mH inductor and a 0.5-µF capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

3.13. A fixed inductance $L = 1.05$ µH is used in series with a variable capacitor in the tuning section of a radio. What capacitance tunes the circuit to the signal from a station, broadcasting at 6.30 MHz?

3.14. Calculate the inductance of an LC circuit that oscillates at 120 Hz when the capacitance is 8 µH.

3.15. What are the dimensions of $\sqrt{LC}$? (Ans. Second.)

3.16. An inductor of inductance 2 mH is connected across a charged capacitor of capacitance 5 µF and the resulting LC-circuit is set oscillating at its natural frequency. It is found that the maximum value of charge on the capacitor is 200 µC. (a) When $q = 100$ µC, what is the value of $\left|\frac{di}{dt}\right|$? (b) When $q = 200$ µC, what is the value of i? (c) Find the maximum value of i. (d) When i is equal to one half its maximum value, what is the value of q? (Ans. (a) 10^{4} A/s, (b) 0, (c) 2 A, (d) 173.2 µC.)

3. 17. Capacitor of 9 pF is charged by 12 V battery and then is connected directly across the 2.81 mH inductor. (a) Find the frequency of oscillation of the circuit. (b) What are the maximum value of charge on the capacitor and current in the circuit? (c) What is the total energy stored in the circuit? [Ans. (a) $f = 1 \times 10^{6}$ Hz; (b) $I_{max} = 6.79 \times 10^{-4}$ A; (c) $U = 6.48 \times 10^{-10}$ J.]

3.18. An inductor of inductance L and a capacitor of capacitance C are connected in series. The current in the circuit increases linearly in time as described by $I = kt$. The capacitor is initially uncharged. Determine (a) the voltage across the inductor as a function of time; (b) The voltage across the capacitor as a function of time; and (c) The time when the energy stored in the capacitor first exceeds that in the inductor.

3.19. An LC circuit contains an 82-mH inductor and a 17-μF capacitor that initially carries a 180–μC charge. (a) Find the frequency (in hertz) of the resulting oscillations. At $t = 1$ ms, find (b) the charge on the capacitor and (c) the current in the circuit.

3.4 Electric and Magnetic Energy Oscillations

The electric energy stored in the LC circuit at any time at any t is, from Eqs. (3.1) and (3.15),

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(wt + f). \quad (3.20)$$

From Eqs. (3.2) and (3.17), the magnetic energy is,:

$$U_B = \frac{Li^2}{2} = \frac{Lw^2Q^2}{2}\sin^2(wt + f) = \frac{LI^2}{2}\sin^2(wt + f).$$

The total energy of the LC circuit is

$$U = U_E + U_B = \frac{Q^2}{2C}\cos^2(wt + f) + \frac{LI^2}{2}\sin^2(wt + f). \quad (3.21)$$

This expression contains all the features described qualitatively at the beginning of this section. It shows that the energy of the LC circuit continuously oscillates between the energy stored in the electric field of the capacitor and the energy stored in the magnetic field of the inductor. When the energy stored in the capacitor has its maximum value $Q^2/2C$, the energy stored in the inductor is zero. When the energy stored in the inductor has its maximum value $LI^2/2$, the energy stored in the capacitor is zero.

Plots of the time variations of U_E and U_B are shown in Figure 3.3. The sum $U_E + U_B$ is constant and equal to the total energy $Q^2/2C$ or $LI^2/2$.

Analytical verification of this is fact straightforward. The amplitudes of the two graphs in Figure 3.3 must be equal because the maximum energy stored in the capacitor (when $I = 0$) must equal the maximum energy stored in the inductor (when $Q = 0$). This is mathematically expressed as

$$\frac{Q^2}{2C} = \frac{LI^2}{2}. \quad (3.22)$$

After substituting this expression into Eq. (3.21) for the total energy, we get

$$U = \frac{Q^2}{2C}[\cos^2(wt + f) + \sin^2(wt + f)] = \frac{Q^2}{2C},$$

because $\cos^2(wt + f) + \sin^2(wt + f) = 1$.

In our idealized situation, the oscillations in the circuit persist indefinitely; however, we remember that the total energy U of the circuit remains constant only if energy transfers and transformations are neglected. In actual circuits, there is always some resistance, and, hence, energy is transformed into internal energy. We mentioned at the beginning of this section that we also ignore radiation from the circuit. In reality, radiation is inevitable in this type of circuit, and the total energy in the circuit continuously decreases as a result of this process.

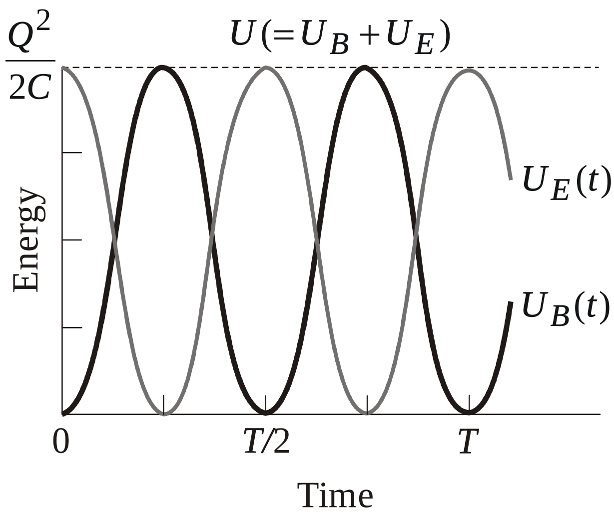


Figure 3.3 The stored magnetic energy and electric energy in the circuit of Figure 3.2 as a function of time. Note that their sum remains constant. T is the period of oscillation

Example 3.2

A 1.5 μF capacitor is charged to 57 V. Then the charging battery is disconnected, and a 12 mH coil is connected in series with the capacitor so that LC oscillations occur.

(a) What is the maximum current in the coil? Assume that the circuit contains no resistance.

Solution.

As the circuit contains no resistance, the electromagnetic energy of the circuit is conserved because the energy is transferred back and forth between the electric field of the capacitor and the magnetic field of the coil (inductor). Then, at any time t, the energy $U_B(t)$ of the magnetic field is related to the current $i(t)$ through the coil by Eq. (3.2): $U_B = \frac{Li^2}{2}$. When all the energy is stored as magnetic energy, the current is at its maximum value I and that energy is $U_B = \frac{LI^2}{2}$. Moreover, at any time t, the energy $U_E(t)$ of the electric field is related to the charge $q(t)$ on the capacitor by Eq. (3.1): $U_B = \frac{q^2}{2C}$. When all the energy is stored as electric energy, the charge is at its maximum value Q and that energy is $U_B = \frac{Q^2}{2C}$. Hence, we can write the conservation of energy as

$$\frac{LI^2}{2} = \frac{Q^2}{2C}.$$

Solving it for I we obtain

$$I = \sqrt{\frac{Q^2}{LC}}\,.$$

We know L and C but not Q. However, with $q = CV$, we can relate Q to the maximum potential difference V across the capacitor which is the initial potential difference of 57 V. Thus, the substitution of $Q = CV$ leads to

$$I = V\sqrt{\frac{C}{L}} = (57\ \text{V})\sqrt{\frac{1.5\times 10^{-6}\,\text{F}}{12\times 10^{-3}\,\text{H}}} = 0.637\,\text{A}\,.$$

(b) What is the potential difference $v_L(t)$ across the inductor as a function of time?

Solution.

We can apply the loop rule to this oscillating circuit. At any time t during the oscillations, the loop rule gives us

$$v_L(t) = v_C(t),$$

that is, the potential difference v_L across the inductor must always be equal to the potential difference v_C across the capacitor, so that the net potential difference around the circuit is zero. Thus, we shall find $v_L(t)$ if we can find $v_C(t)$, and we can find $v_C(t)$ from $q(t)$ with $q = Cv$ and $Q = CV$.

As the potential difference $v_C(t)$ is maximum when the oscillations begin at time $t = 0$, the charge q on the capacitor must also be maximum. Thus, the phase constant ϕ must be zero, so

$$q = Q\cos\omega t\,.$$

Note that this cosine function does indeed yield maximum $q = Q$ when $t = 0$. To get the potential difference $v_C(t)$, we divide both sides of the expression by C and obtain:

$$\frac{q}{C} = \frac{Q}{C}\cos\omega t\,,$$

or $v_C = V_C\cos\omega t$. Here, V_C is the amplitude of the oscillations in the potential difference v_C across the capacitor. As $v_L = v_C$, we find

$$v_L = V_C\cos\omega t\,.$$

We can evaluate the right side of this equation by the first are noting that the amplitude V_C is equal to the initial (maximum) potential difference of 57 V across the capacitor. Then, using the values of L and C, we find ω:

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{[(0.012\,\text{H})(1.5\times 10^{-6}\,\text{F})]} = 7454\,\text{rad/s} \approx 7500\ \text{rad/s}.$$

Thus, potential difference across inductor becomes

$$v_L = (57\,\mathrm{V})\cos(7500\,\mathrm{rad/s})t\,.$$

(c) What is the maximum rate $(di/dt)_{\max}$ at which the current i changes in the circuit?

Solution.

The current is $i = \dfrac{dq}{dt} = -\omega Q\sin\omega t$. Then

$$\frac{di}{dt} = \frac{d}{dt}(-\omega Q\sin\omega t) = -\omega^2 Q\cos\omega t\,.$$

We can simplify this equation by substituting CV_C for Q (because we know C and V_C but not Q and $1/\sqrt{LC}$ for ω). We get

$$\frac{di}{dt} = -\frac{1}{LC}CV_C\cos\omega t = -\frac{V_L}{L}\cos\omega t\,.$$

This tells us that the current changes at a sinusoidal rate, with its maximum rate of change being

$$\frac{V_L}{L} = \frac{57V}{0.012H} = 4750\ \text{A/m.}$$

Exercises

3.20. An LC-circuit contains a 20 mH inductor and a 50 µF capacitor with an initial charge of 10 mC. The resistance of the circuit is negligible. Let the instant at which circuit is closed be $t = 0$. (a) What is the total energy stored initially? Is it conserved during the LC oscillations (Ans. (a) 1 J, yes)? (b) What is the natural frequency of the circuit? (Ans. 159.15 Hz) (c) When is the energy stored that is: (i) completely electrical (i.e. stored in the capacitor)? (Ans. $t = 0, T, T/2, 3T/2, \ldots$, where $T = 2.28\times 10^{-3}$ s); (ii) completely magnetic (i.e. stored in the inductor)? (Ans. $t = T/4,\ 3T/4, 5T/4, \ldots$); (d) When is the total energy shared equally between the inductor and the capacitor? (Ans. $t = T/8,\ 3T/8, 5T/8, \ldots$, where $T = 2.28\times 10^{-3}$ s.)

3.21. Show that in the free oscillations of an LC circuit, the sum of energies stored in the capacitor and the inductor is constant in time.

3.5 Damped Oscillations in an RLC circuit

In our discussion of the LC circuit, we did not include any resistance. This omission is an idealization, of course; every real inductor has resistance in its windings, and there may also be resistance in the connecting wires. The effect of resistance is to dissipate the electromagnetic energy in the circuit and convert it into heat; thus, resistance plays a role in an electric circuit analogous to that of friction in a mechanical system.

To study this situation in greater detail, we consider an inductor with inductance L and a resistor of resistance R connected in series across the terminals of a charged capacitor C. (Figure 3.4). A circuit containing resistance, inductance, and capacitance is called an RLC circuit. As before, the capacitor starts to discharge as soon as the circuit is completed, but, because of i^2R losses in the resistor, there is less energy in the inductor when the capacitor is completely discharged, than there was in capacitor originally. In the same way, the energy of the capacitor is still smaller, when the magnetic field has collapsed, and so on.

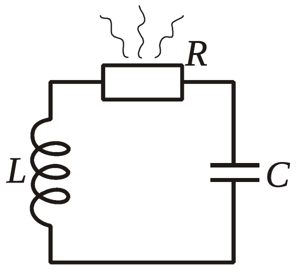


Figure 3.4 A series RLC circuit. As the charge contained in the circuit oscillates back and forth through the resistance, electromagnetic energy is dissipated as thermal energy, damping (decreasing the amplitude of) the oscillations

To analyze the oscillations of this circuit from energy standpoint, we write an equation for the total electromagnetic energy U in the circuit at any instant. Because the resistance does not store electromagnetic energy, we can use Eq. (3.9):

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}.$$

However, now, this total energy decreases as energy is transferred to thermal energy. The rate of this transfer is

$$\frac{dU}{dt} = -i^2R \tag{3.23}$$

where the minus sign indicates that U decreases. By differentiating Eq. (3.9) with respect to time and then substituting the result in Eq. (3.23), we obtain

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

Substituting dq/dt for i and d^2q/dt^2 for di/dt, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0,$$

$$\text{or}\quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0. \tag{3.24}$$

Eq. (3.24) is the differential equation for damped oscillations in an RLC circuit.

Again, as for the case of undumped oscillations, there is an alternative way to obtain the differential equation Eq. (3.24) of dumped oscillations: We apply Kirhgoff's loop rule to the circuit at Figure 3.4 and obtain the equation

$$iR + \frac{1}{C}q = -L\frac{di}{dt} \quad \text{or} \quad iR + L\frac{di}{dt} + \frac{1}{C}q = 0.$$

Replacing i with dq/dt and rearranging, we obtain the same equation as Eq. (3.24)

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 .$$

Note that when $R = 0$, this equation reduces to Eq. (3.11).

If the resistance R is relatively small, the circuit still oscillates, but with *damped oscillations,* as shown in Figure 3.5a. If we increase R, the oscillations die out more rapidly. When R reaches a certain value, the circuit no longer oscillates, and we say that it is *critically damped,* as in Figure 3.5b. For still larger values of R, the circuit is *overdamped,* as in Figure 3.5c.

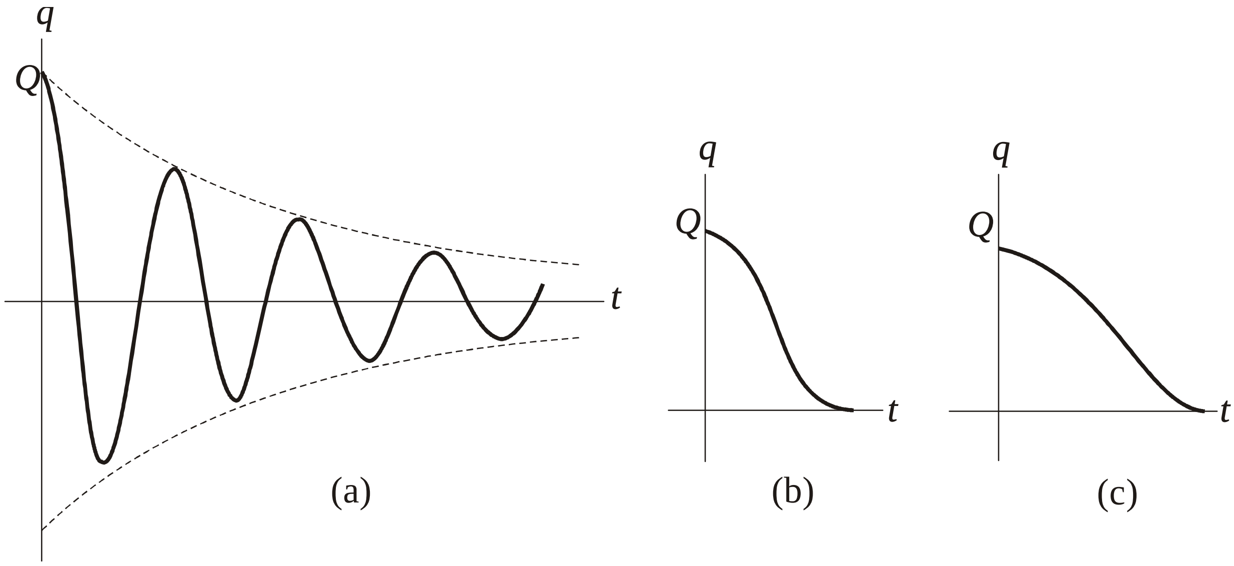


Figure 3.5 Graphs of q versus I in an RLC circuit: (a) Small damping; (b) Critically damped; (c) Overdamped

Solutions of Eq. (3.24) can be obtained by general methods of differential equations. The form of this solution depends on whether R is large or small. When R is less than $2(\sqrt{L/C})$, the solution has the form

$$q = Qe^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t\right) = Qe^{-(R/2L)t}\cos\omega' t . \qquad (3.25)$$

When R is greater than $2\sqrt{L/C}$, the solution is

$$q = e^{-(R/2L)t}\left[Ae^{\left(\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,t\right)} + Be^{\left(-\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,t\right)}\right], \qquad (3.26)$$

where A and B are constants determined by V, R, L, and C.

Eq. (3.25) corresponds to the ***underdamped*** behavior shown in Figure 3.4a; the function represents a sinusoidal oscillation with an exponentially decaying amplitude. Note that the angular frequency of the oscillation is no longer $1/\sqrt{L/C}$ but ***is less*** than this because of the term containing R. The frequency ω' of the damped oscillations is, thus, given by

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\omega^2 - \frac{R^2}{4L^2}}\,.$$

As R increases, ω' becomes smaller and smaller. When $R^2 = 4L/C$, the quantity under the radical becomes zero and the case of ***critical damping*** has been reached (Figure 3.5b). For still larger values of R, the behavior is no longer oscillatory but is described as the sum of two exponential functions, as in Figure 3.5c; the circuit is ***overdamped.***

We emphasize once more that this behavior is completely analogous to that of the damped harmonic oscillator studied in Chapter 1. Eq. (3.25) tells us how the charge on the capacitor oscillates in a damped RLC circuit; the equation is the electromagnetic counterpart of equation which gives the displacement of a damped block-spring oscillator. Similarly, the crossover point between underdamping and overdamping occurs at $b^2 = 4km$ for a mechanical system and at $R^2 = 4L/C$ for an electrical one.

Let us next find an expression for the total electromagnetic energy U of the circuit as a function of time. One way to do so is to monitor the energy of the electric field in the capacitor which is given by Eq. (3.1): $(U_E = q^2/2C)$. By substituting Eq. (3.25) into Eq. (3.1), we obtain

$$U = \frac{q^2}{2C} = \frac{\left[Qe^{-Rt/2L}\cos(\omega' t + \varphi)\right]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \varphi)\,. \tag{3.27}$$

Thus, the energy of the electric field oscillates according to a cosine-squared term and the amplitude of this oscillation decreases exponentially with time.

Example 3.3

A series RLC circuit has the inductance $L = 12$ mH, the capacitance $C = 1.6$ µF, and the resistance $R = 1.5$ Ω. (a) At what time t will the amplitude of the charge oscillations in the circuit be 50% of its initial value?

Solution.

(a) The amplitude of the charge oscillations decreases exponentially with time t. According to Eq. (3.25), the charge amplitude is $Qe^{-Rt/2L}$, at any time t, with Q being the amplitude at time $t = 0$. We want to obtain the time when the charge amplitude has decreased to $0.5Q$, that is, when

$$Qe^{-Rt/2L} = 0.5Q\,.$$

Canceling Q and then taking the natural logarithms of both sides, we have

$$-\frac{Rt}{2L} = \ln 0.5 .$$

Solution for t and then substitution of the given data yield

$$t = -\frac{2L}{R}\ln 0.5 = -\frac{(2)(12\times 10^{-3}\,\text{H})(\ln 0.5)}{1.5\,\Omega} = 0.011\,\text{s} = 11 \text{ ms}.$$

(b) How many oscillations are completed within this time?

Solution.

Time for one complete oscillation is the period $T = 2\pi/\omega'$ where the angular frequency for LC oscillations is given by

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\frac{1}{(12\times 10^{-3}\,\text{H})(1.6\times 10^{-6}\,\text{F})} - \frac{1.5^2}{4(12\times 10^{-3}\,\text{H})^2}} =$$

$$= \sqrt{5.21\times 10^7 - 47} \approx \sqrt{52.1\times 10^6} = 7.2\times 10^3 \approx \omega .$$

Thus, in the time interval $\Delta t = 0.011$ s, the number of complete oscillations N is:

$$N = \frac{\Delta t}{T} = \frac{\Delta t}{2\pi\sqrt{LC}} = \frac{0.011\,\text{s}}{2\pi[(12\times 10^{-3}\,\text{H})(1.6\times 10^{-6}\,\text{F})]^{1/2}} \approx 13 .$$

Thus, the amplitude decays by 50% in about 13 complete oscillations.

Exercises

3.22. Why does a real LC-circuit usually produce damped oscillations? (Ans. An inductor possesses a small resistance. Therefore, each time electric energy converts into magnetic energy during LC-oscillations, a small part of energy or dissipated as heat energy across the resistance of the inductor. As a result, the oscillations produced are damped in nature.)

3.23. In Figure 3.3, let $R = 7.6$ Ω, $L = 2.2$ mH, and $C = 1.8$ μF. (a) Calculate the frequency of the damped oscillation of the circuit; (b) What is the critical resistance?

3.24. Consider an LC circuit in which $L = 500$ mH and $C = 0.1$ μF. (a) What is the resonant frequency ω_0? If a resistance of 1 k Ω is introduced into this circuit, what is the frequency of the (damped) oscillations? What is the percent difference between two frequencies?

3.25. Electrical oscillations are initiated in a series circuit containing a capacitance C, an inductance L, and a resistance R. (a) If $R \ll \sqrt{4L/C}$ (weak damping), how much time elapses before the amplitude of the current oscillation falls to 50.0% of its initial value? (b) How long does it take the energy to decrease to 50.0% of its initial value?

3.26. If a resistor is inserted in the circuit, how much energy is eventually dissipated as heat? (Ans. When resistance is introduced, whole of the energy will be dissipated in the form of the heat. The introduction of resistance produces damped oscillations.)

3.27. What role does the resistance of inductor play in LC-circuit ? (Ans. Due to the resistance of the inductor, the LC-oscillations produced are damped one. It is because, during each oscillation, a part of electric energy is dissipated in the form of heat energy.)

3.6 Forced Oscillations

We have seen that once started, the charge, potential difference, and current in both undamped LC circuits and damped RLC circuits (with small enough R) oscillate at angular frequency $w = 1/\sqrt{LC}$. Such oscillations are said to be *free oscillations* (free of any external EMF), and the angular frequency w is said to be the circuit's *natural angular frequency*.

When the external alternating EMF

$$e = e_m \sin w_d t \qquad (3.28)$$

is connected to an RLC circuit, the oscillations of charge, potential difference, and current are said to be *driven oscillations* or *forced oscillations*. These oscillations always occur at the driving angular frequency w_d.

The oscillations in an RLC circuit will not dumped out if an external EMF device supplies enough energy to make up for the energy dissipated as thermal energy in the resistance R. Circuits in homes, offices, including countless RLC circuits, receive such energy from local power stations. The energy is supplied via oscillating EMFs and current is said to be *alternating current*, or AC for short.

$$i = I_m \sin(w_d t - f). \qquad (3.29)$$

(The nonoscillating current from a battery is said to be a *direct current*, or DC.)

These oscillating EMFs and current vary sinusoidally with time, reversing direction. (In Ukraine 100 times per second and thus having frequency– 50 Hz, in North America 120 times per second and thus frequency is – 60 Hz.)

At first sight this may seem to be a strange arrangement. We have seen that the drift speed of the conduction electrons in household wiring may typically be 10^{-5} m/s. If we now reverse their direction every (1/120) s, such electrons can move only about 3×10^{-7} m in a half-cycle. At this rate, a typical electron can drift past no more than about 10 atoms in the wiring before it is required to reverse its direction. How can the electron ever get anywhere? The answer is as follows: The conduction electrons do not have to “get anywhere”. When we say that the current in a wire is one ampere, we mean that charge passes through any plane cutting across that wire at the rate of one coulomb per second. The speed at which the charge carriers cross that plane does not matter directly: one ampere may correspond to many charge carriers moving very slowly or to a few moving very rapidly. Furthermore, the signal to the electrons to reverse directions – which originates in the alternating EMF is propagated along the conductor at a speed of light. All electrons, no matter where they are located, get their reversal instructions at about the same instant. Finally, we note that for many devices, such as lightbulbs and toasters, the direction of motion is unimportant as long as the electrons do move so as to transfer energy to the device via collisions with atoms in the device.

Whatever the natural angular frequency w of a circuit may be, forced oscillations of charge, current, and potential difference in the circuit always occur at the driving angular frequency w_d .

However, as you will later, the amplitudes of the oscillations very much depend on how close w_d is to w. When the two angular frequencies match – a condition known as resonance - the amplitude I of the current in the circuit is maximum.

3.7 Resistance in an AC Circuit

The simplest problem in the AC-circuit analysis consists of a resistor of resistance R, connected between the terminal of an AC generator with the alternating EMF. (Figure 3.6a). By the loop rule, we have

$$e = v_R .$$

This gives us

$$v_R = e_m \sin w_d t . \tag{3.30}$$

As the amplitude V_R of the alternating potential difference (or voltage) across the resistance is equal to the amplitude e_m of the alternating EMF, we can write this as

$$v_R = V_R \sin w_d t . \tag{3.31}$$

The instantaneous current i_R in the resistance is

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin w_d t \tag{3.32}$$

where $I_R = \dfrac{V_R}{R}$ is the amplitude of the current i_R in the resistance. We also see that the voltage amplitude and the current amplitude are related by

$$V_R = I_R R \qquad \text{(resistor)}. \tag{3.33}$$

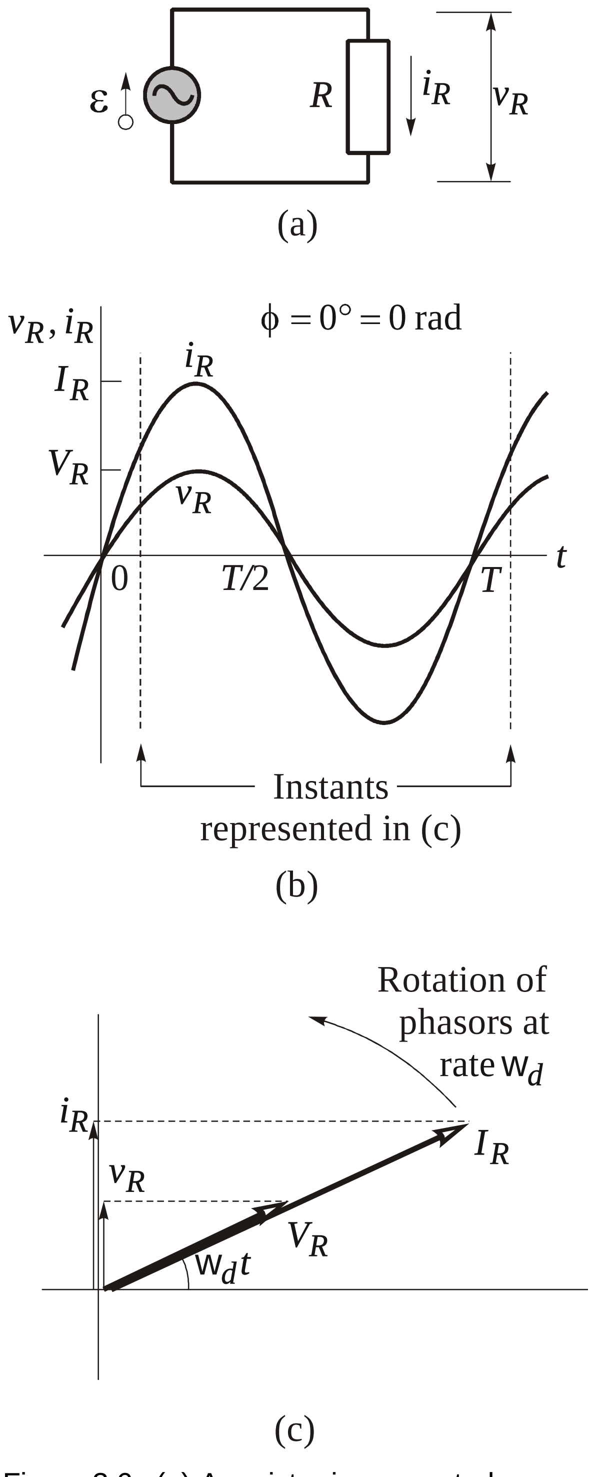


Figure 3.6 (a) A resistor is connected across an alternating-current generator; (b) The current i_R and the potential difference v_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T; (c) A phasor diagram shows the same thing as (b)

The current and voltage are both proportional to $\sin\omega_d t$. Thus, these two quantities are *in phase* which means that their corresponding maxima (and minima) occur at the same time. Figure 3.6b which is a plot of $v_R(t)$ and $i_R(t)$, illustrates this fact. Note that v_R and i_R do not decay here because the generator supplies energy to the circuit to make up for the energy dissipated in R.

Although we found this relation for the circuit of Figure 3.6a, it applies to any resistance in any AC circuit.

The time-varying quantities v_R and i_R can also be represented geometrically by phasors. Recall that phasors are vectors that rotate around an origin. Phasors that represent the voltage across and current in the resistor of Figure 3.6a are shown in Figure 3.6c at an arbitrary time t. Such phasors have the following properties:

Angular speed: Both phasors rotate counterclockwise about the origin with an angular speed equal to the angular frequency ω_d of v_R and i_R.

Length: The length of each phasor represents the amplitude of the alternating quantity: V_R for the voltage and I_R for the current.

Projection: The projection of each phasor on the *vertical* axis represents the value of the alternating quantity at time t: v_R for the voltage and i_R for the current.

Rotation angle: The rotation angle of each phasor is equal to the phase of the alternating quantity at time t. In Figure 3.6c, the voltage and current are in phase, so their phasors always have the same phase $\omega_d t$ and the same rotation angle, and thus they rotate together.

Mentally follow the rotation. Can you see that when the phasors have rotated so that $\omega_d t = 90°$ (they point vertically upward), they indicate that just then $v_R = V_R$ and $i_R = I_R$. Eqs. (3.31) and (3.33) give the same results.

Example 3.4

In Figure 3.6, the resistance R is 200 Ω, and the sinusoidal alternating EMF device operates at the amplitude $\mathscr{E}_m = 36$ V and the frequency $f_d = 60$ Hz. What is the potential difference $v_R(t)$ across the resistance, and what is the amplitude V_R of $v_R(t)$?

Solution.

If we apply loop rule to the circuit, we find that potential difference $v_R(t)$ across the resistance is always equal to the potential difference $\mathscr{E}(t)$ across the EMF device. Thus, $V_R = \mathscr{E}_m = 36$ V, and the potential difference $v_R(t)$

$$v_R(t) = \mathscr{E}(t) = \mathscr{E}_m \sin \omega_d t = 36 \sin(2\pi \times 60)t = 36 \sin(120\pi t).$$

3.8 Effective Values of Current and Voltage

In previous discussion, we have seen that, like the applied voltage, the current varies sinusoidally and has corresponding positive and negative values during each cycle. Thus, the sum of the instantaneous current values over one complete cycle is zero, and the average current is zero. The fact that the average current is zero, however, does not mean that the average power is zero and that there is no dissipation of electrical energy. As we know, joule heating is given by $i^2 R$ and depends on i^2 (which is always positive whether i is positive or negative) and not on i. Thus, there is joule heating and dissipation of electrical energy when an AC current passes through a resistor.

The instantaneous power dissipated in the resistor is

$$P = i^2 R = I^2 R \sin^2 \omega_d t. \tag{3.34}$$

The average value of P over a cycle is

$$\langle P \rangle = \langle i^2 R \rangle = \langle I^2 R \sin^2 \omega_d t \rangle. \tag{3.35}$$

Mathematically, the average value of a function $F(t)$ over a period T is given by

$$\langle F(t) \rangle = \frac{1}{T} \int_0^T F(t)\,dt, \tag{3.36}$$

where the $\langle\ \rangle$ denotes the average of the quantity inside the bracket. Since I^2 and R are constants,

$$\langle P\rangle = I^2R\left\langle \sin^2 \omega_d t\right\rangle .$$

Using the trigonometric identity $\sin^2 \omega_d t = \frac{1}{2}(1 - \cos 2\omega_d t)$, we have

$$\left\langle \sin^2 \omega_d t\right\rangle = \frac{1}{2}(1 - \langle\cos 2\omega_d t\rangle),$$

and since

$$\langle\cos 2\omega_d t\rangle = \frac{1}{T}\int_0^T \cos 2\omega_d t dt = -\frac{1}{T}\left[\frac{\sin 2\omega_d t}{2\omega_d}\right]_0^T =$$

$$= -\frac{1}{2\omega_d T}(\sin 2\omega_d T - 0) = 0,$$

Hence, we have $\left\langle \sin^2 \omega_d t\right\rangle = \frac{1}{2}$. Thus,

$$\langle P\rangle = \frac{1}{2}I^2R. \qquad (3.37)$$

To express AC power in the same form as DC power $P = i^2R$, a special value of current is used. It is called *root mean square* (rms), or *effective current* (Figure 3.7) and is denoted by I_{rms}. It is defined by

$$I_{rms} = \sqrt{\left\langle I^2\right\rangle} = \sqrt{\frac{1}{2}I^2} = \frac{I}{\sqrt{2}} = 0.707I. \qquad (3.38)$$

In terms of I_{rms}, the average power is

$$\langle P\rangle = \frac{1}{2}I^2R = I_{rms}^2R. \qquad (3.39)$$

Similarly, we define the *rms voltage,e* or *effective voltage,* by

$$V_{rms} = \frac{V}{\sqrt{2}} = 0.707V. \qquad (3.40)$$

From Ohm's law we have

$$V = IR,$$

or $\frac{V}{\sqrt{2}} = \frac{I}{\sqrt{2}}R$, or

$$V_{rms} = I_{rms}R. \qquad (3.41)$$

Eq. (3.41) gives us the relation between AC current and voltage and is similar to that in the DC case. This shows the advantage of introducing the concept of rms values. In terms of rms values, the equation for power relation between current and voltage in AC circuits are essentially the same as those for the DC case.

It is customary to measure and specify rms values for AC quantities. For example, the household line voltage of 220 V is an rms value with a peak voltage of

$$V = \sqrt{2} V_{rms} = (1.414)(220) = 311 \text{ V} .$$

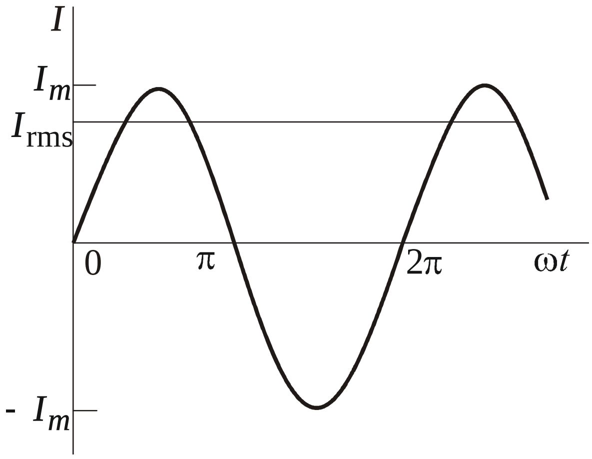


Figure 3.7 The rms current I_{rms} is related to the peak current I as

$$I_{rms} = \frac{I}{\sqrt{2}} = 0.707 I$$

In fact, rms current is the equivalent DC current that would produce the same average power loss as the alternating current. Eq. (3.39) can also be written as

$$\langle P \rangle = \frac{V_{rms}^2}{R} = I_{rms} V_{rms} . \qquad (3.42)$$

Example 3.5

A light bulb is rated at 100 W for a 220 V supply. Find: (a) The resistance of the bulb.

Solution.

(a) We are given $P = 100$ W and $V = 220$ V. The resistance of the bulb is

$$R = \frac{V_{rms}^2}{P} = \frac{(100 \text{ V})^2}{100 \text{ W}} = 484 \text{W}.$$

(b) The peak voltage of the source.

Solution.

The peak voltage of the source is

$$V_m = \sqrt{2} V_{rms} == 311 \text{ V} .$$

(c) The rms current through the bulb.

Solution.

Since $P = I_{rms} V_{rms}$, hence:

$$I_{rms} = \frac{P}{V_{rms}} = \frac{100 \text{W}}{220 \text{V}} = 0.45 \text{A} .$$

Exercises

3.28. A 100 Ω resistor is connected to a 220 V, 50 Hz ac supply. (a) What is the rms value of current in the circuit? (b) What is the net power consumed over a full cycle?

3.29. (a) The peak voltage of an AC supply is 300 V. What is the rms voltage? (b) The rms value of current in an ac circuit is 10 A. What is the peak current?

3.9. Capacitance in an AC Circuit

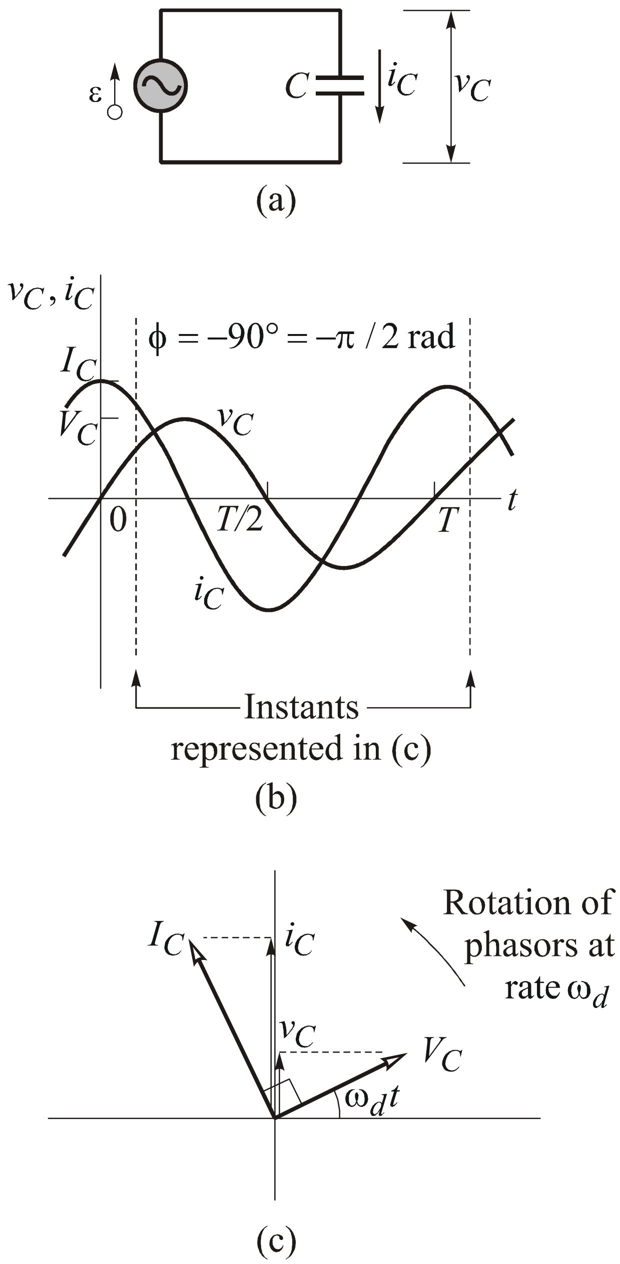


Figure 3.8 (a) A capacitor is connected across an alternating-current generator; (b) The current in the capacitor leads the voltage by 90° ;(c) A phasor diagram shows the same thing

Figure 3.8a shows a circuit containing a capacitance and a generator with the alternating EMF of Eq. (3.28). Using the loop rule, we find that the potential difference across the capacitor is

$$e = v_C = V_C \sin w_d t, \qquad (3.43)$$

where V_c is the amplitude of the alternating voltage across the capacitor.

From the definition of capacitance, we can also write

$$q_C = Cv_C = CV_C \sin w_d t. \qquad (3.44)$$

Now we are interested more in the current than in the charge. Thus, we differentiate Eq. (3.44) to find

$$i_C = \frac{dq_C}{dt} = w_d CV_C \cos w_d t. \qquad (3.45)$$

Then we modify Eq. (3.45) in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_C, called the *capacitive reactance* of a capacitor, defined as

$$X_C = \frac{1}{w_d C}. \qquad (3.46)$$

Its value depends not only on the capacitance but also on the driving angular frequency w_d. We know from the definition of the capacitive time constant ($t = RC$) that the SI unit for C can be expressed as seconds per ohm. By applying this to Eq. (3.46), we show that the SI unit for X_C is the *ohm,* just as for resistance R. Second, we replace $\omega_d t$ in Eq. (3.45) with a phase-shifted sine:

$$\cos\omega_d t = \sin(\omega_d t + 90°).$$

With these two modifications, Eq. (3.45) becomes

$$i_C = \left(\frac{V_C}{X_C}\right)\sin(\omega_d t + 90°).$$

We can also write the current i_C in C as

$$i_C = I_C \sin(\omega_d t + 90°), \qquad (3.47)$$

where I_C is the amplitude of i_C. Comparing Eqs. (3.43) and (3.47), we see that for a purely capacitive load, the phase constant ϕ for the current is -90°. We also see that the voltage amplitude and current amplitude are related by

$$V_C = I_C X_C \qquad (3.48)$$

Although we have found this relation for the circuit of Figure 3.8a, it applies to any capacitance in any ac circuit.

The comparison of Eqs. (3.43) and (4.47), or inspection of Figure 3.8b shows us that the quantities v_C and i_C are 90°, or one-quarter cycle, out of phase. Further we see that i_C *leads* v_C which means that, if you monitored the current i_C and the potential difference v_C in the circuit of Figure 3.8a, you will find that i_C reaches its maximum *before* v_C does, by one-quarter cycle.

This relation between i_C and v_C is illustrated by the phasor diagram of Figure 3.8c. As the phasors representing these two quantities rotate counterclockwise together, the phasor labeled I_C does indeed lead that labeled V_C by an angle of 90°; that is, the phasor I_C coincides with the vertical axis one-quarter cycle before the phasor V_C does. It is clear that the phasor diagram in Figure 3.8c is consistent with Eqs. (3.43) and (3.47).

The instantaneous power supplied to the capacitor is

$$P_C = i_C v_C = (I_C \cos\omega_d t)(V_C \sin\omega_d t) = I_C V_C (\cos\omega_d t)(\sin(\omega_d t) = \frac{I_C V_C}{2}\sin(2\omega_d t).$$

So, the average power is

$$\langle P_C \rangle = \left\langle \frac{IV}{2}\sin(2\omega t) \right\rangle = \frac{IV}{2}\langle \sin(2\omega t)\rangle = 0,$$

since $\sin(2\omega t) = 0$ over a complete cycle. The energy stored by a capacitor in each quarter period is returned to the source in the next quarter period.

Example 3.6

A 15.0 μF capacitor is connected to a 220 V, 50 Hz source. Find the capacitive reactance and the current (rms and peak) in the circuit. If the frequency is doubled, what happens to the capacitive reactance and the current?

Solution.

The capacitive reactance is

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(50\,\text{Hz})(1.5\times 10^{-6}\,\text{F})} = 212\,\Omega.$$

The rms current is

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{220\,\text{V}}{212\,\Omega} = 1.04\,\text{A}.$$

The peak current is

$$I_m = \sqrt{2} I_{rms} = (1.41)(1.04\,\text{A}) = 1.47\,\text{A}.$$

This current oscillates between +1.47 A and -1.47 A and is ahead of the voltage by 90°.

If the frequency is doubled, the capacitive reactance is halved and, consequently, the current is doubled.

Exercises

3.30. A 60 μF capacitor is connected to a 110 V, 60 Hz AC supply. Determine the rms value of the current in the circuit.

3.31. What is the reactance of a 1-μF capacitor at a frequency of 60 Hz? (Ans. 2.65×10^{3} Ω.)

3.32. What is the capacitance of a capacitor whose reactance is 1 Ω at 60 Hz? (Ans. 2.65×10^{-3} F)

3.33. What is the reactance of a 1-μF capacitor at a frequency of 60 Hz?

3.34. What is the capacitance of a capacitor whose reactance is 1 Ω at 60 Hz? (Ans. 2.65×10^{-3} F)

3.35. A 1-μF capacitor is connected across an AC source whose voltage amplitude is kept constant at 50 V but whose frequency can be varied. Find the current amplitude when the angular frequency (a) l00 rad/s; (b) l000 rad/s, (c) 10,000 rad/s. [Ans. (a) 5×10^{-3} A; (b) 5×10^{-2} A); (c) 0.5 A]

3.36. The voltage amplitude of an AC source is 50 V, and its angular frequency is 1000 rad/s. Find the current amplitude if the capacitance of a capacitor connected across the source is (a) 0.01 μF. (b) 1.0 μF, (c) 100 μF.

3.10 Inductance in an AC Circuit

Figure 3.9a shows a circuit containing an inductance and a generator with the alternating EMF of Eq. (3.28). Using the loop rule and proceeding as we did to obtain Eq. (3.30), we find that the potential difference across the inductance is

$$v_L = V_L \sin \omega_d t \tag{3.49}$$

where V_L is the amplitude of v_L. From loop rule we can write the potential difference across an inductance L, in which the current is changing at the rate di_L/dt, as

$$v_L = L\frac{di_L}{dt}. \qquad (3.50)$$

If we combine Eqs. (3.49) and (3.50), we have

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin\omega_d t. \qquad (3.51)$$

We are interested in the current rather than with its time derivative. We find the former by integrating Eq. (3.51), obtaining

$$i_L = \int di_L = \frac{V_L}{L}\int \sin\omega_d t\, dt =$$

$$= -\left(\frac{V_L}{\omega_d L}\right)\cos\omega_d t\, dt. \qquad (3.52)$$

Then we can now modify this equation in two ways. First, for reasons of symmetry of notation, we introduce the quantity X_L, called the *inductive reactance* of an inductor, which is defined as

$$X_L = \omega_d L \qquad (3.53)$$

The value of X_L depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the ohm just as it is for X_C and for R.

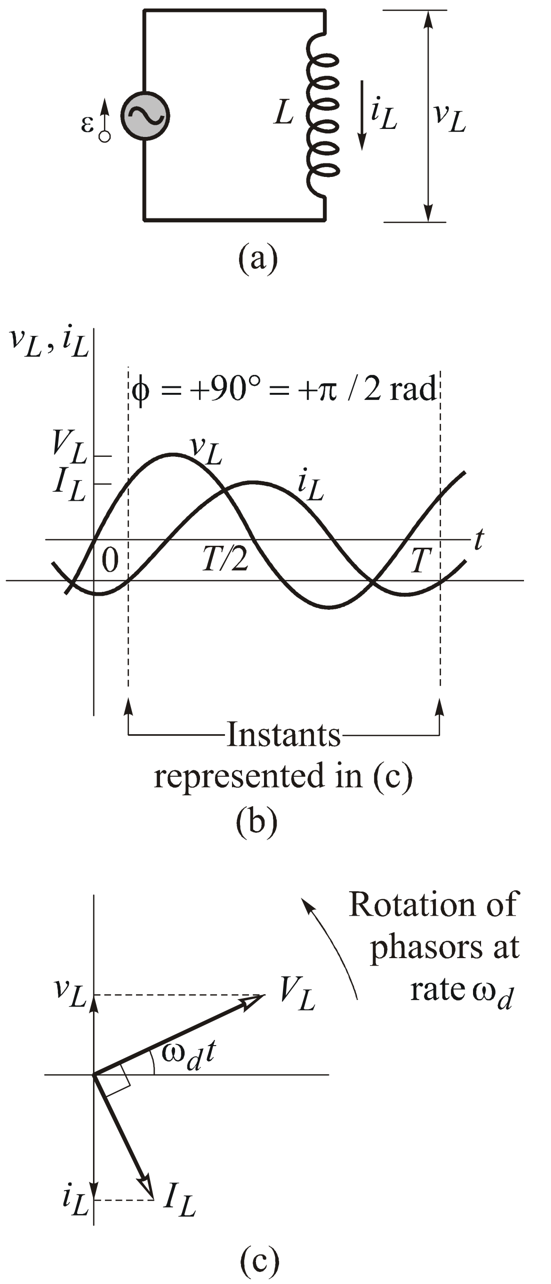


Figure 3.9 (a) An inductor is connected acrtoss an alternating current generator; (b) The current in the inductor lags the voltage by $90°$; (c) A phasor diagram shows the same thing

Second, we replace $-\cos\omega_d t$ in Eq. (3.52) with a phase-shifted sine:

$$-\cos\omega_d t = \sin(\omega_d t - 90°).$$

You can verify this identity by shifting a sine curve in the positive direction by 90°. With these two changes, Eq. (3.52) becomes

$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°). \qquad (3.54)$$

From Eq. (3.54), we can write this current in the inductance as

$$i_L = I_L \sin(\omega_d t - 90°), \qquad (3.55)$$

where I_L is the amplitude of the current i_L. Comparing Eqs. (3.49) and (3.55), we see that for a purely inductive load, the phase constant ϕ for the current is +90º. We also see that the voltage amplitude and current amplitude are related by

$$V_L = I_L X_L . \qquad (3.56)$$

Although we found this relation for the circuit of Figure 3.9a, it applies to any inductance in any AC circuit.

The comparison of Eqs. (3.49) and (3.55) or the inspection of Figure 3.9b, shows us that the quantities i_L and v_L are 90° out of phase. In this case, however, i_L lags v_L: that is, if you monitored the current i_L and the potential difference v_L in the circuit of Figure 3.9a, you would find that i_L reaches its maximum value *after* I_L does, by one-quarter cycle.

The phasor diagram of Figure 3.9c also contains this information. As the phasors rotate counterclockwise in the figure, the phasor labeled I_L does indeed lag that labeled V_L by an angle of 90°. It is clear that Figure 3.9c represents Eqs. (3.49) and (3.55).

We have seen that an inductor has reactance that limits current similar to resistance in a dc circuit.

The instantaneous power supplied to the inductor is

$$P_L = i_L v_L = I \sin(\omega t - \frac{\pi}{2}) V \sin(\omega t) = IV \cos(\omega t) \sin(\omega t) = -\frac{IV}{2} \sin(2\omega t) .$$

So, the average power over a complete cycle, as in the case of a capacitor, is

$$\langle P_L \rangle = \left\langle -\frac{IV}{2} \sin(2\omega t) \right\rangle = -\frac{IV}{2} \langle \sin(2\omega t) \rangle = 0 ,$$

since the average of $\sin(2\omega t)$ over cycle is zero. Thus, the average power supplied to an inductor over one complete cycle is zero.

Physically, this result means the following. During the first quarter of each current cycle, the flux through the inductor builds up and sets up a magnetic field and energy is stored in the inductor. In the following quarter of cycle, as the current decreases, the flux decreases, and the stored energy is returned to the source. Thus, in each half-cycle, the energy which withdrawn from the source is returned to it without any dissipation of power.

Example 3.7

A pure inductor of 25 mH is connected to a source of 220 V. Find the inductive reactance and rms current in the circuit if the frequency of the source is 50 Hz.

Solution.
The inductive reactance,

$$X_L = 2\pi f L = 2 \times 3.14 \times 50 \times 25 \times 10^{-3} = 7.85\,\Omega.$$

The rms current in the circuit is

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{220\,\text{V}}{7.85\,\Omega} = 28.03\,\text{A}.$$

Exercises

3.37. A 44 mH inductor is connected to a 220 V, 50 Hz AC supply. Determine the rms value of the current in the circuit.

3.38. What is the reactance of a 1-H inductor at a frequency of 60 Hz? (Ans. 377 Ω.)

3.39. What is the inductance of an inductor whose reactance is 1 Ω at 60 Hz? (Ans. 2.65×10^{-3} H.)

3.40. Compute the reactance of a 10-H inductor at frequencies of 60 Hz and 600 Hz.

3.41. At what frequency is the reactance of a 10-H inductor equal to that of a 10-μF capacitor?

3.42. An inductor of self-inductance 10 H and of negligible resistance is connected across the AC source whose voltage amplitude is kept constant at 50 V but whose frequency can be varied. Find the current amplitude when the angular frequency is (a) 100 rad/s; (b) 1000 rad/s; (c) 10,000 rad/s. [Ans. (a) 5×10^{-2} A; (b) 5×10^{-3} A; (c) 5×10^{-4} A.]

3.43. Find the current amplitude if the self-inductance of a resistanceless inductor connected across the source of AC source whose voltage amplitude is kept constant at 50 V but whose frequency can be varied, is (a) 0.01 H; (b) 1.0 H; (c) 100 H.

3.11 The Series RLC Circuit

We are now ready to apply the alternating EMF

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t \tag{3.57}$$

to the full RLC circuit of Figure 3.10. As $R, L,$ and C are in series, the same current

$$i = I \sin(\omega_d t - \phi) \tag{3.58}$$

is driven in all three of them. We wish to find (a) the current amplitude I and (b) the phase constant ϕ. The solution is simplified by the use of phasor diagrams.

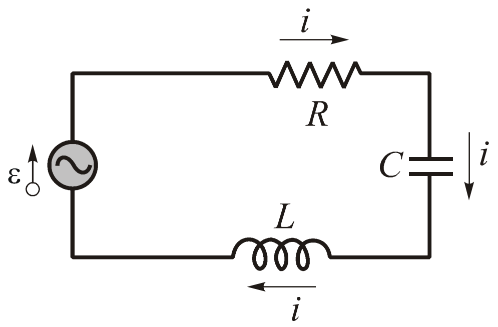


Figure 3.10 A single-loop circuit containing a resistor, a capacitor, and an inductor. A generator produces an alternating EMF that establishes an altemating current; the directions of the EMF and current are indicated here at only one instant

a) The Current Amplitude. We start with Figure 3.11a, which shows the phasor representing the current of Eq. (3.58) at an arbitrary time t. The length of the phasor is the current amplitude I, the projection of the phasor on the vertical axis is the current i at time t, and the angle of rotation of the phasor is the phase $w_d t - f$ of the current at time t.

Figure 3.11 shows the phasors representing the voltages across R, L, and C at the same time t. Each phasor is oriented relative to the angle of rotation of current phasor I in Figure 3.11a, based on the following information:

Resistor: Current and voltage are in phase, so the angle of rotation of the voltage phasor V_R is the same as that of the phasor I.

Capacitor: Current leads voltage by 90°, so the angle of rotation of voltage phasor V_C is 90° less than that of phasor I.

Inductor: Current lags voltage by 90°, so the angle of rotation of the voltage phasor V_L is 90° greater than that of the phasor I,

Figure 3.11b also shows the instantaneous voltages V_R, V_C, and V_L across R, L, and C at time t; those voltages are the projections of the corresponding phasors on the vertical axis of the figure.

Figure 3.11c shows the phasor representing the applied EMF of Eq. (3.57). The length of the phasor is the EMF amplitude e_m, the projection of the phasor on the vertical axis is the emf e at time t, and the angle of rotation of the phasor is the phase $w_d t$ of the EMF at time t.

From the loop rule, we know that at any instant, the sum of the voltages V_R, V_C, and V_L is equal to the applied EMF e

$$e = v_R + v_C + v_L . \tag{3.59}$$

Thus, at time t, the projection e_{emf} in Figure 3.11c is equal to the algebraic sum of the projections V_R, V_C, and V_L in Figure 3.I1b. In fact, as the phasors rotate together, this equality always holds. This means that phasor e_{emf} in Figure 3.11c must be equal to the vector sum of the three voltage phasors V_R, V_C, and V_L in Figure 3.11b.

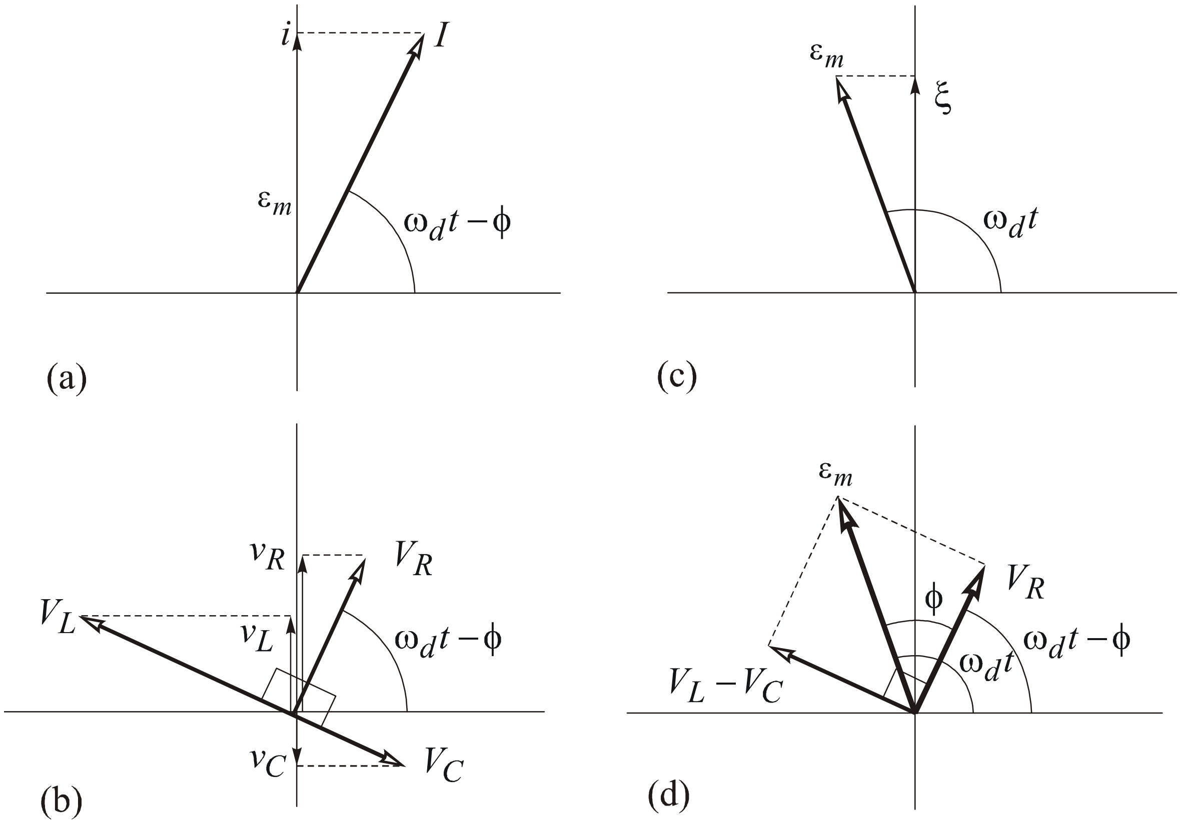


Figure 3.11 (a) A phasor representing the alternating current in the driven RLC at time t. The amplitude I, the instantaneous value i, and the phase $w_d t - j$ are shown; (b) Phasor representing the voltages across the inductor, the resistor, and the capacitor, oriented with respect to the current vector in (a); (c) A phasor representing EMF that drives the current of (a); (d) The EMF phasor is equal to the vector sum of the three voltage phasors of (b). Here, voltage phasors V_L and V_C have been added to yield their net phasor ($V_L - V_C$)

Requirement of Eq. (3.59) is indicated in Figure 3.11d, where the phasor e is the sum of phasors V_R, V_L, and V_C. Because phasors V_L and V_C have opposite directions in the figure, we simplify the vector sum by first combining V_L and V_C to form the single phasor $V_L - V_C$ and the obtained that single phasor with V_R to find the net phasor. Again, the net phasor must coincide with phasor $e_{\max}$ as shown.

Both triangles in Figure 3.11d are right triangles. The application of the Pythagorean Theorem to either yields

$$e_m^2 = V_R{}^2 + (V_C - V_L)^2 .$$

From the amplitude information, we can rewrite this as

$$e_m^2 = V_R{}^2 + (IX_L - IX_C)^2 ,$$

and then rearrange it to the form

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}. \qquad (3.60)$$

The denominator in Eq. (3.60) is called the *impedance* Z of the circuit for the driving angular frequency ω_d*:*

$$Z = \sqrt{R^2 + (X_L - X_C)^2}. \qquad (3.61)$$

Then we can write Eq. (3.60) as

$$I = \frac{\varepsilon_m}{Z}. \qquad (3.62)$$

If we substitute for X_C and X_L from Eqs. (3.46) and (3.53), we can write Eq. (3.60) more explicitly as

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + \left(\omega_d L - \dfrac{1}{\omega_d C}\right)^2}}. \qquad (3.63)$$

We have now accomplished half of our goal: We have obtained an expression for the current amplitude I in terms of the sinusoidal driving EMF and the circuit elements in a series RLC circuit.

The value of I depends on the difference between $\omega_d L$ and $1/\omega_d C$, or, equivalently, the difference between X_L and X_C in Eq. (3.60). In either equation, it does not matter which of two quantities is greater because the difference is always squared.

The current that we described in this section is the *steady-state* current which occurs after the alternating EMF has been applied for some time. When the EMF is first applied to a circuit, a brief *transient current* occurs. Its duration (before settling down into the steady-state current) is determined by the time constants $\tau_L = L/R$ and $\tau_C = RC$ as the inductive and capacitive elements turn on. This transient current can be large, for example, to destroy a motor on start up if it is not properly taken into account in the motor circuit design.

b) The Phase Constant. From the right-hand phasor triangle in Figure 3.11d, we can write

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR}, \qquad (3.64)$$

which gives us

$$\tan\phi = \frac{X_L - X_C}{R}. \qquad (3.65)$$

This is the other half of our goal to obtain an equation to calculate the phase constant ϕ in a sinusoidally driven series RLC circuit. In essence, it gives us three different results for the phase constant, depending on the relative values of X_L and X_C*:*

1. $X_L > X_C$*:* The circuit is said to be ***more inductive than capacitive.*** Eq. (3.65) tells us that f is positive for such a circuit, which means that the phasor I rotates behind the phasor EMF (Figure 3.12a). A plot of EMF and i versus time is like that in Figure 3.12b. (Figures 3.11c and d were drawn assuming $X_L > X_C$.)

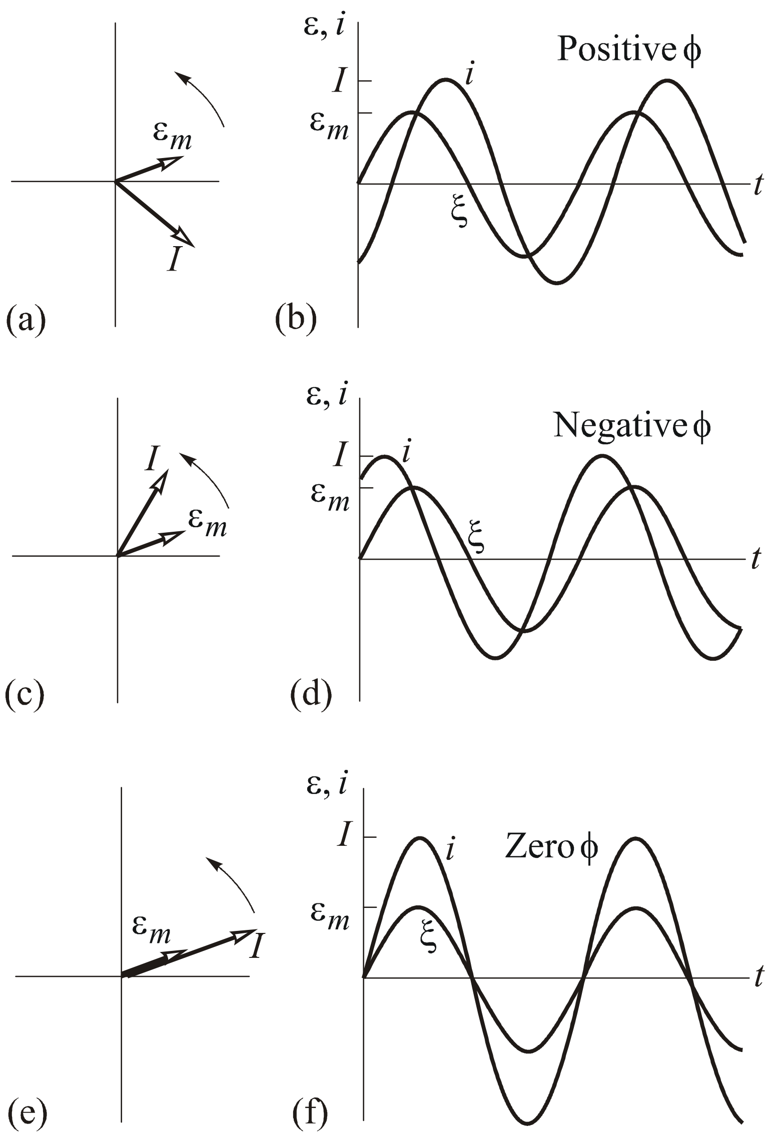


Figure 3.12 Phasor diagrams and graphs of the alternating emf and i for the driven RLC circuit of Figure 3.11. In the phasor diagram of (a) and the graph of (b), the current i lags the driving emf the current's phase constant f is positive. In (c) and (d), the current i leads the driving emf and its phase constant f is negative. In (e) and (f), the current i is in phase with driving emf and its phase constant f is zero

2. $X_L < X_C$ The circuit is said to be ***more capacitive than inductive.*** Eq. (3.65) tells us that φ is negative for such a circuit, which means that the phasor I rotates ahead of the phasor EMF (Figure 3.12c). A plot of EMF and i versus time is like that in Figure 3.12d.

3. $X_L = X_C$*:* The circuit is said to be in ***resonance,*** a state that is discussed next. Eq. (3.65) tells us that $\varphi = 0$ for such a circuit which means that the phasors EMF and i rotate together (Figure 3.12e). A plot of EMF and i versus time is like that in Figure 3.12f.

As an illustration, let us reconsider two extreme circuits: In the *purely inductive circuit* of Figure 3.9a, where X_L is nonzero, and $X_C = R = 0$, Eq.(3.65) tells us that $\varphi = +90°$ which is consistent with Figure 3.9c. In the *purely capacitive circuit* of Figure 3.8a, where X_C is nonzero and $X_L = R = 0$, Eq.(3.65) tells us that $\varphi = -90°$ which is consistent with Figure 3.8c.

Example 3.8

A resistor of 200 Ω and a capacitor of 15.0 µF are connected in series to a 220 V, 50 Hz AC source.

(a) Calculate the current in the circuit.

Solution.

To calculate the current, we need to know the impedance of the circuit. It is

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + (2\pi fC)^{-2}} = \sqrt{(200\,\Omega)^2 + (2\times 3.14\times 50\,\text{Hz}\times 10^{-6}\,\text{F})^{-2}} =$$
$$= \sqrt{(200\,\Omega)^2 + (212\,\Omega)^2} = 291.5\ \ \Omega.$$

Therefore, the current in the circuit is

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{220\,\text{V}}{291.5\,\Omega} = 0.755\ \text{A}.$$

(b) Calculate the voltage (rms) across the resistor and the capacitor. Is the algebraic sum of these voltages greater than the source voltage? If yes, resolve the paradox.

Solution.

Since the current is the same throughout the circuit, we have

$$V_R = I_{rms}R = (0.755\,A)(200\,\Omega) = 151\,\text{V},$$
$$V_C = I_{rms}X_C = (0.755\,A)(212.3\,\Omega) = 160.3\,\text{V}.$$

The algebraic sum of the two voltages, V_R and V_C, is 311.3 V which is more than the source voltage of 220 V. How to resolve this paradox?

As we have leaned, the two voltages are not in the same phase. Therefore, *they cannot be added like ordinary numbers.* The two voltages are out of phase by ninety degrees. Therefore, the total of these voltages must be obtained using the Pythagorean theorem:

$$V_{R+C} = \sqrt{V_R^2 + V_C^2} = 220\,\text{V}.$$

Thus, if the phase difference between two voltages is properly taken into account, the total voltage across the resistor and the capacitor is equal to the voltage of the source.

Example 3.9

In a series circuit, let $R = 300\ \Omega, L = 0.9\ \text{H}$, $C = 2\ \mu\text{F}$, and $\omega = 1000$ rad/s. Calculate the reactance, the impedance, the current amplitude, the phase angle and the voltage amplitudes across the inductor and capacitor.

Solution.

The inductive and the capacitive reactances are:

$$X_L = \omega L = 900\ \Omega,$$

$$X_C = \frac{1}{\omega C} = 500\ \Omega.$$

Then the reactance X of the circuit is

$$X = X_L - X_C = 400\ \Omega,$$

and the impedance Z is

$$Z = \sqrt{R^2 + Z^2} = 500\ \Omega.$$

If the circuit is connected across an AC source of the voltage amplitude 50 V, the current amplitude is

$$I = \frac{V}{Z} = 0.1\ \text{A}.$$

The phase angle φ is

$$\varphi = \arctan\frac{X}{R} = 53^\circ.$$

The voltage amplitude across the resistor is

$$V_R = IR = 30\ \text{V}.$$

The voltage amplitudes across the inductor and capacitor are, respectively,

$$V_L = IX_L = 90\ \text{V},\ V_C = IX_C = 50\ \text{V}.$$

Exercises

3.44. Why is the expression for the impedance Z of an RL series circuit obtained from Eq. (3.61) by setting $X_C = 0$ which corresponds to $C = \infty$, whereas for an RC series circuit, one obtains the impedance Z from Eq. (3.61) by setting $L = 0$? Explain. (Ans. $V_C = Q/C$ whereas $\varepsilon = L\dfrac{di}{dt}$.)

4.45. In an RLC series circuit, the source has a constant voltage amplitude of 50 V and a frequency of 1000 rad/s, $R = 300\ \Omega$, $L = 0.9$ H, $C = 2\ \mu$F. Suppose a series circuit contains only a resistor and an inductor in series. (a) What is the impedance of the circuit? (b) What is the current amplitude? (c) What are the voltage amplitudes across the resistor and across the inductor? (d) What is the phase angle f of the source voltage with respect to the current? (e) Does the source voltage lag or lead the current? (f) Construct the phasor diagram.

3.46. Consider the circuit from the Ex. 3.45, except that it consists of the resistor and the capacitor in series only. For part (c) calculate the voltage amplitudes across the resistor and across the capacitor. (Ans. (a) 583 Ω; (b) 0.0857 A; (c) 25.7 V; 42.9 V; (d) 59.1º)

3.47. Consider the circuit from the Ex. 3.45, except that it consists of the capacitor and the inductor in series only. For part (c), calculate the voltage amplitudes across the capacitor and across the inductor.

3.48. A 400-Ω resistor is in series with a 0.1-H inductor and a 0.5-μF capacitor. Compute the impedance of the circuit and draw the phasor diagram (a) at a frequency of 500 Hz. (b) at a frequency of 1000 Hz. In each case, compute the phase angle of the source voltage with respect to the current, and state whether the source voltage lags or leads the current. (Ans. (a) 514 Ω), (b) 506 Ω; 37.8º)

3.49. (a) Compute the impedance of an RLC series circuit at angular frequencies of 1000, 750, and 500 rad/s. Take $R = 300$ Ω, $L = 0.9$ H, and $C = 2.0$ μF. (b) Describe how the current amplitude varies as the frequency of the source is slowly reduced from l000 rad/s to 500 rad/s. (c) What is the phase angle of the source voltage with respect to the current when $w = 1000$ rad/s? Construct the phasor diagram when $w = 1000$ rad/s. Repeat part (c) for $w = 500$ rad/s.

3.12 Series Resonance

Eq. (3.63) gives the current amplitude I in an RLC circuit as a function of the driving angular frequency w_d of the external alternating EMF. For a given resistance R, this amplitude is a maximum when the quantity $w_d L - 1/w_d C$ in the denominator is zero – that is, when

$$w_d L = \frac{1}{w_d C} \quad \text{or}$$

$$w_d = \frac{1}{\sqrt{LC}}. \tag{3.66}$$

Because the natural angular frequency w_0 of the RLC circuit is also equal to $1/\sqrt{LC}$, the maximum value of I occurs when the driving angular frequency matches the natural angular frequency. This peaking of the current amplitude at a certain frequency is called *resonance.* Thus, in an RLC circuit, resonance and maximum current amplitude I occur when

$$w_d = w_0 = \frac{1}{\sqrt{LC}}. \tag{3.67}$$

If the inductance L or the capacitance C of a circuit can be varied, the resonant frequency can be varied as well. This is the procedure by which a radio or television receiving set may be tuned to receive the signal of the desired station. In the early days of radio, this was accomplished by the use of capacitors with movable metal plates whose overlap could be varied to change C. Nowadays it is more common to use a variable inductor with a ferrite core that slides in and out of a coil to vary L.

Figure 3.13 shows three resonance curves for sinusoidally driven oscillations in three series RLC circuits differing only in R. Each curve peaks at its maximum current amplitude I when the ratio w_d / w_0 is 1.0, but the maximum value of I decreases with the increasing R. (The maximum I is always EMF / R). In addition, the curves increase in width with increasing R.

To make the physical sense of Figure 3.13 consider how the reactance X_L and X_C change as we increase the driving angular frequency w_d, starting with a value much less than the natural frequency w_0. For small w_d, the reactance X_L $(= w_d L)$ is small and the reactance X_C $(= 1/w_d C)$ is large. Thus, the circuit is mainly capacitive, and the impedance is dominated by the large X_C which keeps the current low.

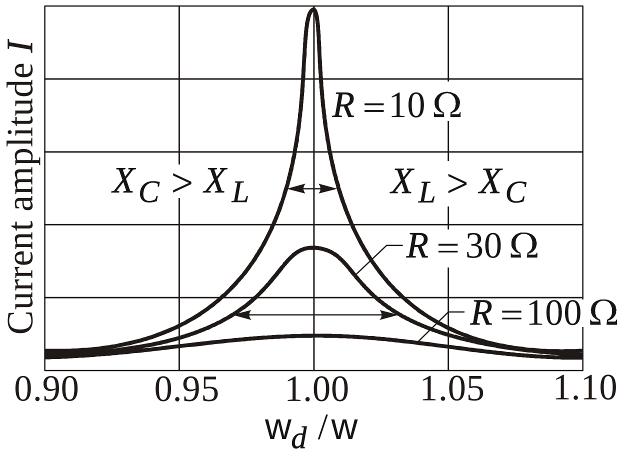


Figure 3.13 Resonance curves for the driven RLC circuit of Figure 3.10 with $L = 100$ μH, $C = 100$ pF, and three values of R. The current amplitude I of the alternating current depends on how close the driving angular frequency w_d is to the natural angular frequency w_0. The horizontal arrow on each curve measures the width of the curve at the half-maximum level, a measure of the sharpness of the resonance. To the left of $w_d / w = 1.0$, the circuit is mainly capacitive, with $X_C > X_L$; to the right it is mainly inductive, with $X_C < X_L$

As we increase w_d the reactance X_C remains dominant but decreases while the reactance X_L increases. The decrease in X_C, decreases the impedance allowing the current to increase, as we see on the left side of any resonance curve in Figure 3.13. When the increasing X_L, and the decreasing X_C reach equal values, the current is the greatest and the circuit is in resonance, with $w_d = w_0$. As we continue to increase w_d, the increasing reactance X_L becomes progressively more dominant over the decreasing reactance X_C. The impedance increases because of X_L, and the current decreases, as on the right side of any resonance curve in Figure 3.13. In summary, the low-frequency side of resonance curve is dominated by the capacitor reactance, the high-frequency side is dominated by the inductor reactance, and resonance occurs in the middle.

Figure 3.14 shows the graphs of R, X_L, X_C, and Z as functions of w. We have used a logarithmic frequency scale because of the wide range of frequencies covered. Note that at one particular frequency, X_L and X_C are numerically equal and $X = X_L - X_C$ is zero. Hence, at this frequency the impedance Z has its *minimum* value, equal simply to the resistance R. By inspecting Eq. (3.63), we must conclude that, when $R = 0$, the current becomes infinite at resonance. Although the equation predicts this, real circuit always has a resistance, which limits the value of the current.

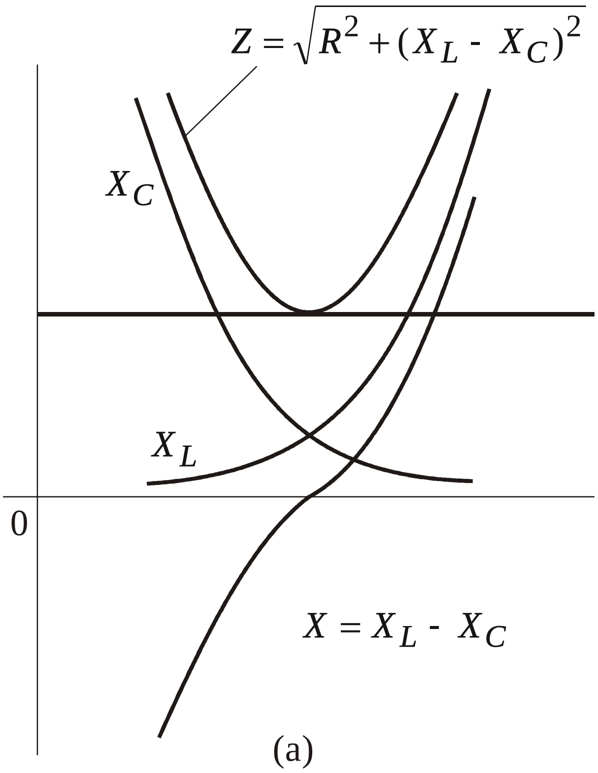


Fig. 3.14 (a) Reactance, resistance, and impedance as functions of frequency (logarithmic frequency scale); (b) Impedance, current, and phase angle as functions of frequency (logarithmic frequency scale)

To understand the resonance peak of the current amplitude more fully, consider the voltages in the circuit of Figure 3.10. At any instant the current is the same in L and C. As we have learned, the voltage across an inductor always *leads* the current by 90°, or a quarter-cycle, and the voltage across a capacitor always *lags* the current by 90°. Thus, comparing the phase of the instantaneous voltage v_L across L and the voltage v_C across C, we find that these two voltages always differ in phase by 180°, or a half-cycle. Therefore, they have opposite signs at each instant. If, in addition, the *amplitudes* of the two voltages are equal, then they add to zero at each instant, and the *total* voltage v_{CL} across the LC combination is exactly zero. As we have found above, this phenomenon occurs only at one particular frequency which we call the resonant frequency. Depending on the numerical values of R, L, and C, the voltages across L and C individually can be larger than that across R, so at frequencies sufficiently close to the resonant frequency, the voltages across L and C individually can be much *larger* than the source voltage!

An airport metal detector is essentially a resonant circuit. The portal you step through is an inductor (a large loop of conducting wire) that is part of the circuit. The frequency of the circuit is tuned to the resonant frequency of the circuit when there is no metal in the inductor. Any metal on your body increases the effective inductance of the loop and changes the current in it.

Resonance phenomena occur in all areas of physics; we have already seen it in the forced oscillation of the harmonic oscillator. In that case the amplitude of a mechanical oscillation peaked at a driving-force frequency close to the natural frequency of the system, and the behavior of the RLC circuit is analogous to this. Other important examples of resonance occur in acoustics, in atomic and nuclear physics, and in the study of fundamental particles (high-energy physics).

Example 3.10

A series RLC ac circuit has $R = 425\ \Omega$. $L = 1.25$ H, $C = 3.5$ μF. $\omega = 377$ Hz, and $\Delta V_{max} = 150$ V.

(a) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit.

Solution.

The reactances are $X_L = \omega L = 471\ \Omega$ and $X_C = 1/\omega C = 758\ \Omega$. The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(425)^2 + (471 - 758)^2} = 513\ \Omega.$$

(b) Find the maximum current in the circuit.

Solution.

$$I_m = \frac{V_m}{Z} = \frac{150}{513} = 0.292\ \text{A}.$$

(c) Find the phase angle between the current and voltage.

Solution.

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{471 - 758}{425}\right) = -34^\circ.$$

Because the circuit is more capacitive than inductive, ϕ is negative and the current leads the applied voltage.

(d) Find both the maximum voltage and the instantaneous voltage across each element.

Solution.

The maximum voltages are

$$\Delta V_R = I_m R = (0.292)(425) = 124\ \text{V},$$
$$\Delta V_L = I_m X_L = (0.292)(471) = 138\ \text{V},$$
$$\Delta V_C = I_m X_C = (0.292)(758) = 221\ \text{V}.$$

We can write the instantaneous voltages across the three elements as

$$\Delta v_R = (124)\sin 377t\ \ \text{V},$$
$$\Delta v_L = (138)\cos 377t\ \ \text{V},$$
$$\Delta v_R = (-221)\cos 377t\ \ \text{V},$$

Comments. The sum of the maximum voltages across the elements is $\Delta v_R + \Delta v_L + \Delta v_C = 483$ V. Note that this sum is much greater than the maximum voltage of the generator, 150 V. As we saw, the sum of the maximum voltages is a meaningless quantity because when sinusoidally varying quantities are added, *both their amplitudes and their phases* must be taken into account. We know that the maximum voltages across the various elements occur at different times. That is, the voltages must be added in a way that takes into account of the different phases. When it is done, Eq. (3.59) is satisfied.

Exercises

3.50. Show that a series RLC circuit driven by an ac source exhibits resonance at $w_r = 1/\sqrt{LC}$.

3.51. In series RLC circuit let $R = 200$ Ω, $C = 15.0$ μF, $L = 230$ mH, $f_d = 60$ Hz and $e_m = 36$ V. (a) What is the current amplitude I? (b) What is the phase constant? (Ans. -0.424 rad, 0.164 A.)

3.52. Find Z, f, and I for the circuit of Ex. 3.51 with the capacitor removed from the circuit, all other parameters remaining unchanged. (Ans. $X_C = 0$ Ω, $X_L = 86.7$ Ω, $Z = 182$ Ω, $I = 198$ mA, $f = 28.5°$.)

3.53. Find Z, f, and I for the circuit of Ex. 3.51 with the inductor removed from the circuit, all other parameters remaining unchanged.

3.54. Find Z, f, and I for the circuit of Ex. 3.51 with $C = 70.0$ μF, the other parameters remaining unchanged. (Ans. $X_C = 37.9$ Ω, $X_L = 86.7$ Ω, $Z = 167$ Ω, $I = 216$ mA, $f = 17.1°$.)

3.55. In an RLC circuit, can the amplitude of the voltage across an inductor be greater than the amplitude of the generator EMF? Consider an RLC circuit with $e_m = 10$ V, $R = 10$ Ω, $L = 1.0$ H, and $C = 1.0$ μF. Find the amplitude of the voltage across the inductor at resonance. (Ans. 1000 V.)

3.56. When the generator EMF in Ex. 3.51 is a maximum, what is the voltage across (a) the generator, (b) the resistance, (c) the capacitance, and (d) the inductance? (e) By summing these with appropriate signs, verify that the loop rule is satisfied.

3.57. A coil of inductance 88 mH and unknown resistance and a 0.94 μF capacitor are connected in series with an alternating EMF of frequency 930 Hz. If the phase constant between the applied voltage and the current is 75°, what is the resistance of the coil? (Ans. 89 Ω.)

3.58. An ac generator with $e_m = 220$ V and operating at 400 Hz causes oscillations in a series RLC circuit having $R = 220$ Ω, $L = 150$ mH, and $C = 24.0$ μF. Find (a) the capacitive reactance X_C, (b) the impedance Z, and (c) the current amplitude I. A second capacitor of the same capacitance is then connected in series with the other components. (d) Determine whether the values of X_C, (e) Z, and I increase, decrease, or remain the same.

3.59. An RLC circuit such as that of Figure 3.10 has $R = 5.0\ \Omega$, $C = 20.0$ μF, $L = 1.0$ mH, and $\varepsilon_m = 3.0$ V. (a) At what angular frequency ω_d will the current amplitude have its maximum value, as in the resonance curves of Figure 3.10? (b) What is this maximum value? (c) At what two angular frequencies ω_{d1} and ω_{d2} will the current amplitude be half this maximum value? (d) What is the fractional half-width $[= (\omega_{d1} - \omega_{d2})/\omega_0]$ of the resonance curve for this circuit? [Ans. (a) 224 rad/s, (b) 6.0 A, (c) 228 rad/s, 219 rad/s, (d) 0.040.]

3.60. At an airport, a person is made to walk through the doorway of a metal detector, for security reasons. If she or he is carrying anything made of metal, the metal detector emits a sound. On what principle does this detector work?

3.13 Q-factor of the Circuit

Let's return to Figure 3.13. We see, that the curves increase in width with the increasing R. Thus, a small R gives a sharply peaked response curve, and a large R gives a broad flat curve. This distinction is crucially important in the design of radio and television receiving circuits. The sharply peaked curve is what makes it possible to discriminate between two stations broadcasting on adjacent frequency bands. But if the peak is too sharp, some information is lost in the received signal. Finally, note that the shape of the resonance curve is related to the over-damped and underdamped oscillations. A sharply peaked resonance curve corresponds to a small value of R and a lightly damped oscillating system; a broad flat curve goes with a large value of R and a heavily damped system.

As the resistance is made smaller, the curve becomes sharper in the vicinity of the resonance frequency. This curve sharpness is usually described by a dimensionless parameter known as the *quality factor,* denoted by Q:

$$Q = \frac{\omega_0}{2\Delta\omega}, \tag{3.68}$$

where ω_0 is the resonance frequency, $2\Delta\omega$ is the width of the curve measured between the two values of ω for which P_{av} has half its maximum value, called the *half-power points* (see Figure 3.15.)

To understand the physical meaning of the quality factor more fully, return again to the circuit of Figure 3.10. For values of ω_d other than ω_0, the amplitude of the current is smaller than the maximum value. Suppose we choose a value of ω_d for which the current amplitude is $I = \frac{1}{\sqrt{2}} I_m$, that is $\frac{1}{\sqrt{2}}$ times its maximum value. From the curve in Figure 3.15, we see that there are two such values of ω_d, say, ω_1, and ω_2, one greater and the other smaller than ω_0 and symmetrical about ω_0. We can write

$$\omega_1 = \omega_0 + \Delta\omega \quad \text{and} \quad \omega_2 = \omega_0 - \Delta\omega.$$

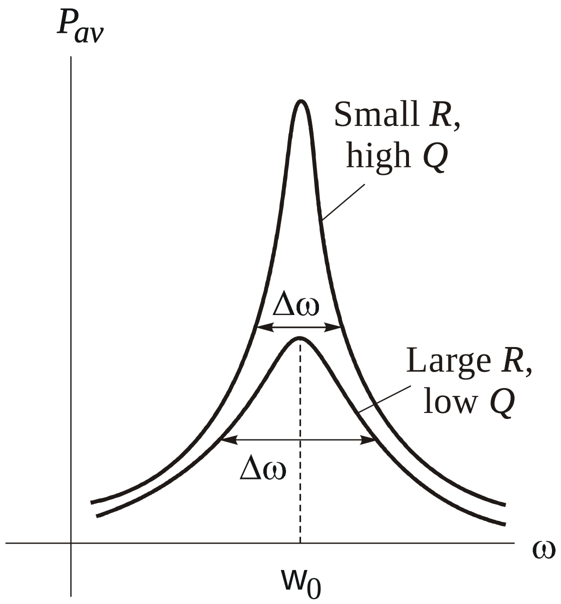


Figure 3.15 Average power versus frequency for a series RLC circuit. The width $\Delta\omega$ of each curve is measured between the two points where the power is half its maximum value. The power is a maximum at the resonance frequency

The difference $\omega_1 - \omega_2 = 2\Delta\omega$ is often called the *bandwidth* of the circuit. The quantity $\omega_0 / 2\Delta\omega$ is regarded as a measure of the sharpness of resonance. The smaller the $\Delta\omega$, the sharper, or narrower, is the resonance.

To get an expression for $\Delta\omega$, we note that the current amplitude is $I = \frac{1}{\sqrt{2}} I_m = \frac{\varepsilon_m}{R\sqrt{2}}$ for $\omega_1 = \omega_0 + \Delta\omega$. Therefore, at ω_1, current I is on one hand,

$$I = \frac{\varepsilon_m}{\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}},$$

and, on the other hand,

$$I = \frac{\varepsilon_m}{R\sqrt{2}}.$$

It is clear, that denominators of both expressions must be equal, hence

$$\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2} = R\sqrt{2}, \text{ or } R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2 = 2R^2,$$

and $\omega_1 L - \frac{1}{\omega_1 C} = R$. As $\omega_1 = \omega_0 + \Delta\omega$, we can rewrite the latest expression as

$$(\omega_0 + \Delta\omega)L - \frac{1}{(\omega_0 + \Delta\omega)C} = R, \quad \text{or}$$

$$\omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{1}{\omega_0 C\left(1 + \frac{\Delta\omega}{\omega_0}\right)} = R.$$

Using $\omega_0^2 = \frac{1}{LC}$ in the second term on the left hand side, we get

$$\omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{\omega_0 L}{\left(1 + \frac{\Delta\omega}{\omega_0}\right)} = R.$$

We can approximate $\left(1+\frac{\Delta\omega}{\omega_0}\right)^{-1}$ as $\left(1-\frac{\Delta\omega}{\omega_0}\right)$ since $\frac{\Delta\omega}{\omega_0} << 1$. Therefore,

$$\omega_0 L\left(1+\frac{\Delta\omega}{\omega_0}\right) - \omega_0 L\left(1-\frac{\Delta\omega}{\omega_0}\right) = R, \text{ or}$$

$$\omega_0 L \frac{2\Delta\omega}{\omega_0} = R, \text{ and, finally,}$$

$$\Delta\omega = \frac{R}{2L}.$$

The sharpness of resonance is given by

$$\frac{\omega_0}{2\Delta\omega} = \frac{\omega_0 L}{R}. \qquad (3.\ 69)$$

The ratio

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0}{2\Delta\omega}$$

is the *quality factor* (Q-factor) of the circuit, defined by Eq. (3.68).

From Eqs. (3.69) and (3.70), we see that

$$2\Delta\omega = \frac{\omega_0}{Q}.$$

So, the larger the value of Q, the smaller is the value of $2\Delta\omega$, or the bandwidth, and the sharper the resonance. Using $\omega_0^2 = \frac{1}{LC}$, Eq. (3.70) can be equivalently expressed as

$$Q = \frac{1}{\omega_0 CR}. \qquad (3.70)$$

We see from Figure 3.15, that if the resonance is less sharp, not only is the maximum current less, but also the circuit is close to resonance for a larger range $\Delta\omega$ of frequencies and the tuning of the circuit will not be good. So, the less sharp the resonance, the less the selectivity of the circuit, or vice versa. From Eq. (3.70) we see that if the quality factor is large, i.e., R is low or L is large, the circuit is more selective.

The quality factor is also defined as the ratio

$$Q = \frac{2\pi E}{\Delta E} \qquad (3.71)$$

where E is the energy stored in the oscillating system, and ΔE is the energy lost per cycle of oscillation.

The curves plotted in Figure 3.15 show that a high-Q circuit responds to only a very narrow range of frequencies, whereas a low-Q circuit can detect a much broader range of frequencies. In electronic circuits, typical values of Q range from 10 to 100.

The receiving circuit of a radio is an important application of a resonant circuit. One tunes the radio to a particular station (which transmits a specific electromagnetic wave or signal) by varying a capacitor or an inductor, which changes the resonant frequency of the receiving circuit. When the resonance frequency of the circuit matches that of the incoming electromagnetic wave, then the current in the receiving circuit increases. This signal caused by the incoming wave is then amplified and fed to a speaker. Because many signals are often present over a range of frequencies, it is important to design a high-Q circuit to eliminate unwanted signals. In this manner, stations whose frequencies are near but not equal to the resonance frequency give signals at the receiver that are negligibly small relative to the signal that matches the resonance frequency.

Exercises

3.61. Obtain the resonant frequency w_r of a series RLC circuit with $L = 2$, H, $C = 32$ µF, and $R = 10$ Ω. What is the Q-value of this circuit?

3.13 Power in Alternating-Current Circuits

No power losses are associated with pure capacitors and pure inductors in an AC circuit. To see why this is true, let us first analyze the power in an AC circuit containing only a generator and a capacitor.

When the current begins to increase in one direction in an AC circuit, the charge begins to accumulate on the capacitor, and a voltage drops across it. When this voltage drop reaches its maximum value, the energy stored in the capacitor is $CV^2/2$. However, this energy storage is only momentary. The capacitor is charged and discharged twice during each cycle: The charge is delivered to the capacitor during two quarters of the cycle and is returned to the voltage source during the remaining two quarters. Therefore, *the average power supplied by the source is zero.* In other words, no power losses occur in a capacitor in an AC circuit.

Similarly, the voltage source must do work against the back EMF of the inductor. When the current reaches its maximum value, the energy stored in the inductor is a maximum and is given by $LI^2/2$. When the current begins to decrease in the circuit, this stored energy is returned to the source as the inductor attempts to maintain the current in the circuit.

The power delivered by a battery to a DC circuit is equal to the product of the current and the EMF of the battery. Likewise, the instantaneous power delivered by an AC generator to a circuit is the product of the generator current and the applied voltage. For the *RLC* circuit shown in Figure 3.10, we can express the instantaneous power P as

$$P = iv = I\sin(\omega t - \phi)V\sin\omega t = IV\sin\omega t\sin(\omega t - \phi), \tag{3.72}$$

Clearly, this result is a complicated function of time and, therefore, is not very useful from a practical viewpoint. What is generally of interest is the average power over one or more cycles. Such an average can be computed by using the trigonometric identity $\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$. The substitution of this into Eq. (3.72) gives

$$P = IV\sin^2\omega t\cos\phi - IV\sin\omega t\cos\omega t\sin\phi\ . \tag{3.73}$$

We now take the time average of P over one or more cycles, noting that I, V, ϕ and ω are all constants. The time average of the first term in the right side of Eq. (3.73) involves the average value of $\sin^2\omega t$, which is a $\frac{1}{2}$. The time average of the second term on the right is identically zero because $\sin\omega t\cos\omega t = \frac{1}{2}\sin 2\omega t$ and the average value of $\sin 2\omega t$ is zero. Therefore, we can express the average power P_{av} as

$$P_{av} = \frac{1}{2}IV\cos\phi\ . \tag{3.74}$$

It is convenient to express the average power in terms of the rms current and rms voltage defined by Eq. (3.39) and (3.41):

$$P_{av} = I_{rms}V_{rms}\cos\phi\ , \tag{3.75}$$

where the quantity $\cos\phi$ is called the *power factor*. The maximum voltage drop across the resistor is given by $V_m\cos\phi = I_m R$. Using Eq. (3.75) and the fact that $\cos\phi = IR/V$, we find that we can express P_{av} as

$$P_{av} = I_{rms}V_{rms}\cos\phi = I_{rms}\left(\frac{V}{\sqrt{2}}\right)\frac{IR}{V} = I_{rms}\frac{IR}{\sqrt{2}}.$$

After making the substitution $I_m = \sqrt{2}I_{rms}$ from Eq. (3.39), we have

$$P_{av} = I_{rms}^2 R\ . \tag{3.76}$$

In words, the *average power delivered by the generator is converted into the internal energy in the resistor*, just as in the case of a DC circuit. No power loss occurs in an ideal inductor or capacitor. When the load is purely resistive, then $\phi = 0$, $\cos\phi = 1$, and from Eq. (3.76), we see that

$$P_{av} = I_{rms}V_{rms}. \tag{3.77}$$

Eq. (3.76) shows that the power delivered by an AC source to any circuit depends on the phase, and this result has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, technicians introduce capacitance in the circuits to shift the phase.

So, as we have understood, the average power dissipated depends not only on the voltage and current but also on the power factor. Let us discuss the following cases:

Case (a). *Resistive circuit:* If the circuit contains only pure R, it is called resistive. In that case, $\phi = 0$, $\cos\phi = 1$. There is maximum the power dissipation.

Case (b). Purely *inductive or capacitive circuit:* If the circuit contains only an inductor or capacitor, we know that the phase difference between the voltage and the current is $\pi/2$. Therefore, $\cos\phi = 0$, and no power is dissipated even though a current is flowing in the circuit. This current is sometimes referred to as *wattless current.*

Case (c). *RLC series circuit.* In an RLC series circuit, the power dissipated is given by Eq. (3.75) with $\phi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right)$. So, ϕ may be non-zero (except $\pi/2$) in a RL, RC, or RLC circuit. Even in such cases, power is dissipated only in the resistor.

Case (d). *Power dissipated at resonance in RLC circuit:* At resonance, $X_C - X_L = 0$ and $\phi = 0$. Therefore, $\cos\phi = 1$ and $P = \frac{I_m}{Z} = \frac{I_m}{R}$. That is, maximum power is dissipated in a RLC circuit (through R) at resonance.

Example 3.11

A sinusoidal voltage of the peak value 283 V and the frequency 50 Hz is applied to a series RLC circuit in which $R = 3$ Ω, $L = 25.48$ mH, and $C = 796$ μF.

(a) Find the impedance of the circuit.

Solution.

(a) To find the impedance of the circuit, we first calculate X_L and X_C:

$$X_L = 2\pi f L = 2 \times 3.14 \times 50 \times 25.48 \times 10^{-3}\Omega = 8\Omega,$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2 \times 3.14 \times 50 \times 796 \times 10^{-6}} = 4\Omega.$$

Therefore,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{3^2 + (8 - 4)^2} = 5\Omega.$$

(b) Find the phase difference between the voltage across the source and the current.

Solution.

Phase difference

$$\varphi = \tan^{-1}\left(\frac{X_C - X_L}{R}\right) = \tan^{-1}\left(\frac{4-8}{3}\right) = -53.1^\circ .$$

Since φ is negative, the current in the circuit lags the voltage across the source.

(c) Find the power dissipated in the circuit.

Solution.

The power dissipated in the circuit is

$$P = I_{rms}^2 R ,$$

Now, $I_{rms} = \dfrac{I_m}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\left(\dfrac{283}{5}\right) = 40\,A,$

Therefore,

$$P = (40\,A)^2 \times 3\Omega = 4800\,\mathrm{W} .$$

(d) Find the power factor.

Solution.

Power factor: $\cos\varphi = \cos 53.1^\circ = 0.6$.

Example 3.12

Suppose the frequency of the source in the previous example can be varied.

(a) What is the frequency of the source at which resonance occurs?

Solution.

(a) The frequency at which the resonance occurs is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{25.48 \times 10^{-3} \times 796 \times 10^{-6}}} = 222.1\,\mathrm{rad/s} ,$$

$$f_r = \frac{\omega_0}{2\pi} = \frac{221.1}{2 \times 3.14}\,\mathrm{Hz} = 35.4\,\mathrm{Hz} .$$

(b) Calculate the impedance, the current and the power dissipated at the resonant condition.

Solution.

The impedance Z at resonant condition is equal to the resistance:

$$Z = R = 3\Omega.$$

The rms current at resonance is

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{V_{rms}}{R} = \left(\frac{283}{\sqrt{2}}\right)\frac{1}{3} = 66.7\,\mathrm{A} .$$

The power dissipated at resonance is

$$P = I_{rms}^2 R = (66.7)^2 \times 3 = 13.35\,\mathrm{kW} .$$

3.62. A series RLC circuit with $R = 20$ Ω, $L = 1.5$ H, and $C = 35$ μF is connected to a variable-frequency 200 V AC supply. When the frequency of the supply equals the natural frequency of the circuit, what is the average power transferred to the circuit in one complete cycle?

3.63. A series RLC circuit is connected to a variable frequency 230 V source. $L = 5$ H, $C = 80$ μF, $R = 40$ Ω; (a) Determine the source frequency which drives the circuit in resonance. (b) Obtain the impedance of the circuit and the amplitude of current at the resonating frequency. (c) Determine the rms potential drops across the three elements of the circuit. Show that the potential drop across the LC combination is zero at the resonating frequency.

3.64. A 400-Ω resistor is in series with a 0.1-H inductor and a 0.5-μF capacitor. The circuit carries an rms current of 0.25 A with a frequency of 100 Hz. (a) What average power is delivered by the source? (b) What average power is consumed in the resistor? (c) In the capacitor? (d) In the inductor? (e) Compare your answers in (a) to the sum of (b), (c), and (d) (Ans. 25 W), (b) 25 W, (c) 0, (d) 0, (e) (a) =(b)+ (c)+ (d).)

3.65. A series circuit has a resistance of 75 Ω and an impedance of 150 Ω. What power is consumed in the circuit when a voltage of 120 V (rms) is impressed across it?

3.14 Parallel Resonance

A different kind of resonance occurs when L, R, and C are connected in *parallel,* as shown in Figure 3.16a. We can analyze this circuit by using the same procedure as for the series circuit. In this case, the instantaneous potential difference v is the same for all elements and is equal to the source voltage. Figure 3.16b shows a phasor diagram; the single phasor V represents this common voltage. There are three separate currents, one in each branch, and the three corresponding current phasors are also shown. The phasor I_R, (with amplitude V/R and in phase with V) represents the current in the resistor. Phasor I_L (with amplitude V/X_L and lagging V by 90°) represents the current in the inductor. Phasor I_C, with amplitude V/X_C and leading V by 90°, represents the current in the capacitor.

By Kirchhoff's point rule, the instantaneous current i, equals the (algebraic) sum of the instantaneous currents i_R, i_L, and i_C and is represented by the phasor I, the vector sum of phasors I_R, I_L, and I_C. Angle f is the phase angle of current with respect to source voltage (the negative of the phase angle of voltage with respect to current).

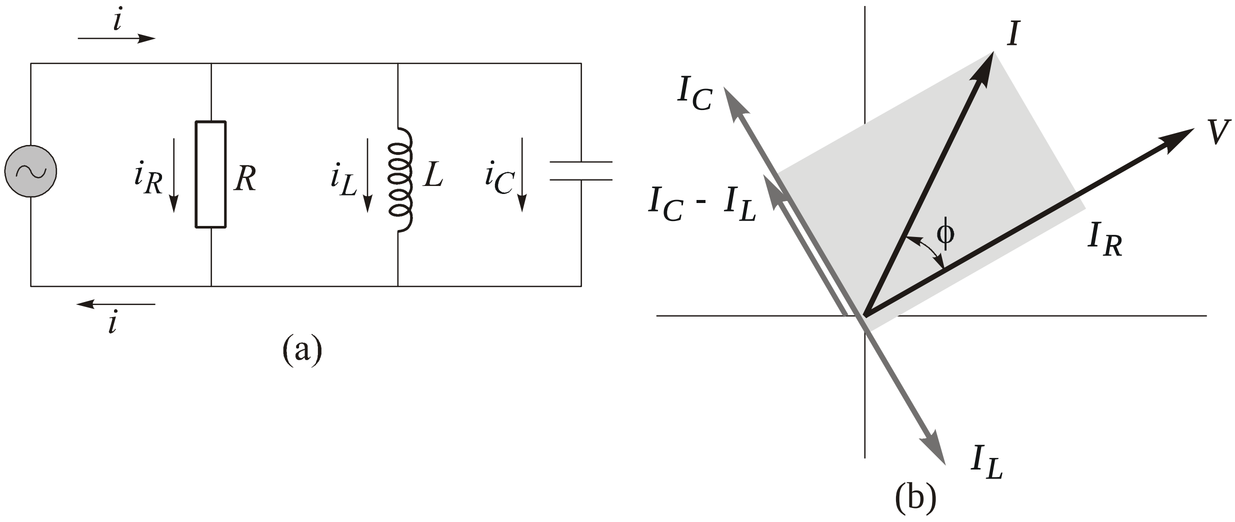


Figure 3.16 (a) Parallel RLC circuit; (b) Phasor diagram showing current phasors for the three branches. The single voltage phasor *V* represents the voltage across all three branches

From Figure 3.16,

$$I=\sqrt{I_R^2+(I_C-I_L)^2}=\sqrt{\left(\frac{V}{R}\right)^2+\left(\omega CV-\frac{V}{\omega L}\right)^2}=$$

$$=V\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}\ . \tag{3.78}$$

The maximum current I is frequency-dependent, as expected. It is *minimal* when the second factor in the radical is zero; this occurs when the two reactances have equal magnitudes, at the resonant frequency ω_0.

Comparing this equation with Eq. (3.61), we see that the *impedance* Z of a parallel RLC circuit is given by

$$\frac{1}{Z}=\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}\ . \tag{3.79}$$

At resonance $1/Z$, is minimum, so Z itself has its *maximum* value at

$$\omega=\omega_0=1/\sqrt{LC}\ . \tag{3.80}$$

Thus, at resonance, the total current in the parallel RLC circuit is *minimum,* in contrast to the RLC series circuit which has *maximum* current at resonance. This distinction can be understood by noting that, in the parallel circuit, the currents in L and C are always exactly a half-cycle out of phase. When they also have equal magnitudes, they cancel each other completely, and the total current is simply the current through R. Indeed, when $\omega C=1/\omega L$, Eq. (3.78) becomes simply $I=V/R$. This does not mean that there is no current in L or C at resonance, but only that the two currents cancel. If R is large, the equivalent impedance of the circuit near resonance is much *larger* that the individual reactances X_L and X_C.

Exercises

3.66. For the circuit of Figure 3.16a, let $V = 120$ V, $R = 200$ Ω, $L = 0.5$ H, and $C = 0.2$ μF. (a) What is the resonant angular frequency of the circuit? Sketch the phasor diagram at the resonant frequency. (b) At the resonant frequency, what is the current through the source? (c) At the resonant frequency, what is the current through the resistor? (Ans. (a) 3160 rad/s, (b) 0.6 A, (c) 0.6 A,)

3.67. Consider the circuit of Figure 3.16a, with the same numerical values as in Exercise 3.51. At resonance, what is: (a) the average rate at which the electrical energy is delivered by the source? (b) the average rate at which electrical energy is dissipated in the resistor? Compare to the result of (a). (c) Is the current through the inductor, and, hence, the energy stored in its magnetic field, zero at all times? If not, how can the result obtained in (b) be explained? (d) Calculate the maximum energy stored in the inductor. (e) Calculate the maximum energy stored in the capacitor.

Summary

A circuit containing an inductance L and a capacitance C undergoes electromagnetic oscillations, with angular frequency w given by

$$w = \frac{1}{\sqrt{LC}}.$$

Such a circuit is analogous to a mechanical harmonic oscillator with the mass m analogous to the inductance L, the force constant k to the reciprocal of the capacitance $1/C$, the displacement x to the charge q, and the velocity v to the current i. A series circuit containing inductance, resistance, and capacitance undergoes damped oscillations for sufficiently small resistance. As R increases, the damping increases; at a certain value of R, the behavior becomes overdamped, and the circuit no longer oscillates. The crossover between the underdamping and the overdamping occurs when

$$R = \sqrt{\frac{4L}{C}},$$

and the frequency w' of damped oscillations when R is smaller than this critical value is

$$w' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

There is a direct analogy between every aspect of the behavior of the RLC circuit and the mechanical damped harmonic oscillator. This analogy is widely used in analog computers.

An alternator or AC source produces an EMF that varies sinusoidally with time. Voltages and currents that vary sinusoidally with time can be represented by vectors called phasors. A phasor rotates counterclockwise with constant angular velocity w equal to the angular frequency of the sinusoidal quantity, and its projection on the horizontal axis at any instant represents the instantaneous value of the quantity.

The voltage across a resistor R is in phase with the current, and the amplitudes are related by

$$V = IR.$$

The voltage across a capacitor C lags the current by 90°; the amplitudes are related by

$$V = IX_C.$$

where $X_C = 1/wC$ is the capacitive reactance of the capacitor.

The voltage across an inductor L leads the current by 90°; the amplitudes are related by

$$V = IX_L.$$

where $X_L = wL$ is the inductive reactance of the inductor.

In an RLC series circuit, the voltage and current amplitudes are related by

$$V = IZ.$$

where Z is the impedance of the circuit, given by

$$Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + [wL - (1/wC)]^2}.$$

The phase angle q of the voltage relative to the current is given by

$$\tan q = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{I} = \frac{X}{R}.$$

The SI unit of capacitive or inductive reactance, or impedance is the ohm.

The root-mean-square (rms) value of a sinusoidally varying quantity is $1/\sqrt{2}$ times the amplitude; thus $I_{rms} = I/\sqrt{2}$ and $V_{rms} = V/\sqrt{2}$.

The average power input P to an ac circuit is given by

$$P = V_{rms} I_{rms} \cos q,$$

where q is the phase angle of voltage with respect to current. The quantity $\cos q$ is called the power factor.

The current in an RLC series circuit becomes maximum, and the impedance minimum, at a frequency $w_0 = 1/\sqrt{LC}$ called the resonant frequency. This phenomenon is called resonance. At resonance the voltage and current are in phase, and the impedance Z is equal to the resistance R.

The current in an RLC parallel circuit becomes minimum, and the impedance maximum, at this same resonant frequency w_0. The impedance Z of this circuit at any frequency is given by

$$\frac{1}{Z} = \sqrt{R^2 + [wL - (1/wC)]^2}.$$

At resonance, $Z = R$.

Key Terms

Electro–magnetic oscillations – электромагнитные колебания
damped oscillations – затухающие колебания
alternating current – переменный ток
ac source – источник переменного тока
phasor diagrams – фазорная диаграмма
phase angle – фазовый угол
capacitive reactance – емкостное сопротивление
inductive reactance – индуктивное сопротивление
reactance – реактивное сопротивление
impedance – импеданс
root-mean-square (rms) value – среднеарифметическое значение
resonance – резонанс
resonant frequency – резонансная частота

Chapter 4

Electromagnetic Waves

The waves we discussed in Chapter 2 require a medium (some material) through which or along which to travel. We had waves traveling along a string, through Earth, and through the air. However, an electromagnetic wave is different in that it requires no medium for its travel. It can, indeed, travel through a medium such as air or glass, but it can also travel through the vacuum of space between a star and us.

Once the special theory of relativity became accepted, long after Einstein published it in 1905, the speed of light waves was realized to be special. One reason is that light has the same speed regardless of the frame of reference from which it is measured. If you send a beam of light along an axis and ask several observers to measure its speed while they move at different speeds along that axis, either in the direction of the light or opposite it, they will all measure the same speed for the light. This result is an amazing one and quite different from what would have been found if those observers had measured the speed of any other type of wave; for other waves, the speed of the observers relative to the wave would have affected their measurements.

4.1 Nature of Electromagnetic Waves

We have studied various aspects of electric and magnetic fields, falling in two general categories. The first category includes fields that *do not* vary with time. The electrostatic field of a distribution of charges at rest and the magnetic field of a steady current in a conductor are examples of fields that do not vary with time at any individual point, although they may vary from point to point in space. For these situations we could treat the electric and magnetic fields independently, without worrying much about interactions between them. The second category includes fields that *do* vary with time, and in all such cases, we found that it is *not* possible to treat the fields independently.

James Clerk Maxwell's crowning achievement was to show that a beam of light is a traveling wave of electric and magnetic fields — an electromagnetic wave. In Maxwell's time (the mid 1800s), the visible, infrared, and ultraviolet forms of light were the only electromagnetic waves known. Using Maxwell's work, however, Heinrich Hertz discovered what we now call radio waves and verified that they move through the laboratory at the same speed as visible light.

We have given a brief description of Maxwell's equations which form the theoretical basis of all electromagnetic phenomena. The consequences of Maxwell's equations are far-reaching. The Ampere-Maxwell law predicts that a time-varying electric field produces a magnetic field, just as Faraday's law tells us that a time-varying magnetic field produces an electric field. Maxwell's introduction of the concept of displacement current as a new source of a magnetic field provided the final important link between electric and magnetic fields in classical physics. Maxwell's equations also predict the existence of electromagnetic waves that propagate through space with the speed of light c.

In his unified theory of electromagnetism, Maxwell showed that electromagnetic waves are a natural consequence of the fundamental laws expressed in the following four equations:

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

$$div\,\vec{B} = 0,$$

$$rot\,\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t},$$

$$div\,\vec{D} = \rho .$$

Some electromagnetic waves, including X-rays, gamma rays, and visible light, are radiated (emitted) from sources that are of atomic or nuclear size where quantum physics rules. Here we discuss how other electromagnetic waves are generated. To simplify matters, we restrict ourselves to that region of the spectrum (wavelength $\lambda = 1$ m) in which the source of the radiation is both macroscopic and of manageable dimensions.

Devises for generating such waves contain an LC oscillator which establishes an angular frequency $\omega = \sqrt{LC}$. Charges and currents in this circuit vary sinusoidally at this frequency. An external source – possibly an AC generator – must be included to supply energy to compensate both for thermal losses in the circuit and for energy carried away by the radiated electromagnetic wave.

The LC oscillator is coupled by a transformer and a transmission line to an antenna which consists essentially of two thin solid conducting rods. Through this coupling, the sinusoidally varying current in the oscillator causes charge to oscillate sinusoidally along the rods of the antenna at the angular frequency ω of the LC oscillator. The current in the rods associated with this movement of charge also varies sinusoidally, in magnitude and direction, at angular frequency ω. The antenna has the effect of an electric dipole whose electric dipole moment varies sinusoidally in magnitude and direction along the length of the antenna.

Because the dipole moment varies in magnitude and direction, the electric field produced by the dipole varies in magnitude and direction. As well, because the current varies, the magnetic field produced by the current varies in magnitude and direction. However, the changes in the electric and magnetic fields do not happen everywhere instantaneously; rather, the changes travel outward from the antenna at the speed of light c. Together the changing fields form an electromagnetic wave that travels away from the antenna at speed c. The angular frequency of this wave is ω, the same as that of the LC oscillator.

We are bathed in electromagnetic waves of different wavelengths. The Sun, whose radiations define the environment in which we as a species have evolved and adapted, is the dominant source. We are also crisscrossed by radio and television signals. Microwaves from radar systems and from telephone relay systems reach us. There are electromagnetic waves from lighlbulbs, from the heated engine blocks of automobiles, from X-ray machines, from lightning flashes, and from buried radioactive materials. Beyond this, radiation reaches us from stars and other objects in our galaxy and from other galaxies. Electromagnetic waves also travel in the other direction. Television signals, transmitted from Earth since about 1950, have now taken news about us to whatever technically sophisticated inhabitants there may be on whatever planets may encircle the nearest 400 or so stars.

4.2 Hertz's Experiment

In 1887, Heinrich Hertz produced electromagnetic waves with the aid of oscillating circuits, and received and detected these waves with other circuits tuned to the same frequency.

The experimental apparatus that Hertz used to generate and detect electromagnetic waves is shown schematically in Figure 4.1. An induction coil is connected to a transmitter made up of two spherical electrodes separated by a narrow gap. The coil provides short voltage surges to the electrodes, making one positive and the other negative. A spark is generated between the spheres when the electric field near either electrode surpasses the dielectric strength for air (3×10^{6} V/m). In a strong electric field, the acceleration of free electrons provides them with enough energy to ionize any molecules they strike. This ionization provides more electrons which can accelerate and cause further ionization. As the air in the gap is ionized, it becomes a much better conductor, and the discharge between the electrodes exhibits an oscillatory behavior at a very high frequency. From an electric-circuit viewpoint, this is equivalent to a LC circuit in which the inductance is that of the coil and the capacitance is due to the spherical electrodes.

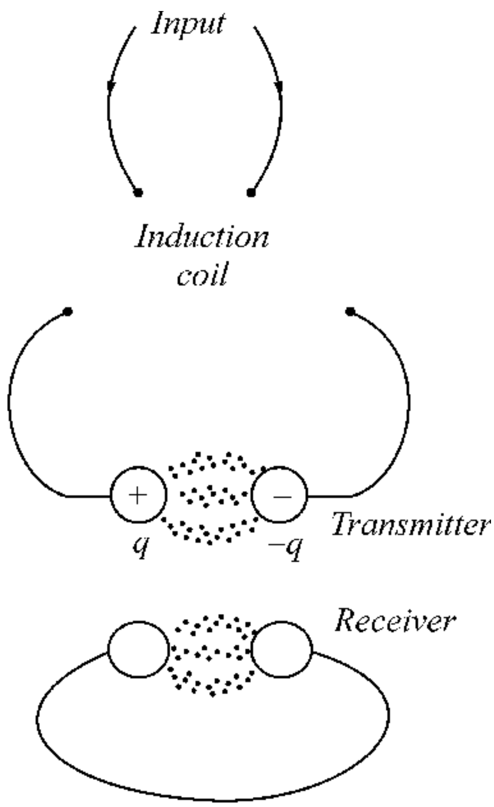


Figure 4.1 The experimental apparatus that Hertz used to generate and detect electromagnetic waves

Because L and C are quite small in Hertz's apparatus, the frequency of oscillations is very high, $f = 100$ MHz. (Recall that $\omega = \frac{1}{\sqrt{LC}}$ for a LC circuit). Electromagnetic waves are radiated at this frequency as a result of the oscillation (and hence acceleration) of free charges in the transmitter circuit. Hertz was able to detect these waves by using a single loop of wire with its own spark gap (the receiver). Such a receiver loop, placed several metres from the transmitter, has its own effective inductance, capacitance, and natural frequency of oscillation. In Hertz's experiment, sparks were induced across the gap of the receiving electrodes when the frequency of the receiver was adjusted to match that of the transmitter. Thus, Hertz demonstrated that the oscillating current induced in the receiver was produced by electromagnetic waves radiated by the transmitter. His experiment is analogous to the mechanical phenomenon in which a tuning fork responds to acoustic vibrations from an identical tuning fork that is oscillating.

Additionally, Hertz showed in a series of experiments that the radiation generated by his spark-gap device was transverse and exhibited the wave properties of interference, diffraction, reflection, refraction, and polarization, all of which are properties exhibited by light. Thus, it became evident that the radio-frequency waves Hertz had generated had properties similar to those of light waves and differed only in frequency and wavelength. Perhaps his most convincing experiment was the measurement of the speed of this radiation. Radio-frequency waves of known frequency were reflected from a metal sheet and created a standing-wave interference pattern whose nodal points could be detected. The measured distance between the nodal points enabled determination of the wavelength λ. Using the relationship $v = \lambda f$, Hertz found that v was close to 3×10^8 m/s, the known speed c of visible light.

4.3 Wave Equation for Plane Electromagnetic Waves

The existence and properties of electromagnetic waves can be deduced from Maxwell's equations. One approach to deriving these properties is to solve the second-order differential equation obtained from Maxwell's equations. To circumvent this problem, we assume that the vectors for a electric field and magnetic field in an electromagnetic wave have a specific space-time behavior that is simple but consistent with Maxwell's equations.

To understand the prediction of electromagnetic waves more fully, let us focus on a plane electromagnetic wave that travels in the x direction (the direction of propagation). In this wave, the electric field $\vec{E}$ is in the y direction, and the magnetic field $\vec{B}$ is in the z direction, as shown in Figure 4.2. It means that their components on coordinate axes does not depend on coordinates y and z.

We can relate $\vec{E}$ and $\vec{H}$ to each other. In an empty space, where $\rho = 0$ and $j = 0$, $\varepsilon = 1$, $\mu = 1$, Maxwell's equations become:

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad (4.1)$$

$$div\,\vec{B} = 0, \qquad (4.2)$$

$$rot\,\vec{H} = \frac{\partial \vec{D}}{\partial t}, \qquad (4.3)$$

$$div\,\vec{D} = 0. \qquad (4.4)$$

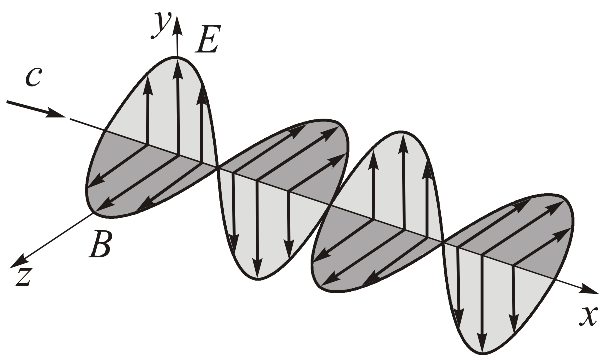


Figure 4.2 At some instant, a plane electromagnetic wave moving in the x direction has a maximum electric field in the positive y direction. At that point the corresponding magnetic field has a magnitude E/c and is in the z direction

Consider the Eq. (4.1) which is the differential form of law of electromagnetic induction. As

$$rot\,\vec{E} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = i\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - j\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + k\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right),$$

then Eq. (4.1) can be rewritten as

$$0 = \mu_0 \frac{\partial H_x}{\partial t}, \qquad (4.5)$$

$$\frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_y}{\partial t}, \qquad (4.6)$$

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}. \qquad (4.7)$$

After the same procedure, the Eq. (4.3) can be written in the form:

$$0 = \varepsilon_0 \frac{\partial E_x}{\partial t}, \qquad (4.8)$$

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t}, \qquad (4.9)$$

$$\frac{\partial H_y}{\partial x} = \varepsilon_0 \frac{\partial E_z}{\partial t}. \qquad (4.10)$$

Eqs. (4.2) and (4.4) take form

$$\frac{\partial B_x}{\partial x} = \mu_0 \frac{\partial H_x}{\partial x} = 0 \quad \text{and} \quad . \frac{\partial D_x}{\partial x} = \varepsilon_0 \frac{\partial E_x}{\partial x} = 0,$$

respectively.

Eqs (4.4) and (4.8) show that E_x does not depend either on coordinate x or time t. Eqs (4.2) and (4.5) give the same information about H_x. Hence the field itself has no components along the x–axis. This means that vectors $\vec{E}$, and $\vec{H}$ are perpendicular to the direction of wave propagation, that is, that electromagnetic waves are *transverse* waves.

The Eqs (4.7), (4.9), and (4.6), (4.10) can be combined into two independent systems:

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t}, \qquad \frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t}, \tag{4.11}$$

and

$$\frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_y}{\partial t}, \qquad \frac{\partial H_y}{\partial x} = \varepsilon_0 \frac{\partial E_z}{\partial t}. \tag{4.12}$$

The Eqs. (4.11) interrelate the components E_y and H_z, and the Eqs. (4.12) interrelate the components E_z and H_y.

Suppose that initially the time-varying electric field E_y directed along the y-axis was created. According to the second of Eq. (4.11), this field produces the magnetic field H_z which is directed along z-axis. Furthermore, according to the first of Eqs. (4.11), the field H_z produces the electric field E_y and so on. Neither the field E_z not field H_y are induced in this case. Similarly, if initially the electric field E_z was produced, then the field H_y appears, and this H_y field induces electric field E_z and so on, according to Eqs. (4.12). Hence, for description of plane wave, we can use only one of the above systems, putting the components of the second system equal zero.

Let's choose system (4.11), putting $E_z = H_y = 0$. Taking derivative of the first of Eqs. (4.9) with respect to x, substituting $\frac{\partial}{\partial x}\frac{\partial H_z}{\partial t} = \frac{\partial}{\partial t}\frac{\partial H_z}{\partial x}$, and combining the result with Eq. (4.11), we obtain:

$$\frac{\partial^2 E_y}{\partial x^2} = -\mu_0 \frac{\partial}{\partial x}\left(\frac{\partial H_z}{\partial t}\right) = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial H_z}{\partial x}\right) = -\mu_0 \frac{\partial}{\partial t}\left(-\varepsilon_0 \frac{\partial E_y}{\partial t}\right) = \varepsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}.$$

That is,

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}. \tag{4.13}$$

In the same manner, taking the derivative of the second of Eqs. (4.11) with respect to x, we get:

$$\frac{\partial^2 H_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_z}{\partial t^2}. \tag{4.14}$$

We obtain very important result: Eqs. (4.13) and (4.14) both have a form of the general wave equation. It is known that any function which satisfies the wave equation describes the wave motion, which can be realized in nature. Moreover, the root of quantity, which is reciprocal to the coefficient at time derivative, gives squared phase speed of the wave. Hence, Eqs. (4.13) and (4.14) state that electromagnetic fields can exist in the form of electromagnetic waves and their phase speed c in vacuum is determined as

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \qquad (4.15)$$

Recall again, that $E_x = E_z = 0$ and $H_x = H_y = 0$, that is $E_y = E$ and $H_z = H$. We keep indexes y and z in Eqs. (4.13) and (4.14) to show that vectors $\vec{E}$ and $\vec{H}$ are directed along mutually perpendicular axes y and z.

The simplest solutions of Eqs.(4.13) and (4.14) are sinusoidal waves for which the field magnitudes E and H vary with time according to expression:

$$E_y = E_m \cos(\omega t - kx + \varphi_1), \qquad (4.16)$$

$$H_z = H_m \cos(\omega t - kx + \varphi_2), \qquad (4.17)$$

where E_m and H_m are the amplitudes of the fields. Here, as usual, k is a wave number, $k = 2\pi/\lambda$, λ is a wavelength, φ_1 and φ_2 are the phase constants of oscillations for points with coordinates $x = 0$. The angular frequency is $\omega = 2\pi f$ where f is the wave frequency.

The substitution of functions (4.16) and (4.17) into Eqs.(4.11) leads to the following relationships:

$$kE_m \sin(\omega t - kx + \varphi_1) = \mu_0 \omega H_m \sin(\omega t - kx + \varphi_2),$$

$$kH_m \sin(\omega t - kx + \varphi_2) = \varepsilon_0 \omega E_m \sin(\omega t - kx + \varphi_1).$$

As these relationships must be satisfied at any of t and x, certain requirements are necessary:

a) initial phases must be equal, $\varphi_1 = \varphi_2$, and

b) following equalities must hold:

$$kE_m = \mu_0 \omega H_m \quad \text{and} \quad \varepsilon_0 \omega E_m = kH_m.$$

Multiplying these equations, we obtain:

$$\varepsilon_0 E_m^2 = \mu_0 H_m^2.$$

Thus, electric and magnetic fields oscillate with the same phase ($\varphi_1 = \varphi_2$), and amplitudes of the fields are related as

$$E_m \sqrt{\varepsilon_0} = H_m \sqrt{\mu_0}.$$

It is clear from the last expression that for a wave, propagating through empty space, the ratio of electric field intensity E to magnetic field intensity is

$$E_m / H_m = \sqrt{\mu_0 / \varepsilon_0} = \sqrt{(4\pi \times 10^{-7})(4\pi \times 9 \times 10^{9})} = 120\pi = 377\ \Omega.$$

Another interesting and useful result is:

$$\frac{E_m}{B_m} = \frac{E}{B} = c\,, \tag{4.18}$$

that is, at every instant of time, the ratio of the magnitude of the electric field E to the magnitude of the magnetic field $B = \mu_0 H$ in electromagnetic wave equals the speed of light c.

Finally, note that electromagnetic waves obey the superposition principle because the differential equations involving $\vec{E}$ and $\vec{B}$ are linear equations. For example, we can add two waves with the same frequency simply by adding the magnitudes of the two electric fields algebraically.

Let us summarize the properties of electromagnetic waves as we have described them:

1. The solutions of Maxwell's first and second (or third and fourth) equations are wave-like, with both E and H satisfying the wave equation.

2. Electromagnetic waves travel through empty space at the speed of light

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\,.$$

3. The components of the electric and magnetic fields of plane electromagnetic waves are perpendicular to each other and perpendicular to the direction of wave propagation. We can summarize the latter property by saying that electromagnetic waves are transverse waves.

4. The magnitudes of E and B in empty space are related by the expression

$$\frac{E}{B} = c\,.$$

5. Electromagnetic waves obey the principle of superposition.

Example 4.1

A sinusoidal electromagnetic wave of frequency 40.0 MHz travels in free space in the x direction, as shown in Figure 4.2.

a) Determine the wavelength and period of the wave.

Solution.

Using equation $c = \lambda f$ and given that $f = 40.0\ \text{MHz} = 4.0 \times 10^7\ \text{s}^{-1}$, we have

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8\ \text{m/s}}{4.0 \times 10^7\ \text{s}^{-1}} = 7.5\ \text{m}.$$

The period T of the wave is the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{4.0 \times 10^7\,\text{s}^{-1}} = 2.5 \times 10^{-8}\ \text{s}.$$

b) At some point and at some instant, the electric field has its maximum value of 750 N/C and is along the y axis. Find the magnitude and direction of the magnetic field at this position and time.

Solution.

We know that

$$B_m = \frac{E_m}{c} = \frac{750 \quad \text{N/C}}{3.0 \times 10^8 \text{ m/s}} = 2.5 \times 10^{-6} \text{ T.}$$

Because $\vec{E}$ and $\vec{B}$ must be perpendicular to each other and perpendicular to the direction of wave propagation (x in this case), we conclude that B is in the z direction.

c) Write expressions for the space-time variation of the components of the electric and magnetic fields for this wave.

Solution.

We can apply Eqs.(4.16) and (4.17) directly;

$$E = E_m \cos(\omega t - kx) = 750 \cos(\omega t - kx);$$

$$B = B_m \cos(\omega t - kx) = 2.50 \times 10^{-6} \cos(\omega t - kx).$$

Here,

$$\omega = 2\pi f = 2\pi(4.0 \times 10^{-7}) = 2.51 \times 10^8 \text{ rad/s,}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{7.50 \text{ m}} = 0.838 \text{ rad/m}.$$

Example 4.2

The electric field of a plane electromagnetic wave in vacuum is represented by

$$E_x = 0. \; E_y = 0.5 \cos[2\pi \times 10^8 (t - x/c)] \text{ and } E_z = 0.$$

a) What is the propagation direction of electromagnetic waves?

Solution.

a) The equation

$$E_y = 0.5 \cos[2\pi \times 10^8 (t - x/c)] \qquad \text{(a)}$$

indicates that the electromagnetic waves are propagating along the positive direction of x-axis.

b) Determine the wavelength of the wave.

Solution.

Comparing the equation (a) with the equation in standard form i.e. $E_y = E_0 \cos \omega(t - x/c)$, we get

$$\omega = 2\pi \times 10^8 \text{ or } 2\pi f = 2\pi \times 10^8, \text{ or } f = 10^8 \text{ Hz.}$$

Now,

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^8} = 3 \text{ m.}$$

c) Compute the component of associated magnetic field.

Solution.

The associated magnetic field is perpendicular to both the electric field and the direction of propagation. Since the wave is propagating along the x-axis and the electric field is along the y-axis, the magnetic field must be along z-axis. Hence, the components of associated magnetic field are

$$B_x = 0,\ B_y = 0 \text{ and } B_z = \frac{0.5}{3\times 10^8}\cos[2\pi\times 10^8(t - x/c)].$$

Exercies

4.1. We are surrounded by electromagnetic waves emitted by many radio and television stations. How are radio or television receivers able to select a single station among all this mishmash of waves? What happens inside a radio receiver when the dial is turned to change stations?

4.2. Write down expressions for the electric and magnetic fields of a sinusoidal plane electromagnetic wave having a frequency of 3.0 GHz and traveling in the positive x direction. The amplitude of the electric field is 300 V/m.

4.3. List as many similarities and differences between sound waves and light waves as you can.

4.4 The maximum electric field in the vicinity of a certain radio transmitter is $1.0\times 10^{-3}\ \text{V}\cdot\text{m}^{-1}$. What is the maximum magnitude of the $\vec{B}$ field? How does this compare in magnitude with the Earth's field?

4.5. A certain radio station broadcasts at a frequency of 1020 kHz. At a point some distance from the transmitter, the maximum magnetic field of the electromagnetic wave it emits is found to be 1.6×10^{-11} T.

a) What is the wavelength of the wave? (Ans. 294 m).

b) What is the maximum electric field? (Ans. 4.80×10^{-3} V/m).

4.6. Consider each of the electric and magnetic-field orientations given below. In each case, what is the direction of propagation of the wave?

$$\vec{E} = E\vec{i},\quad \vec{B} = B\vec{j};$$
$$\vec{E} = -E\vec{j},\quad \vec{B} = B\vec{i};$$
$$\vec{E} = E\vec{k},\quad \vec{B} = -B\vec{i};$$
$$\vec{E} = E\vec{j},\quad \vec{B} = -B\vec{k}.$$

4.7. A radar pulse returns to the receiver after a total travel time of 4.0×10^{-4} s. How far away is the object that reflected the wave?

4.8. If the North Star, Polaris, were to burn out today, in what year would it disappear from our vision? Take the distance from the Earth to Polaris as 6.44×10^{18} m (Ans. 2680 A.D.)

4.9. The amplitude of the electric field is 300 V/m. In SI units, the electric field in an electromagnetic wave is described by

$$E_y = 100\sin(10^7 x - \omega t).$$

4.10. Find (a) the amplitude of the corresponding magnetic field; (b) the wavelength λ; and (c) the frequency f. [Ans. (a) 0.333 μT; (b) 0.628 μm; (c) 477 THz].

4.4 Electromagnetic Waves in Matter

Thus far, we have discussed only electromagnetic waves in vacuum, but it is easy to extend our analysis to include electromagnetic waves in dielectrics. The wave speed now is not the same as in vacuum, so we denote it by v instead of c. Faraday's law is unaltered, but $E = cB$ is replaced by $E = vB$. In Ampere's law, the displacement-current density is given not by $\varepsilon_0 \frac{d\vec{E}}{dt}$ but by $\varepsilon\varepsilon_0 \frac{d\vec{E}}{dt}$. In addition, the constant μ_0 in Ampere's law must be replaced by $\mu\mu_0$ and the wave speed is given by

$$v = \frac{1}{\sqrt{\varepsilon\mu\varepsilon_0\mu_0}} = \frac{c}{\sqrt{\varepsilon\mu}}.$$

For many dielectrics, the permeability μ is practically equal to unity; in such a case, we have:

$$v = \frac{1}{\sqrt{\varepsilon\mu}}\frac{1}{\sqrt{\varepsilon_0\mu_0}} = \frac{c}{\sqrt{\varepsilon}}.$$

As permittivity ε is always greater than unity, the speed v of electromagnetic waves in a dielectric is always smaller than the speed c in vacuum by a factor of $\frac{1}{\sqrt{\varepsilon}}$. The ratio of the speed c in vacuum to the speed v in a material is known in optics as the *index of refraction* n of the material. For most dielectrics, where $\mu \approx 1$, n is given by

$$\frac{c}{v} = n = \sqrt{\varepsilon}.$$

Example 4.3

Electromagnetic waves travel in a medium with a speed of 2×10^8 m/s. The permeability μ of the medium is 1. Find the permittivity ε.

Solution.

Here, $v=2\times 10^8$ m/s and $\mu=1$. The speed of electromagnetic waves in a medium is given by

$$v=\frac{1}{\sqrt{\varepsilon_0\varepsilon\mu_0\mu}}, \qquad \text{(a)}$$

where μ and ε are permeability and permittivity of the medium.

Therefore, Eq. (a) becomes

$$v=\frac{1}{\sqrt{\mu_0\mu\varepsilon_0\varepsilon}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\times\frac{1}{\sqrt{\mu\varepsilon}}=\frac{c}{\sqrt{\mu\varepsilon}}.$$

$$\text{Or, } \varepsilon=\frac{c^2}{v^2\mu}=\frac{(3\times 10^8)^2}{(2\times 10^8)^2\times 1}=2.25.$$

Exercises

4.11. Determine the speed of light in water, which has a dielectric constant at optical frequencies of 1.78.

4.12. An electromagnetic wave in vacuum has electric field amplitude of 220 V/m. Calculate the amplitude of the corresponding magnetic field. (Ans. 733 nT).

4.13. Calculate the maximum value of the magnetic field of an electromagnetic wave in a medium where the speed of light is two thirds of the speed of light in vacuum and where the electric field amplitude is 7.60 mV/m.

4.14. Choose the correct answer: Maxwell's electromagnetic theory of light suggests that the light consists of oscillation of:

a) magnetic vector along;
b) electric vector along;
c) electric and magnetic vectors perpendicular to each other;
d) parallel electric and magnetic vectors.

4.5 Energy Carried by Electromagnetic Waves

Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to object placed in their path. Two simple examples are the energy of the Sun's radiation and cooking with microwave oven. To derive detailed relationship for the energy in an electromagnetic wave, we begin with the expression for the ***energy densities*** associated with electric and magnetic fields. Recall that the energy per unit volume, which is the instantaneous energy density associated with an electric field, for vacuum is given by

$$u_E = \frac{1}{2}\varepsilon_0 E^2. \tag{4.19}$$

And that the instantaneous energy density associated with magnetic fields is

$$u_B = \frac{B^2}{2\mu_0}. \tag{4.20}$$

Because E and B vary with time for an electromagnetic wave, the energy densities also vary with time. When we use the relationships

$$B = \frac{E}{c} \quad \text{and} \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}},$$

we obtain

$$u_B = \frac{\left(\frac{E}{c}\right)^2}{2\mu_0} = \frac{\mu_0\varepsilon_0}{2\mu_0}E^2 = \frac{1}{2}\varepsilon_0 E^2. \tag{4.21}$$

Comparing this result with the expression for u_E, we see that

$$u_B = u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{B^2}{2\mu_0}. \tag{4.22}$$

That is, for an electromagnetic wave, the instantaneous energy density associated with the magnetic field equals the instantaneous energy density associated with the electric field. Hence, in a given volume, the energy is equally shared by two fields.

The *total instantaneous energy density* u is equal to the sum of the energy densities associated with the electric and magnetic fields:

$$u = u_E + u_B = \varepsilon_0 E^2 = \frac{B^2}{\mu_0}. \tag{4.23}$$

As the E and B fields in a simple wave considered above advance with time into regions where originally no fields were present, it is clear that the wave transports energy from one region to another. We can describe this energy transfer in terms of energy transferred per unit time, per unit cross-sectional area for an area perpendicular to the direction of wave travel. This quantity will be denoted by S. It is analogous to the concept of current density, which is the charge per unit time transferred across unit area perpendicular to the direction of flow.

To see how the energy flow is related to the fields, consider a stationary plane perpendicular to the x-axis that coincides with the wave front at a certain time, Figure 4.3.

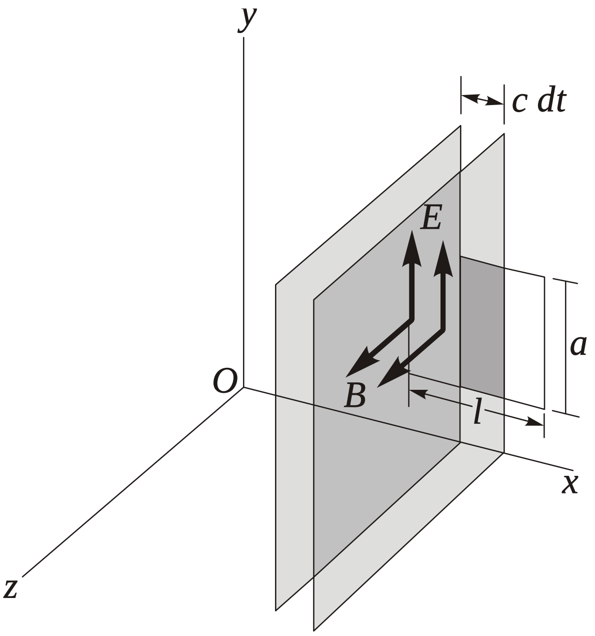


Figure 4.3 At time dt the wave front moves to the right a distance cdt

In a time dt after this, the wave front moves a distance cdt to the right. Considering an area A on the stationary plane, we note that the energy in the space to the right of this area must have passed through it to reach its new location. The volume dV of the relevant region is the base area A times the length cdt, and the total energy dU in this region is the energy density u times this volume:

$$dU = \varepsilon_0 E^2 Acdt\,. \tag{4.24}$$

Since this much energy passed through area A in time dt, the energy flow S per unit time, per unit area, is

$$S = \frac{1}{A}\frac{dU}{dt} = \varepsilon_0 c E^2\,.$$

Using Eq. (4.18)

$$B = \frac{E}{c} = \sqrt{\varepsilon_0 \mu_0}\,E\,,$$

we obtain the alternative forms

$$S = \frac{\varepsilon_0}{\sqrt{\varepsilon_0 \mu_0}} E^2 = \sqrt{\frac{\varepsilon_0}{\mu_0}} E^2 = \frac{EB}{\mu_0} = uc\,. \tag{4.25}$$

The unit of S is energy per unit time, per unit area. The SI unit of S is $1\,J/s\cdot m^2$ or $1\,W/m^2$.

We can define a vector quantity that describes both the magnitude and the direction of the energy-flow rate:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}\,. \tag{4.26}$$

$\vec{S}$ is called the *Pointing vector*; its magnitude is given by Eq. (4.25), and its direction is the direction of propagation of the wave. The magnitude $\frac{EB}{\mu_0}$ gives the flow of energy through a cross-section perpendicular to the direction of propagation, per unit area and per unit time. The total energy flow per unit time (power, P) through any surface is given by the integral

$$P = \oint_A \vec{S}\cdot d\vec{A} \tag{4.27}$$

over the surface. The electric and magnetic fields at any point in a wave vary with time, so the Pointing vector at any point is also a function of time.

What is of greater interest for a sinusoidal plane electromagnetic wave is the time average of S over one or more cycles, which is called the *wave intensity* I. When this average is taken, we obtain an expression involving the time average of $\cos^2(\omega t - kx)$ which equals $1/2$. Hence, the average value of S (in other words, the intensity of the wave) is

$$I = S_{av} = \frac{E_m B_m}{2\mu_0} = \frac{E_m^2}{2\mu_0 c} = \frac{c}{2\mu_0} B_m^2. \tag{4.28}$$

Comparing this result with expression for value of S, we see that

$$I = S_{av} = c u_{av}. \tag{4.29}$$

In other words, the intensity of an electromagnetic wave equals the average energy density multiplied by the speed of light.

Example 4.4

For a plane wave, suppose $E = 100$ V/m $= 100$ N/C. Find the value of B, the energy density, and the rate of energy flow per unit area A.

Solution.

From Eq. (4.18)

$$B = \frac{E}{c}$$

we obtain

$$B = \frac{E}{c} = \frac{100 \text{ V/m}}{3.0 \times 10^8 \text{ m/s}} = 3.33 \times 10^{-7} \text{ T}.$$

From Eq. (4.23) $u = u_E + u_B = \varepsilon_0 E^2$, it follows:

$$u = \varepsilon_0 E^2 = (8{,}85 \times 10^{-12} \text{C}^2/(\text{N}\cdot\text{m}^2))(100\text{N/C})^2 = 8.85 \times 10^{-8} \text{J/m}^3$$

$$A = EB/\mu_0 = (100 \text{ V/m})(3.33 \times 10^{-7})/(4\pi \times 10^{-7} \text{Wb/(A}\cdot\text{m}) =$$

$$= 26.5 \text{ V}\cdot\text{A/m} = 26.5 \text{ W/m}.$$

Example 4.5

The Sun delivers about 1000 W/m^2 of energy to the Earth's surface via electromagnetic radiation. Calculate the total power that is incident on a roof of dimensions $8.0 \text{ m} \times 20.0 \text{ m}$.

Solution.

The magnitude of the Pointing vector for solar radiation at the surface of the Earth is $S = 1000$ W/m^2; this represents the power per unit area, or the light intensity. Assuming that the radiation is incident normal to the roof, we obtain

$$P = SA = (1000 \text{ W/m}^2)(8.0 \times 20.0 \text{ m}^2) = 1.60 \times 10^5 \text{ W}.$$

If all this power could be converted to electrical energy, it would provide more than enough power for an average house. However, solar energy is not easily harnessed, and the prospects for large-scale conversion are not as bright as may appear from this calculation. For example, the efficiency of conversion from solar to electrical energy is typically 10% for photovoltaic cells. Roof systems for converting solar energy to internal energy are approximately 50% efficient; however, solar energy is associated with other practical problems, such as overcast days, geographic location, and methods of energy storage.

Exercises

4.15. A certain plane electromagnetic wave emitted by a microwave antenna has a wavelength of 3.0 cm and a maximum magnitude of electric field of $2.0\times 10^{2}\ \mathrm{V\cdot m^{-1}}$.

a) What is the frequency of the wave?

b) What is the maximum magnetic field?

c) What is the intensity (average power per unit area) of the wave, if the wave is sinusoidal?

4.16. A plane sinusoidal electromagnetic wave has a wavelength of 3.0 cm and an E-field amplitude of $30\ \mathrm{V\cdot m^{-1}}$.

a) What is the frequency?

b) What is the B-field amplitude?

c) What is the intensity?

4.17. Suggest reasons, why (a) food in metal containers cannot be cooked in a microwave oven; (b) an empty glass container does not get hot in a microwave oven. (Ans. In a microwave oven, the frequency of microwaves is selected to match the resonance frequency of water molecules so that the energy from the waves is transferred efficiently to the kinetic energy of the molecules. This rises the temperature of any food containing water.

(a) The atoms of the metallic container are set into forced vibrations by the microwaves. Due to this, energy of the microwaves is not efficiently transferred to the metallic containers. Owing to this, food in metallic containers cannot be cooked in a microwave oven.

(b) The molecules of the glass container do not respond to the frequency of microwaves. Due to this, energy from the microwaves is not transferred to the glass container and, hence, it does not get hot in a microwave oven.)

4.18. Describe the physical significance of the Pointing vector.

4.19. The energy flow to the Earth associated with sunlight is about 1.4 $\mathrm{kW/m^2}$. (a) Find the maximum values of E and B for a wave of this intensity. (b) Find the total power radiated by the Sun.

4.20. How much electromagnetic energy per cubic metre is contained in sunlight if the intensity of sunlight at the Earth's surface under a fairly clear sky is $1000\ \mathrm{W/m^2}$? (Ans. $3.33\ \mu\mathrm{J/m^2}$).

4.21. Some neodymium-glass lasers can provide 100 TW of power in 1.0 ns pulses at a wavelength of 0.26 μm. How much energy is contained in a single pulse?

4.22. What is the intensity of a plane traveling electromagnetic wave if B_m is 1.0×10^{-4} T?

4.23. In a plane radio wave, the maximum value of the electric field component is 5.0 V/m. Calculate (a) the maximum value of the magnetic field component and (b) the wave intensity. (Ans. (a) 16.7 nT; (b) 33.1 mW/m^2)

4.6 Intensity Variation of Spherical Waves with Distance

The intensity variation of electromagnetic waves varies with distance from a real source of electromagnetic radiation is usually a complex matter – especially when the source beams the radiation in a particular direction. However, in some situations we can assume that the source is a point source that emits the light *isotropically;* that is, with equal intensity in all directions. The spherical wavefronts spreading from such an isotropic point source S at a particular instant are shown in cross section in Figure 4.4.

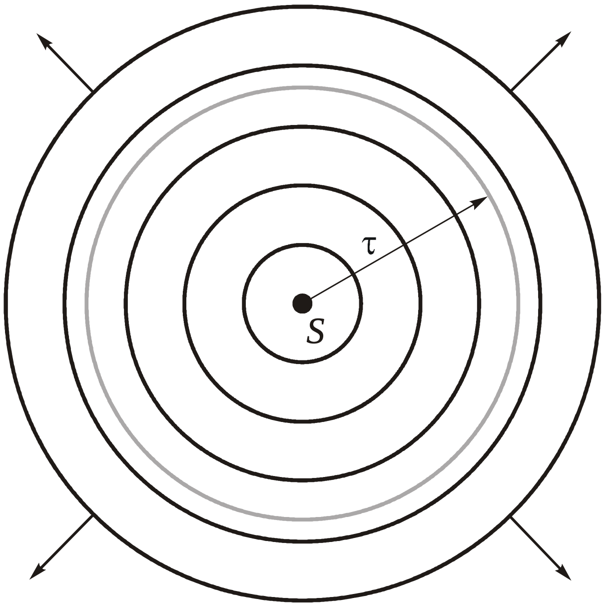


Figure 4.4 A point source S emits electromagnetic waves uniformly in all directions. The spherical wavefronts pass through an imaginary sphere of radius r that is centered on S

Let us assume that the energy of the waves is conserved as they spread from this source. Let us also center an imaginary sphere of radius r on the source, as shown in Figure 4.4. All the energy emitted by the source must pass through the sphere. Thus, the rate at which energy is transferred through the sphere by the radiation must equal the rate at which energy is emitted by the source – that is, the power P_s of the source. The intensity I at the sphere must then be

$$I = \frac{P_s}{4\pi r^2}, \qquad (4.30)$$

where $4\pi r^2$ is the area of the sphere. Eq. (4.30) tells us that the intensity of the electromagnetic radiation from an isotropic point source decreases with the square of the distance r from the source.

Example 4.6

Estimate the maximum magnitudes of the electric and magnetic fields of the light that is incident on this page because of the visible light coming from your desk lamp. Treat the bulb as a point source of electromagnetic radiation that is about 5% efficient at convening electrical energy to visible light.

Solution.

From Eq. (4.30), the wave intensity I at the distance r from a point source is $I = P_{sv}/4\pi r^2$ where P_s is the average power output of the source and $4\pi r^2$ is the area of a sphere of radius r centered on the source. Because the intensity of an electromagnetic wave is also given by Eq. (4.28), we have

$$I = \frac{P_s}{4\pi r^2} = \frac{E_m^2}{2\mu_0 c}.$$

We must now make some assumptions about numbers to enter in this equation. If we have a 60-W lightbulb, its output at 5% efficiency is approximately 3.0 W in the form of visible light. (The remaining energy transfers out of the bulb by conduction and invisible radiation.) A reasonable distance from the bulb to the page might be 0.30 m. Thus, we have

$$E_m = \sqrt{\frac{\mu_0 c P_{av}}{2\pi r^2}} = \sqrt{\frac{(4\pi \times 10^7\, T\cdot m/A)(3\times 10^8\, m/s)(3.0W)}{2\pi(0.30\,m)^2}} = 45 \text{ V/m.}$$

From Eq. (4.18)

$$B_m = \frac{E_m}{c} = \frac{45\, V/m}{(3\times 10^8\, m/s)} = 1.5\times 10^{-7} \text{ T.}$$

This value is two orders of magnitude smaller than the Earth's magnetic field, which, unlike the magnetic field in the light wave from your desk lamp, is not oscillating.

Exercises

4.24. At a distance of 50 km from a rádio station antenna, the electric-field amplitúde is measured to be $E_{\max} = 2\times 10^{-2}$ V/m.

a) What is the magnetic-field amplitúde $B_{\max}$ at this same point?

b) Assuming that the antenna radiates equally in all directions (which is probably not the case), what is the total power output of the station?

c) At what dištance from the antenna would $E_{\max} = 1\times 10^{-2}$ V/m, half the above value?

4.25. Estimate the energy density of the light wave just before it strikes this page. (Ans. 9.0×10^{-9} J/m^3.)

4.26. An airplane flying at a distance of 10 km from a radio transmitter receives a signal of intensity 10 μW/m^2. Calculate (a) the amplitude of the electric field at the airplane due to this signal, (b) the amplitude of the magnetic field at the airplane, and (c) the total power of the transmitter, assuming the transmitter to radiate uniformly in all directions. (Ans. (a) 87 mV/m; (b) 0.30 nT; (c) 13 kW.)

4.7 Momentum and Radiation Pressure

Electromagnetic waves transport linear momentum as well as energy. It follows that, as this momentum is absorbed by some surface, pressure is exerted on this surface. It can be shown by the following example. Let the electromagnetic wave strike the surface at normal incidence and transports a total energy U to the surface with $\varepsilon = 1$ and $\mu = 1$ in a time t. The electric field of the wave creates current of density $\vec{j} = \sigma \vec{E}$ in the body. The magnetic field of the wave will exert on every charge carrier of the body with the force $\vec{F} = q\vec{v} \times \vec{B}$, where q is the charge of the charge carrier, $\vec{v}$ is its speed. The force exerted on all n charge carries in a unit volume is defined by the expression:

$$\vec{F} = nq\vec{u} \times \vec{B} = (nq\vec{u}) \times \vec{B} = \vec{j} \times \vec{B},$$

where $\vec{j} = nq\vec{u}$ is a current density. As it is clear, the direction of the force is the same as direction of the wave propagation.

The momentum dp delivered to the surface layer of the unit area and the thickness dl per unit time

$$dp = Fdl = \mu_0 jHdl \tag{4.31}$$

(vectors $d\vec{l}$ and $\vec{B}$ are mutually perpendicular). During the unit time this layer absorbs the energy

$$dU = jEdl \tag{4.32}$$

which transforms in the form of heat.

The momentum dp (4.31) and the energy dU (4.32) are delivered to the layer by that part of the wave which is absorbed by this layer. Taking their ratio, we obtain

$$\frac{dp}{dU} = \frac{\mu_0 jHdl}{jEdl} = \mu_0 \frac{H}{E}.$$

Recalling that $\mu_0 H^2 = \varepsilon_0 E^2$, we can write:

$$\frac{dp}{dU} = \sqrt{\varepsilon_0 \mu_0} = \frac{1}{c}.$$

Hence, the electromagnetic wave of the energy dU has the momentum dp as well, and the relation between them is

$$dp = \frac{dU}{c} \text{ or } p = \frac{U}{c}. \tag{4.33}$$

From Eq. (4.33), it follows that the momentum density (i.e., momentum per unit volume) of electromagnetic field is

$$p_{un.vol.} = \frac{1}{c} u. \tag{4.34}$$

The energy density u is related to the module of Pointing vector by the relation $S = uc$. Substituting $u = S/c$ in Eq. (4.34) and taking into account that vectors $\vec{p}$ and $\vec{S}$ coincide in direction, we obtain

$$p_{un.vol.} = \frac{1}{c^2}S = \frac{1}{c^2}\vec{E}\times\vec{H}.$$

Suppose the incident wave is absorbed by the body completely. Then a unit surface of the body obtains additional energy per unit time, which is included in the cylinder of unit base and height c. According to Eq. (4.34), this momentum equals $(u/c)c = u$. The momentum delivered to the unit surface per unit time equals the pressure P at the surface, $P = u$. This quantity oscillates with high frequency, and hence, of practical importance, is its average in time magnitude:

$$P = \langle u \rangle .$$

Maxwell showed that, if the surface absorbs all the incident energy U in this time, the total momentum p transported to the surface has a magnitude

$$p = \frac{U}{c} \qquad \text{(Complete absorption)} \qquad (4.35)$$

The pressure exerted on the surface is defined as force per unit area F/A. Let us combine this with Newton's second law to obtain

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt}.$$

If we now replace p, the momentum transported to the surface by light, from (4.33), we get

$$P = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{U}{c}\right) = \frac{1}{c}\frac{\left(dU/dt\right)}{A}.$$

We recognize $(dU/dt)/A$ as the rate at which energy is arriving at the surface per unit area, which is the magnitude of the Pointing vector. Thus, the radiation pressure P exerted on the perfectly absorbing (with reflectivity $r = 0$) surface is

$$P = \frac{S}{c} = \frac{I}{c}. \qquad (4.36)$$

If the surface is a perfect reflector (such as mirror, with reflectivity $r = 1$) and the incidence is normal, then the momentum transported to the surface at time t is twice that given by Eq. (4.35). That is, the momentum transferred to the surface by the incoming light is $p = U/c$, and that transferred by the reflected light also is $p = U/c$. Therefore:

$$p = \frac{2U}{c}. \qquad \text{(Complete reflection)} \qquad (4.37)$$

Finally, the radiation pressure exerted on a perfectly reflecting surface for normal incidence of the wave is:

$$P=\frac{2S}{c}=\frac{2I}{c}. \qquad (4.38)$$

In a general case, the momentum delivered to a surface having a ***reflectivity*** ρ somewhere between these two extremes has a value between U/c and $2U/c$, depending on the properties of the surface, namely, on the reflectivity ρ. Finally, the radiation pressure exerted on a surface for arbitrary incidence is:

$$P=u(1+\rho)\cos^2\alpha=\frac{I}{c}(1+\rho)\cos^2\alpha \qquad (4.39)$$

where u is the volume energy density, ρ is the reflectance coefficient of the surface, α is the angle of incidence, I is the intensity of light.

Radiation pressure is important in the structure of stars. Gravitational attractions tend to shrink a star, but this tendency is balanced by radiation pressure in maintaining the size of the star through most stages of its evolution.

The pressure of the sun's radiation is responsible for pushing the tail of a comet away from the Sun (Figure 4.5).

Figure 4.5 The comet. The tail of the comet is pushed away from the Sun and split into two distinct parts by radiation pressure from the Sun's electromagnetic radiation and by the "solar wind," a stream of particles emitted by the Sun

Although radiation pressures are very small (about 5×10^{-6} N/m^2 for direct sunlight), they have been measured with torsion balances such as the one shown in Figure 4.6. A mirror (a perfect reflector) and a black disk (a perfect absorber) are connected by a horizontal rod suspended from a fine fiber. Normal-incidence light striking the black disk is completely absorbed, so all of the momentum of the light is transferred to the disk. Normal-incidence light striking the mirror is totally reflected, and, hence, the momentum transferred to the mirror is twice as great as that transferred to the disk. The radiation pressure is determined by measuring the angle through which the horizontal connecting rod rotates. The apparatus must be placed in a high vacuum to eliminate the effects of air currents.

Example 4.7

A great amount of dust exists in interplanetary space. Although in theory, these dust particles can vary in size from that of a molecule to much larger, very little dust in our solar system is smaller than about 0.2 μm. Why?

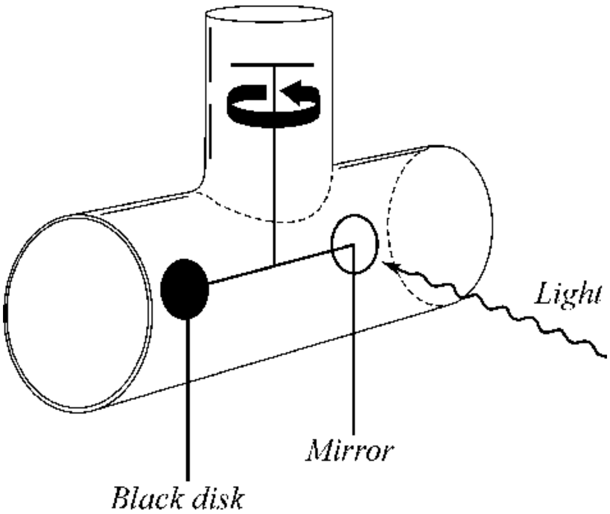


Figure 4.6 An apparatus for measuring the pressure exerted by light. In practice, the system is contained in a high vacuum

Solution.

Dust particles are subject to two significant forces, the gravitational force that draws them toward the Sun and the radiation-pressure force that pushes them away from it. The gravitational force is proportional to the cube of the radius of a spherical dust particle because it is proportional to the mass and, therefore, to the volume $4\pi r^3/3$ of the particle. The radiation pressure is proportional to the square of the radius because it depends on the planar cross-section of the particle. For large particles, the gravitational force is greater than the force from radiation pressure. For particles having radii less than about 0.2 μm, the radiation-pressure force is greater than the gravitational force, and, as a result, these particles are swept out of the Solar System.

Example 4.8

Many people giving presentations use a laser pointer to direct the attention of the audience. If a 3.0 mW pointer creates a spot that is 2.0 mm in diameter, determine the radiation pressure on a screen that reflects 70% of the light that strikes it. The power of 3.0 mW is a time-averaged value.

Solution.

We certainly do not expect the pressure to be very large. Before we can calculate it, we must determine the Pointing vector of the beam by dividing the time-averaged power delivered via the electromagnetic wave by the cross-sectional area of the beam:

$$S = \frac{P}{A} = \frac{P}{\pi r^2} = \frac{3.0\times 10^{-3}\,\text{W}}{\pi\left(\dfrac{2.0\times 10^{-3}\,\text{m}}{2}\right)^2} = 955\ \text{W/m}^2.$$

This is about the same as the intensity of sunlight at the Earth's surface. (Thus, it is not safe to shine the beam of a laser pointer into a person's eyes; that may be more dangerous than looking directly at the Sun.)

Now we can determine the radiation pressure from the laser beam. Eq. (4.38) indicates that a completely reflected beam would apply a pressure of $P = 2S/c$. We can model the actual reflection as follows: Imagine that the surface absorbs the beam, resulting in pressure $P = S/c$. Then the surface emits the beam, resulting in additional pressure $P = S/c$. If the surface emits only a fraction r of the beam (so that r is the amount of the incident beam reflected), then the pressure due to the emitted beam is $P = rS/c$. Thus, the total pressure on the surface due to absorption and re-emission (reflection) is

$$P = \frac{S}{c} + r\frac{S}{c} = (1+r)\frac{S}{c}.$$

Notice that if $r = 1$, which represents complete reflection, this equation to reduces to Eq.(4.38). For a beam 70% reflected, the pressure is

$$P = (1+0.70)\frac{955\ \text{W/m}^2}{3.0\times 10^8\ \text{m/s}} = 5.4\times 10^{-6}\ \text{N/m}^2.$$

This is an extremely small value, as expected. (Recall that atmospheric pressure is approximately 10^5 N/m^2).

Exercises

4.27. For a given incident energy of an electromagnetic wave, why is the radiation pressure on a perfectly reflecting surface twice is great as that on a perfectly absorbing surface?

4.28. If the intensity of direct sunlight is 1.4 kW/m^2, find:

a) The momentum density (momentum per unit volume); (Ans. $1.56\times 10^{-14}\ \text{kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$).

b) The momentum flow rate (momentum carried through a surface area A in unit time) in the sunlight. (Note: This equals the radiation pressure.) (Ans. 4.67×10^{-6} Pa).

4.29. The intensity of a bright light source is 900 W/m^2. Find the radiation pressure (in paschal) on

a) a totally absorbing surface,

b) a totally reflecting surface.

4.30. A radio wave transmits 25.0 W/m^2 of power per unit area. A flat surface of area A is perpendicular to the propagation direction of the wave. Calculate the radiation pressure on it if the surface is a perfect absorber.

4.31. A plane electromagnetic wave of intensity 6.00 W/m^2 strikes a small pocket mirror, of area 40.0 cm^2, held perpendicular to the approaching wave. (a) What momentum does the wave transfer to the mirror each second? (b) Find the force that the wave exerts on the mirror.

4.32. A 100-mW laser beam is reflected back upon itself by a mirror. Calculate the force on the mirror.

4.33. Given that the intensity of solar radiation incident on the upper atmosphere of the Earth is 1 340 W/m^2, determine (a) the solar radiation incident on Mars; (b) the total power incident on Mars; and (c) the total force acting on the planet. (d) Compare this force to the gravitational attraction between Mars and the Sun.

4.34. A plane electromagnetic wave has an intensity of 750 W/m^2. A flat rectangular surface of dimensions $50.0\,\text{cm}\times 100\,\text{cm}$ is placed perpendicular to the direction of the wave. If the surface absorbs half of the energy and reflects half, calculate

a) the total energy absorbed by the surface in 1.0 min; (Ans. 11.3 kJ).

b) the momentum absorbed in this time. (Ans. 1.13×10^{-4} kg·m/s).

4.35. A plane sinusoidal electromagnetic wave has a wavelength of 3.0 cm and an E-field amplitude of 30 V/m.

a) What is the frequency?

b) What is the B-field amplitude?

c) What is the intensity?

d) What average force does this radiation exert on a totally absorbing surface of area 0.5 m^2 perpendiculars to the direction of propagation?

4.36. The energy flow to the Earth associated with sunlight is about $1.4\ \text{kW/m}^2$.

a) Find the maximum values of E and B for a wave of this intensity.

b) The distance from the Earth to the Sun is about 1.5×10^{11} m. Find the total power radiated by the sun.

4.37. For a sinusoidal electromagnetic wave in vacuum, show that the average density of energy in the electric field is the same as that in the magnetic field.

4.8 Standing Waves

Electromagnetic waves can be reflected; a conducting surface can serve as a reflector. The superposition principle holds for electromagnetic waves just as for all electric and magnetic fields, and the superposition of an incident wave and a reflected wave can form a *standing wave.* The situation is analogous to standing waves on a stretched string.

Suppose a sheet of an ideal conductor, having zero resistivity, is placed in the yz-plane, and the wave traveling in the negative x-direction is incident on it. The essential characteristic of an ideal conductor is that no electric field can ever exist within it; any attempt to establish a field is immediately canceled by the rearrangement of the mobile charges in the conductor. Thus, $\vec{E}$ must always be zero everywhere in this plane, and the $\vec{E}$ field of the incident wave induces sinusoidal currents in the conductor so that $\vec{E}$ is zero inside it.

These induced currents produce a reflected wave, traveling out from the plane to the right. From the superposition principle, the total $\vec{E}$ field at any point to the right of the plane is the vector sum of the $\vec{E}$ fields of the incident and reflected waves; the same is true for the total $\vec{B}$ field.

Suppose the incident wave is described by the wave functions of equation

$$E = E_m \sin(\omega t - kx)$$

and the reflected wave by the wave functions of equation

$$E = -E_m \sin(\omega t + kx)].$$

From the superposition principle, the total fields at any point are given by

$$E = E_m[-\sin(\omega t + kx) + \sin(\omega t - kx)],$$

$$B = B_m[\sin(\omega t + kx) + \sin(\omega t - kx)].$$

These expressions may be expanded and simplified by using the identity

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

The results are

$$E = -2E_m \cos\omega t \sin kx, \qquad (4.40)$$

$$B = 2B_m \sin\omega t \cos kx \qquad (4.41)$$

The former is analogous to the equation for a stretched string. We see that at $x = 0$, E is *always* zero; this condition is required by the nature of the ideal conductor, which plays the same role as a fixed point at the end of the string. Furthermore, E is zero at all times in those planes for which $\sin kx = 0$; that is,

$$kx = 0, \pi, 2\pi, \ldots, \text{ or}$$

$$x = 0. \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots. \qquad (4.42)$$

These are called the *nodal planes* of the $\vec{E}$ field.

The total magnetic field is zero at all times in those planes for which $\cos kx = 0$. or at which

$$x = \frac{\lambda}{4}, 3\frac{\lambda}{4}, 5\frac{\lambda}{4}, \ldots \qquad (4.43)$$

These are the nodal planes of the $\vec{B}$ field. The magnetic field is *not* zero at the conducting surface ($x = 0$), and there is no reason it should be. The nodal planes of one field are midway between those of the other, and the nodal planes of either field are separated by one-half wavelength. Figure 4.7 shows a standing-wave pattern at one instant of time.

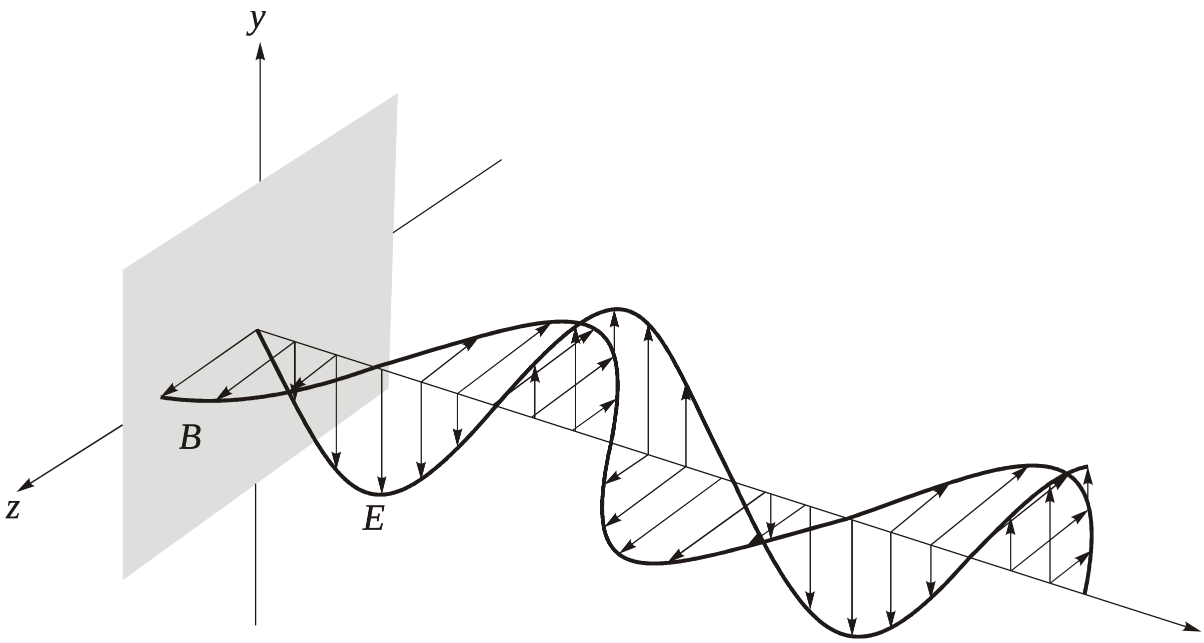


Figure 4.7 $\vec{E}$ and $\vec{B}$ vectors in a standing wave. The pattern does not move along the x-axis, but the $\vec{E}$ and $\vec{B}$ vectors grow and diminish with time at each point. At each point $\vec{E}$ is maximum when $\vec{B}$ is minimum, and conversely. The position of the wave at time $t = 0$ is shown

The total electric field is a cosine function of t, and the total magnetic field is a sine function of t. The fields are, therefore. 90° out of phase. At times when $\cos \omega t = 0$, the electric field is zero *everywhere* and the magnetic field is maximum. When $\sin \omega t = 0$, the magnetic field is zero *everywhere* and the electric field is maximum.

Pursuing the stretched-string analogy, we may now insert a second conduction plane (parallel to the first and a distance L from it) along the $+x$-axis. This is analogous to the stretched string held at the points $x = 0$ and $x = L$. A standing wave can exist only when L is an integer multiple of $\lambda / 2$. Hence, the possible wavelengths are

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \ldots \tag{4.44}$$

and the corresponding frequencies are

$$f_n = \frac{c}{\lambda_n} = n\frac{c}{2L}, \quad n = 1, 2, 3, \ldots \tag{4.45}$$

Thus, there is a set of *normal modes,* each with a characteristic frequency, wave shape, and node pattern. The measurement of the node positions makes it possible to measure the wavelength. If the frequency is known, the wave speed can be determined. This technique was, in fact, used by Hertz in his pioneering investigations of electromagnetic waves.

Conducting surfaces are not the only reflectors of electromagnetic waves; reflections also occur at an interface between two insulating materials having different dielectric or magnetic properties. The mechanical analog is a junction of two strings with equal tension but different linear mass density. In general, a wave incident on such a boundary surface is partly transmitted into the second material and partly reflected back into the first one. The partial transmission and reflection of light at a glass surface is a familiar phenomenon; light is transmitted through a sheet of glass, but its surfaces also reflect light.

Exercises

4.38. For a standing wave given by Eqs. (4.40) and (4.41),

a) Plot the energy density as a function of x, $0 < x < \pi / k$ for the times $t = 0, \pi / 4\omega, \pi / 2\omega, 3\pi / 4\omega$ and π / ω.

b) Find the direction of $\vec{S}$ in the regions $0 < x < \pi / 2k$ and $\pi / 2k < x < \pi / k$ at the times $t = \pi / 4\omega$ and $t = 3\pi / 4\omega$.

c) Use your results in (b) to explain the plots obtained in (a).

4.9 Radiation from a Dipole

Plane waves are the simplest of all electromagnetic waves to be described and analyzed, but they are not the simplest to produce experimentally. Any charge or current distribution that oscillates sinusoidally with time produces sinusoidal electromagnetic waves, but in general there is no reason to expect them to be plane waves.

The simplest example of an oscillating charge distribution is an *oscillating dipole* which is a pair of electric charges of equal magnitude and opposite sign, $+q$ and $-q$, separated by distance l and characrerized by electric dipole moment

$$\vec{p} = q\vec{l} \tag{4.46}$$

where $\vec{l}$ is a position-vector joining negative and positive charges (arm of dipole) and q is magnitude of charges.

Such an oscillating dipole can be constructed in various ways, but we need not be concerned with the details. The radiation from an oscillating dipole is not a plane wave, but it travels out in all directions from the source. Because the dipole fields fall off as $1/r^3$, they are not important at great distances from the antenna. However, at these great distances, something else causes a type of radiation different from that close to the antenna. The source of this radiation is the continuous induction of an electric field by the time-varying magnetic field and the induction of a magnetic field by the time-varying electric field. The electric and magnetic fields produced in this manner are in phase with each other and vary as $1/r$. The result is an outward flow of energy at all times.

At points far from the source, the $\vec{E}$, and $\vec{B}$ fields are perpendicular to the direction from the source and to each other; in this sense the wave is still transverse. The value of $\vec{S}$ drops off as the square of the distance from the source. The average value of $\vec{S}$ (intensity) depends on the direction from the source; it is greatest at directions perpendicular to the dipole axis, and $S = 0$ at directions parallel to the axis.

Let the electric dipole moment of system vary with time according to harmonica law:

$$p = ql\cos\omega t = p_m \cos\omega t \tag{4.47}$$

where p_m is the amplitude of vector $\vec{p}$.

Consider radiation from dipole in assumption that its size is much smaller than the wavelength of radiation ($l << \lambda$). Such a dipole is called an *elementary dipole,* or a *point dipole*. In the vicinity of the dipole, the character of the electromagnetic field is complex enough, but it simplifies in a so-called *wave zone,* that is, at distances $r >> \lambda$. In any points vectors $\vec{E}$ and $\vec{B}$ are perpendicular to the ray, that is, to the position-vector from center of dipole to the given point. In any point, vectors $\vec{E}$ and $\vec{B}$ vary according to $\cos(\omega t - kx)$. The amplitudes E_m and H_m depend on the distance r from the radiator and the angle θ between the vector $\vec{r}$ and axis of dipole. For vacuum, this dependence has the form

$$E_m \propto \frac{\sin\theta}{r} \quad \text{and} \quad H_m \propto \frac{\sin\theta}{r}.$$

Average value of Pointing vector $\langle S\rangle$, that is intensity, is proportional to the product $E_m H_m$, hence,

$$\langle S\rangle \propto \frac{\sin^2\theta}{r^2}. \tag{4.48}$$

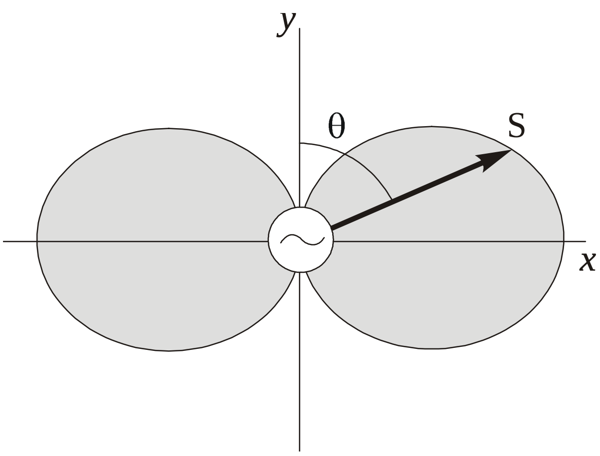


Figure 4.8 Angular dependence of the intensity of radiation produced by an oscillating electric dipole

From Eq. (4.48), it follows that the intensity drops along the ray as $1/r^2$. Next, the intensity is maximum in a plane which is perpendicular to the axis of antenna ($\theta = \pi/2$) and passes through its midpoint. Furthermore, it is zero along the axis of an antenna where $\theta = \pi$ or $\theta = 0$. It means that in the direction of the axis, dipole doesn't radiate. The dependence of the intensity on the angle θ, which is known as *directional pattern*, or *antenna pattern*, is shown on the Figure 4.8.

Mathematical analysis proves that radiation power of the dipole (that is, energy radiated over all the directions per unit time) is proportional to the squared second derivative of dipole moment with time

$$P = \frac{\mu_0}{6\pi c}\left|\frac{d^2 p}{dt^2}\right|^2 . \qquad (4.49)$$

In accordance with Eq. (4.47)

$$\frac{d^2 p}{dt^2} = -\omega^2 p = -\omega^2 p_m \cos^2 \omega t .$$

Therefore,

$$P = \frac{\mu_0}{6\pi c}\omega^4 p_m^2 \cos^2 \omega t^2 ,$$

that is,

$$P \propto p_m^2 \omega^4 \cos^2 \omega t .$$

During one complete oscillation, the average power of the dipole radiation is:

$$\langle P \rangle = \frac{1}{T}\int_0^T P\,dt = \frac{\mu_0}{12\pi c}\omega^4 p_m^2 ,$$

that is,

$$\langle P \rangle \propto p_m^2 \omega^4 . \qquad (4.50)$$

From the above expression, it is clear that the average power of the dipole radiation is proportional to the p_m^2 and ω^4. Thus, at low frequencies, radiation of an electrical system (for example, AC transmitting lines) is insignificant.

The radiation pattern from a dipole source is shown schematically in Figure 4.9. The figure shows a cross section of the radiation pattern at one instant. The oscillating dipole P is located at the centres of the spheres. At all points in the plane of the figure, the $\vec{E}$ field lies in the plane and the $\vec{B}$ field is perpendicular to it. The $\vec{E}$ field is shown by arrows, and the direction of $\vec{B}$ by crosses (where it points into the plane) and circles with dots (where it points out of the plane). It is easy to verify that the direction of the Poynting vector $\vec{S}$ is radially outward from the source at every point.

As we have discussed, electromagnetic waves can be reflected by conducting surfaces. When the surface is large compared to the wavelength of the radiation, the reflection behaves like reflection of light rays from a mirror. Large parabolic mirrors several meters in diameter are used as both transmitting and receiving antennas for microwave communications signals; typical wavelengths are a few centimeters. A transmitting reflector produces a wave that radiates in a

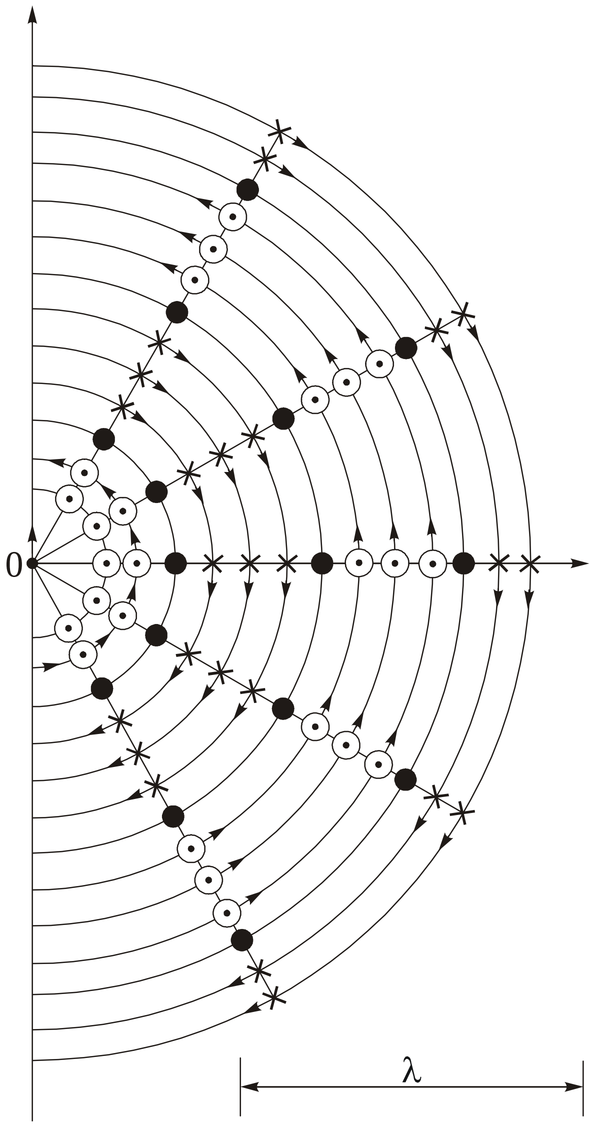


Figure 4.9 Cross section in the xz-plane of radiation from an oscillating electric dipole P. The wave fronts are expanding concentric spheres centered at P. At every point the $\vec{E}$ field lies in the plane, and the $\vec{B}$ field is perpendicular to it. At points with circles, $\vec{B}$ comes out of the plane, and at points with crosses, it is into the plane. The direction of the Poynting vector $\vec{S}$ is radially outward at every point

narrow, well-defined beam; a receiving reflector gathers wave energy over its whole area and reflects it to the focus of parabola where a detecting device is placed.

According to (4.46), $\frac{d^2\vec{p}}{dt^2} = q\frac{d^2\vec{l}}{dt^2} = -q\vec{a}$ where $\vec{a}$ is acceleration of oscillating charge. Substituting above expression into Eq. (4.49), we obtain

$$P \propto q^2 a^2. \qquad (4.51)$$

Eq. (4.51) defines the radiation power not only for oscillations but for arbitrary motion of charged particle. It follows that any charge moving with acceleration excites electromagnetic waves and its radiation power is proportional to the squared charge and squared acceleration.

The charge executing SHM radiates a monochromatic wave of frequency which matches the frequency of charge oscillation. But if the charge acceleration $\vec{a}$ varies according to the nonharmonic law, radiation consists of a set of waves with different frequencies.

According to Eq. (4.51), intensity drops to zero at $\vec{a} = 0$. Hence, an electron moving uniformly does not emit electromagnetic waves. Nevertheless, this is valid only for situations when a speed of electron v_e does not exceed the speed of light v_l in the medium $v_l = c/\sqrt{n} = c/\sqrt{\varepsilon\mu}$ where electron moves. For the case $v_e > v_l = c/\sqrt{\varepsilon\mu}$, a very special kind of radiation, called Vavilov-Cherenkov radiation, is observed. This kind of radiation we will discuss in our book “Wave Optics".

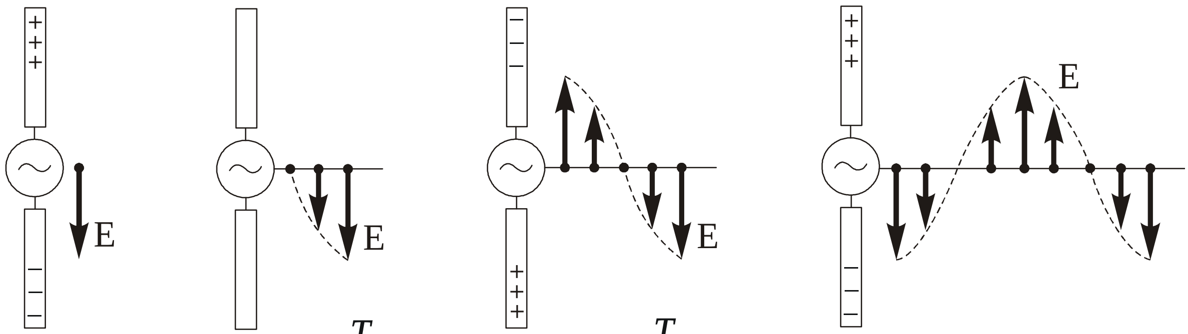
4.39. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron with a radius of 0.500 m and a magnetic field with a magnitude of 0.350 T.

4.40. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron of radius R and magnetic field B.

4.10 Production of Electromagnetic Waves by an Antenna

Neither stationary charges nor steady currents can produce electromagnetic waves. However, whenever the current going through a wire changes with time, the wire emits electromagnetic radiation. The fundamental mechanism responsible for this radiation is the acceleration of a charged particle. Whenever a charged particle accelerates, it must radiate energy.

An alternating voltage applied to the wires of an antenna forces an electric charge in the antenna to oscillate. This is a common technique for accelerating charges and is the source of the radio waves emitted by the transmitting antenna of a radio station. Figure 4.10 shows how this is done. Two metal rods are connected to a generator that provides a sinusoidally oscillating voltage. This causes charges to oscillate in two rods. At $t = 0$, the upper rod is given a maximum positive charge and the bottom rod an equal negative charge, as shown in Figure 4.10a. The electric field near the antenna at this instant is also shown in Figure 4.10a.



(*a*) $t = 0$ (*b*) $t = \dfrac{T}{4}$ (*c*) $t = \dfrac{T}{2}$ (*d*) $t = T$

Figure 4.10 The electric field set up by charges oscillating in an antenna. The field moves away from the antenna with the speed of light

As positive and negative charges decrease from their maximum values, the rods become less charged, the field near the rods decreases in strength, and the downward-directed maximum electric field produced at $t = 0$ moves away from the rod (A magnetic field oscillating in a direction perpendicular to the plane of the diagram in Figure 4.10 accompanies the oscillating electric field, but it is not shown for the sake of clarity.) When the charges on the rods are momentarily zero (Figure 4.10b), the electric field at the rod dropps to zero. This occurs at a time equal to one quarter of the period of oscillation.

As the generator charges the rods in the opposite sense from that at the beginning, the upper rod soon obtains a maximum negative charge and the lower rod a maximum positive charge (Figure 4.10c); this results in an electric field near the rod that is directed upward after a time equal to one-half the period of oscillation. The oscillations continue as indicated in Figure 4.10d. The electric field near the antenna oscillates in phase with the charge distribution. That is, the field points down when the upper rod is positive and up when the upper rod is negative. Furthermore, the magnitude of the field at any instant depends on the amount of charge on the rods at that instant.

As the charges continue to oscillate (and accelerate) between the rods, the electric field they set up moves away from the antenna at the speed of light. As you can see from Figure 4.10, one cycle of charge oscillation produces one wavelength in the electric-field pattern.

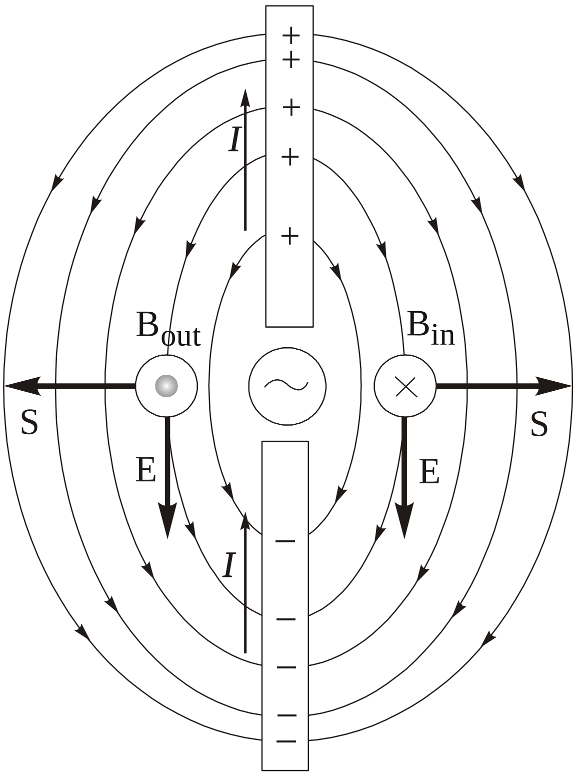


Figure 4.11 A half-wave antenna consists of two metal rods connected to an alternating voltage source. This diagram shows $\vec{E}$ and $\vec{B}$ at an instant when the current is upward. Note that the electric field lines resemble those of a dipole

Now let us consider the production of electromagnetic waves by a *half-wave antenna.* In this arrangement, two conducting rods are connected to a source of alternating voltage (such as an *LC* oscillator), as shown in Figure 4.11. The length of each rod is equal to one quarter of the wavelength of the radiation that will be emitted when the oscillator operates at frequency f. The oscillator forces charges to accelerate back and forth between the two rods. Figure 4.11 shows the configuration of the electric and magnetic fields at some instant when the current is upward. The electric field lines resemble those of an electric dipole. (As a result, this type of antenna is sometimes called a *dipole antenna.)* Because these charges are continuously oscillating between the two rods, the antenna can be approximated by an oscillating electric dipole. The magnetic field lines form concentric circles around the antenna and are perpendicular to the electric field lines at all points. The magnetic field is zero at all points along the axis of the antenna. Furthermore, $\vec{E}$ and $\vec{B}$ are 90° out of phase at time because the current is zero when the charges at the outer ends of the rods are at a maximum.

At the two points where the magnetic field is shown in Figure 4.11, the Pointing vector $\vec{S}$ is directed radially outward. This indicates that energy is flowing away from the antenna at this instant. At later times, the fields and the Poynting vector change direction as the current alternates.

The electric field lines produced by a dipole antenna at some instant are shown in Figure 4.12 as they propagate away from the antenna. Note that the intensity and the power radiated are a maximum in a plane that is perpendicular to the antenna and passing through its midpoint. Furthermore, the power radiated is zero along the antenna's axis.

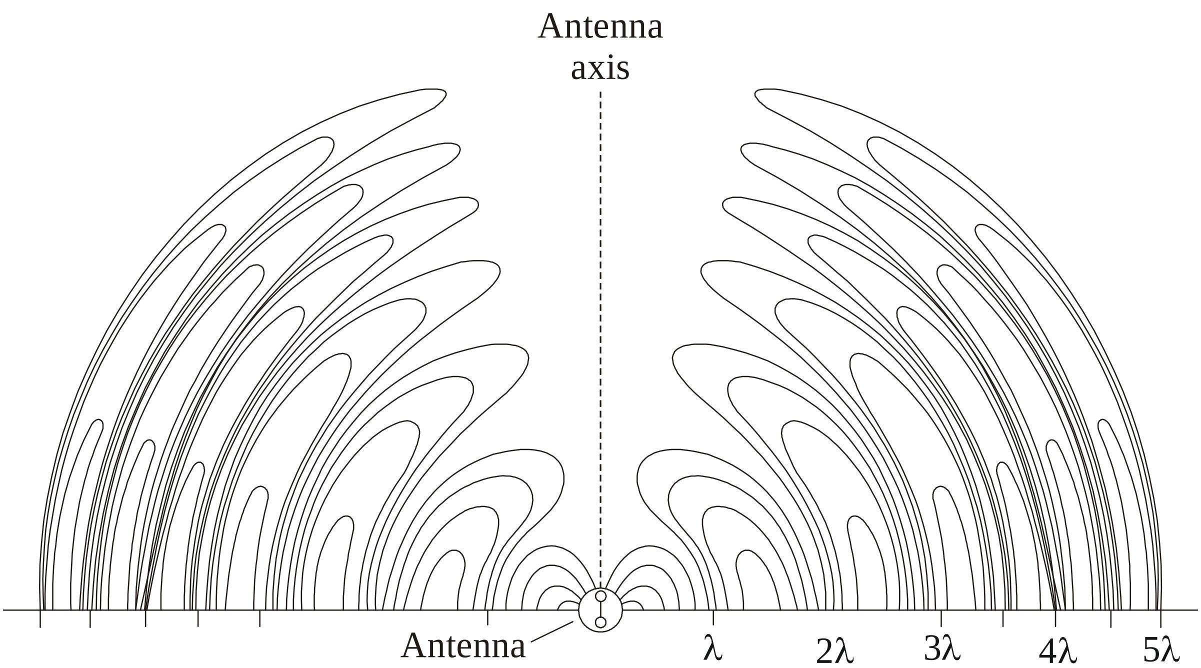


Figure 4.12 Electric field lines surrounding a dipole antenna at a given instant. The radiation fields propagate outward from the antenna with a speed c

Electromagnetic waves can also induce currents in a receiving antenna. The response of a dipole receiving antenna at a given position is a maximum when the antenna axis is parallel to the electric field at that point and zero when the axis is perpendicular to the electric field.

Example 4.9

A half-wave antenna works on the principle that its optimal length is one-half the wavelength of the radiation being received. What is the optimal length of a car antenna when it receives a signal of frequency 94.0 MHz?

Solution.

Wavelength of the signal is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{8}\,\text{m/s}}{9.4 \times 10^{7}\,\text{Hz}} = 3.19\ \text{m.}$$

Thus, to operate most efficiently, the antenna should have a length of (3.19 m)/2 = 1.60 m.

For practical reasons, car antennas are usually one-quarter wavelength in size.

Example 4.10

What is the order of magnitude of the minimum frequency of electromagnetic waves that can be used to detect the presence of (a) the planet Venus; (b) an aircraft 50 m long; (c) a bird 0.1 m long? From what sources of electromagnetic radiation, are you able to generate radiation of these wavelengths?

Solution.

In order to use a wave phenomenon to detect the presence of some object, the wavelength of the waves used must be comparable to or smaller than the dimensions of the object to be detected.

a) the planet Venus.

Solution.

Venus is about 10^{7} m in diameter. The frequency of electromagnetic waves of wavelength 10^{7} m is given by

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{10^{7}} = 30\ \text{Hz.}$$

The waves of frequency 30 Hz correspond to very low audio-frequency radio waves. Practically, it will not be possible to detect Venus by employing such waves, because of the following reasons:

1. It would be almost impossible to get much enough power into such a low frequency wave.

2. Such a radio wave would be absorbed completely in the upper atmosphere.

3. Even if we could send such a radiowave to Venus and receive an echo, its beam would be so broad that we will not be able to pinpoint the direction of the Venus.

b) An aircraft 50 m long.

Solution.

The waves of wavelength 50 m required to detect an aircraft possess frequency

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{50} = 6 \times 10^{6}\ \text{Hz} = 6\ \text{MHz}.$$

This frequency is higher than the frequency of radiowaves used by A.M. broadcasting stations. Primitive radars operated at frequencies about 20 times this.

c) a bird 0.1 m long.

Solution.

A 0.1 m long bird will require frequency,

$$f = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{0.1} = 3000\ \text{MHz}.$$

It is very close to the popular radar frequency. Indeed, radars occasionally detect birds.

Exercises

4.41. Figure 4.11 shows a Hertz antenna (also known as a half-wave antenna since its length is $\lambda/2$). The antenna is far enough from the ground that reflections do not significantly affect its radiation pattern. Most AM radio stations, however, use a Marconi antenna which consists of the top half of a Hertz antenna. The lower end of this (quarter-wave) antenna is connected to the earth ground, and the ground itself serves as the missing lower half. What are the heights of the Marconi antennas for radio stations broadcasting at (a) 360 kHz and (b) 1 600 kHz?

4.42. Two hand-held radio transceivers with dipole antennas are separated by a great fixed distance. Assuming that the transmitting antenna is vertical, what fraction of the maximum received power will occur in the receiving antenna when it is inclined from the vertical by (a) 15.0^{c}? (b) 45.0°? (c) 90.0°?

4.43. Two radio-transmitting antennas are separated by half the broadcast wavelength and are driven in phase with each other. In which directions are (a) the strongest and (b) the weakest signals radiated?

4.11 Radiation from an Infinite Current Sheet

In this section, we describe the electric and magnetic fields radiated by a flat conductor carrying a time-varying current. In the symmetric plane geometry employed here, the mathematics is less complex than that required in lower-symmetry situations.

Consider an infinite conducting sheet lying in the yz-plane and carrying a surface current in the y direction, as shown in Figure 4.13.

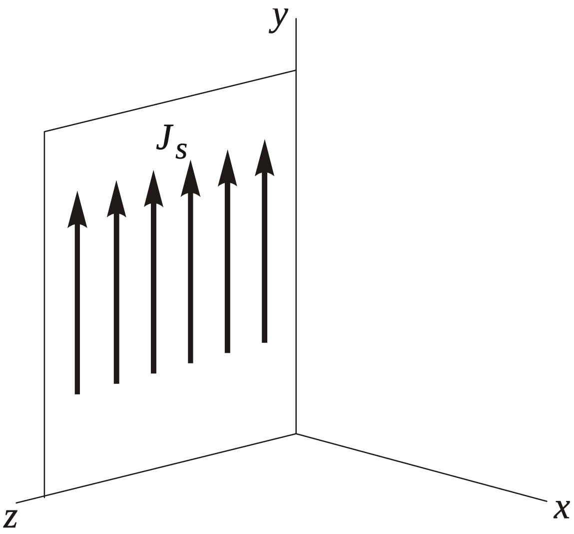


Figure 4.13 A portion of an infinite current sheet lying in the yz plane. The current density is sinusoidal and is given by the expression $J = J_{mx} \cos \omega t$. The magnetic field is everywhere parallel to the sheer and lies along z

The current is distributed across the z direction such that the current per unit length is $\vec{J}$. Let us assume that J varies sinusoidally with time as

$$J = J_m \cos \omega t \qquad (4.52)$$

where J_m is the amplitude of the current variation and ω is the angular frequency of the variation. The magnetic field outside the sheet is everywhere parallel to the sheet and lies along the z axis. It can be shown that in the present situation, where J varies with time, the magnetic field B_z can be described as:

$$B_z = \frac{\mu_0}{2} J_m \cos \omega t \text{ (for small values of } x\text{)}$$

To obtain the expression valid for B_z for arbitrary values of x, we can investigate the solution:

$$B_z = \frac{\mu_0}{2} J_m \cos(kx - \omega t)\,. \qquad (4.53)$$

You should note two things about this solution, which is unique to the geometry under consideration. First, when x is very small, it agrees with our original solution. Second, it satisfies the wave equation. We conclude that the magnetic field lies along the z axis, varies with time, and is characterized by a transverse traveling wave having an angular frequency ω and an angular wave number $k = 2\pi / \lambda$.

We can obtain the electric field radiating from our infinite current sheet using Eq.(4.18):

$$E_z = cB_z = \frac{c\mu_0}{2} J_{\max} \cos(kx - \omega t)\,. \qquad (4.54)$$

That is, the electric field is in the y direction, perpendicular to $\vec{B}$, and has the same space and time dependencies. These expressions for B_z and E_y show that the radiation field of an infinite current sheet earning a sinusoidal current is a plane electromagnetic wave propagating with a speed c along the x axis, as shown in Figure 4.14.

We can calculate the Poynting vector for this wave from Eqs. (4.25), (4.46), and (4.27):

$$S = \frac{EB}{\mu_0} = \frac{c\mu_0}{4} J_m^2 \cos^2(kx - \omega t)\,. \qquad (4.55)$$

Intensity of the wave, which equals the average value of S is

$$I = S_{av} = \frac{c\mu_0}{8} J_m^2. \qquad (4.56)$$

This intensity represents the power per unit area of the outgoing wave on each side of the sheet. The total rate of energy emitted per unit area of the conductor is

$$2S_{av} = c\mu_0 J^2{}_{\max}/4. \qquad (4.57)$$

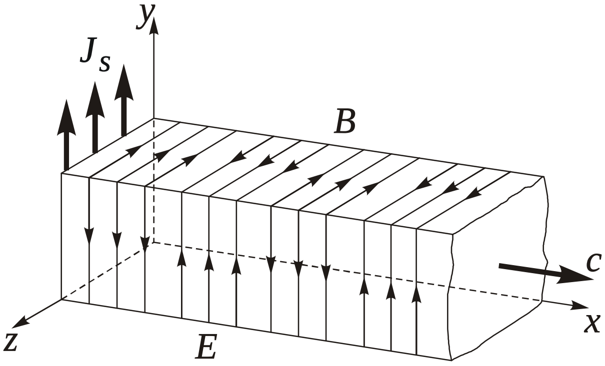


Figure 4.14 Representation of the plane electromagnetic wave radiaied by an infinite current sheet lying in the yz plane. The vector $\vec{B}$ is in the z; direction, the vector $\vec{E}$ is in the y direction, and the direction of wave motion is along x

Example 4.11

An infinite current sheet lying in the yz plane carries a sinusoidal current that has a maximum density of 5.00 A/m^2.

a) Find the maximum values of the radiated magnetic and electric fields.

Solution.

From Eq. (4.53) and (4.54), we see that the maximum values of B_z and E_y are

$$E_m = \frac{c\mu_0}{2} J_m \quad \text{and} \quad B_m = \frac{\mu_0}{2} J_m.$$

Using the values $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}$, and $c = 3 \times 10^8\,\mathrm{m/s}$, we get

$$B_m = \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(5\,\mathrm{A/m})}{2} = 3.14 \times 10^{-6}\,\mathrm{T},$$

$$E_m = \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(5\,\mathrm{A/m})(3 \times 10^8\,\mathrm{m/s})}{2} = 942\ \mathrm{V/m}.$$

b) What is the average power incident on a flat surface that is parallel to the sheet and has an area of 3.0 m^2? (The length and width of this surface are both much greater than the wavelength of the radiation.)

Solution.

The intensity, or power per unit area, radiated in each direction by the current sheet is given by Eq. (4.56):

$$I = \frac{\mu_0 c}{8} J_m^2 = \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(3 \times 10^8\,\mathrm{m/s})(5\,\mathrm{A/m})^2}{8} = 1.18 \times 10^3\ \mathrm{W/m^2}.$$

Multiplying this by the area of the surface, we obtain the incident power:

$$P = IA = (1.18 \times 10^3\,W/m^2)(3m^2) = 3.54 \times 10^2\,W.$$

The result is independent of the distance from the current sheet because we are dealing with a plane wave.

Exercises

4.44. A large current-earning sheet emits radiation in each direction (normal to the plane of the sheet) with an intensity of 570 W/m^2. What maximum value of sinusoidal current density is required?

4.45. A rectangular surface of dimensions $120\,cm \times 40\,cm$ is parallel to and 4.40 m away from a much larger conducting sheet in which a sinusoidally varying surface current exists that has a maximum value of 10.0 A/m. (a) Calculate the average power that is incident on the smaller sheet. (b) What power per unit area is radiated by the larger sheet?

4.12 Electromagnetic Spectrum

The various types of electromagnetic waves are listed in Figure 4.15, which shows the electromagnetic spectrum. No sharp dividing point exists between one type of wave and the other. Remember that all forms of the various types of radiation are produced by the same phenomena – accelerating charges. The names given to the types of waves are simply for convenience in describing the region of the spectrum in which they lie.

Radio waves are the result of charges accelerating through conducting wires. Ranging from more than 104 m to about 0.1 m in wavelength, (in the frequency range from 500 kHz to about 1000 MHz), they are generated by such electronic devices as *LC* oscillators and are used in radio and television communication systems. The AM (amplitude modulated) band is from 530 kHz to 1710 kHz. Higher frequencies up to 54 MHz are used for “short wave” bands. TV waves range from 54 MHz to 890 MHz. The FM (frequency modulated) radio band extends from 88 MHz to 108 MHz. Cellular phones use radio waves to transmit voice communication in the ultrahigh frequency (UHF) band.

Microwaves have wavelengths ranging from approximately 0.3 m to 10^{-4} m and are also generated by electronic devices. Because of their short wavelengths, they are well suited for radar systems and for studying the atomic and molecular properties of matter. Microwave ovens (in which the wavelength of the radiation is $\lambda = 0.122$ m) are an interesting domestic application of these waves.

Infrared waves have wavelengths ranging from 10^{-3} m to the longest wavelength of visible light, 7×10^{-7} m. These waves, produced by molecules and room-temperature objects, are readily absorbed by most materials. The infrared (*IR*) energy absorbed by a substance appears as internal energy because the energy agitates the atoms of the object, increasing their vibration or translational motion, which results in a temperature increase. Infrared radiation has practical and scientific applications in many areas, including physical therapy, *IR* photography, and vibrational spectroscopy. Infrared radiation also plays an important role in maintaining the Earth’s warmth or average temperature through the greenhouse effect. Incoming visible light (which passes relatively easily through the atmosphere) is absorbed by the earth's surface and re-radiated as infrared (longer wavelength) radiations. This radiation is trapped by greenhouse gases such as carbon dioxide and water vapor.

Visible light, the most familiar form of electromagnetic waves, is the part of the electromagnetic spectrum that the human eye can detect. Light is produced by the rearrangement of electrons in atoms and molecules. Various wavelengths of visible light correspond to different colours and range from red ($\lambda = 7\times 10^{-7}$ m) to violet ($\lambda = 4\times 10^{-7}$ m). The sensitivity of the human eye is a function of wavelength, being a maximum at a wavelength of about 5.5×10^{-7} m. Different animals are sensitive to different range of wavelength. For example, snakes can detect infrared waves, and the visible range of many insects extends well into the ultraviolet.

Because of the very small magnitudes of light wavelengths, it is convenient to measure them in small units of length. Three such units are commonly used: the micrometer (1 μm), the nanometer (1 nm) and the angstrom (1 Å):

$1\mu\text{m} = 10^{-6}$ m,

$1\text{nm} = 10^{-9}$ m,

1Å = 10^{-10} m.

The color of light depends on its wavelength or frequency. Different parts of the visible spectrum evoke the sensations of different colors. Wavelengths for colors in the visible spectrum are (very approximately) as follows:

Violet	400 to 440 nm
Blue	440 to 480 nm
Green	480 to 530 nm
Yellow	530 to 590 nm
Orange	590 to 630 nm
Red	630 to 700 nm

Ultraviolet waves cover wavelengths ranging from approximately 4×10^{-7} m to 6×10^{-10} m. The ultraviolet radiation (UV) is produced by special lamps and very hot bodies. The Sun is an important source of UV radiation, which is the main cause of sunburn.

Sunscreen lotions are transparent to visible light but absorb most UV light. The higher a sunscreen's solar protection factor (SPF), the greater the percentage of UV light absorbed. Ultraviolet rays have also been implicated in the formation of cataracts, a clouding of the lens inside the eye. Wearing sunglasses that do not block UV light is worse for your eyes than wearing no sunglasses. Lenses of any sunglasses absorb some visible light, thus causing the wearer's pupils to dilate. If the glasses do not also block UV light, then more damage may be done to the lens of the eye because of the dilated pupils. If you wear no sunglasses at all, your pupils are contracted, you squint, and less UV light enters your eyes. High-quality sunglasses block nearly all the eye-damaging UV light.

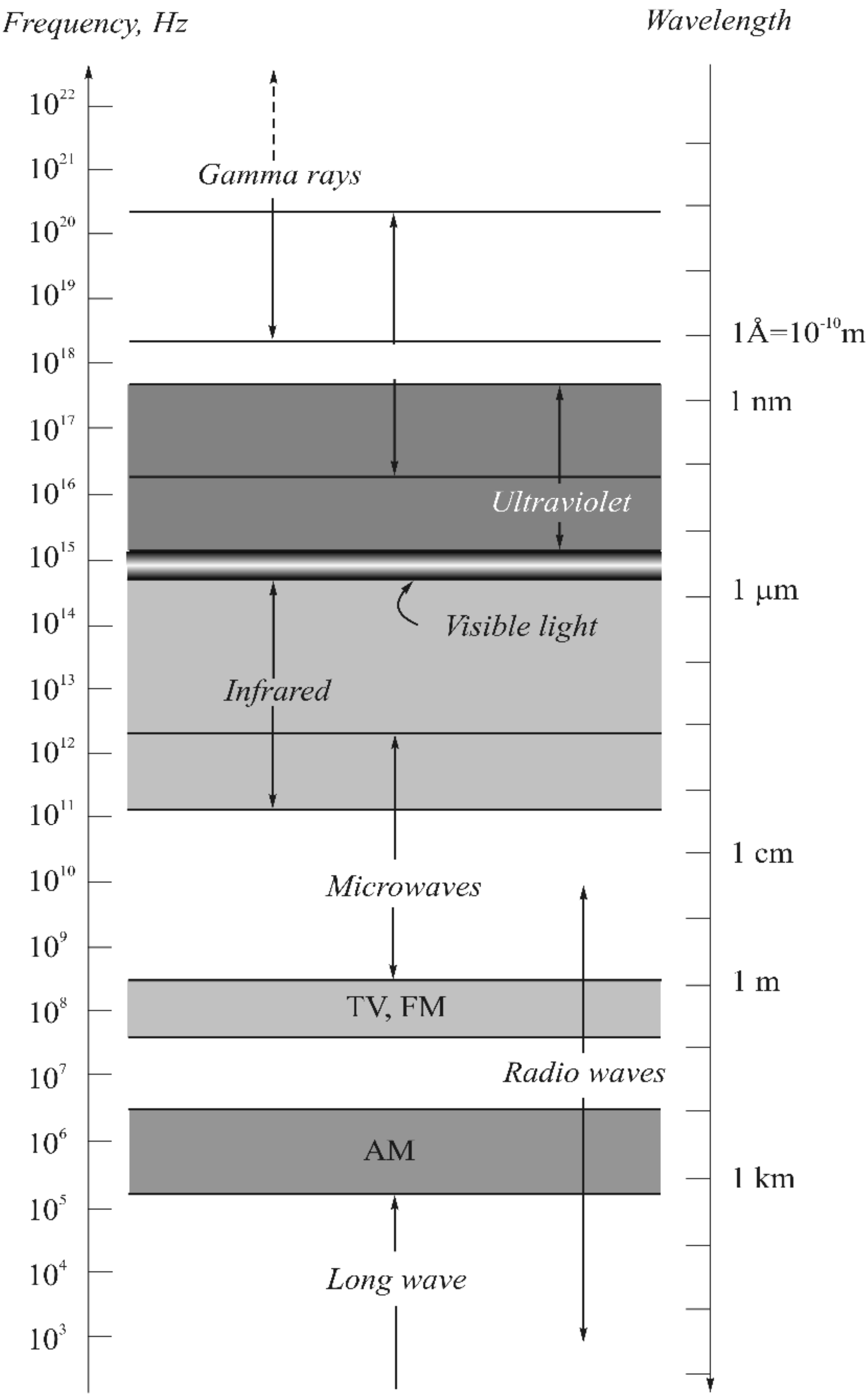


Figure 4.15   A chart of the electromagnetic spectrum

Most of the *UV* light from the Sun is absorbed by ozone (O_3) molecules in the Earth's upper atmosphere, in a layer called the stratosphere. This ozone shield converts lethal high-energy *UV* radiation to infrared radiation which in turn warms the stratosphere. Recently, a great deal of controversy has arisen concerning the possible depletion of the protective ozone layer as a result of the chemicals emitted from aerosol spray cans and used as refrigerants.

X-rays have wavelengths in the range from approximately 10^{-8} m to 10^{-12} m. The most common source of X-rays is the deceleration of high-energy electrons bombarding a metal target. X-rays are used as a diagnostic tool in medicine and as a treatment for certain forms of cancer. Because X-rays damage or destroy living tissues and organisms, care must be taken to avoid unnecessary exposure or overexposure. X-rays are also used in the study of crystal structure because X-ray wavelengths are comparable to the atomic separation distances in solids (about 0.1 nm).

Gamma rays are electromagnetic waves emitted by radioactive nuclei (such as ^{60}Co and ^{137}Cs) and during certain nuclear reactions. High-energy gamma rays are a component of cosmic rays that enter the Earth's atmosphere from space. They have wavelengths ranging from approximately 10^{-10} m to less than 10^{-14} m. They are highly penetrating and produce serious damage when absorbed by living tissues. Consequently, those working near such dangerous radiation must be protected with heavily absorbing materials, such as thick layers of lead.

Exercises

4.46. Give several examples of electromagnetic waves that you encounter in everyday life. How are they all alike? How do they differ?

4.47. We are surrounded by electromagnetic waves emitted by many radio and television stations. How is a radio or television receiver able to select a single station among all this mishmash of waves? What happens inside a radio receiver when the dial is turned to change stations?

4.48. The ionosphere is a layer of ionized air 100 km or so above the Earth's surface. It acts as a reflector of radio waves of frequency less than about 30 MHz, but not of higher frequency. How does this reflection occur? Why does it work better for lower frequencies than for higher?

4.49. What is the fundamental cause of electromagnetic radiation?

4.50. Suppose a creature from another planet had eyes that were sensitive to infrared radiation. Describe what the creature would see if it looked around the room you are now in. That is, what would be bright and what would be dim?

4.51. What is the wavelength in meters, microns, nanometers, and angstrom units of:

a) Soft X-rays of frequency 2·1017 Hz? (Ans. 1.5×10^{-9} m).

b) Green light of frequency 5.6·1014 Hz? (Ans. 5.35×10^{-7} m).

4.52. Classify waves with frequencies of 2 Hz, 2 kHz,2 MHz, 2 GHz, 2 THz, 2 PHz, 2 EHz, 2 ZHz, and 2 YHz on the electromagnetic spectrum, (b) with wavelengths of 2 km, 2 m, 2 mm,2 μm, 2 nm, 2 pm.

4.53. Compute an order-of-magnitude estimate for the frequency of an electromagnetic wave with a wavelength equal to (a) your height; (b) the thickness of this sheet of paper. How is each wave classified on the electromagnetic spectrum?

4.54. A human eye is most sensitive to light having a wavelength of 550 nm, which is in the green-yellow region of the visible electromagnetic spectrum. What is the frequency of this light? (Ans. 545 THz).

4.55. Suppose you are located 180 m from a radio transmitter.

a) How many wavelengths are you from the transmitter if the station calls itself 1150 AM? (Ans. The AM band frequencies are in kilohertz.)

b) What if this station were 98.1 FM? (Ans. The FM band frequencies are in megahertz.)

4.56. What are the wavelengths of electromagnetic waves in free space that have frequencies of (a) 5.0·1019 Hz and (b) 4.0·109 Hz?

4.57. What are the wavelengths ranges in:

(a) the AM radio band (Ans. 540- 600 kHz):

(b) the FM radio band (Ans. 88.0-108 MHz)?

4.13 Doppler Effect

We have seen in Einstein theory of relativity that the same speed is measured for light no matter what the relative speeds of the light sources and the observer are. The measured frequency and wavelength will change, but always in such a way that their product, which is the velocity of light, remains constant. Such frequency shifts are called *Doppler shift*, after Johann Doppler (1803-1853) who first predicted them.

In section “Sound waves” we showed that if a source of sound is moving away from an observer or if observer is moving away from the source, the frequency heard by the observer is

$$f' = f_0 \frac{1}{1+u/v}, \tag{4.58}$$

where u is speed of sound source, v is the speed of sound.

For light the “source receding from observer” and “observer receding from source” is physically identical situations and must exhibit exactly the same Doppler frequency. The Doppler frequency predicted by the theory of relativity, is

$$f' = f_0 \frac{1-u/c}{\sqrt{1-(u/c)^2}}. \tag{4.59}$$

When $u << c$, formula (4.59) can be rewritten as

$$f' = f_0\left[1 - \frac{u}{c} + \frac{1}{2}\left(\frac{u}{c}\right)^2 + \ldots\right]. \tag{4.60}$$

The ratio u/c for all available monochromatic light sources, even those of atomic dimensions, is small. In this equation, this means that successive terms becomes small rapidly and, depending on the accuracy required, only a limited number of terms need to be retained.

From (4.60) we can obtain relative change of frequency:

$$\frac{\Delta f}{f_0} = -\frac{u}{c}. \tag{4.61}$$

A police radar unit employs the Doppler effect with microwaves to measure the speed u of a car. A source in the radar unit emits a microwave beam at a certain (proper) frequency f_0 along the road. A car that is moving toward the unit intercepts that beam but at a frequency that is shifted upward by the Doppler effect due to the car's motion toward the radar unit. The car reflects the beam back towards the radar unit. Because the car is moving towards the radar unit, the detector in the unit intercepts a reflected beam that is further shifted up in the frequency. The unit compares that detected frequency with f_0 and computes the speed u of the car.

The Doppler effect for light finds many applications in astronomy where it is used to determine the speeds at which luminous heavenly bodies are moving toward or receding from us. Such Doppler shifts measure only the radial or line-of-sight components of the relative velocity. Almost all galaxies for which such measurements have been made appear to be receding from us, the recession velocity being greater for the more distant galaxies; these observations form the basis of the concept of the expending Universe.

Let us assume that the radial speed u of a certain light source is low enough for us to neglect the $\left(\frac{u}{c}\right)^2$ term in Eq. (4.59). Let us also explicitly show a $\pm$ option in front of the $\left(\frac{u}{c}\right)$ term – the minus sine corresponding to radial motion away from us and the plus sign corresponding to radial motion towards us. Then Eq. (4.59) becomes

$$f = f_0\left(1 \pm \frac{u}{c}\right). \tag{4.62}$$

Astronomical measurements involving light are usually done in wavelengths rather then in frequencies, so let us replace f with c/λ and f_0 with c/λ_0 where λ is the measured wavelength and λ_0 is the proper wavelength. Then Eq. (4.62) is written as

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}\left(1 \pm \frac{u}{c}\right),$$

which leads to

$$u = \pm \frac{\lambda - \lambda_0}{\lambda} c .$$

This is conventionally written as

$$u = \frac{\Delta\lambda}{\lambda} c$$

where $\Delta\lambda = |\lambda - \lambda_0|$ is the wavelength Doppler shift of the light source. If the source is moving away from us, λ is greater than λ_0 and the Doppler shift is called a *red shift.* (The term means that the wavelength increases). Similarly, if the source is moving toward us, λ is less than λ_0 and the Doppler shift is called a *blue shift*.

Example 4.12

Certain characteristic wavelength in the light from a galaxy in the constellation Virgo are observed to be increased in wavelength, as compared with terrestrial sources, by about 0.4%. What is the radial speed of this galaxy with respect to the Earth? Is it approaching or receding?

Solution.

If λ_0 is the wavelength for a terrestrial source, then

$$\lambda' = 1.004\lambda_0 .$$

Since we must have $\lambda' f' = \lambda f_0 = c$, we can write this as

$$f' = 0.0996 f_0 .$$

This frequency shift is so small that, in calculating the source speed, we can use Eq. (4.62). As a result, we obtain

$$f' = 0.996 f_0 = f_0 \left(1 - \frac{u}{c}\right).$$

Solving yields $u/c = 0.004$, or $u = (0.004)(3 \times 10^8 \text{m/s}) = 1.2 \times 10^6$ m/s. The galaxy is receding from us; had u turned out to be negative, the galaxy would have been moving towards us.

Exercises

4.58. Can a galaxy be so distant that its recession speed equals c? If so, will its light ever reach us?

4.59. The “red shift” of radiation from a distant nebula consists of the light (H_g), known to have a wavelength of 4340×10^{-8} cm when observed in the laboratory. It appears to have a wavelength of 6540×10^{-8} cm. What is the speed of the nebula in the line of sight relative to the Earth? Is it approaching or receding?

4.60. The difference in wavelength between an incident microwave beam and one reflected from an approaching or receding car is used to determine automobile speeds on the highway. Show that if u, the speed, and f, the frequency of the incident beam, the change of frequency is approximately $\frac{2uf}{c}$.

4.61. Show that for low speed, the Doppler shift can be written in the appropriate form

$$\frac{\Delta\lambda}{\lambda} = \frac{u}{c},$$

where $\Delta\lambda$ is the change in wavelength.

4.62. The rotation period of the Sun at its equator is 24.7 days; its radius is 7.0×10^8 m. What Doppler wavelength shifts are expected for characteristic wavelength in the vicinity of 5500 Å emitted from the edge of the Sun disk? (3.8×10^{-2} Å).

4.63. A rocket ship is receding from the Earth at a speed of $0.2c$. A light in the rocket ship appears blue to passengers on the ship. What color would it appear to be to an observer on the Earth? (yellow-orange).

4.14 Propagation of Electromagnetic Waves in Atmosphere

Before we discuss the propagation of EM waves in the atmosphere, it is necessary to learn a few things about the atmosphere and its various layers. The atmosphere is the gaseous envelope surrounding our Earth. It is retained to the earth due to gravitational attraction. As we go up, the air thins out gradually and air pressure decreases. The atmosphere can be divided into various layers as shown in Figure 4.16. The layers are known by different names.

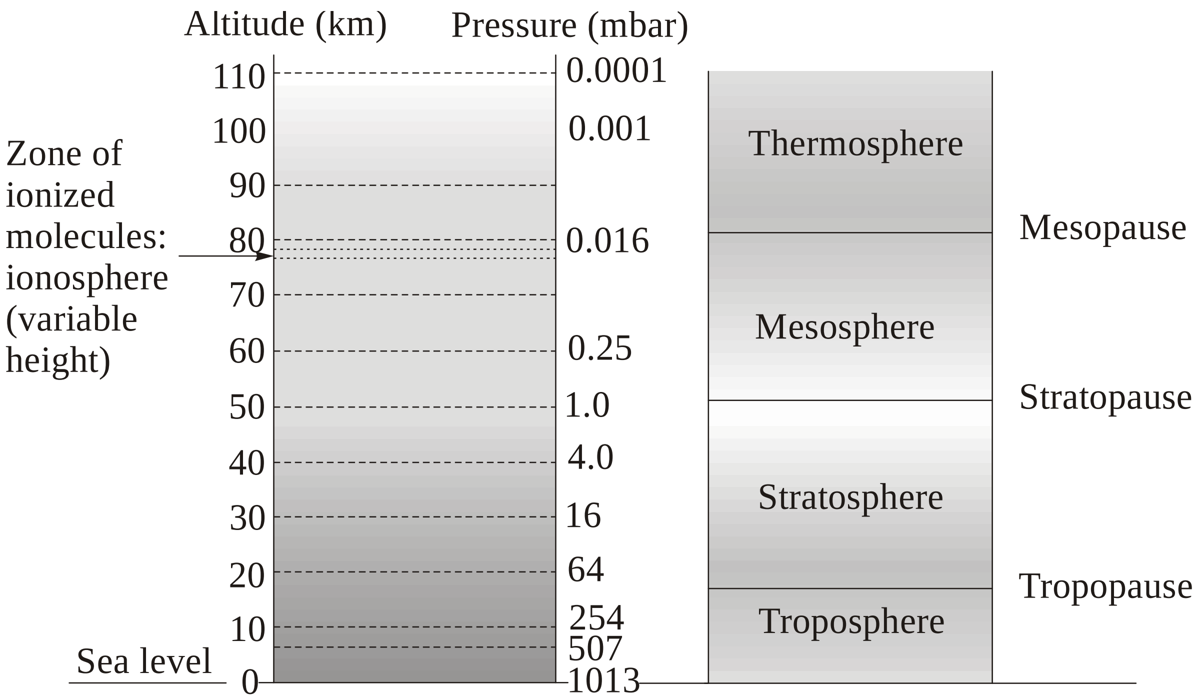


Figure 4.16 The Earth's atmosphere

The *troposphere* includes the layer close to the earth and extends up to about 12 km. This layer is responsible for all the important weather phenomena affecting our environment. The next layer, called the *stratosphere*, extends from about 10-16 km to about 50 km. The *mesosphere* extends from about 50 km to about 80 km. The *thermosphere* extends from 80 km to the edge of the atmosphere. It receives energy directly from the solar radiation. The ozone layer is in the lower stratosphere. This ozone results from the dissociation of molecular oxygen by solar ultraviolet radiation in the upper atmosphere. Except for the layer in the upper atmosphere, called *ionosphere*, which is composed partly of electrons and positive ions, the rest of the atmosphere is composed mostly of neutral molecules.

The atmosphere is transparent to visible radiation and we can see the Sun and the stars through it clearly. However, most infrared radiation is not able to pass through, as it is absorbed by the atmosphere. Low lying clouds in the atmosphere also prevent infrared radiation from passing through. The ozone layer blocks the passage of ultraviolet radiation from the sun.

The behavior of electromagnetic waves of wavelength 10^{-3} m and higher (called radio waves) in their propagation through the atmosphere is an important consideration in all modern forms of communication: radio, television, microwaves etc. At low frequencies, radio waves radiated by an antenna near the earth travel directly following the surface of the earth. This is called *wave along ground propagation* (Figure 4.17).

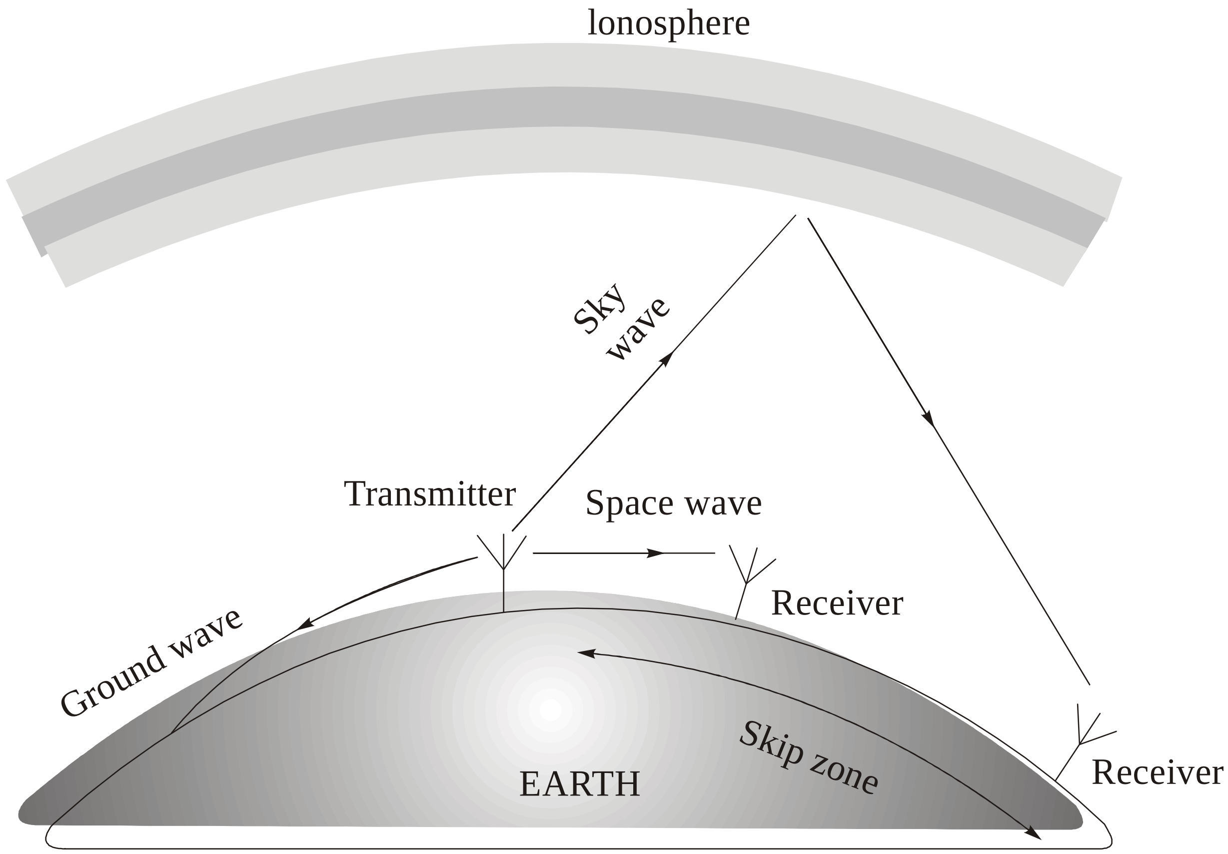


Figure 4.17 The three main modes of propagation of electromagnetic waves

During the daytime, broadcast from medium waveband station can travel nearly 200 km like this. Above 2 MHz, such waves weaken rapidly with distance.

Radio waves of frequencies between 2 MHz and about 20 MHz are reflected off the ionosphere. So, in this frequency range, radio waves radiated from a certain point and reflected by the ionosphere can be received at another point on the surface. This is known as *sky wave*, or ionosperic propagation. In this way, radio waves travel very large distances and can even travel round the earth.

Ionosphere does not help in the propagation of waves of frequencies higher than 30 MHz. Television signals have frequencies in the 100 - 200 MHz range and penetrate ionosphere (no reflection), therefore, their propagation is not possible through the sky wave, Such waves can be reseaved only if the receiver antenna directly intercepts the signal. Thus, television broadcasts are made from tall antenna to get larger coverage. This is *space wave propagation.* Radiowaves with frequencies higher than television signals are the microwaves. In recent times, microwaves have revolutionized telecommunications. The signals (in this range) from the broadcasting station are beamed towards a geostationary satellite which in turn broadcasts it back to the earth. In this way, signals can be propagated over a the earth's surface.

The Sun is the main source of electromagnetic radiation. It sends out electromagnetic waves of different wavelengths towards the Earth. As the electromagnetic waves propagate through the Earth's atmosphere, a major part of them is absorbed. Most of the infrared radiation is absorbed by the atmosphere, and the atmosphere gets heated. The visible light is only slightly absorbed. The electromagnetic radiation from the Sun is quite rich in ultra-violet radiation. Owing to its small wavelength (high energy), ultraviolet radiation is harmful to plants and living cells. However, the ozone layer absorbs most of the ultra-violet radiation and other harmful radiations of lower wavelengths. The ozone layer converts the ultraviolet radiation into the infrared which further heats up the atmosphere and the earth's surface

Green–House Effect. The Sun is the source of energy. It emits energy in the form of visible light, infrared and ultraviolet radiations. The behavior of atmosphere is different towards different types of radiations. Whereas the ultra violet radiation and other low wavelength waves are absorbed by the ozone layer, a large part of the infrared radiation is not allowed by the atmosphere to pass through it. The earth's atmosphere is transparent to visible light. Therefore, only visible light and a part of infrared radiation reach the Earth's surface. These radiations keep the earth's surface warm even at night due to the green house effect of the atmosphere, as explained below

The Earth gets heated to low temperature only due to the solar energy reaching its surface. At such a low temperature, the energy emitted from the Earth lies mostly in the infrared region. Since the Earth's atmosphere is not transparent to infra-red radiations, these radiations are reflected back. The low lying clouds and heavy gases like CO_2 present in the atmosphere reflect infrared radiation back towards the earth surface. Due to this, the earth's atmosphere becomes richer in infrared radiation. As this radiation is absorbed by the objects readily, they get heated in this process. This phenomenon is called the *green house effect.*

Questions

4.64. Static crashes are heard on radio when a lightning flash occurs – even if the lightning occurs far away. Why does this happen? (Ans. A lightning flash involves tremendous electrical fields and currents which oscillate between the earth and the clouds or between two groups of clouds. In this electrical activity, many charges oscillate and produce a wide variety of electromagnetic waves. The flashes of light we see are emitted by atoms during this intense activity. Those electromagnetic waves, which have frequencies in radio-wave range, interfere with radio waves. Since light and radiowaves travel with the same speed, they arrive at the same time as does the light.)

4.65. Optical telescopes are built on the ground, but X-ray astronomy is possible only from satellites orbiting the Earth. Why? (Ans. The Earth's atmosphere is transparent to visible light and radio waves, but absorbs X-rays. Therefore, X-astronomy is possible only from satellites orbiting the Earth.)

4.66. Some scientists have predicted that a global nuclear war on the Earth would be followed by a severe "nuclear winter" with a devastating effect on life on Earth. What might be the basis of this prediction? (Ans. Scientists estimate that in case of global nuclear war, the clouds produced will cover probably the whole of the sky In that case, solar radiation would be prevented from reaching the earth and it will result in what they call *nuclear winter* on the Earth.)

4.67. Explain the "green house effect" of Earth's atmosphere.

4.68. What is the role of ozone layer in the atmosphere?

4.69. If the Earth did not have atmosphere, would its average surface temperature be higher or lower than what it is now? (Ans. The infrared radiation, emitted by earth is retained by the Earth's atmosphere due to the green house effect, and this keeps the earth warm. If the Earth did have atmosphere, its average temperature would have been low.)

4.70. Discuss the significance of the greenhouse effect in the atmosphere.

Summary

Maxwell's equations, which incorporate all the basic relationships of electric and magnetic fields and their sources (charges and currents), predict the existence of electromagnetic disturbances that can propagate through empty space and travel with a speed equal to the measured value of the speed of light. The simplest such a wave is a plane wave in which $\vec{E}$ and $\vec{B}$ are uniform over any plane perpendicular to the propagation direction, so that $\vec{E}$ and $\vec{B}$ are zero everywhere to the left of a certain plane and have constant values everywhere to the right of it. For such a wave disturbance to be consistent with Faraday's law, the two field magnitudes must be related by

$$E = cB,$$

or

$$B = \varepsilon_0 \mu_0 cE$$

where c is the propagation speed. For both of these requirements to be satisfied, c must be given by

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s.}$$

Electromagnetic waves are transverse; the $\vec{E}$ and $\vec{B}$ fields are perpendicular to the direction of propagation and to each other. There is a definite ratio between $\vec{E}$ and $\vec{B}$ in a wave, and the waves travel in vacuum with a definite and unchanging speed c.

The energy density in an electromagnetic wave can be expressed as

$$u = \frac{1}{2}\varepsilon_0 \mu_0 E^2 + \frac{1}{2\mu_0} B^2 = \varepsilon_0 E^2.$$

The energy-flow rate (power per unit area) is given by the Pointing vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}.$$

The time-average value of the magnitude EB/μ_0 of the Pointing vector is called the intensity of the wave. These waves also carry momentum; the momentum per unit volume has magnitude

$$\frac{EB}{\mu_0 c^2} = \frac{S}{c^2},$$

and the rate of transfer of momentum per unit cross-sectional area is

$$\frac{1}{A}\frac{dp}{dt} = \frac{S}{c} = \frac{EB}{\mu_0 c}.$$

When an electromagnetic wave travels through a dielectric, the wave speed v is given by

$$v = \frac{c}{n}.$$

For a sinusoidal electromagnetic wave traveling in the $+x$-direction, both $\vec{E}$ and $\vec{B}$ are sinusoidal functions of the quantity $(\omega t - kx)$, and at each point the sinusoidal variations of $\vec{E}$ and $\vec{B}$ are in phase. For a wave in the $-x$ direction, $\vec{E}$ and $\vec{B}$ are sinusoidal functions of $(\omega t + kx)$.

The electromagnetic spectrum covers a range of frequencies from at least 1 to 1024 Hz and a correspondingly broad range of wavelengths. Visible light is a very small part of this spectrum, with wavelengths of 4×10^{-7} to 7×10^{-7} m or 400 to 700 nm.

The Doppler frequency predicted by the theory of relativity, is

$$f' = f_0 \frac{1 - u/c}{\sqrt{1 - (u/c)^2}}.$$

Key Terms

Speed of light – скорость света
Energy density – плотность энергии
Pointing vector – вектор Пойнтинга
Intensity – интенсивность
Index of refraction – показатель преломления
Doppler effect – эффект Допплера
Red shift – красное смещение
Electromagnetic wave – электромагнитная волна
Gap – зазор, промежуток
Spark – искра
Spark gap – искровой промежуток
Standing wave – стоячая волна
Natural frequency – собственная частота
Loop – контур, петля
To match – подходить, согласовываться
Tuning fork – камертон
Restricted – связанный, ограниченный
Plane wave – плоская волна
Index of refraction – коэффициент преломления
Energy flow – поток энергии
Radiation – излучение
Perfect reflector – идеальный отражатель
Reflectivity – коэффициент отражения
Tail – хвост
Torsion balance – крутильные весы
Fiber – волокно, нить
Beam – луч
Absorption – поглощение
Electromagnetic spectrum – спектр электромагнитных волн
Infrared – инфракрасный
Visible – видимый
Ultraviolet – ультрафиолетовый
X-rays – рентгеновские лучи
Gamma-rays – g-лучи
Doppler effect – эффект Допплера
Doppler shift –допплеровский сдвиг
Galaxy – галактика
Radiation pressure – давление излучения (света)

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ФІЗИКА ДЛЯ ІНЖЕНЕРІВ
Частина IV
КОЛИВАННЯ І ХВИЛІ
(Англійською мовою)

Редактор І.В. Пизіна
Технічний редактор Л.О. Кузьменко

Комп'ютерне верстання: А.І. Безхутрий

Зв. план, 2011
Підписано до друку 05.12.2011
Формат $60\times 84\ 1/16$. Папір офс. № 2. Офс. друк
Ум. друк. арк. 14,7. Обл.-вид. арк. 16,56. Наклад 300 пр.
Замовлення 298. Ціна вільна

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