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# **THEORETICAL MECHANICS. STATICS**

100  
Т44

**MINISTRY OF EDUCATION AND SCIENCE,  
YOUTH AND SPORT OF UKRAINE**  
**National aerospace university named after N. Ye. Zhukovskiy**  
**“Kharkov Aviation Institute”**

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# **THEORETICAL MECHANICS.**

# **STATICS**

**Tutorial for self-education**

Научно-техническая  
библиотека  
“ХАИ”



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**НАУКОВО-ТЕХНІЧНА  
БІБЛІОТЕКА**  
Національного аерокосмічного  
університету ім. М.Є.Жуковського  
«Харківський авіаційний інститут»

Kharkov “KhAI” 2012

UDK [531+531.1] (075.8)  
Т33

Розглянуто дві основні проблеми статички: еквівалентне перетворення систем сил, що діють на тверде тіло, та визначення умов, які мусить задовольняти система сил для забезпечення рівноваги твердого тіла.

Наведено короткі відомості з теоретичного курсу, основні формули і пояснення до них. Подано розв'язання задач різної складності.

Для студентів механічних та інших спеціальностей (з повною та скороченою програмою з теоретичної механіки).

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The textbook considers two basic problems of statics: the composition of forces and reduction of force system acting on rigid bodies, and determination of the equilibrium conditions of force system acting on rigid bodies.

The information from the theoretical course, the basic formulas and their explanations are given in the workbook. The problems of different complexity are presented.

For college students studying theoretical mechanics (for full-time and for reduced courses of study).

Fig. 82. Table 4. Bibliography: 3 names

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# 1. STATICS QUANTITIES AND AXIOMS. FORCE PROJECTION ON AN AXIS AND A PLANE. RESULTANT FORCE. MOMENT OF THE FORCE ABOUT THE POINT

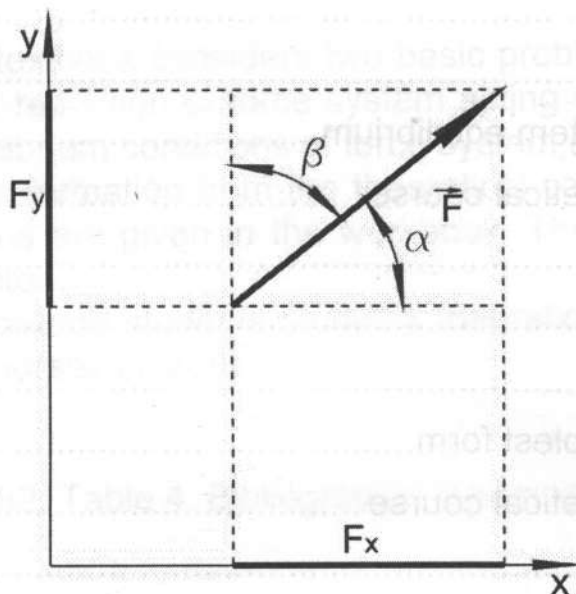
## 1.1. Main information from the theoretical course

There are **two general problems** of rigid body statics. They are:

- the composition of forces and the reduction of the force system acting on rigid bodies to the simplest equivalent form;
- the determination of conditions for the equilibrium of the force system acting on rigid bodies.

**Force** is a fundamental quantity of mechanics. In mechanics, a force can be defined as measure of mechanical interaction between bodies. Force is **vector** and determined by three characteristics: its magnitude, direction and point of application. Operations with forces obey the rules of vector algebra. In rigid body statics a force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts (force is sliding vector).

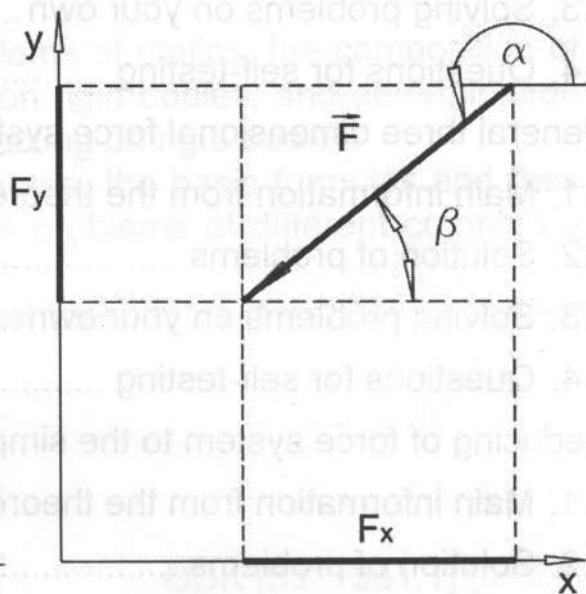
**Projection of force  $\vec{F}$  on axis** is **scalar** quantity that is equal to product of force magnitude and cosine between the axis positive direction and the direction of the force  $\vec{F}$  (Fig. 1.1, 1.2).



$$F_x = F \cdot \cos \alpha$$

$$F_y = F \cdot \cos \beta = F \cdot \sin \alpha$$

Fig. 1.1



$$F_x = F \cdot \cos \alpha = -F \cdot \cos \beta$$

$$F_y = -F \cdot \sin \beta$$

Fig. 1.2

A composition (addition) of applied at the same point forces is performed in accordance with parallelogram law: two forces applied at one point of a body have as their resultant a force applied at the same point and represented by the diagonal of a parallelogram constructed with the two given forces as its sides (Fig. 1.3):

$$\vec{R} = \vec{F}_1 + \vec{F}_2,$$

$$R = \sqrt{F_1^2 + F_2^2 + 2 \cdot F_1 \cdot F_2 \cdot \cos \alpha},$$

where  $\alpha$  is an angle between vectors  $\vec{F}_1$  and  $\vec{F}_2$ .

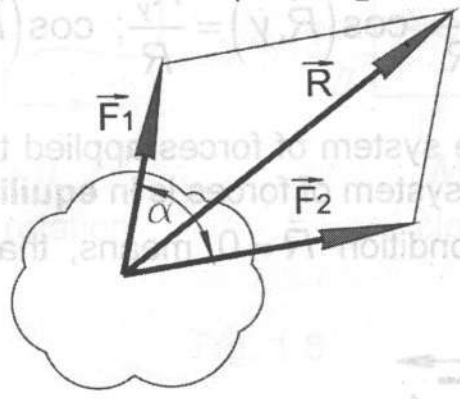


Fig. 1.3

If lines of action of several forces intersect at the same point, the force system is called concurrent.

A concurrent force system is equivalent to a single force (resultant). The resultant equals the vector sum of the system forces with its line of action passing through the point where all the lines of action of the system forces intersect.

$$\vec{R} = \sum_{k=1}^n \vec{F}_k.$$

The composition of forces can be accomplished:

- geometrically, constructing a vector force polygon drawing every next vector from an end of previous one; closing vector will be resultant (Fig. 1.4);

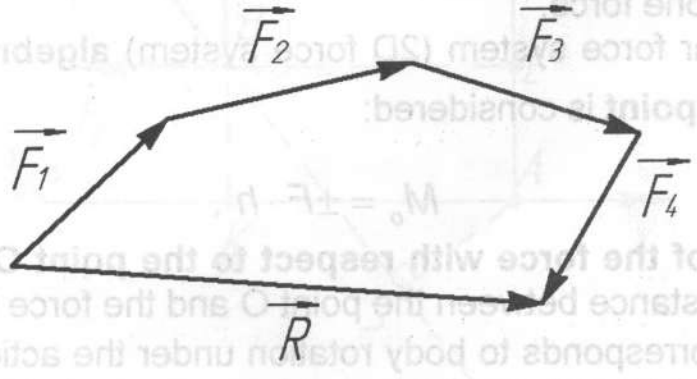


Fig. 1.4

- analytically, summing projections of all forces on selected coordinate axes:

$$R_x = \sum_{k=1}^n F_{xk}; R_y = \sum_{k=1}^n F_{yk}; R_z = \sum_{k=1}^n F_{zk},$$

then  $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$ .

The resultant  $\vec{R}$  direction can be determined by using direction cosines

$$\cos(\vec{R}, x) = \frac{R_x}{R}; \cos(\vec{R}, y) = \frac{R_y}{R}; \cos(\vec{R}, z) = \frac{R_z}{R}.$$

If the resultant of the system of forces applied to a rigid body is equal to zero, it means that such a system of forces is in **equilibrium**.

Geometrically the condition  $\vec{R} = 0$  means, that force polygon of given forces is closed (Fig. 1.5).

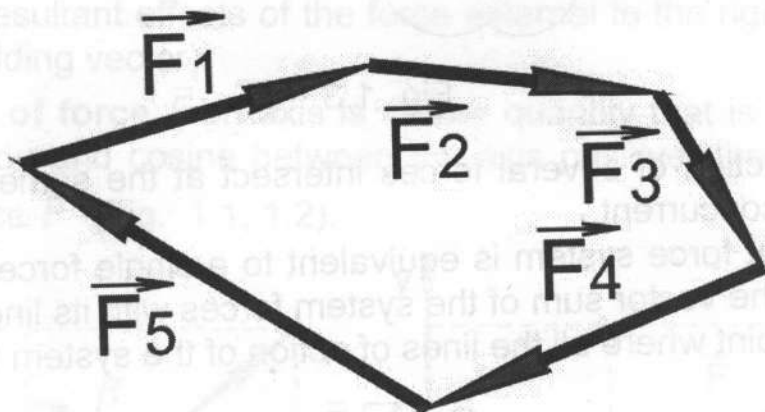


Fig. 1.5

General system (non-concurrent) can not be reduced to resultant force in general case, i. e. an effect of such system on a body can not be equivalent to an action of only one force.

For coplanar force system (2D force system) **algebraic moment of the force  $\vec{F}$  about a point** is considered:

$$M_o = \pm F \cdot h ,$$

where  $h$  is **arm of the force with respect to the point O** that is the shortest (perpendicular) distance between the point O and the force line of action.

Sign plus corresponds to body rotation under the action of force  $\vec{F}$  counterclockwise (Fig. 1.6) and sign minus conforms to clockwise one.

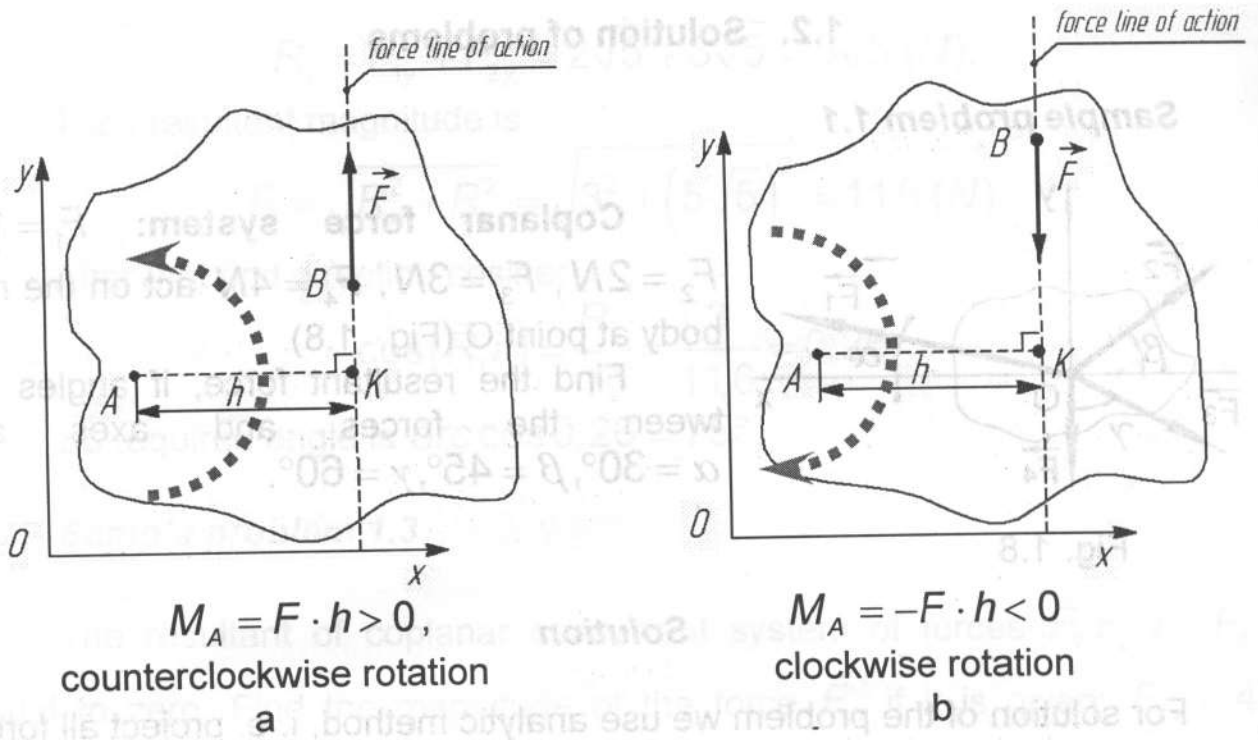


Fig. 1.6

Very often it is convenient to determine a moment of force about a point by decomposing of the force into two components parallel to the coordinate axes (Fig. 1.7). Then the moment of force about the point is equal to the sum of the component moments about this point (in accordance with Varignon's theorem):

$$M_A(\vec{F}) = M_A(\vec{F}_1) + M_A(\vec{F}_2) = \pm F_1 h_1 \pm F_2 h_2$$

or for the sketch shown (see Fig. 1.7)

$$M_A(\vec{F}) = -F_1 \cdot KA + F_2 \cdot LA.$$

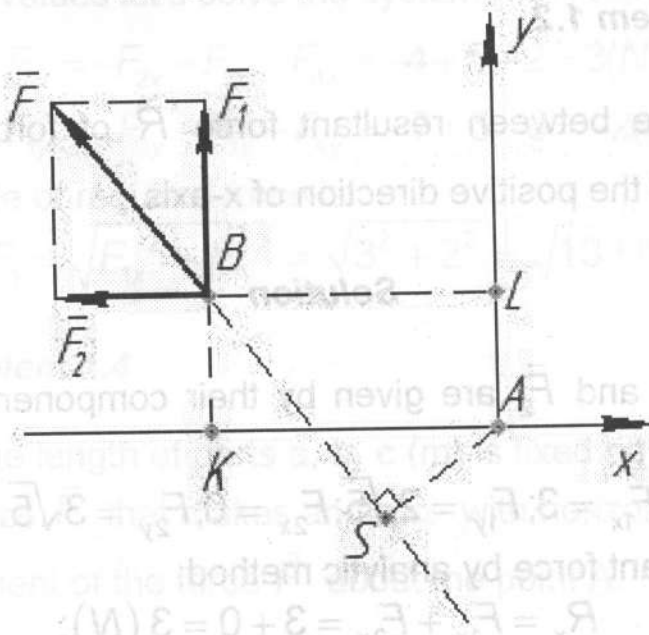


Fig. 1.7

## 1.2. Solution of problems

### Sample problem 1.1

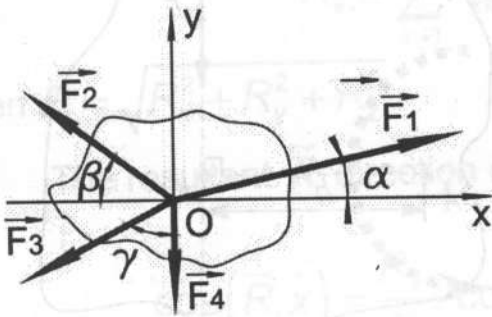


Fig. 1.8

**Coplanar force system:**  $F_1 = 1N$ ,  $F_2 = 2N$ ,  $F_3 = 3N$ ,  $F_4 = 4N$  act on the rigid body at point O (Fig. 1.8).

Find the resultant force, if angles between the forces and axes are:  $\alpha = 30^\circ$ ,  $\beta = 45^\circ$ ,  $\gamma = 60^\circ$ .

### Solution

For solution of the problem we use analytic method, i. e. project all forces on two orthogonal axes and find projections of resultant force on these axes:

$$R_x = \sum F_{kx} = F_1 \cos 30^\circ - F_2 \cos \beta - F_3 \sin \gamma,$$

$$R_x = 1 \cdot 0,86 - 2 \cdot 0,7 - 3 \cdot 0,86 = -3,15 (N);$$

$$R_y = \sum F_{ky} = F_1 \sin 30^\circ + F_2 \sin \beta - F_3 \cos \gamma - F_4,$$

$$R_y = 1 \cdot 0,5 + 2 \cdot 0,7 - 3 \cdot 0,5 - 4 = -3,2 (N).$$

Then

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(-3,15)^2 + (-3,2)^2} = 4,48 (N).$$

### Sample problem 1.2

Find the angle between resultant force  $\vec{R}$  of forces  $\vec{F}_2 = 3\sqrt{5}\vec{j}$  and  $\vec{F}_1 = 3\vec{i} + 2\sqrt{5}\vec{j}$  and the positive direction of x-axis.

### Solution

The forces  $\vec{F}_1$  and  $\vec{F}_2$  are given by their components along coordinate axes:

$$F_{1x} = 3; F_{1y} = 2\sqrt{5}; F_{2x} = 0; F_{2y} = 3\sqrt{5}.$$

Find the resultant force by analytic method:

$$R_x = F_{1x} + F_{2x} = 3 + 0 = 3 (N);$$



$$R_y = F_{1y} + F_{2y} = 2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5} \text{ (N)}.$$

Then resultant magnitude is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{3^2 + (5\sqrt{5})^2} = 11,6 \text{ (N)}.$$

After that find direction cosine:

$$\cos(\vec{R}, \vec{i}) = \frac{R_x}{R} = \frac{3}{11,6} = 0,26.$$

So required angle is  $\arccos 0,26 = 75^\circ$ .

### Sample problem 1.3

The resultant of coplanar concurrent system of forces  $\vec{F}_1, \vec{F}_2, \vec{F}_3, \vec{F}_4$  is equal to zero. Find the magnitude of the force  $\vec{F}_1$ , if it is given:  $F_{2x} = 4N, F_{2y} = 7N, F_{3x} = -5N, F_{3y} = -5N, F_{4x} = -2N, F_{4y} = 0$ .

### Solution

If the resultant  $\vec{R} = 0$ , its x- and y-components are  $R_x = R_y = 0$ . Therefore the sums of projections of all forces also are equal to zero, i. e.

$$\begin{cases} R_x = \sum F_{xk} = F_{1x} + F_{2x} + F_{3x} + F_{4x} = 0, \\ R_y = \sum F_{yk} = F_{1y} + F_{2y} + F_{3y} + F_{4y} = 0. \end{cases}$$

Substituting known values let's solve the system with respect to  $F_{1x}$  and  $F_{1y}$ :

$$\begin{cases} F_{1x} = -F_{2x} - F_{3x} - F_{4x} = -4 + 5 + 2 = 3(N), \\ F_{1y} = -F_{2y} - F_{3y} - F_{4y} = -7 + 5 - 0 = -2(N). \end{cases}$$

The magnitude of required force is

$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2} = \sqrt{3^2 + 2^2} = \sqrt{13} \text{ (N)}.$$

### Sample problem 1.4

Frame with the length of parts **a, b, c** (m) is fixed rigidly at the point A and is loaded by the force  $\vec{F}$  that makes angle  $\alpha$  with horizontal  $\alpha$  (Fig. 1.9).

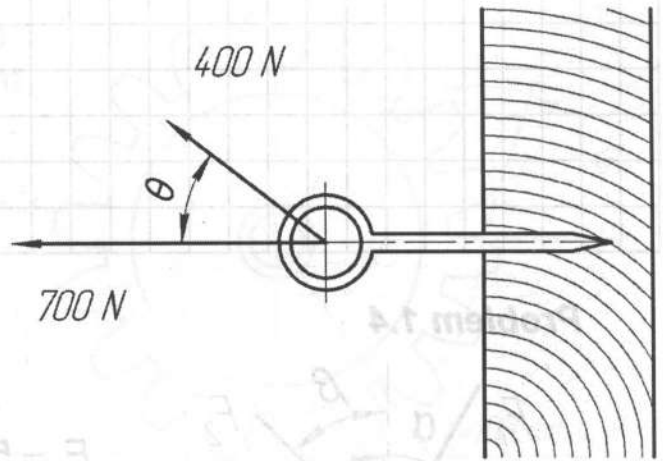
Find the moment of the force  $\vec{F}$  about the point A.



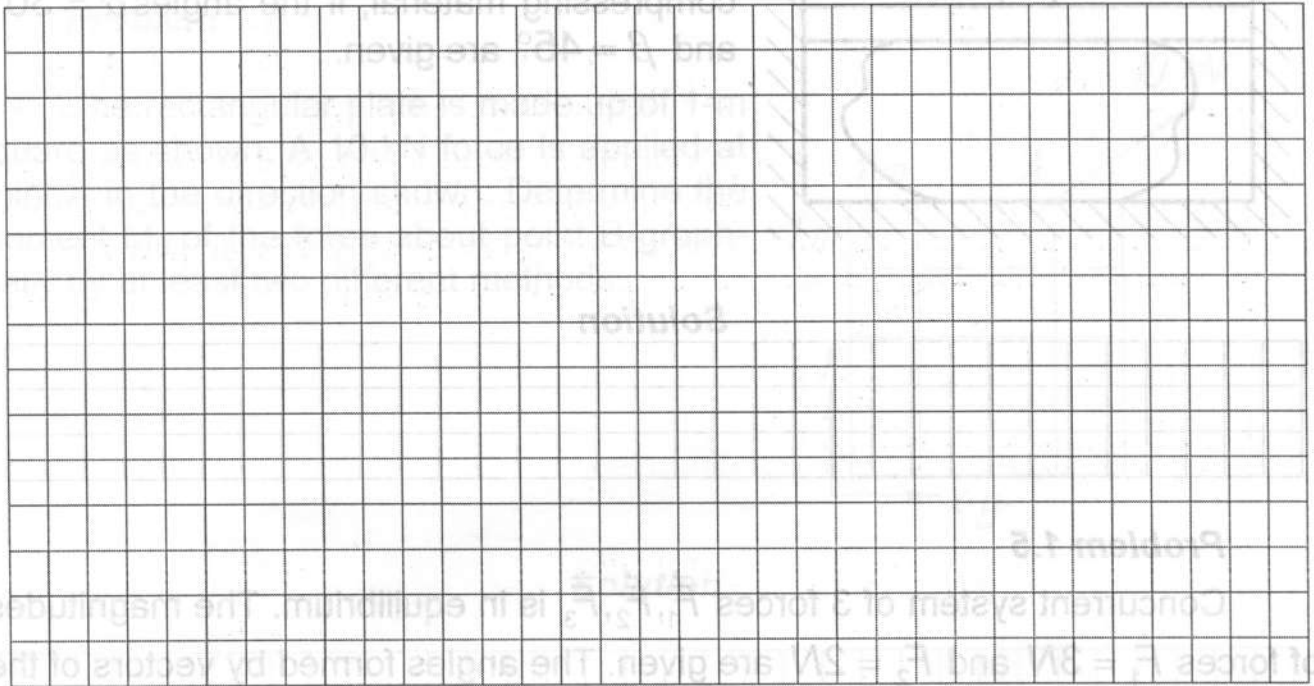


### Problem 1.2

At what angle  $\theta$  must the 400-N force be applied in order to that the resultant  $R$  of the two forces will have a magnitude of 1000 N? For this condition what will be the angle  $\beta$  between  $R$  and the horizontal?

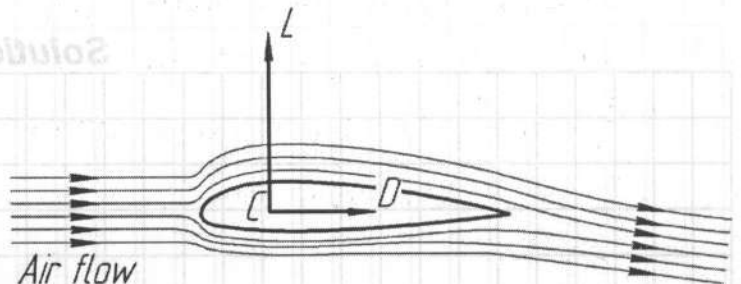


### Solution



### Problem 1.3

The ratio of the lift force  $L$  to the drag force  $D$  for the simple airfoil is  $L/D=10$ . If the lift force on a short section of the airfoil is 200 N, compute the magnitude of the resultant force  $R$  and the angle  $\theta$  which it makes with the horizontal.

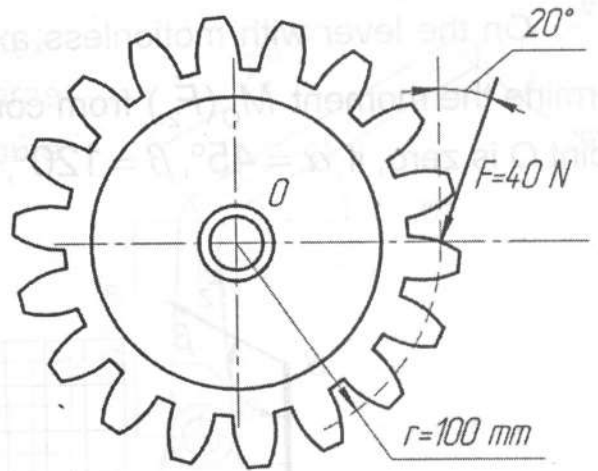
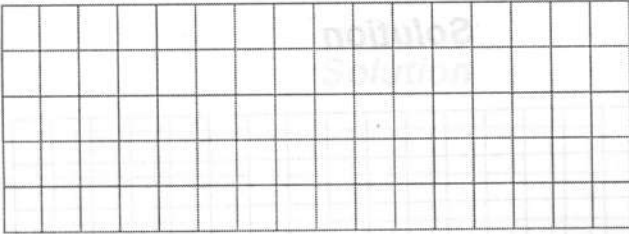




### Problem 1.6

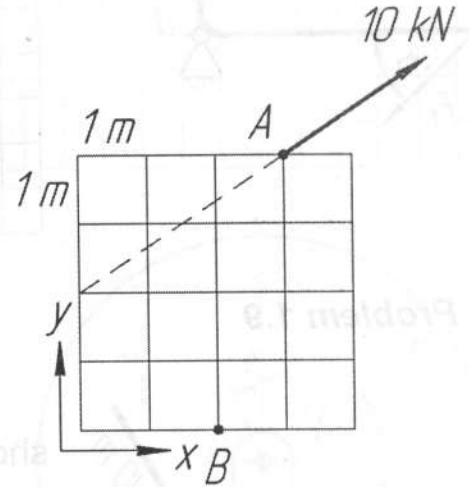
A force  $F$  of magnitude 40 N is applied to the gear. Determine the moment of  $F$  about point  $O$ .

#### Solution

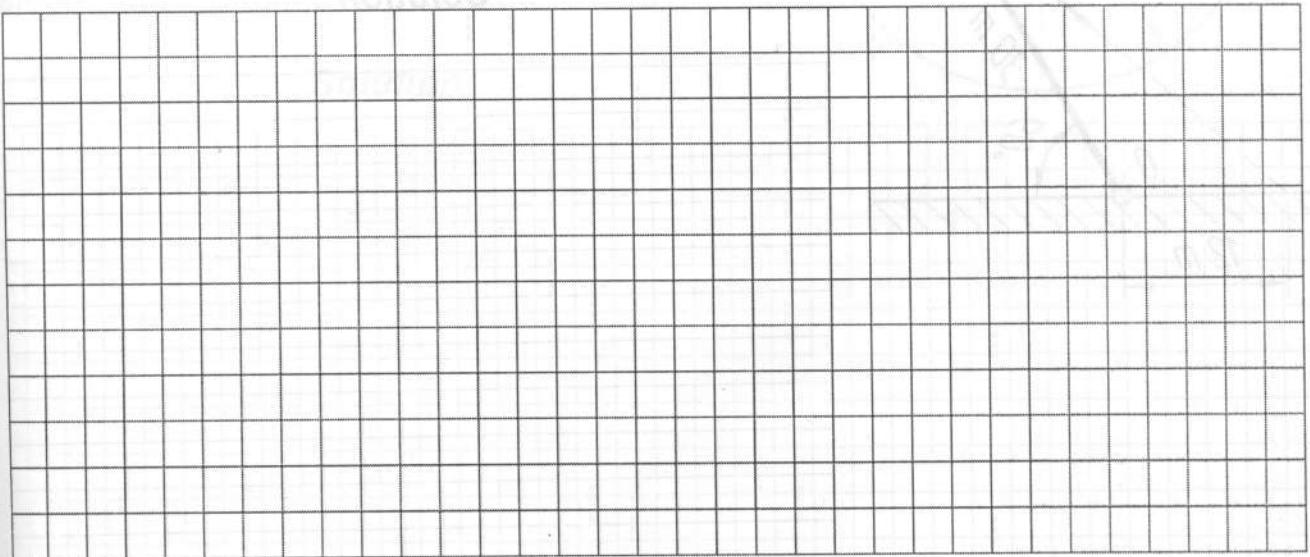


### Problem 1.7

The rectangular plate is made up of 1-m square as shown. A 10-kN force is applied at point  $A$  in the direction shown. Determine the moment  $M_B$  of the force about point  $B$  graphically by at least two different methods.



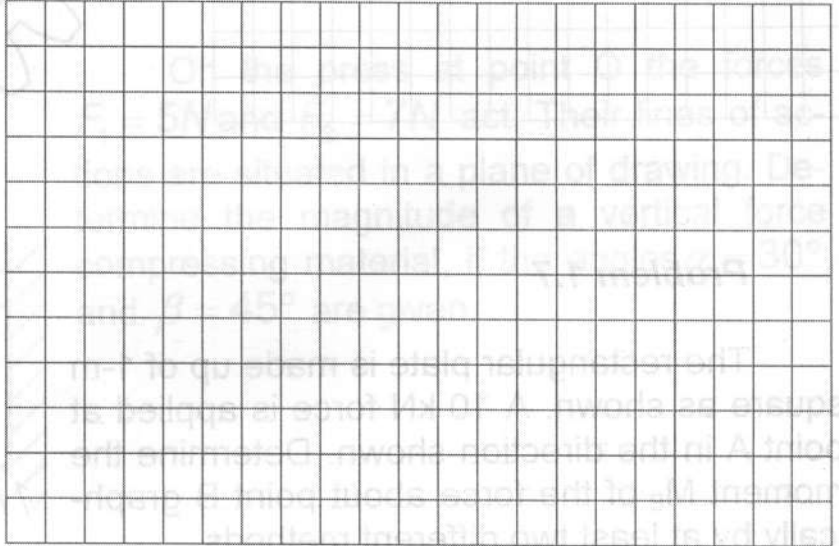
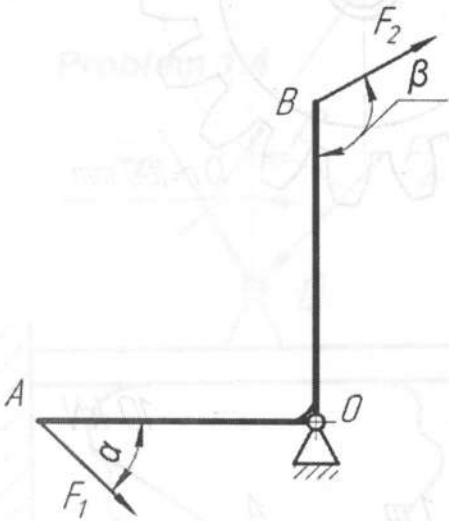
#### Solution



### Problem 1.8

On the lever with motionless axis  $O$  the forces  $F_1 = 4N$  and  $F_2$  act. Determine the moment  $M_O(\vec{F}_2)$  from condition that the sum of moments about the point  $O$  is zero, if  $\alpha = 45^\circ$ ,  $\beta = 120^\circ$ , lengths  $AO = 0,5\text{ m}$ ,  $BO = 0,6\text{ m}$ .

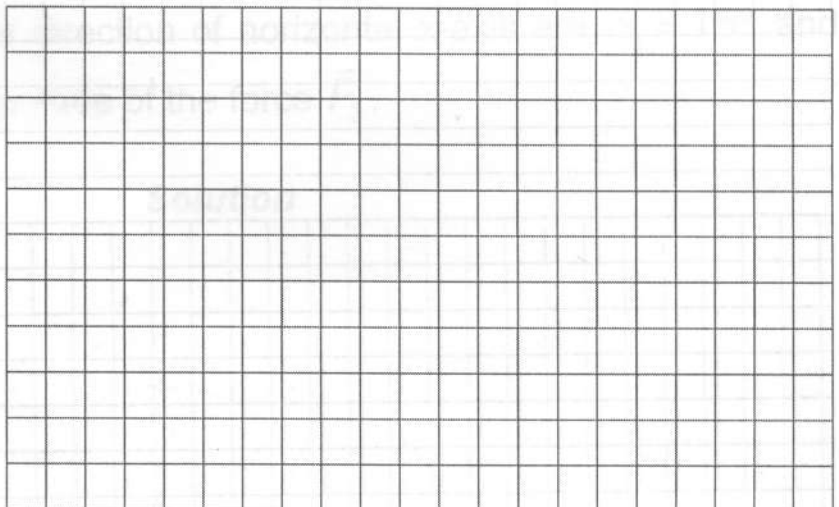
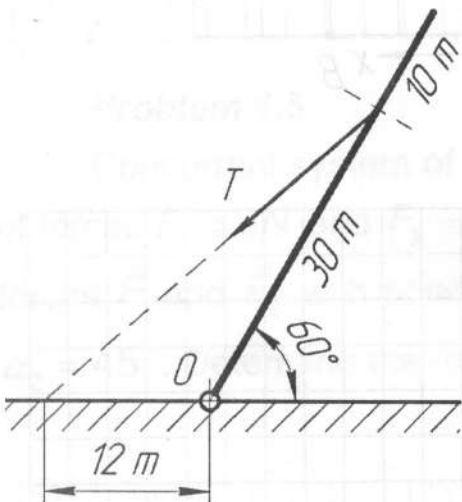
**Solution**



### Problem 1.9

In raising the flagpole from the position shown, the tension  $T$  in the cable must supply a moment about  $O$  of  $72\text{ kN}\cdot\text{m}$ . Determine  $T$ .

**Solution**

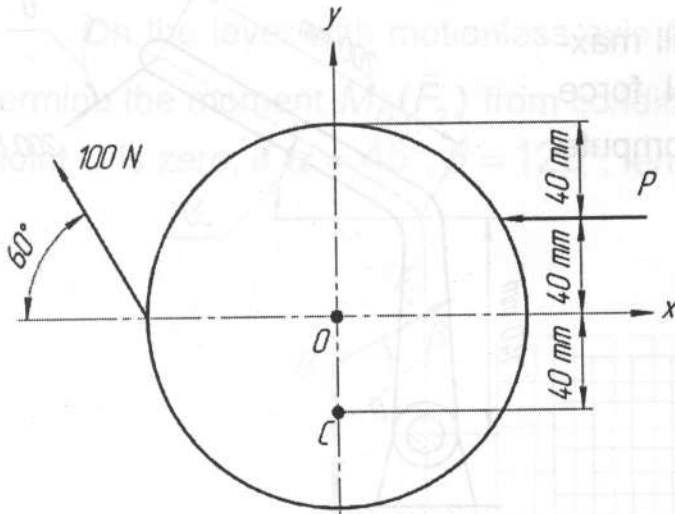






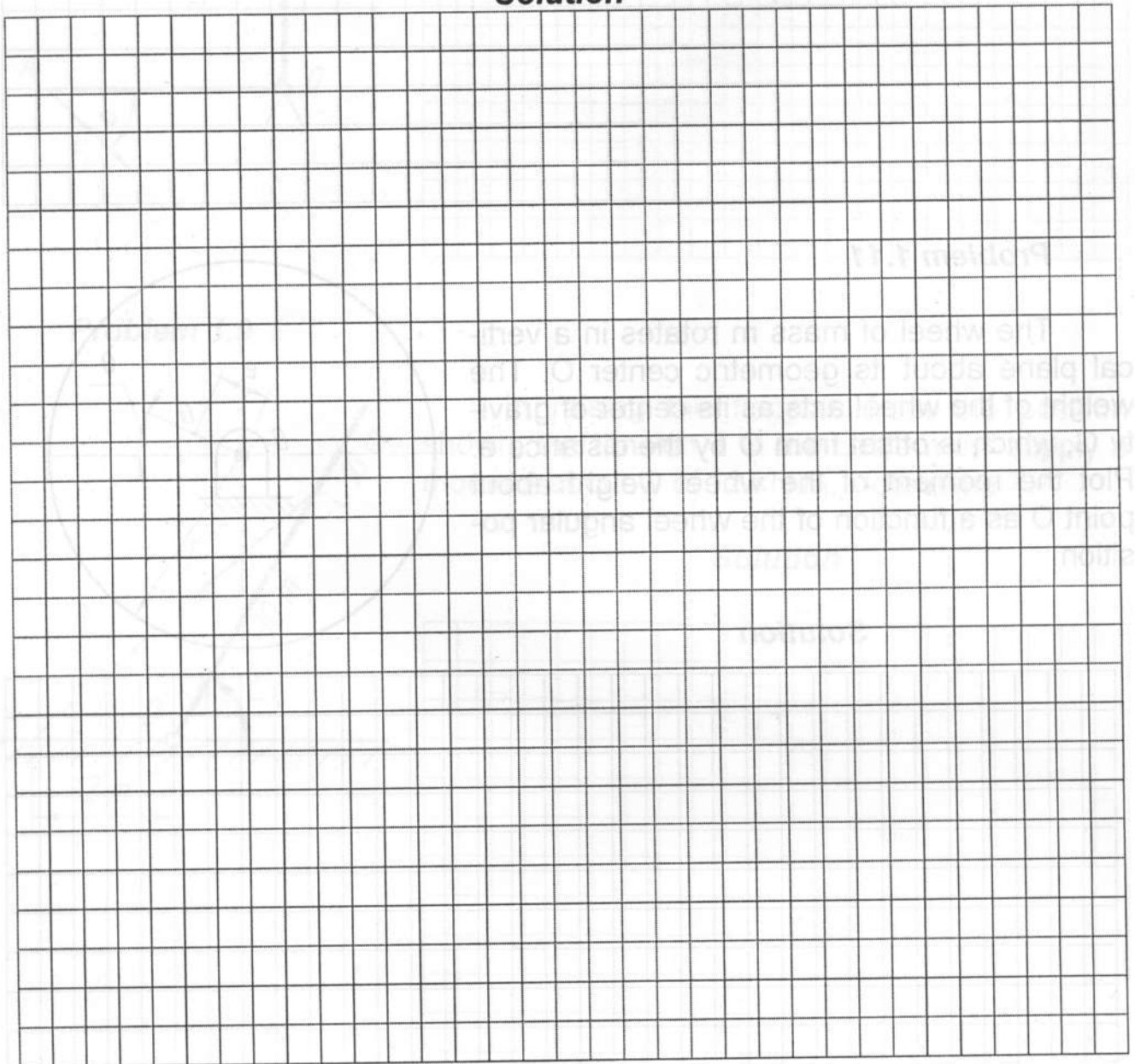


### Problem 1.12



If the combined moment of the 2 forces about point  $C$  is zero. Determine: the magnitude of the force  $P$ ; the magnitude  $R$  of the resultant of the two forces; the coordinates  $x$  and  $y$  of the point  $A$  on the rim of the wheel about which the combined moment of the two forces is a maximum; the combined moment  $M_A$  of the two forces about  $A$ .

### Solution



## 1.4. Questions for self-testing

2. What does statics learn?
3. Give the definition of a material point and an absolutely rigid body.
4. What is a force? What characteristics does it have?
5. Enumerate the axioms of statics and their contents.
6. What is the law of forces parallelogram?
7. How can projections of a force a) on a plane; b) on an axis be determined?
8. What is the resultant of a system of forces?
9. Does an order of forces position matter when constructing a force polygon?
10. In what case is a force polygon closed?
11. Give the definition of a vector moment of a force about a point. What are the methods of the vector moment determination?
12. Does a moment of a force about a point change when moving the force along a line of its action?
13. In what case is a moment of a force about a point equal to zero?

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## 2. EQUILIBRIUM OF COPLANAR FORCE SYSTEM APPLIED TO A RIGID BODY. STATICALLY DETERMINATE AND INDETERMINATE PROBLEMS

### 2.1. Main information from the theoretical course

In this chapter the action of coplanar force system (for example, **XOY**) on body is considered. There are two special coplanar force systems:

- **concurrent** with forces lines of action intersecting at the same point,
- **parallel** with parallel forces lines of action.

A system of force is called **equilibrated (balanced)** or system of force is **equivalent to zero** if under the action of this system rigid body is at rest:

$$(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n) \sim 0.$$

The sufficient and necessary **equilibrium conditions** for coplanar force system are that the projections of all forces on coordinate axes situated in a plane of forces action are equal to zero and the algebraic sum of the same forces moments about arbitrary point in this plane is equal to zero. These conditions have view:

$$\begin{cases} \sum_{k=1}^n F_{xk} = 0; \\ \sum_{k=1}^n F_{yk} = 0; \\ \sum_{k=1}^n M_O(\vec{F}_k) = 0. \end{cases}$$

Equilibrium conditions for general coplanar force system can be written in other forms:

$$\begin{cases} \sum_{k=1}^n F_{xk} = 0, \\ \sum_{k=1}^n M_A(\vec{F}_k) = 0, \\ \sum_{k=1}^n M_B(\vec{F}_k) = 0, \end{cases}$$

where the segment **AB** is not perpendicular to the axis **Ox**;

$$\sum_{k=1}^n M_A(\vec{F}_k) = 0,$$

$$\sum_{k=1}^n M_B(\vec{F}_k) = 0,$$

$$\sum_{k=1}^n M_C(\vec{F}_k) = 0,$$

where the points **A**, **B**, **C** belong to different lines.

For concurrent coplanar force system the equilibrium conditions are:

$$\sum_{k=1}^n F_{xk} = 0, \quad \sum_{k=1}^n F_{yk} = 0.$$

For parallel coplanar force system (for example parallel to y-axis) the equilibrium conditions are:

$$\sum_{k=1}^n F_{yk} = 0, \quad \sum_{k=1}^n M_A(\vec{F}_k) = 0.$$

To be soluble (or **statically determinate**) the statics problem must have equal number of unknowns (for example, unknown reactions) and the number of equations of equilibrium. If the number of unknowns is greater than the number of equilibrium equations the problem is called **statically indeterminate** and can not be solved solely by means of rigid body statics.

**Algorithm of problems solution is following:**

1. Construct a Free body Diagram (FBD)
  - 1.1. Determine the rigid body under consideration, i.e. the body the equilibrium of which should be considered for finding unknown quantities.
  - 1.2. Draw the body under consideration.
  - 1.3. Draw all applied forces acting on the body. Replace distributed applied loads by their resultants.
  - 1.4. Substitute constraints acting on the body for their reactions. Figure on a scheme all reactions acting on the body.
  - 1.5. Determine the type of force system obtained: coplanar concurrent, coplanar parallel or coplanar general force system.
  - 1.6. Choose the set of coordinate axes to be used in solving the problem and indicate these direction on the FBD.
2. Check statically determinacy of a problem.
3. If the problem is statically determinate choose appropriate form of equilibrium equations, write down them and solve regarding to unknown parameters.
4. Run a check of solution. For this, for example, write down an equation of moments about any another point not used in making equilibrium equation.

To simplify your solution, it may be helpful to use one of the following solution techniques if applicable:

- by summing moments about the point of intersection of the lines of action of two unknown forces, you will obtain an equation in a single unknown;
- by summing components in a direction perpendicular to two unknown parallel forces, you will obtain an equation in a single unknown.

### **Reactions of constraints (supports and connections) for two-dimensional structure**

A body is considered **free** if its displacements are not restricted by any other bodies; otherwise a body is **constrained**. The bodies that prevent the motion of the first body are called **constraints** imposed upon the body.

The mechanical effect of a constraint is the same as the action of a force. Therefore the action of a constraint in the body may be replaced by the forces that are called **reactions**.

Now let us consider some constraints and show their reaction lines of actions, supposing that contacting surfaces are smooth enough to neglect friction completely (Table 2.1).

## 2.2. Solution of problems

### **Sample problem 2.1**

The cylinder with weight  $G=200\text{ N}$  (Fig. 2.1, a) is held by a rope  $OA$  on ideal smooth inclined plane  $MK$  making with horizon an angle  $\beta = 45^\circ$  and exert on plane pressure  $Q=60\text{ N}$ . Determine an angle  $\alpha$  and rope tension  $T$ .

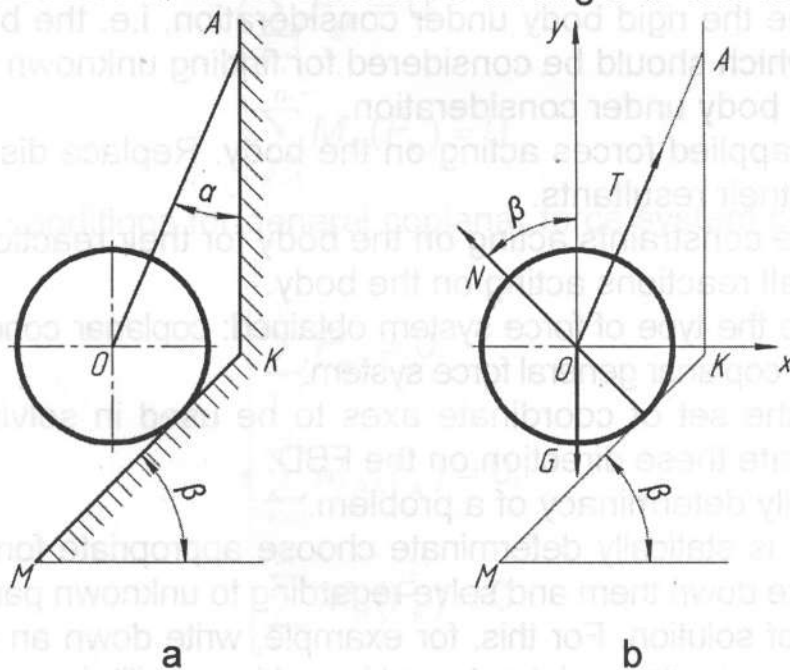


Fig. 2.1



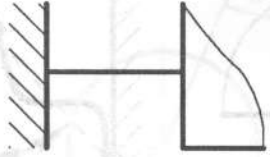
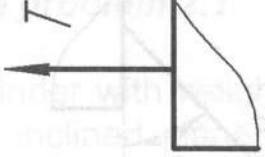
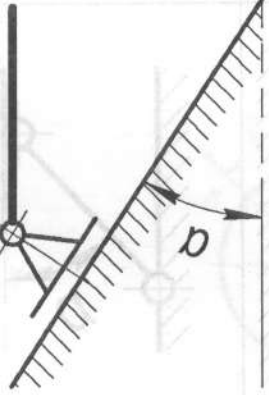
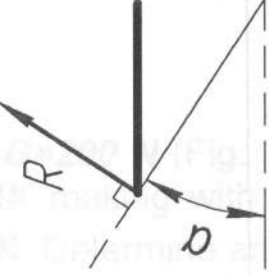
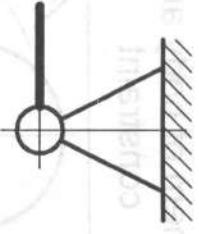
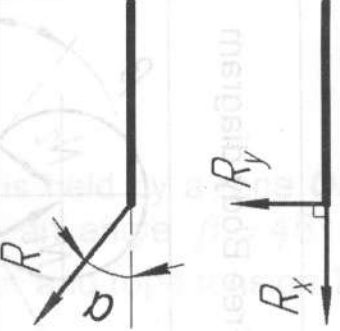
Reactions at supports and connections for two-dimensional structure

Constraint	Schem of body and constraint	Free Body diagram	Number of unknowns	Unknowns characteristics	Additional information
Smooth surfaces contact			<b>1</b>	One force with known line of action	Contact force is compressive and normal to the surface.
Rigid weightless link (rectilinear or curvilinear)			<b>1</b>	One force with known line of action and unknown sense	Rigid link reaction is directed along the line connecting the centers of the link pins.

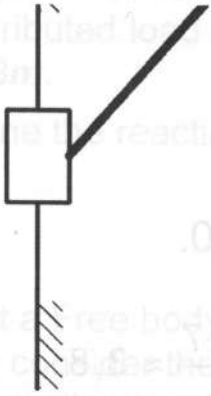
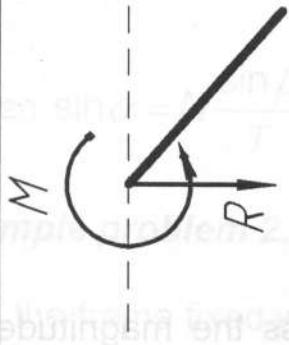


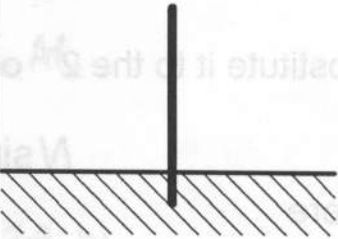
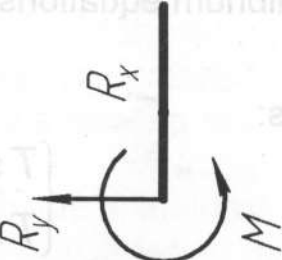


Continuation of Table 2.1

Reactions at supports and connections for two-dimensional structure

<p>Weightless flexible, nonstretchable cord</p>			<p>1</p>	<p>One force with known line of action</p>	<p>Force exerted by a cord is always a tension away from the body in the direction of the cord.</p>
<p>Sliding joint or roller support</p>			<p>1</p>	<p>One force with known line of action</p>	<p>Sliding joint impedes motion of the beam end A in direction that is perpendicular to the plane where the joint is installed.</p>
<p>Pinned joint, pin</p>			<p>2</p>	<p>Force with unknown magnitude and direction or two mutually perpendicular forces. Application point is at pin center.</p>	<p>Pinned joint prevents the motion of the beam end in two mutually perpendicular directions: along axes x and y.</p>

Reactions at supports and connections for two-dimensional structure

Collar or sliding support			2	One force perpendicular to collar axis and one couple with unknown moment	Collar restricts the motion of body in the direction normal to the guide and body's rotation.
Double stage sliding support			1	One couple with unknown moment	Double stage sliding support prevents rotation about axis perpendicular to the plane of sketch
Fixed support, clamped joint			3	Two mutually perpendicular forces and one couple with unknown moment	Clamped joint, or fixed support prevents motion of the beam end in all directions (along axes x and y, rotation about axis z).

## Solution

1. Construct a Free body Diagram (FBD) (see Fig. 2.1, b).
  - 1.1. Equilibrium of the cylinder is considered.
  - 1.2. Figure the considering cylinder on a certain drawing.
  - 1.3. Gravity force is the single applied force acting at mass center of the cylinder.
  - 1.4. Two constraints restrict the cylinder motion: smooth inclined surface and rope. Substituting constraints for their reactions we obtain two reactions  $\vec{N}$  and  $\vec{T}$ . Here we note that normal reaction of the surface on the cylinder is equal by the magnitude to pressure force  $Q$  on a surface and opposite in direction  $\vec{N} = -\vec{Q}$ .
  - 1.5. Force system obtained is coplanar concurrent.
  - 1.6. We choose a coordinate system, put its origin at the center of the cylinder.
2. The problem is statically determinate because for the coplanar concurrent force system there are two equilibrium conditions and our problem has two unknown quantities.
3. The first method (analytical).

Equilibrium equations

$$\sum F_{xk} = 0, \quad \sum F_{yk} = 0$$

will view as:

$$\begin{cases} T \sin \alpha - N \sin \beta = 0; \\ T \cos \alpha + N \cos \beta - G = 0. \end{cases}$$

From the 1<sup>st</sup> equation of the system we express the magnitude of the force

$$T = N \frac{\sin \beta}{\sin \alpha}$$

and substitute it to the 2<sup>nd</sup> one. Then

$$N \sin \beta \operatorname{ctg} \alpha + N \cos \beta - G = 0.$$

From here

$$\operatorname{ctg} \alpha = \frac{G - N \cos \beta}{N \sin \beta} = \frac{200 - 60 \cdot 0,7}{60 \cdot 0,7} \approx 3,8,$$

$$\alpha = \operatorname{arcctg} 3,8 = 15^\circ.$$

Rope tensile load will be

$$T = 60 \frac{\sin 45^\circ}{\sin 15^\circ} = 60 \frac{0,7}{0,26} \approx 163 \text{ (N)}.$$

4. The second method (graphical, Fig. 2.2).

We apply graphical interpretation of rigid body equilibrium condition under action of concurrent system of forces, i. e. if the sum of vectors  $\vec{G}, \vec{N}, \vec{T}$  is equal to zero the vector force polygon is closed. Let's figure on a scale known by the magnitude and the direction vector of the force  $\vec{G}$ . From its end at an angle  $\beta$  we put on the same scale vector  $\vec{N}$ . A segment connecting the end of vector  $\vec{N}$  and the origin of vector  $\vec{G}$  will be required magnitude of vector  $\vec{T}$ . Measured the length of this segment and multiplied it on a scale coefficient we will find the magnitude of the force  $T$ .

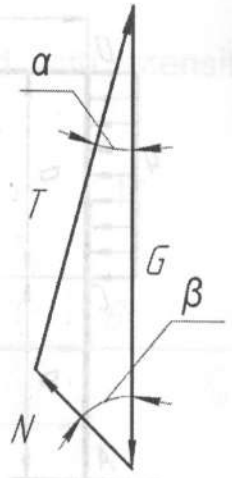


Fig. 2.2

The same value can be found also using cosine theorem:

$$T = \sqrt{G^2 + N^2 - 2G \cdot N \cos \beta} \approx 163 \text{ (N)}.$$

The angle  $\alpha$  can be found directly by measurement on the drawing or using sine theorem:

$$\frac{N}{\sin \alpha} = \frac{T}{\sin \beta}.$$

Then  $\sin \alpha = N \frac{\sin \beta}{T} = 0,26$  and  $\alpha = 15^\circ$ .

**Sample problem 2.2**

On the frame fixed with the use of clamped joint (Fig. 2.3, a), the following loads act: couple of forces with moment  $M = 4 \text{ N}\cdot\text{m}$ , concentrated force  $P = 10 \text{ N}$ , distributed load with intensity  $q = 1,5 \text{ N/m}$ . The distances are known:  $a = 2 \text{ m}$ ,  $b = 3 \text{ m}$ .

Determine the reaction of clamped joint -  $\bar{X}_A, \bar{Y}_A, M_A$ .

**Solution**

1. Construct a Free body Diagram (FBD) (see Fig. 2.3, b).
  - 1.1. We consider the frame equilibrium.
  - 1.2. Figure the considering frame on a certain drawing.
  - 1.3. Applied forces acting on the frame: concentrated force  $P$ , couple of forces with moment  $M$  and distributed load with intensity  $q$  that is changed on concentrated force  $Q = q \cdot a = 1,5 \cdot 2 = 3 \text{ (N)}$ . The force  $Q$  is the resultant of parallel force system and because  $q = \text{const}$  the resultant  $Q$  passes through the middle of the segment  $CD$ .

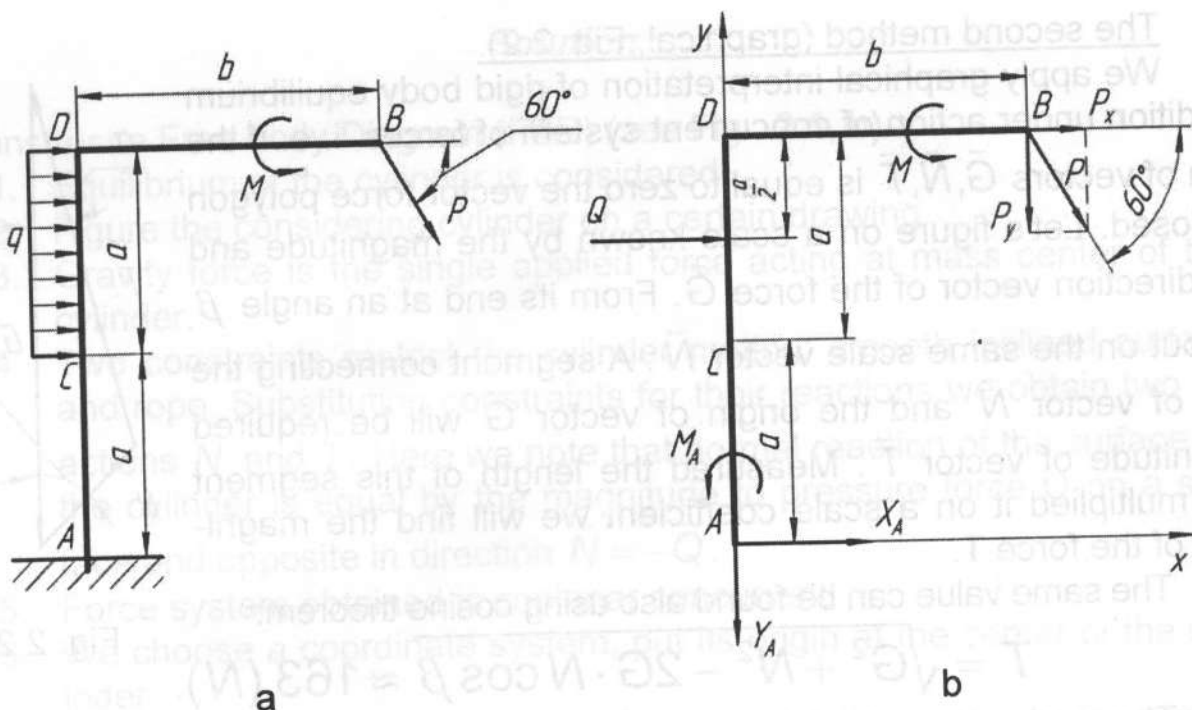


Fig. 2.3

- 1.4. Motion of the frame is limited by clamped joint. Substituting constraint for its reactions we obtain two reactions  $\vec{X}_A, \vec{Y}_A$  and reactive moment  $M$ .
- 1.5. Obtained system of forces is general coplanar one.
- 1.6. We choose a coordinate system such as its origin coincides with the point A.
2. The problem is statically determinate because the conditions of body equilibrium under action of general coplanar forces include three equations and our problem has three unknown quantities.
3. Equilibrium equations will be:

$$\begin{cases} \sum F_{xk} = X_A + Q + P \cos 60^\circ = 0; \\ \sum F_{yk} = Y_A - P \sin 60^\circ = 0; \\ \sum M_A(\vec{F}_k) = M_A + M - Q \cdot 3 - P \cos 60^\circ \cdot 2a - P \sin 60^\circ \cdot b = 0. \end{cases}$$

We put into these equations known values:

$$\begin{cases} X_A + 3 + 10 \cdot 0,5 = 0; \\ Y_A - 10 \cdot 0,85 = 0; \\ M_A + 4 - 3 \cdot 3 - 10 \cdot 0,5 \cdot 4 - 10 \cdot 0,85 \cdot 3 = 0. \end{cases}$$

From here

$$X_A = -8N, \quad Y_A = 8,5N, \quad M_A = 50,5N \cdot m.$$



### Sample problem 2.3

On the double-beat frame (Fig. 2.4, a) distributed load with intensity  $q_{max}=20\text{ N/m}$  acts. The lengths of sections are  $l=0,3\text{ m}$ . Determine the reaction of constraint A ( $R_A$ ).

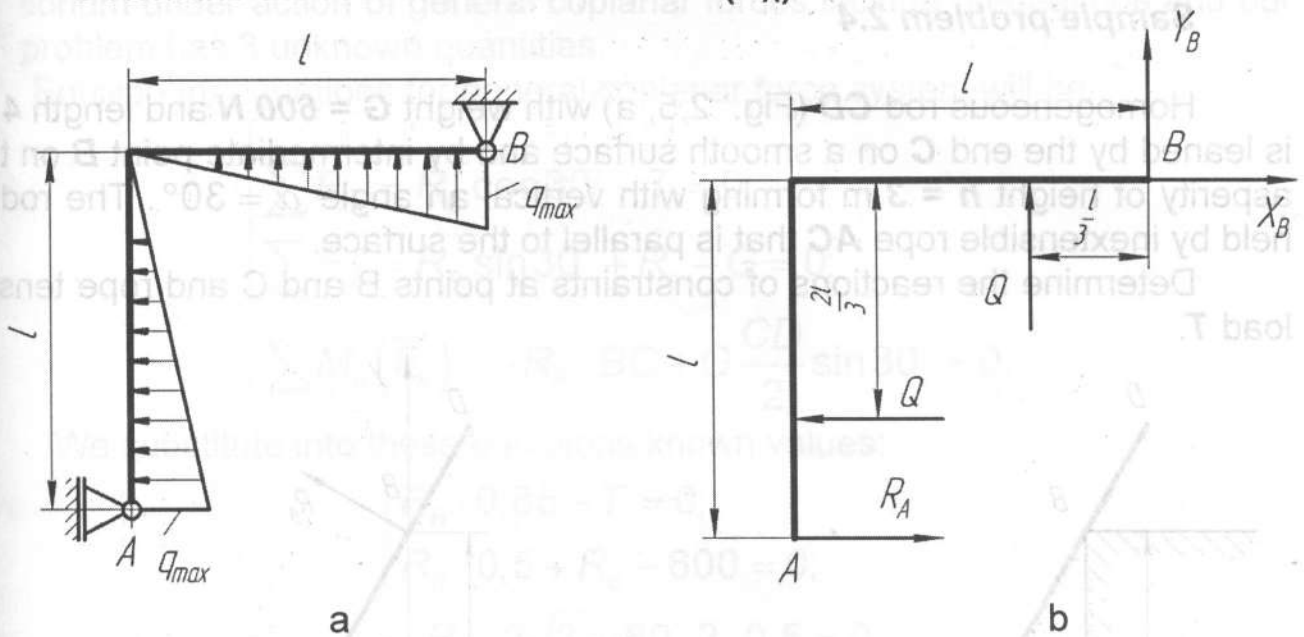


Fig. 2.4

### Solution

1. Construct a Free body Diagram (FBD) (Fig. 2.4, b).
  - 1.1. We consider the frame equilibrium.
  - 1.2. Figure the considering frame on a certain drawing.
  - 1.3. Applied forces acting on the frame: two distributed loads with intensity  $q$  on the segments AC and BC that are changed on concentrated forces  $Q = \frac{1}{2}(q_{max} \cdot l) = \frac{1}{2}(20 \cdot 0,3) = 3(N)$ . In given case the lines of their actions is on the distance of  $1/3l$  from the point A and B, where  $q = q_{max}$ .
  - 1.4. Motion of the frame is limited by fixed pin at the point B and roller at the point A. Exchanging constraints by their reactions we obtain 3 reactions  $\vec{X}_B, \vec{Y}_B, \vec{R}_A$ .
  - 1.5. Obtained system of forces is general coplanar one.
2. The problem is statically determinate because the conditions of body equilibrium under action of general coplanar forces include 3 equations and our problem has 3 unknown quantities.
3. For finding unknown reaction  $R_A$  it is sufficient to make only one equation (the sum of moments about point B):



$$\sum M_B(\vec{F}_k) = -Q \cdot \frac{1}{3}l - Q \cdot \frac{2}{3}l + R_A \cdot l = 0.$$

After solving this equation we will find that  $R_A = Q = 3N$ .

### Sample problem 2.4

Homogeneous rod **CD** (Fig. 2.5, a) with weight  $G = 600 \text{ N}$  and length  $4 \text{ m}$  is leaned by the end **C** on a smooth surface and by intermediate point **B** on the asperity of height  $h = 3 \text{ m}$  forming with vertical an angle  $\alpha = 30^\circ$ . The rod is held by inextensible rope **AC** that is parallel to the surface.

Determine the reactions of constraints at points B and C and rope tensile load  $T$ .

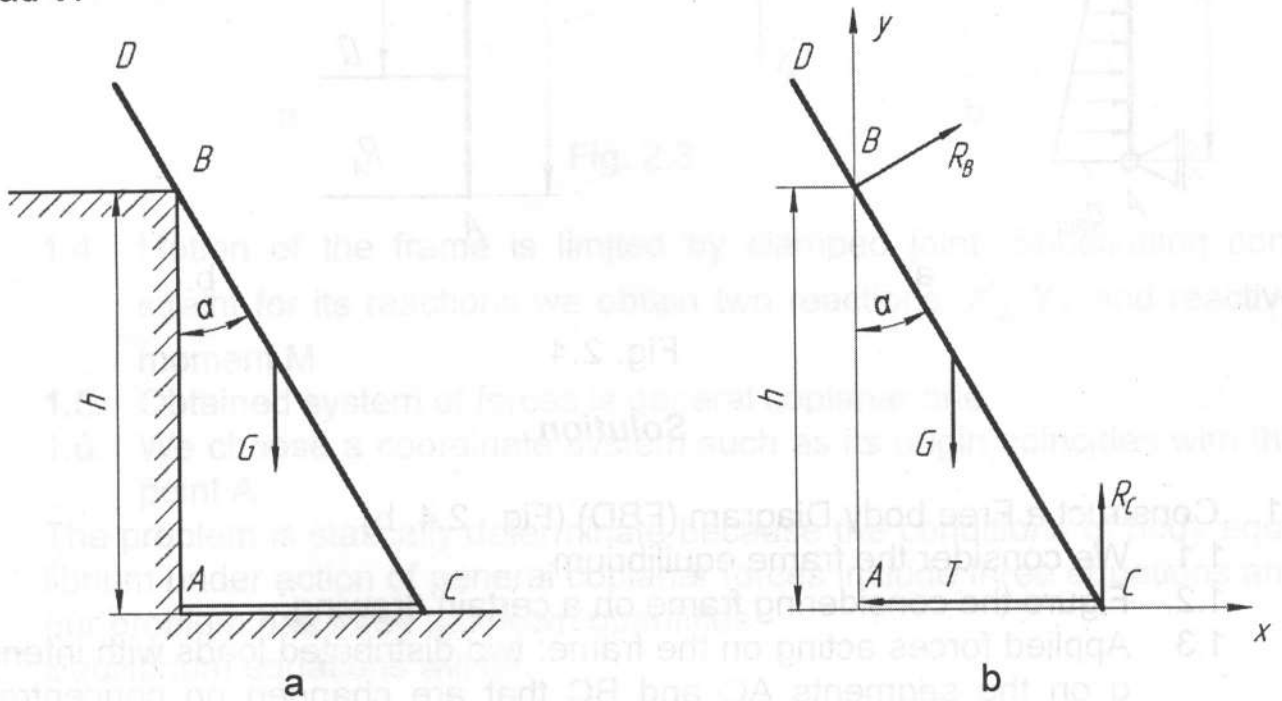


Fig. 2.5

### Solution

1. Construct a Free body Diagram (FBD) (Fig. 2.5, b).
  - 1.1. We consider the rod equilibrium.
  - 1.2. Figure the considering rod on a certain drawing.
  - 1.3. Gravity force is the only applied force acting on the rod applied in a mass center of the rod.
  - 1.4. The rod is leaned by the end **C** on a smooth surface and by intermediate point **B** on the asperity. The rod is held by inextensible rope **AC**. Exchanging these constraints by their reactions we obtain two reactions  $\vec{R}_B, \vec{R}_C$  and tensile load  $\vec{T}$ .

1.5. Obtained system of forces is general coplanar one.

1.6. We choose a coordinate system such as its origin coincides with the point A.

The problem is statically determinate because the conditions of body equilibrium under action of general coplanar forces include 3 equations and our problem has 3 unknown quantities.

Equilibrium equations for general coplanar force system will be:

$$\begin{cases} \sum F_{xk} = R_B \cos 30^\circ - T = 0; \\ \sum F_{yk} = R_B \sin 30^\circ + R_C - G = 0; \\ \sum M_c(\bar{F}_k) = -R_B \cdot BC + G \frac{CD}{2} \sin 30^\circ = 0. \end{cases}$$

We substitute into these equations known values:

$$\begin{cases} R_B \cdot 0,85 - T = 0; \\ R_B \cdot 0,5 + R_C - 600 = 0; \\ -R_B \cdot 2\sqrt{3} + 60 \cdot 2 \cdot 0,5 = 0, \end{cases}$$

where  $BC = \frac{h}{\cos 30^\circ} = \frac{3 \cdot 2}{\sqrt{3}} = 2\sqrt{3} \text{ (m)}$ .

Solving this system of equations, we will find:

$$T = 150 \text{ N}, R_B = 173 \text{ N}, R_C = 513 \text{ N}.$$

### Sample problem 2.5

On cantilever beam AB (Fig. 2.6, a) the couple of forces with the moment  $M = 2 \text{ N}\cdot\text{m}$ , the concentrated force  $P = 4 \text{ N}$ , distributed load  $q = 1,5 \text{ N/m}$ . Dimensions in meters shown on the figure. Determine the reaction of clamped point.

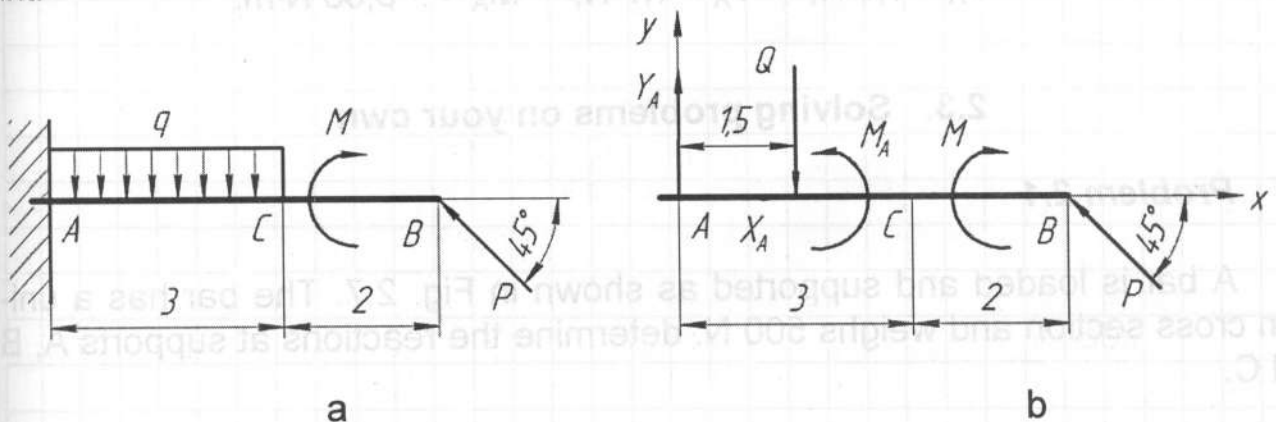


Fig. 2.6

## Solution

1. Construct a Free body Diagram (FBD) (Fig. 2.6, b).
  - 1.1. We consider the beam equilibrium.
  - 1.2. Figure the considering beam on a certain drawing.
  - 1.3. Applied forces acting on the beam: concentrated force  $P$ , couple of forces with moment  $M$  and distributed load with intensity  $q$  that is changed on concentrated force  $Q = q \cdot l = 1,5 \cdot 3 = 4,5(N)$ .  
The force  $Q$  is the resultant of parallel force system and because  $q = \text{const}$  the resultant  $Q$  passes through the middle of the segment  $AC$ .
  - 1.4. Motion of the beam is limited by clamped joint. Exchanging constraint by its reactions we obtain two reactions  $\vec{X}_A, \vec{Y}_A$  and reactive moment  $M_A$ .
  - 1.5. Obtained system of forces is general coplanar one.
  - 1.6. We choose a coordinate system superposed its center with the point  $A$ .
2. The problem is statically determinate because the conditions of body equilibrium under action of general coplanar forces include 3 equations and our problem has 3 unknown quantities.
3. Equilibrium equations for general coplanar force system will be:

$$\begin{cases} \sum F_{xk} = X_A - P \cos 45^\circ = 0; \\ \sum F_{yk} = Y_A - Q + P \sin 45^\circ = 0; \\ \sum M_A(F_k) = M_A - Q \cdot 1,5 + P \cdot 5 \sin 45^\circ - M = 0. \end{cases}$$

Solving the system we'll find:

$$X_A = 2,8 \text{ N}, \quad Y_A = 1,7 \text{ N}, \quad M_A = -5,35 \text{ N}\cdot\text{m}.$$

### 2.3. Solving problems on your own

#### Problem 2.1

A bar is loaded and supported as shown in Fig. 2.7. The bar has a uniform cross section and weighs 500 N. determine the reactions at supports A, B and C.







**Problem 2.3**

The mass of the cylinder shown in Fig. 2.9 is 100 kg. Determine the reactions at contact points A and B. All surfaces are smooth.

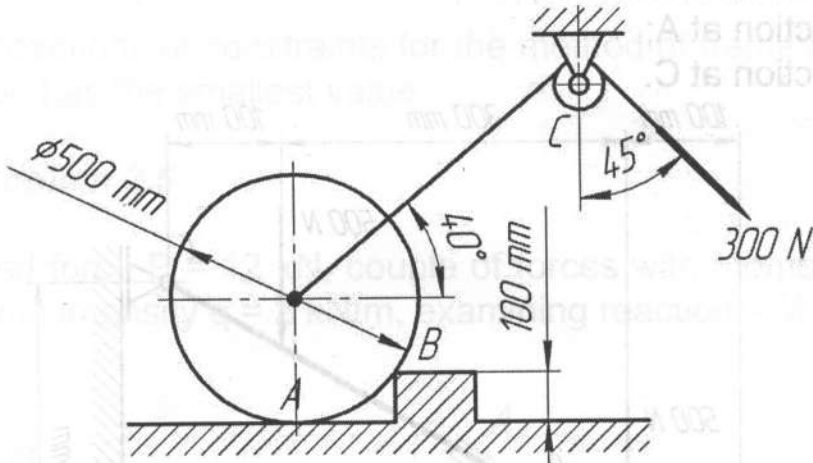


Fig. 2.9

**Solution**

### Problem 2.4

Bar AC supports two 500 N loads as shown in Fig.2.10. Rollers A and C rest against frictionless surfaces and a cable BD is attached at B. determine:

- the tension in cable BD;
- the reaction at A;
- the reaction at C.

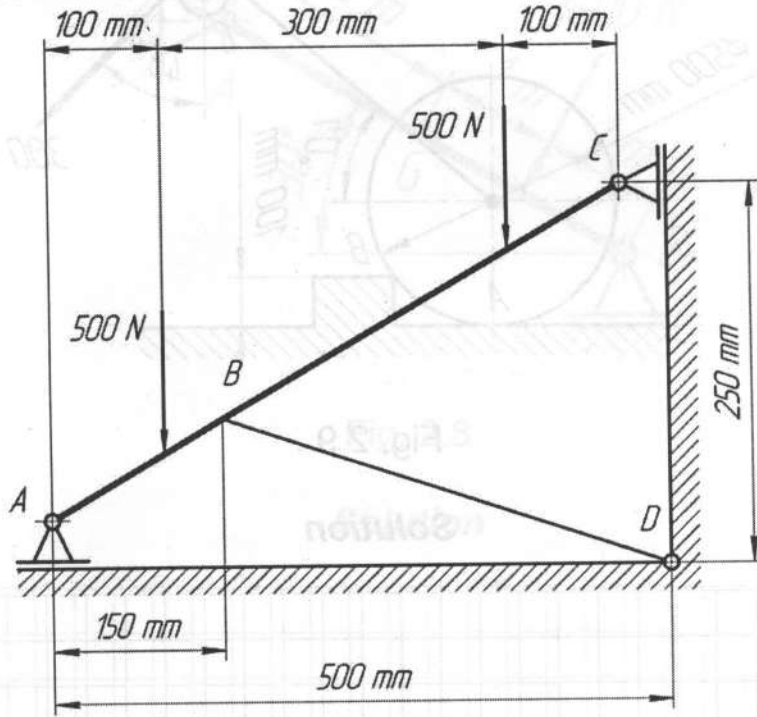
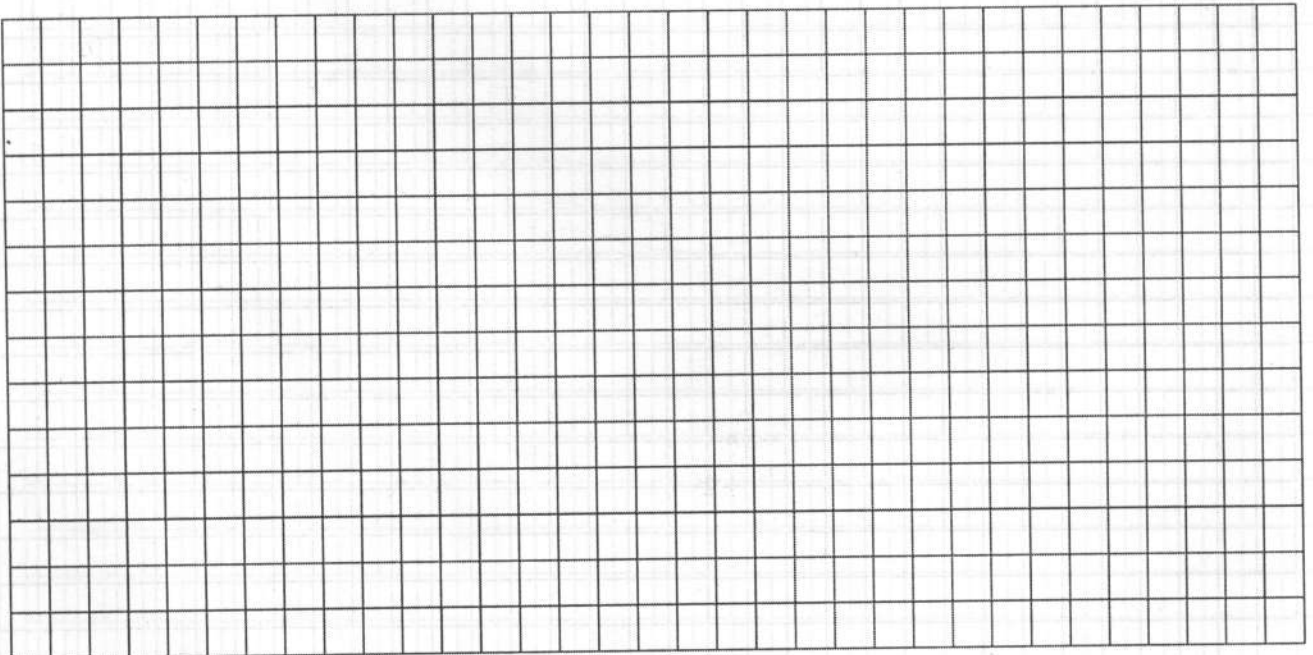


Fig. 2.10

### Solution



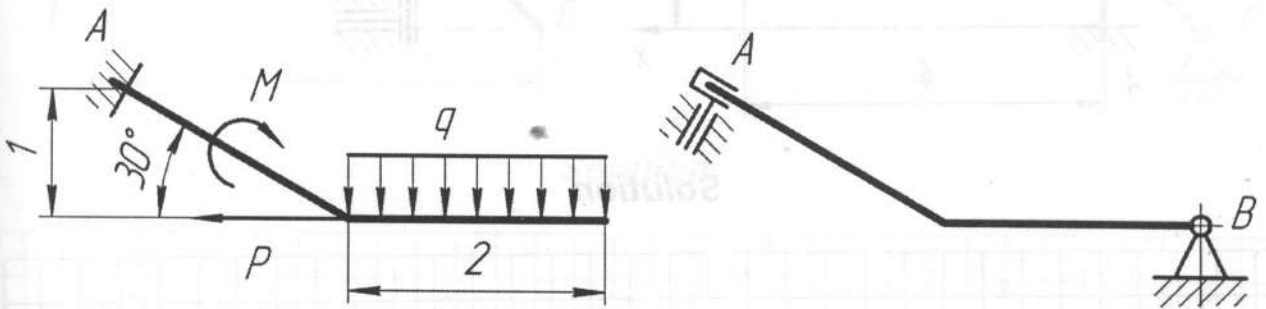
### Problem 2.5 – 2.11

In the problems (2.5 – 2.11) two methods of frame fixing are shown on the schemes. Axis of a frame is polyline. Specified loads and dimensions in 2 cases are equal.

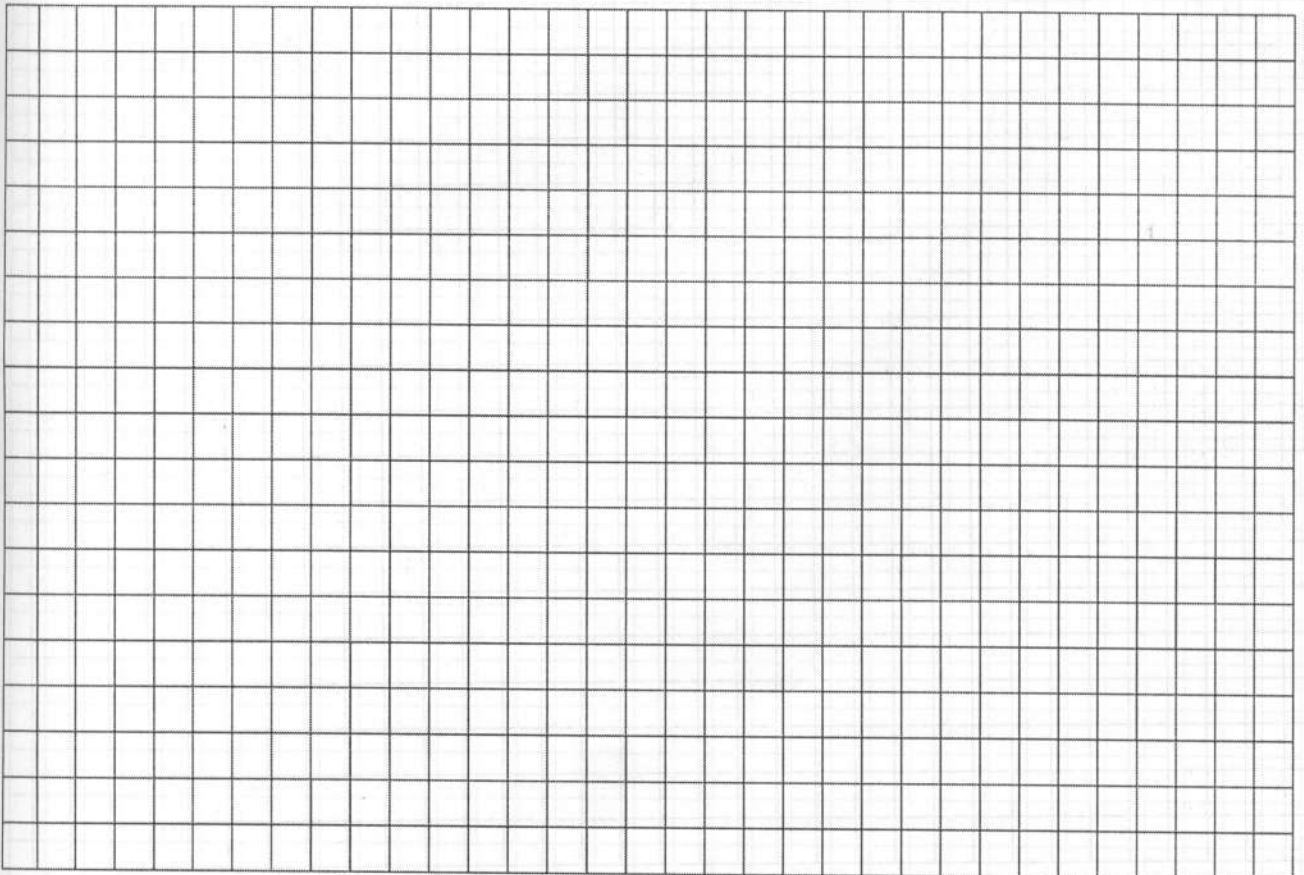
Determine reactions of constraints for the method of frame fixing at which examining reaction has the smallest value.

#### Data for problem 2.5

Concentrated force  $P = 12 \text{ kN}$ , couple of forces with moment  $M = 6 \text{ kN}\cdot\text{m}$ , distributed load with intensity  $q = 2 \text{ kN/m}$ , examining reaction –  $M_A$ .

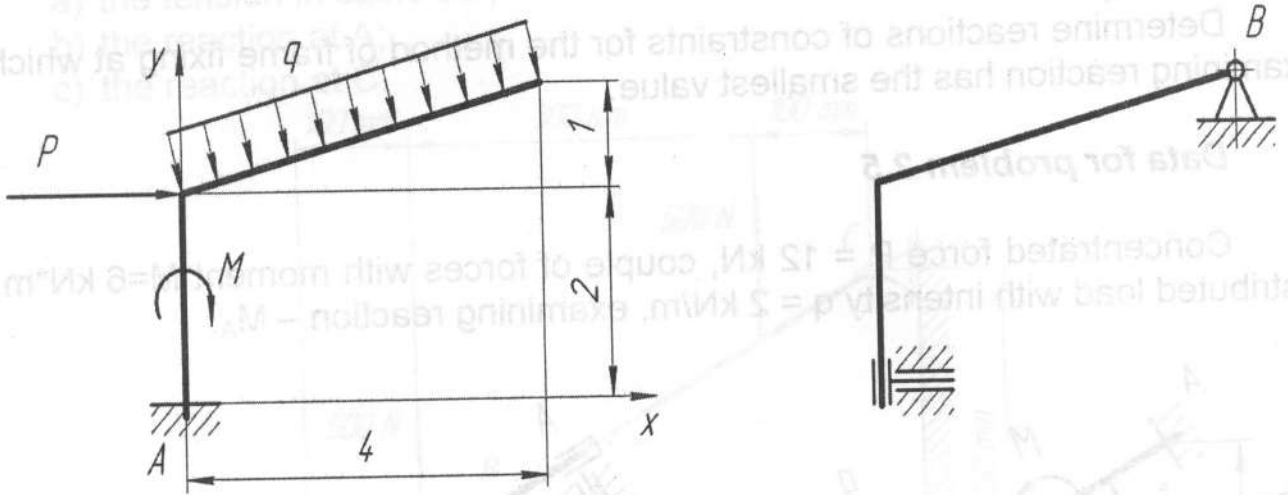


#### Solution

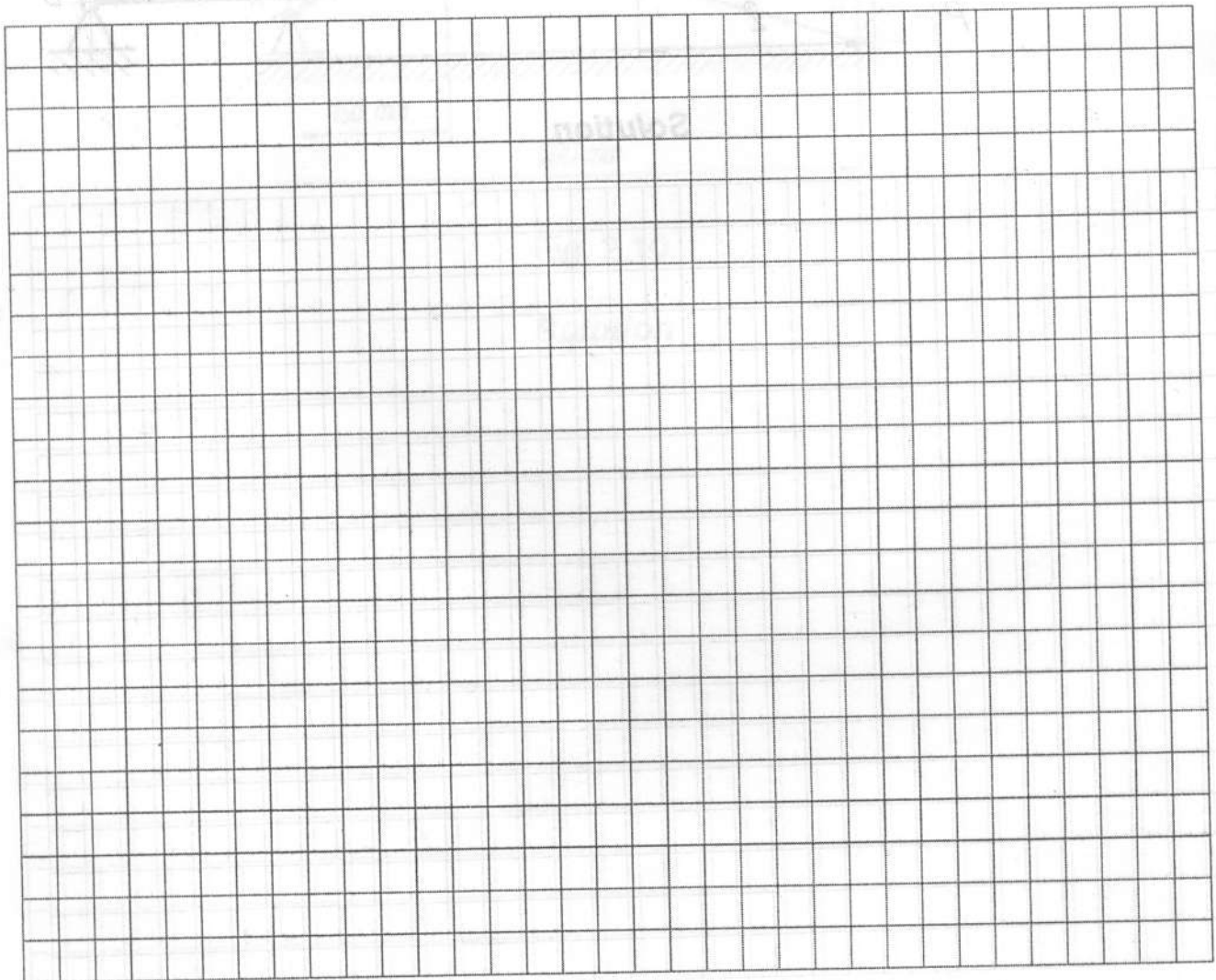


**Data for problem 2.6**

Concentrated force  $P = 6 \text{ kN}$ , couple of forces with moment  $M = 2 \text{ kN}\cdot\text{m}$ , distributed load with intensity  $q = 1 \text{ kN/m}$ , examining reaction is  $M_A$ .

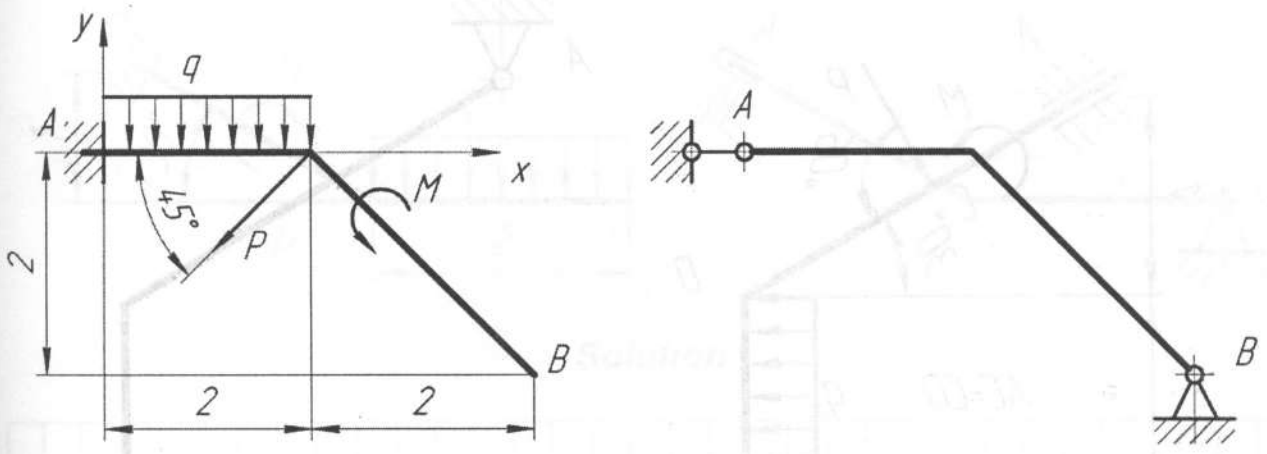


**Solution**

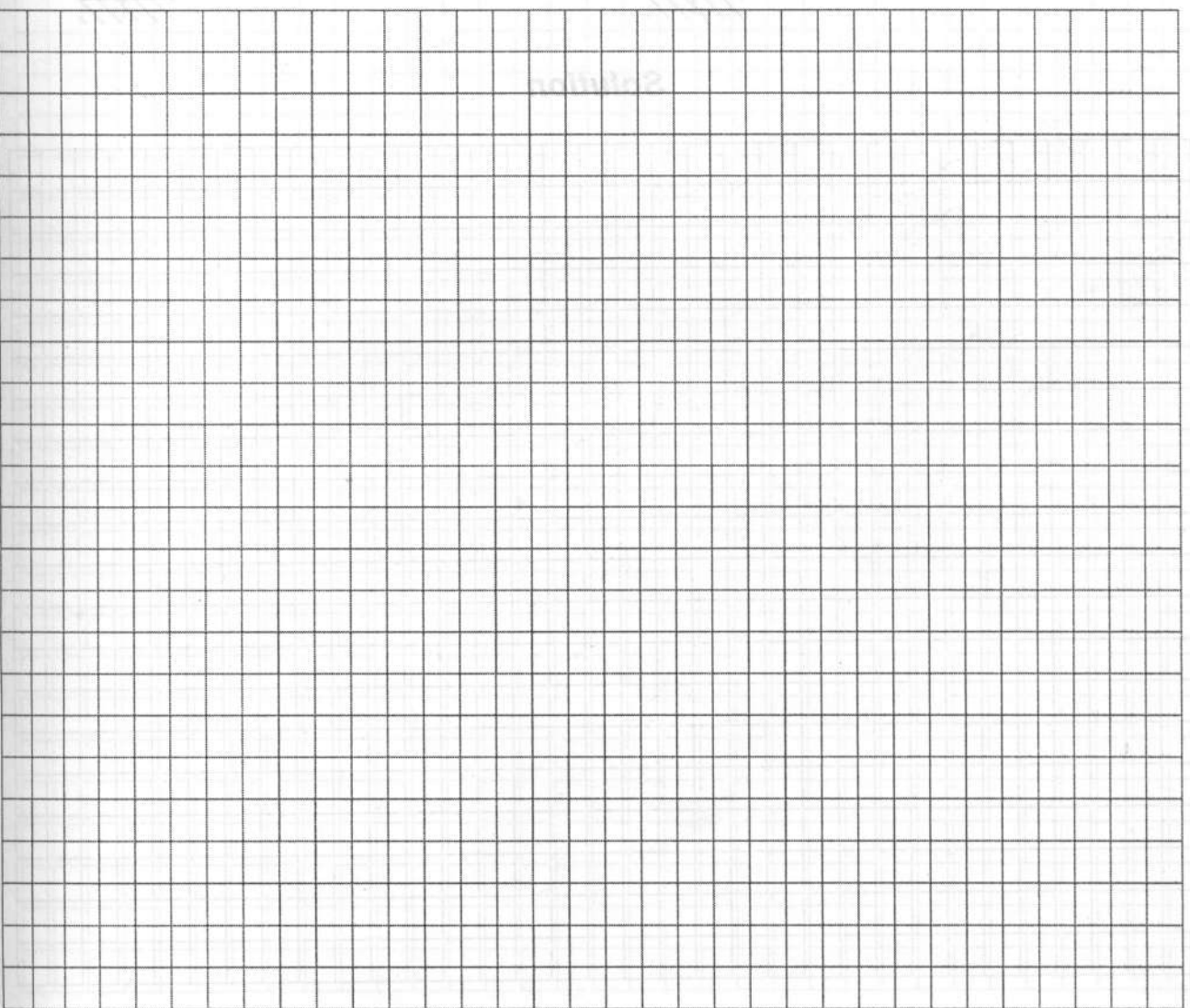


**Data for problem 2.7**

Concentrated force  $P = 2 \text{ kN}$ , couple of forces with moment  $M = 4 \text{ kN}\cdot\text{m}$ , distributed load with intensity  $q = 2 \text{ kN/m}$ , examining reaction is  $X_A$ .



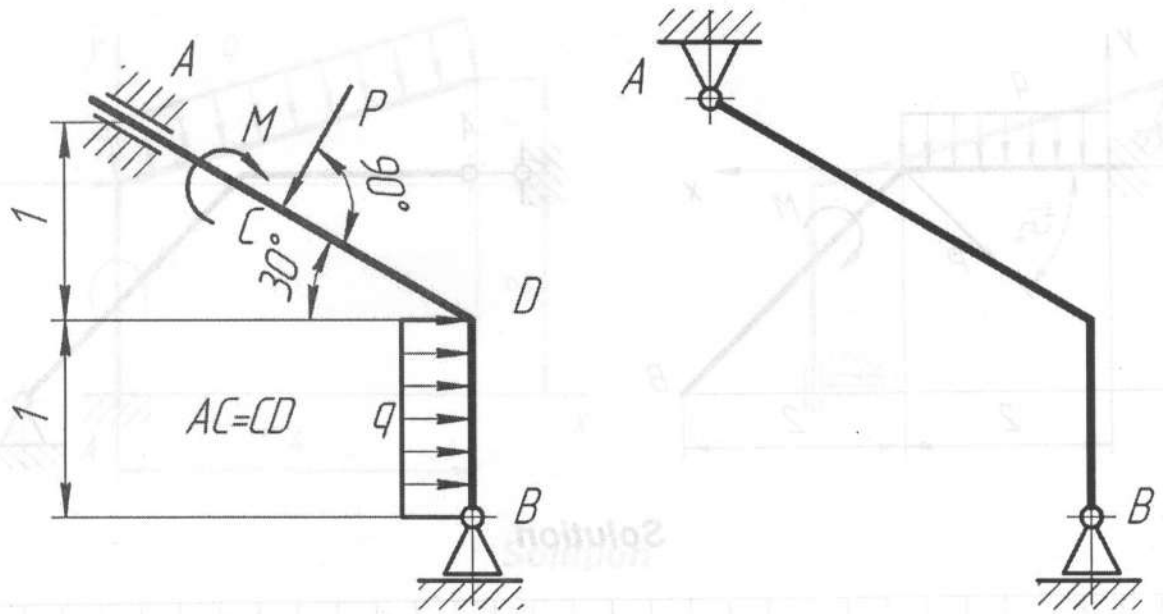
**Solution**



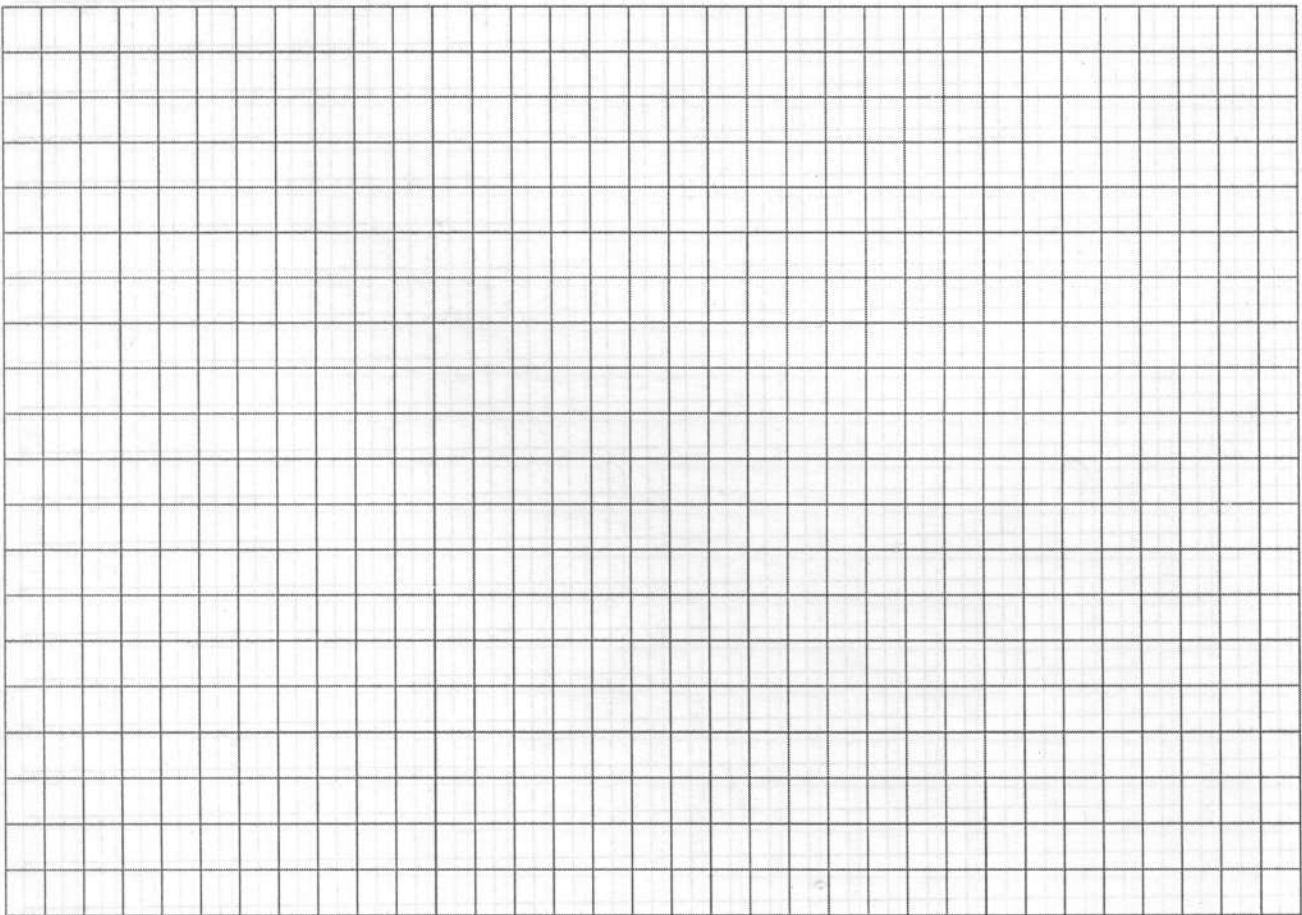


**Data for problem 2.8**

Concentrated force  $P = 20$  kN, couple of forces with moment  $M = 10$  kN\*m, distributed load with intensity  $q = 4$  kN/m, examining reaction is  $R_B$ .



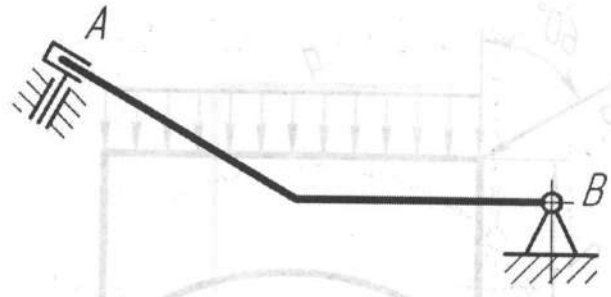
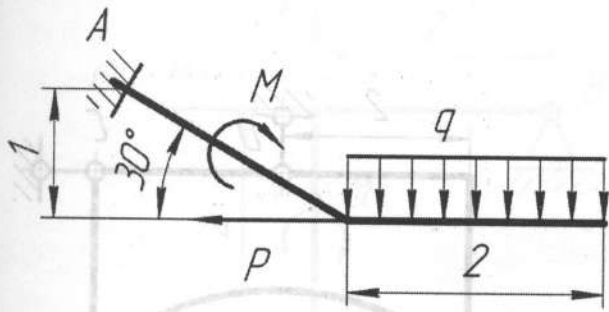
**Solution**



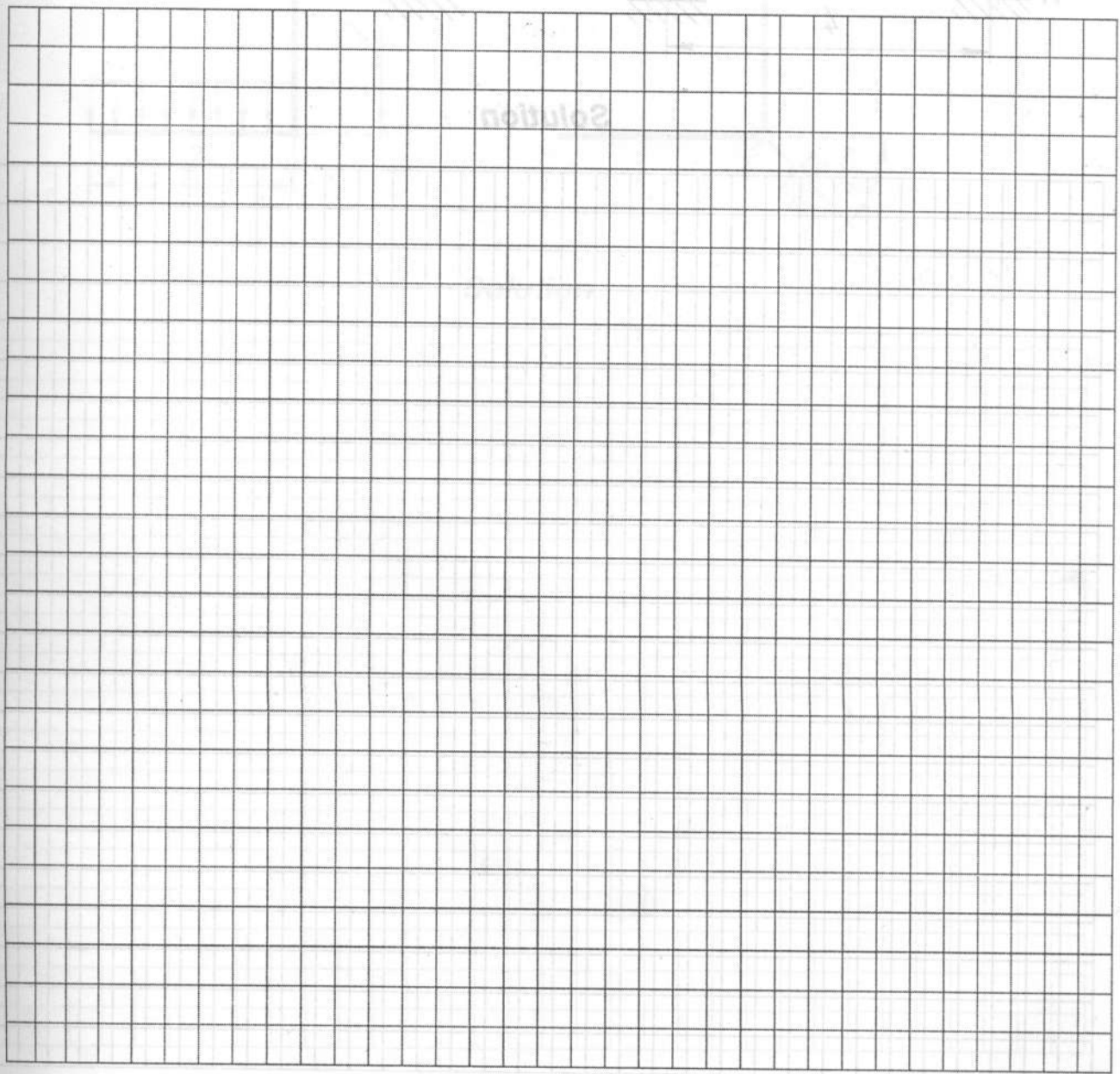
**Data for problem 2.9**

Data for problem 2.10

Concentrated force  $P = 12 \text{ kN}$ , couple of forces with moment  $M = 6 \text{ kN}\cdot\text{m}$ , distributed load with intensity  $q = 2 \text{ kN/m}$ , examining reaction is  $M_A$ .



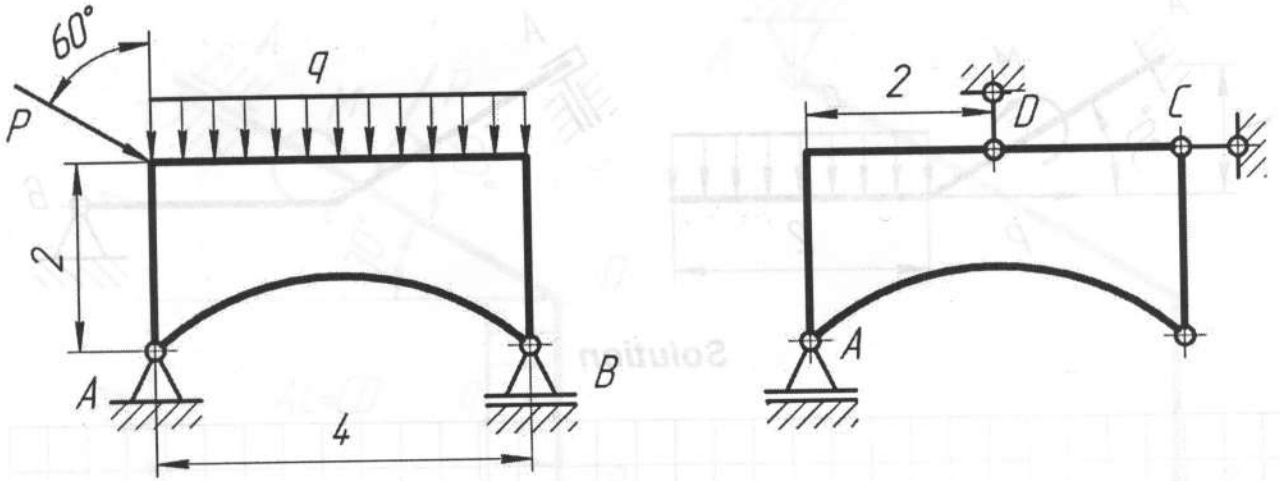
**Solution**



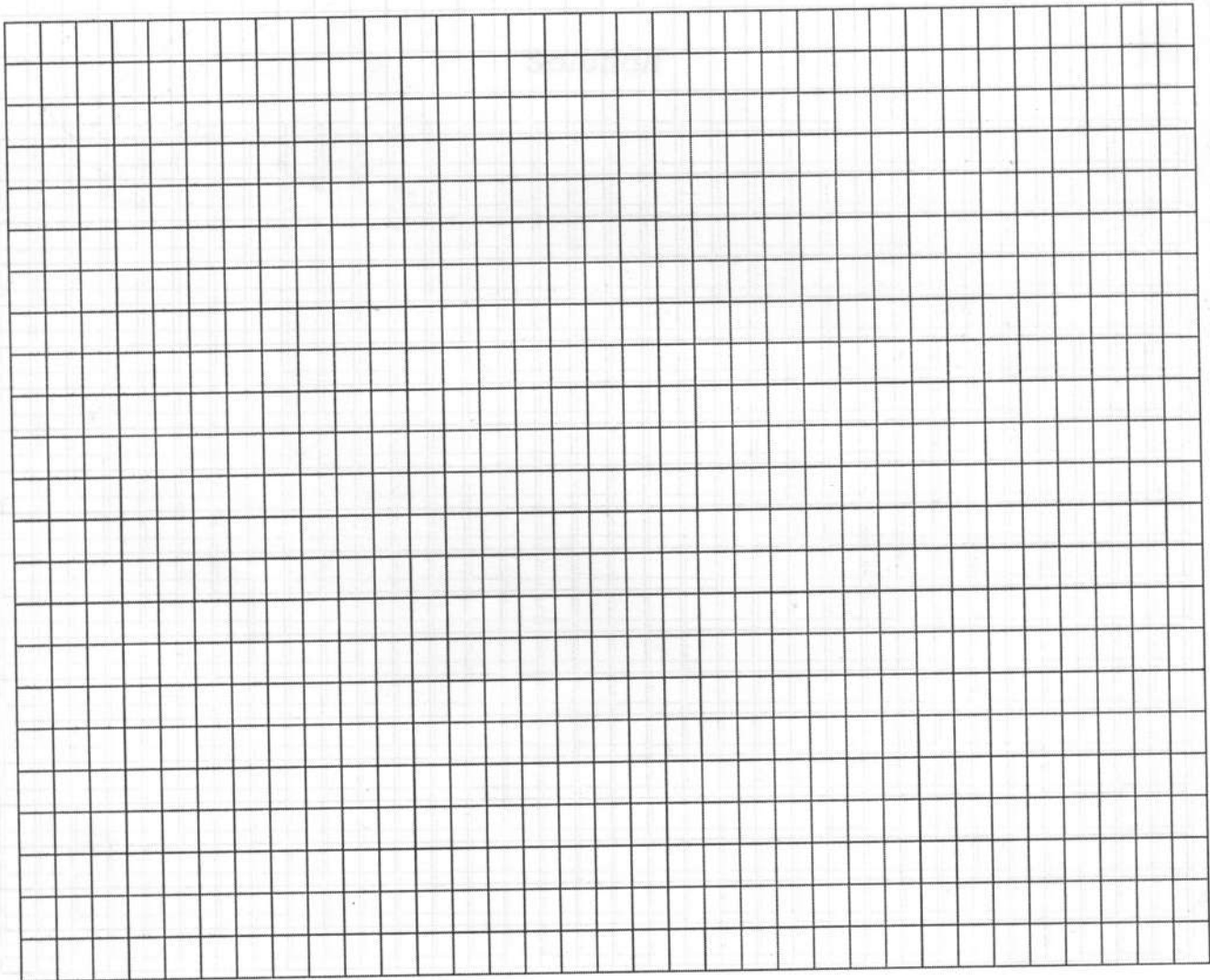
**Data for problem 2.10**

Data for problem 2.9

Concentrated force  $P = 10$  kN, distributed load with intensity  $q = 4$  kN/m, examining reaction  $- Y_A$ .

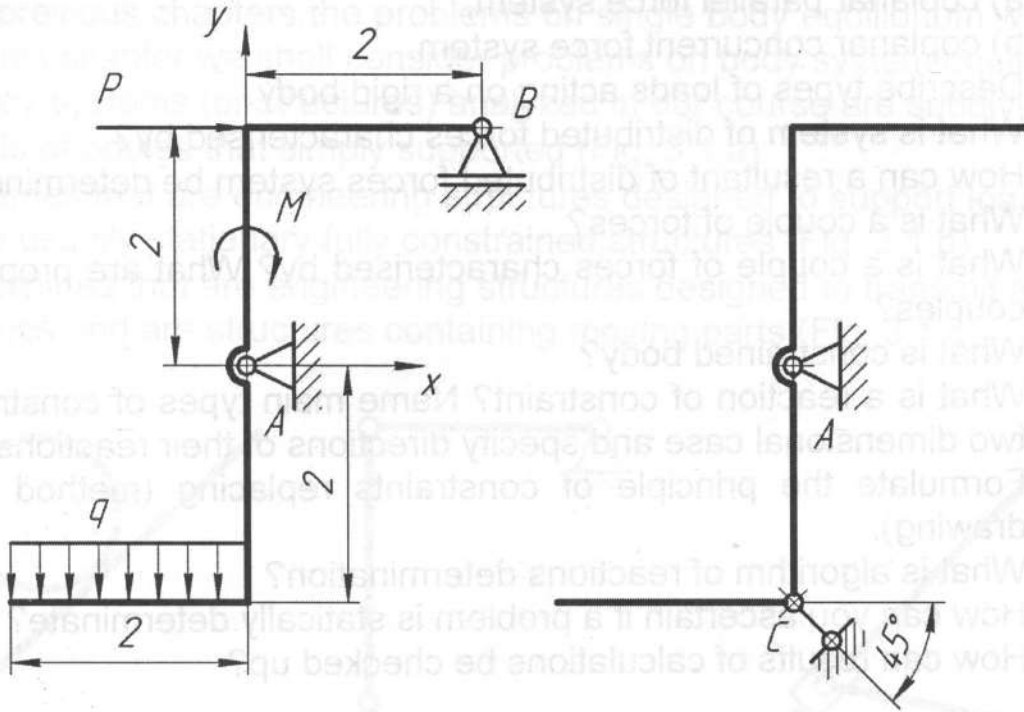


**Solution**

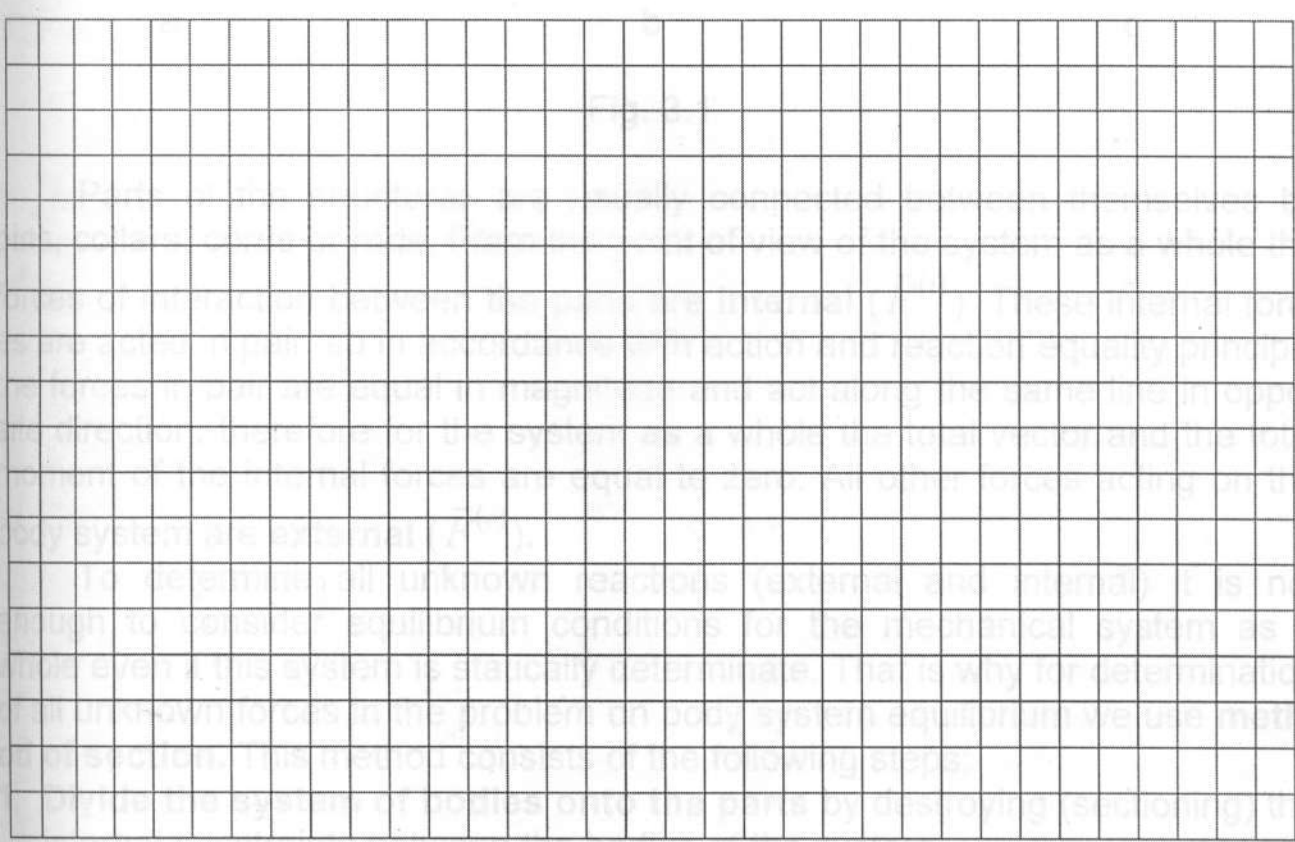


**Problem 2.11**

Concentrated force  $P = 20$  kN, couple of forces with moment  $M = 10$  N\*m, distributed load with intensity  $q = 2$  kN/m, examining reaction is  $Y_A$ .



**Solution**



## 2.4. Questions for self-testing

1. Write an equilibrium equation for general coplanar force system.
2. Write equilibrium equations for:
  - a) coplanar parallel force system;
  - b) coplanar concurrent force system.
3. Describe types of loads acting on a rigid body.
4. What is system of distributed forces characterised by?
5. How can a resultant of distributed forces system be determined?
6. What is a couple of forces?
7. What is a couple of forces characterised by? What are properties of couples?
8. What is constrained body?
9. What is a reaction of constraint? Name main types of constraints for two dimensional case and specify directions of their reactions.
10. Formulate the principle of constraints replacing (method of FBD drawing).
11. What is algorithm of reactions determination?
12. How can you ascertain if a problem is statically determinate?
13. How can results of calculations be checked up?



### 3. BODIES SYSTEM EQUILIBRIUM. METHOD OF SECTION

#### 3.1. Main information from the theoretical course

In previous chapters the problems on single body equilibrium were examined, in this chapter we shall consider problems on body system equilibrium.

Body systems (or structures) analyzed in our course are subdivided on:

- Sets of bodies that simply supported (Fig. 3.1,a)
- Frames that are engineering structures designed to support loads, frames are usually stationary fully constrained structures (Fig. 3.1,b)
- Machines that are engineering structures designed to transmit and modify forces and are structures containing moving parts (Fig. 3.1,c).

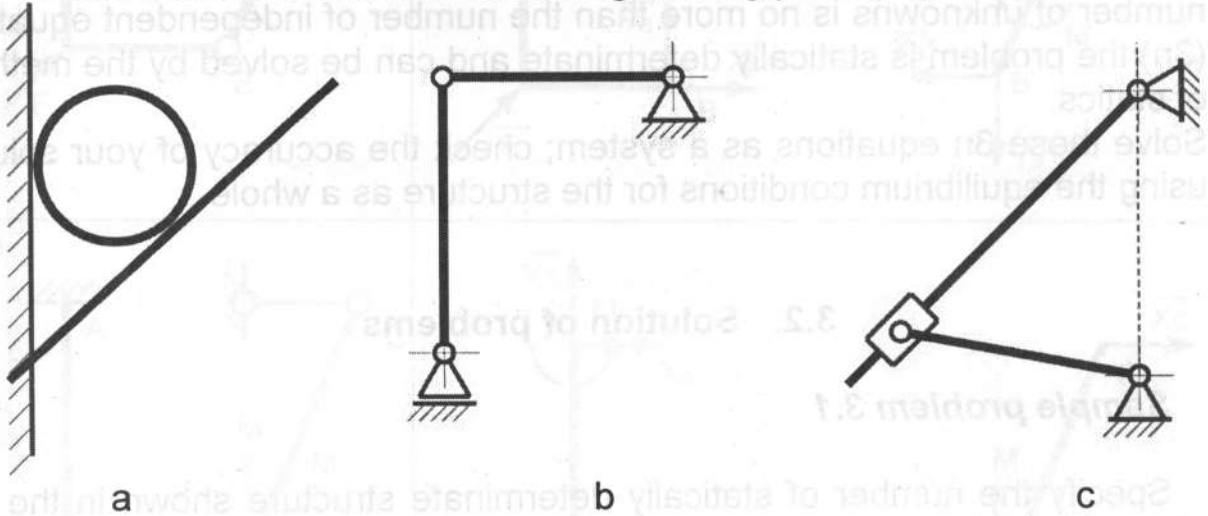


Fig. 3.1

Parts of the structures are usually connected between themselves by pins, collars, cords or rods. From the point of view of the system as a whole the forces of interaction between the parts are **internal** ( $\vec{F}^{(i)}$ ). These internal forces are acted in pair, so in accordance with action and reaction equality principle the forces in pair are equal in magnitude and act along the same line in opposite direction, therefore for the system as a whole the total vector and the total moment of the internal forces are equal to zero. All other forces acting on the body system are **external** ( $\vec{F}^{(e)}$ ).

To determine all unknown reactions (external and internal) it is not enough to consider equilibrium conditions for the mechanical system as a whole even if this system is statically determinate. That is why for determination of all unknown forces in the problem on body system equilibrium we use **method of section**. This method consists of the following steps:

1. **Divide the system of bodies onto the parts** by destroying (sectioning) the internal constraints between the bodies of the system.

2. **Form FBD for each part of the system** (use the first step of the algorithm of a problem solution in the chapter 2). For each adjacent part at the section **apply the reactions** corresponded to the type of the destroyed constraint. The direction you choose for each of the internal force components exerted on the first part is arbitrary, but you must apply equal and opposite force components of the same name to the other parts.
3. For each part **write the equilibrium conditions** taking into consideration all forces acting on the part with including reactions of destroyed internal constraints.
4. **Analyze the static determinacy of the problem.** For the structure consisting of  $n$  members under the action of general coplanar force system it is possible to form  $3n$  independent equations of equilibrium. If the total number of unknowns is no more than the number of independent equations ( $3n$ ) the problem is statically determinate and can be solved by the methods of statics.
5. Solve these  $3n$  equations as a system; check the accuracy of your solution using the equilibrium conditions for the structure as a whole.

### 3.2. Solution of problems

#### Sample problem 3.1

Specify the number of statically determinate structure shown in the first column of Table 3.1.

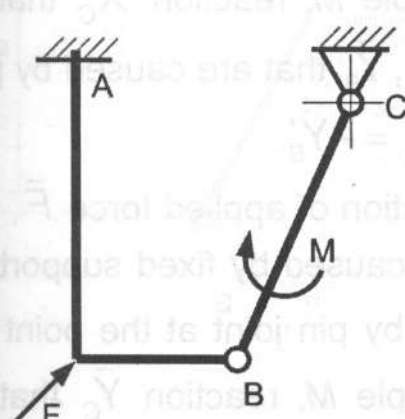
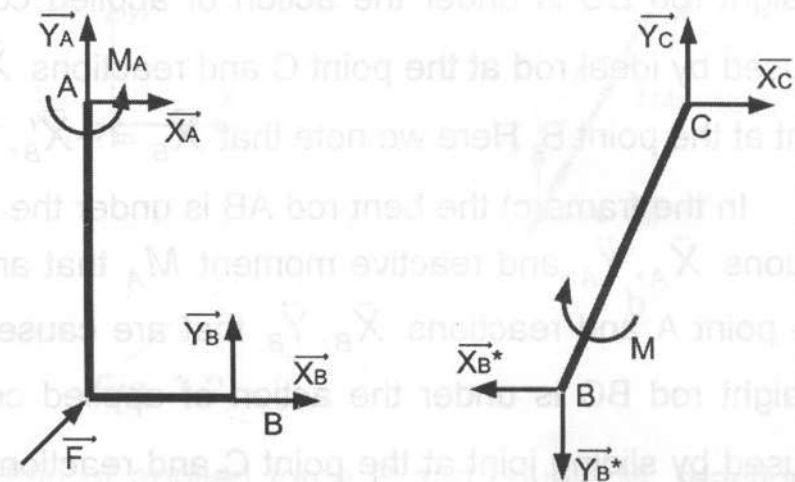
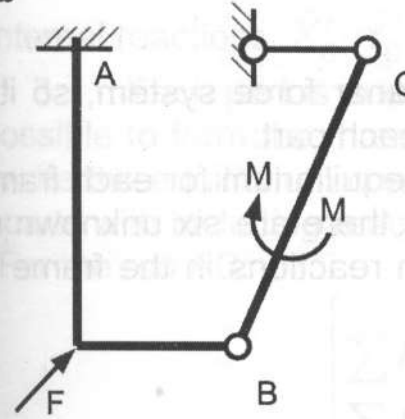
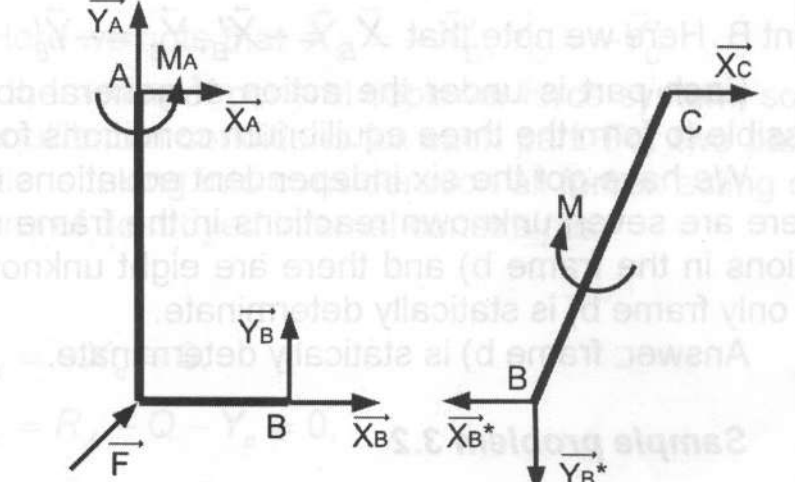
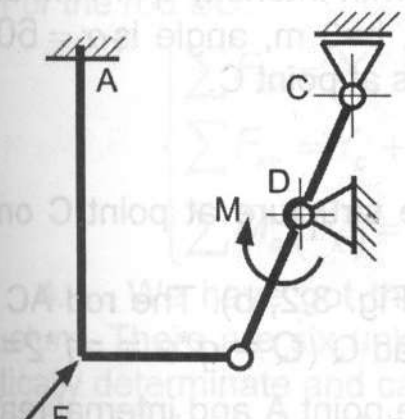
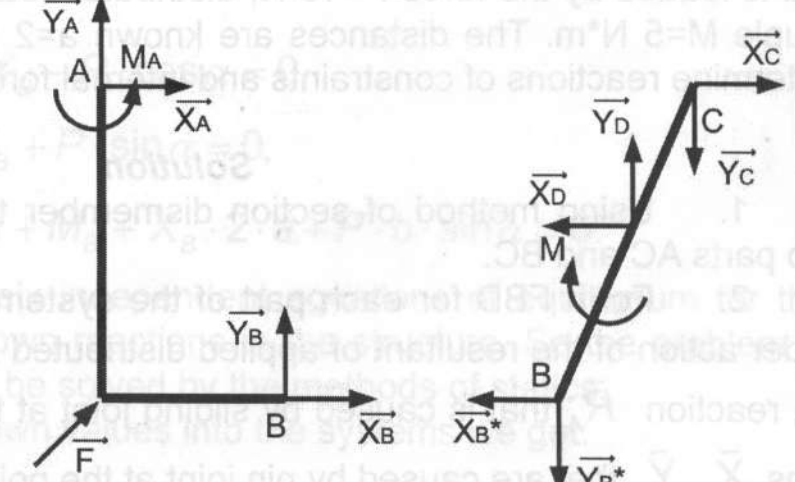
#### Solution

Dismember each frame onto the two parts: bent rod AB and straight rod BC, that are connected by the pin **B** (separated part of each structure are shown in the second and third column of the Table 3.1).

Form FBD for each part of the system.

In the frame a) the bent rod AB is under the action of applied force  $\vec{F}$ , reactions  $\vec{X}_A, \vec{Y}_A$  and reactive moment  $M_A$  that are caused by fixed support at the point A and reactions  $\vec{X}_B, \vec{Y}_B$  that are caused by pin joint at the point B; straight rod BC is under the action of applied couple  $M$ , reactions  $\vec{X}_C, \vec{Y}_C$  that are caused by pin at the point C and reactions  $\vec{X}'_B, \vec{Y}'_B$  that are caused by pin joint at the point B. Here we note that  $\vec{X}_B = -\vec{X}'_B, \vec{Y}_B = -\vec{Y}'_B$ .

Table 3.1

Structure	FBD for parts of the structure
<p>a</p> 	
<p>b</p> 	
<p>c</p> 	

In the frame b) the bent rod AB is under the action of applied force  $\vec{F}$ , reactions  $\vec{X}_A, \vec{Y}_A$  and reactive moment  $M_A$  that are caused by fixed support at the point A and reactions  $\vec{X}_B, \vec{Y}_B$  that are caused by pin joint at the point B; straight rod BC is under the action of applied couple  $M$ , reaction  $\vec{X}_C$  that is caused by ideal rod at the point C and reactions  $\vec{X}'_B, \vec{Y}'_B$  that are caused by pin joint at the point B. Here we note that  $\vec{X}_B = -\vec{X}'_B, \vec{Y}_B = -\vec{Y}'_B$ .

In the frame c) the bent rod AB is under the action of applied force  $\vec{F}$ , reactions  $\vec{X}_A, \vec{Y}_A$  and reactive moment  $M_A$  that are caused by fixed support at the point A and reactions  $\vec{X}_B, \vec{Y}_B$  that are caused by pin joint at the point B; straight rod BC is under the action of applied couple  $M$ , reaction  $\vec{Y}_C$  that is caused by sliding joint at the point C and reactions  $\vec{X}_D, \vec{Y}_D$  that are caused by pin joint at the point D and reactions  $\vec{X}'_B, \vec{Y}'_B$  that are caused by pin joint at the point B. Here we note that  $\vec{X}_B = -\vec{X}'_B, \vec{Y}_B = -\vec{Y}'_B$ .

Each part is under the action of general coplanar force system, so it is possible to form the three equilibrium conditions for each part.

We have got the six independent equations of equilibrium for each frame. There are seven unknown reactions in the frame a), there are six unknown reactions in the frame b) and there are eight unknown reactions in the frame c). So only frame b) is statically determinate.

Answer: frame b) is statically determinate.

### Sample problem 3.2

The structure (Fig. 3.2, a) consists of two parts joined by a pin at point C and is loaded by the force  $P=10$  N, distributed load with intensity  $q=1$  N/m and couple  $M=5$  N\*m. The distances are known  $a=2$  m,  $b=3$  m, angle is  $\alpha = 60^\circ$ . Determine reactions of constraints and internal forces at point C.

#### Solution

- Using method of section dismember the structure at point C onto two parts AC and BC.
- Form FBD for each part of the system (Fig. 3.2, b). The rod AC is under action of the resultant of applied distributed load  $Q$  ( $Q = q \cdot a = 1 \cdot 2 = 2$  N), reaction  $\vec{R}_A$  that is caused by sliding joint at the point A and internal reactions  $\vec{X}_C, \vec{Y}_C$  that are caused by pin joint at the point C.



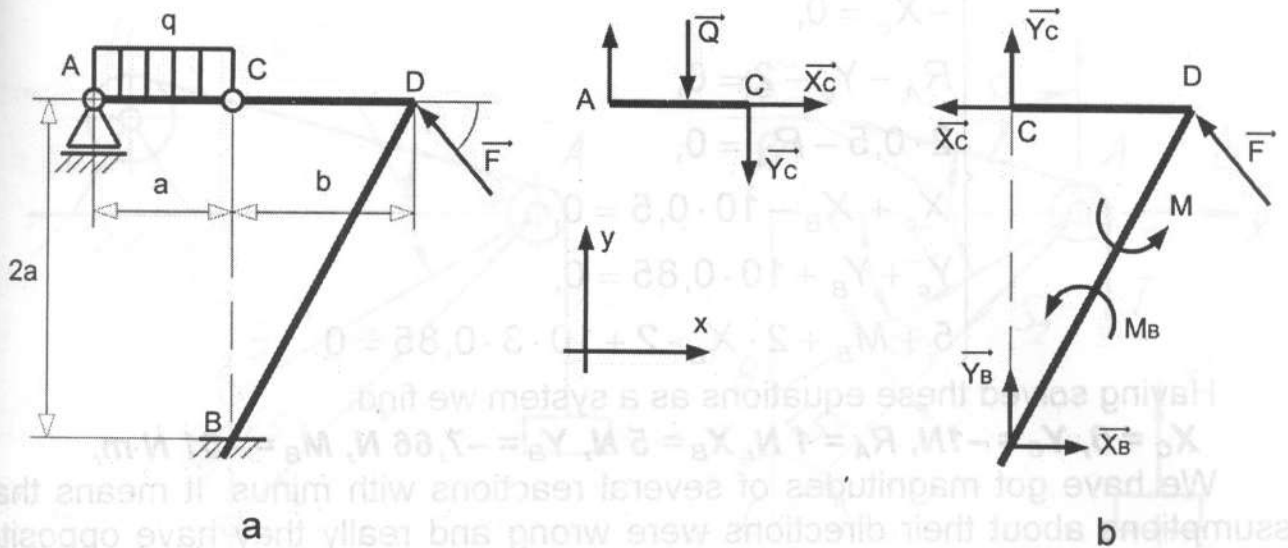


Fig. 3.2

The rod BC is under action of applied force  $P$  and couple  $M$ , reactions  $\vec{X}_B, \vec{Y}_B$  and reactive moment  $M_B$  that are caused by fixed support at the point B, internal reactions  $\vec{X}'_C, \vec{Y}'_C$ . Here we note that  $\vec{X}_C = -\vec{X}'_C, \vec{Y}_C = -\vec{Y}'_C$ .

3. Each part is under the action of general coplanar force system, so it is possible to form the three equilibrium conditions for each part. For two parts we write the equilibrium conditions taking into consideration all forces acting on the parts with including reactions of destroyed internal constraints.

For the rod AC:

$$\begin{cases} \sum F_{xk} = -X_c = 0, \\ \sum F_{yk} = R_A - Q - Y_c = 0, \\ \sum M_c(\vec{F}_k) = Q \cdot \frac{a}{2} - R_A \cdot a = 0. \end{cases}$$

For the rod BC:

$$\begin{cases} \sum F_{kx} = X_c + X_B - P \cdot \cos \alpha = 0, \\ \sum F_{ky} = Y_c + Y_B + P \cdot \sin \alpha = 0, \\ \sum M_c(\vec{F}_k) = M + M_B + X_B \cdot 2 \cdot a + P \cdot b \cdot \sin \alpha = 0. \end{cases}$$

4. We have got the six independent equations of equilibrium for the structure. There are six unknown reactions in the structure. So the problem is statically determinate and can be solved by the methods of statics.

5. Substituting all known values into the systems we get:



$$\begin{cases} -X_c = 0, \\ R_A - Y_c - 2 = 0, \\ 2 \cdot 0,5 - R_A = 0, \\ X_c + X_B - 10 \cdot 0,5 = 0, \\ Y_c + Y_B + 10 \cdot 0,85 = 0, \\ 5 + M_B + 2 \cdot X_B \cdot 2 + 10 \cdot 3 \cdot 0,85 = 0. \end{cases}$$

Having solved these equations as a system we find:

$$X_c = 0, Y_c = -1\text{N}, R_A = 1\text{N}, X_B = 5\text{N}, Y_B = -7,66\text{N}, M_B = -51\text{N}\cdot\text{m}.$$

We have got magnitudes of several reactions with minus. It means that assumptions about their directions were wrong and really they have opposite sense.

6. To check the accuracy of the solution use the equilibrium conditions for the structure as a whole:

$$\begin{cases} \sum F_{kx} = X_B - P \cdot \cos \alpha = 0, \\ \sum F_{ky} = R_A - Q + Y_B + P \cdot \sin \alpha = 0, \\ \sum M_c(\bar{F}_k) = Q \cdot \frac{a}{2} - R_A \cdot a + M + M_B + X_B \cdot 2 \cdot a + P \cdot b \cdot \sin \alpha = 0. \end{cases}$$

After substituting the values of applied forces and reactions and moments we get zero in right side of all equations:

$$\begin{cases} \sum F_{kx} = 5 - 10 \cdot \cos 60^\circ = 0, \\ \sum F_{ky} = 1 - 2 - 7,66 + 10 \cdot \sin 60^\circ = 0, \\ \sum M_c(\bar{F}_k) = 2 \cdot \frac{2}{2} - 1 \cdot 2 + 5 - 51 + 5 \cdot 2 \cdot 2 + 10 \cdot 3 \cdot \sin 60^\circ = 0. \end{cases}$$

We have received identity that's why all reactions have been found correctly.

**Answer.**  $X_c = 0, Y_c = -1\text{N}, R_A = 1\text{N}, X_B = 5\text{N}, Y_B = -7,66\text{N}, M_B = -51\text{N}\cdot\text{m}.$

### Sample problem 3.3

A load of weight  $G = 3000\text{N}$  (Fig. 3.3, a) is hanged with the use of cord threw over block **A** and reeled on the winch **D**.

Determine the forces in rods **AB** and **AC**. Angles are shown on the figure. Neglect block dimensions.

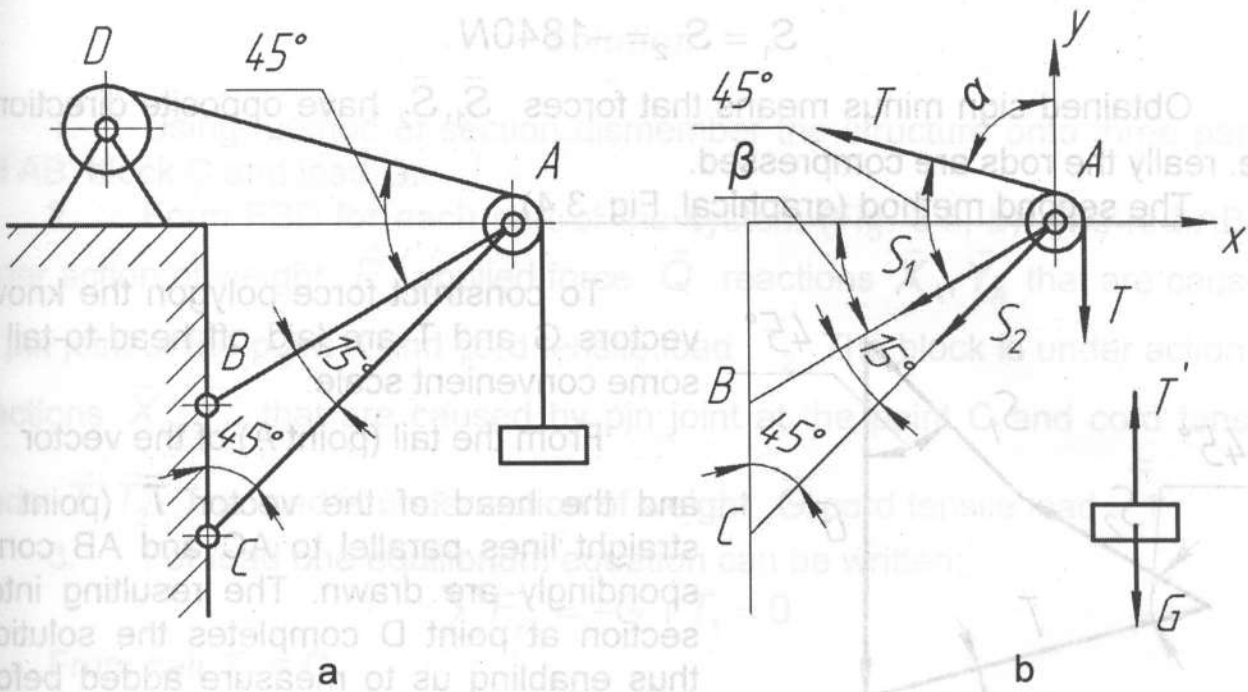


Fig. 3.3

**Solution**

1. Dismember the structure at point A onto two parts: block A with rods AB, AC; and load.
2. Form FBD for each part of the system (Fig. 3.3, b). The block is under action of reactions  $\vec{S}_1, \vec{S}_2$  that are caused by supporting rods (chosen directions of reactions  $\vec{S}_1, \vec{S}_2$  correspond to assumption about stretched state of the rods) and cord tension  $\vec{T}$ . The load is under action of weight  $G$  and cord tension  $\vec{T}' = -\vec{T}$ .

From the load equilibrium conditions we get :

$$\sum F_{yk} = 0: T' - G = 0,$$

$$T' = G = 3000N.$$

3. The first method (analytical). Block is under action of coplanar concurrent force system, so it is possible to form the two equilibrium conditions for it:

$$\sum F_{xk} = 0, \quad \sum F_{yk} = 0;$$

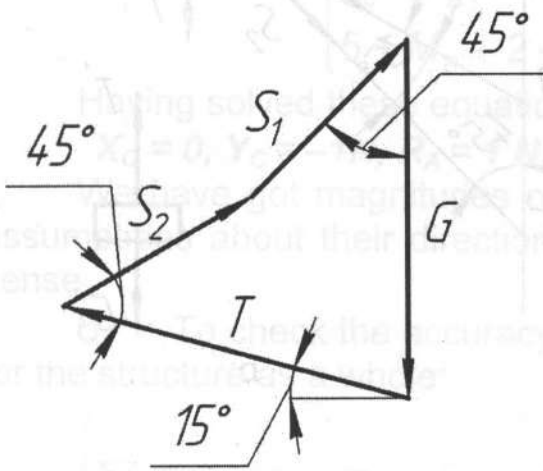
$$\begin{cases} -T \sin 75^\circ - S_1 \cos 30^\circ - S_2 \sin 45^\circ = 0; \\ T \cos 75^\circ - S_1 \sin 30^\circ - S_2 \cos 45^\circ - G = 0. \end{cases}$$

4. We have got the two independent equations of equilibrium. There are two unknown reactions in the structure. So the problem is statically determinate and can be solved by the methods of statics.
5. Having solved these equations as a system we find:

$$S_1 = S_2 = -1840N.$$

Obtained sign minus means that forces  $\vec{S}_1, \vec{S}_2$  have opposite directions, i. e. really the rods are compressed.

The second method (graphical, Fig. 3.4).



To construct force polygon the known vectors  $G$  and  $T$  are laid off head-to-tail to some convenient scale.

From the tail (point A) of the vector  $\vec{G}$  and the head of the vector  $\vec{T}$  (point C) straight lines parallel to AC and AB correspondingly are drawn. The resulting intersection at point D completes the solution, thus enabling us to measure added before the unknown vectors  $\vec{S}_1, \vec{S}_2$ .

**Answer.**  $S_1 = S_2 = -1840N.$

Fig. 3.4

### Sample problem 3.4

The homogeneous rod  $AB$  of length  $l$  and weight  $P = 100 N$  is fixed to a wall by pin  $A$  (Fig. 3.5, a) and is held at an angle  $45^\circ$  with by cord which passes over a fixed pulley  $C$ , the load of weight  $G$  is attached to the cord. The cord branch  $BC$  makes an angle  $30^\circ$  with vertical. At the point  $D$  ( $AD = 0,75 \cdot l$ ) the force  $Q=200 N$  is applied.

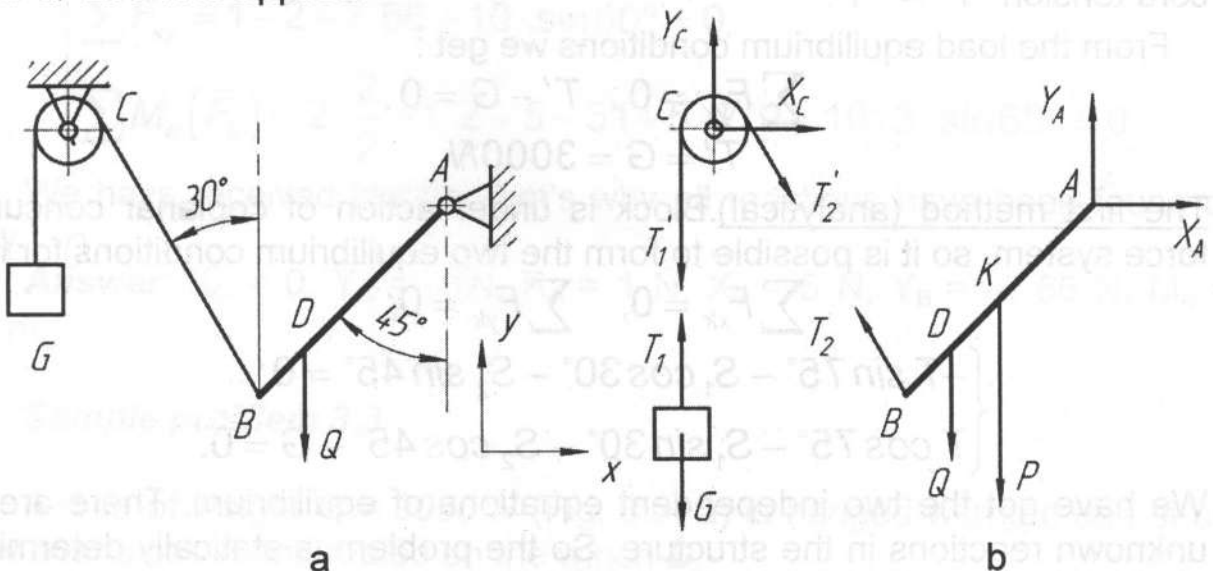


Fig. 3.5

## Solution

1. Using method of section dismember the structure onto three parts: rod AB, block C and load G.

2. Form FBD for each part of the system (Fig. 3.5, b). The rod AB is under action of weight  $\vec{P}$ , applied force  $\vec{Q}$ , reactions  $\vec{X}_A, \vec{Y}_A$  that are caused by pin joint at the point A and cord tensile load  $\vec{T}_2$ . The block is under action of reactions  $\vec{X}_C, \vec{Y}_C$  that are caused by pin joint at the point C and cord tensile loads  $\vec{T}'_1, \vec{T}'_2$ . The load is under action of weight  $\vec{G}$ , cord tensile load  $\vec{T}_1$ .

3. For load one equilibrium equation can be written:

$$\sum F_{yk} = -G + T_1 = 0.$$

From heir  $T_1 = G$ .

The block is under action of general coplanar force system. As far as force of action is equal to counteraction  $\vec{T}'_1 = -\vec{T}_1$  and  $\vec{T}'_2 = -\vec{T}_2$ . It is sufficient to make one equation for finding tensile forces:

$$\sum M_C(\vec{F}_k) = T'_1 \cdot r - T'_2 \cdot r = 0,$$

where  $r$  is radius of the block.

Thus  $T_2' = T_1' = T_1 = G$ . Because of  $T_2 = T_2'$  tensile force is equal to weight of the load, i. e.  $T_2 = G$ .

The rod AB is under action of general coplanar force system, that's why three equations of equilibrium can be written:

$$\begin{cases} \sum F_{xk} = X_A - T_2 \sin 30^\circ = 0; \\ \sum F_{yk} = Y_A - Q - P + T_2 \cos 30^\circ = 0; \\ \sum M_A(\vec{F}_k) = P \cdot \frac{l}{2} \sin 45^\circ + Q \cdot \frac{3}{4} \cdot l \sin 45^\circ - T_2 \cdot l \sin 75^\circ = 0. \end{cases}$$

4. Three independent equations of equilibrium for the rod were obtained and there are three unknown reactions in the system. So the problem is statically determinate and can be solved by the methods of statics.

5. Having solved these equations as a system we'll find:

$$X_A = 73 \text{ N}, Y_A = 173 \text{ N}, G = 146 \text{ N}.$$

**Answer:**  $X_A = 73 \text{ N}, Y_A = 173 \text{ N}, G = 146 \text{ N}.$



### Sample problem 3.5

Homogeneous bar AC (AC weight is  $P_1$ ) is kept in horizontal position by homogeneous rod BD (AC=BD=l, BD weight is  $P_1$ ) (Fig. 3.6, a). Force  $Q=100$  N is applied at point C to the bar AC. It is known that  $AE=CE$ ,  $BF=DF$ ,  $P_1=P_2=P=40$  N and pin joints are at points A, B, D.

Determine reactions of pin joints A and B ( $\vec{R}_A$  and  $\vec{R}_B$ ).

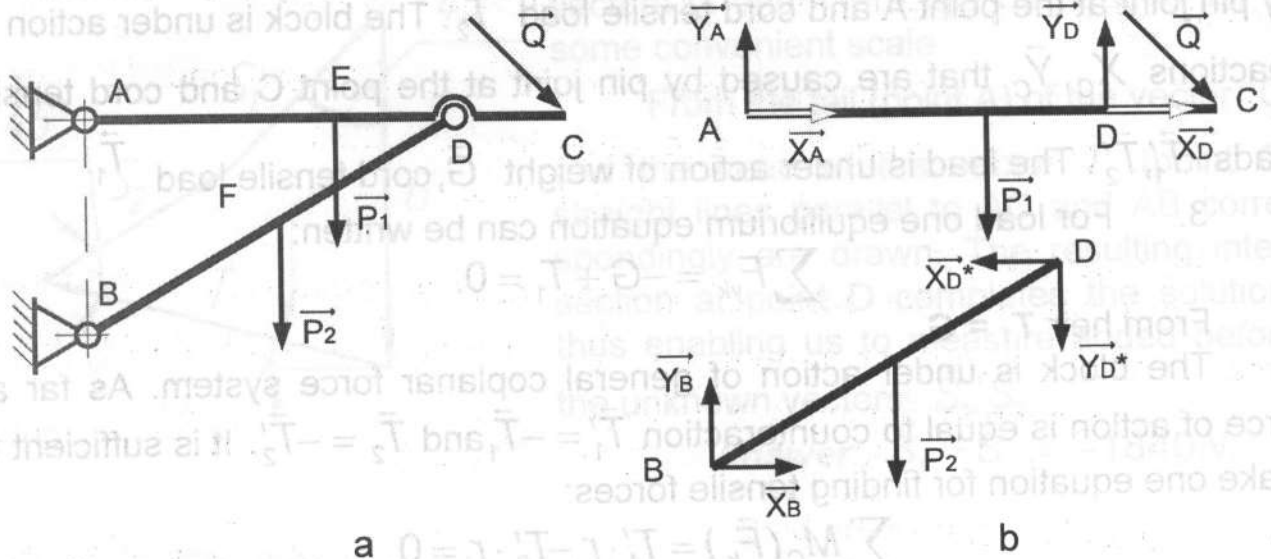


Fig. 3.6

### Solution

- Using method of section dismember the structure at point D onto two parts AC and BD.
- Form FBD for each part of the system (Fig. 3.7, b). The rod AC is under action of weight  $P_1$ , applied force  $Q$ , reactions  $\vec{X}_A, \vec{Y}_A$  that are caused by pin joint at the point A and reactions  $\vec{X}_D, \vec{Y}_D$  that are caused by pin joint at the point D. The rod BD is under action of weight  $P_2$ , reactions  $\vec{X}_B, \vec{Y}_B$  that are caused by pin joint at the point B, reactions of interaction with separated part AC  $\vec{X}'_D, \vec{Y}'_D$ . Here we note that  $\vec{X}_D = -\vec{X}'_D, \vec{Y}_D = -\vec{Y}'_D$ .
- Each part is under the action of general coplanar force system, so it is possible to form the three equilibrium conditions for each part. But as far as we have not to find internal reactions in pin joint D we write three equilibrium conditions for the structure as a whole and one equation for BD.

For the structure as a whole:



$$\sum F_{xk} = X_A + X_B + Q \cos 45^\circ = 0;$$

$$\sum F_{yk} = Y_A + Y_B - P_1 - P_2 - Q \sin 60^\circ = 0;$$

$$\sum M_A(\vec{F}_k) = -P_1 \frac{l}{2} - Q \sin 45^\circ l + X_B l \cos 45^\circ - P_2 \frac{l}{2} \sin 60^\circ = 0.$$

For the rod BD:

$$\sum M_D(\vec{F}_k) = X_B l \cos 60^\circ - Y_B l \sin 60^\circ + P_2 \frac{l}{2} \sin 60^\circ = 0.$$

4. We have got four independent equations of equilibrium for the structure. There are four unknown reactions that we have to find. So the problem is statically determinate and can be solved by the methods of statics.

5. Having solved these equations as a system we'll find:

$$X_A = -287 \text{ N}, Y_A = 6 \text{ N}, X_B = 216 \text{ N}, Y_B = 145 \text{ N}.$$

Then reaction of pin joints will be:

$$R_A = \sqrt{X_A^2 + Y_A^2} = \sqrt{287^2 + 6^2} = 287,1 \text{ (N)},$$

$$R_B = \sqrt{X_B^2 + Y_B^2} = \sqrt{216^2 + 145^2} = 260,2 \text{ (N)}.$$

6. We check the accuracy of the solution using the equilibrium condition for the structure as a whole:

$$\sum M_C(\vec{F}_k) = -Y_A l + P_1 \frac{l}{2} + X_B l \cos 60^\circ - Y_B l + P_2 (l - \frac{l}{2} \sin 60^\circ) = 0.$$

After substituting the known values we'll get zero in right side of the equation:

$$\begin{aligned} \sum M_C(\vec{F}_k) = & -6 \cdot l + 40 \cdot \frac{l}{2} + 216 \cdot l \cos 60^\circ - 145 \cdot l + \\ & + 40 \cdot (l - \frac{l}{2} \sin 60^\circ) = 0. \end{aligned}$$

We have received identity that's why all reactions have been found correctly.

$$\text{Answer. } R_A = 287,1 \text{ (N)}, R_B = 260,2 \text{ (N)}.$$

### Sample problem 3.6

The compound structure is loaded by force  $F=3 \text{ kN}$  and couple with moment  $M=8 \text{ N}\cdot\text{m}$  (Fig. 3.7, a). It is known that  $\alpha = 30^\circ$ ,  $AB=BC=a \text{ (m)}$ .

Determine reaction of constraint A ( $\vec{R}_A$ ).

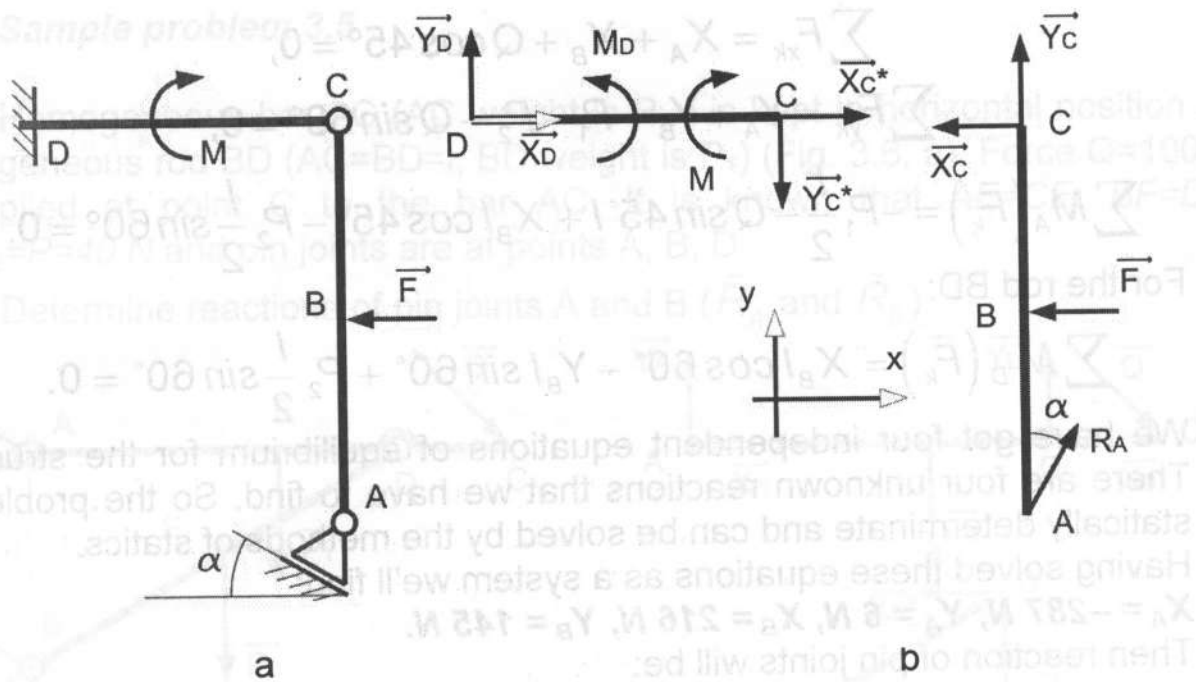


Fig. 3.7

### Solution

- Using method of section dismember the structure at point C onto two parts DC and CA.
- Form FBD for each part of the system (Fig. 3.7, b). The rod DC is under action of applied couple with moment  $M$ , reactions  $\vec{X}_D, \vec{Y}_D$  and reactive moment  $M_D$  that are caused by clamped joint at the point D and reactions  $\vec{X}_C, \vec{Y}_C$  that are caused by pin joint at the point C. The rod CA is under action of applied force  $F$ , reaction  $\vec{R}_A$  that is caused by sliding joint at the point A, reactions of interaction with separated part DC  $\vec{X}'_C, \vec{Y}'_C$ . Here we note that  $\vec{X}_C = -\vec{X}'_C, \vec{Y}_C = -\vec{Y}'_C$ .
- Each part is under the action of general coplanar force system, so it is possible to form the three equilibrium conditions for each part. But as far as we have to find only reaction in sliding joint A we write one equilibrium equation for the rod AC.

$$\sum M_C(\vec{F}_k) = -F \cdot a + R_A \cdot 2 \cdot a \cdot \sin \alpha = 0.$$

- Having solved this equation we'll find:

$$R_A = \frac{F}{2 \sin \alpha} = \frac{3}{2 \cdot 0,5} = 3(\text{kN}).$$

**Answer.**  $R_A = 3\text{kN}$ .

### Sample problem 3.7

Stepladder is on the smooth horizontal surface. It consists of the parts AC and CB ( $AC=BC=3\text{ m}$ ), has weight  $P=120\text{ N}$  of each part and is joined by pin C and rope EF (EF is parallel to AB). Distances  $AE=BF=c=1\text{ m}$ . Centres of gravity of the parts AC and CB are in their middles. At point D on a distance  $a=0,6\text{ m}$  a man with weight  $G=720\text{ kg}$  stands. Parts of the stepladder are connected with inextensible cord (Fig. 3.8, a).

Determine reaction of the surface, pin joint and cord tensile load.

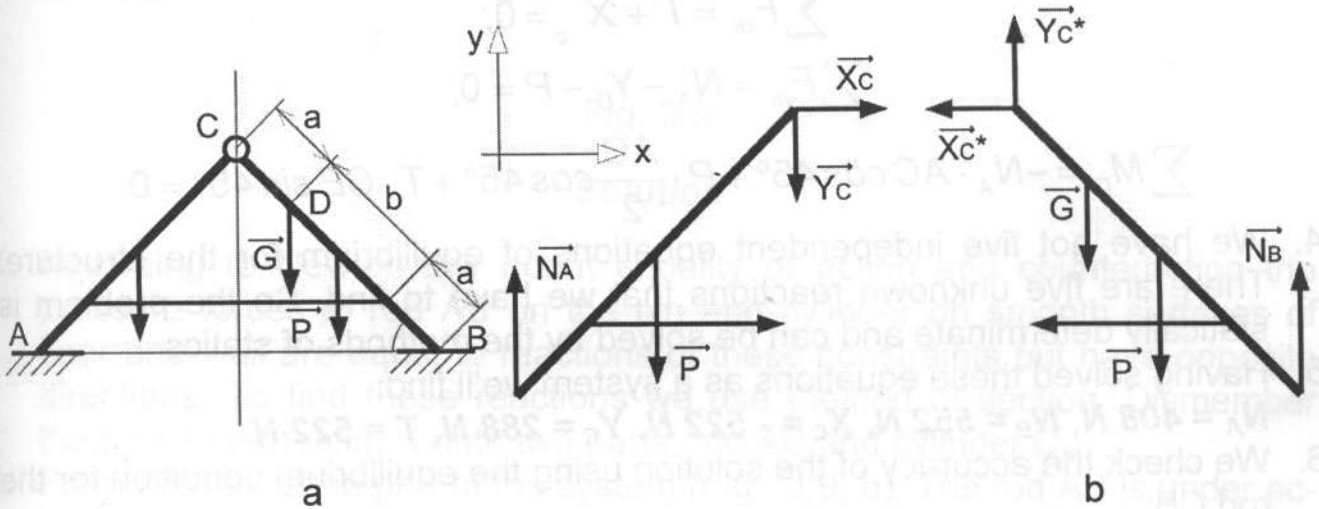


Fig. 3.8

### Solution

- Using method of section dismember the structure at point C onto two parts AC and CB.
- Form FBD for each part of the system (Fig. 3.8, b). The rod AC is under action of weight  $P$ , reaction  $\vec{N}_A$  that is caused by horizontal surface, tensile load  $T$  and reactions  $\vec{X}_C, \vec{Y}_C$  that are caused by pin joint at the point C. The rod CB is under action of weight  $P$ , applied force  $G$  as weight of a man, reaction  $\vec{N}_B$  that is caused by horizontal surface, tensile load  $T$  and reactions  $\vec{X}'_C, \vec{Y}'_C$  of interaction with separated part AC. Here we note that  $\vec{X}_C = -\vec{X}'_C, \vec{Y}_C = -\vec{Y}'_C$ .
- Structure as a whole is under action of parallel coplanar force system so it is possible to form two equilibrium conditions for the structure:

$$\sum F_{yk} = N_A + N_B - 2P - G = 0;$$

$$\sum M_B(\vec{F}_k) = P \frac{BC}{2} \cos 45^\circ + G \cdot BD \cos 45^\circ + P \left( BC + \frac{AC}{2} \right) \cos 45^\circ - N_A (AC + BC) \cos 45^\circ = 0.$$

Each rod is under the action of general coplanar force system, so it is possible to form the three equilibrium conditions for each part. Let's consider rod AC:

$$\sum F_{xk} = T + X_C = 0;$$

$$\sum F_{yk} = N_A - Y_C - P = 0;$$

$$\sum M_C = -N_A \cdot AC \cos 45^\circ + P \cdot \frac{AC}{2} \cos 45^\circ + T \cdot CE \sin 45^\circ = 0.$$

4. We have got five independent equations of equilibrium for the structure. There are five unknown reactions that we have to find. So the problem is statically determinate and can be solved by the methods of statics.
5. Having solved these equations as a system we'll find:  
 **$N_A = 408 \text{ N}$ ,  $N_B = 552 \text{ N}$ ,  $X_C = -522 \text{ N}$ ,  $Y_C = 288 \text{ N}$ ,  $T = 522 \text{ N}$ .**
6. We check the accuracy of the solution using the equilibrium condition for the rod CB:

$$\sum M_D = \vec{X}'_C \cdot CD \cos 45^\circ - \vec{Y}'_C \cdot CD \cos 45^\circ - P \cdot \left( \frac{CB}{2} - CD \right) \cos 45^\circ + N_B \cdot BD \cos 45^\circ - T \cdot DF \cos 45^\circ = 0.$$

After substituting the known values we'll get zero in right side of the equation:

$$\sum M_D = -522 \cdot 0,6 \cos 45^\circ - 288 \cdot 0,6 \cos 45^\circ - 120 \cdot \left( \frac{3}{2} - 0,6 \right) \cos 45^\circ + 552 \cdot (3 - 0,6) \cos 45^\circ - 522 \cdot (3 - 0,6 - 1) \cos 45^\circ = 0.$$

We have received identity that's why all reactions have been found correctly.

**Answer:  $N_A = 408 \text{ N}$ ,  $N_B = 552 \text{ N}$ ,  $X_C = -522 \text{ N}$ ,  $Y_C = 288 \text{ N}$ ,  $T = 522 \text{ N}$ .**

### Sample problem 3.8

Homogeneous rod AB with weight P is fixed with pin at point A and leans on a cylinder C with radius r and weight Q=2P (Fig. 3.9, a). The cylinder is located on a horizontal floor and contacts with vertical wall. It is known that AB=3r,  $\alpha = 60^\circ$ . Determine pressure forces of the bar on the pin A and the cylinder on the horizontal surface if all the surfaces are smooth.

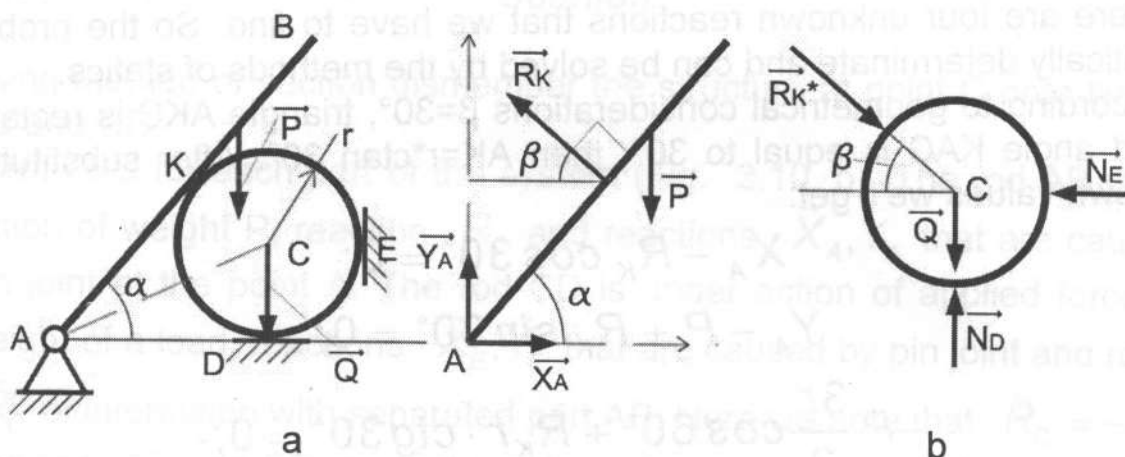


Fig. 3.9

**Solution**

1. According to Newton law about equality of action and counteraction the pressure forces of rod AB on the pin and cylinder on smooth surfaces of floor and wall are equal to reactions of these constraints but have opposite directions. To find these reactions we use method of section. Dismember the structure at point K onto two parts: rod AB and cylinder C.
2. Form FBD for each part of the system (Fig. 3.9, b). The rod AB is under action of weight P, reactions  $\vec{X}_A, \vec{Y}_A$  that are caused by pinned joint and reaction  $\vec{R}_K$  ( $\vec{R}_K$  is perpendicular to AB) that is caused by cylinder pressure at point K. The cylinder C is under action of weight Q, reactions  $\vec{N}_D, \vec{N}_E$  that are caused by horizontal and vertical surfaces and reaction  $\vec{R}'_K$  of interaction with separated part AB. Here we note that  $\vec{R}_K = -\vec{R}'_K$ .
3. The rod AB is under action of general coplanar force system and the cylinder is under action of concurrent coplanar force system. We have not to find reaction  $\vec{N}_E$  so it is possible to form one equilibrium condition for the cylinder.

For the cylinder:

$$\sum F_{yk} = N_D - R_k \sin \beta - Q = 0.$$

For the rod AB:

$$\sum F_{kx} = X_A - R_k \cos \beta = 0,$$

$$\sum F_{ky} = Y_A - P + R_k \sin \beta = 0,$$

$$\sum M_A(\vec{F}_k) = -P \cdot \frac{3r}{2} \cos 60^\circ + R_k \cdot AK = 0.$$



4. We have got four independent equations of equilibrium for the structure. There are four unknown reactions that we have to find. So the problem is statically determinate and can be solved by the methods of statics.
5. According to geometrical considerations  $\beta=30^\circ$ , triangle AKC is rectangular and angle KAC is equal to  $30^\circ$ , then  $AK=r \cdot \text{ctan } 30^\circ$ . After substituting all known values we'll get:

$$\begin{aligned} X_A - R_K \cos 30^\circ &= 0; \\ Y_A - P + R_K \sin 30^\circ &= 0; \\ -P \frac{3r}{2} \cos 60^\circ + R_K r \cdot \text{ctg} 30^\circ &= 0; \\ N_D - R_K \sin 30^\circ - 2P &= 0. \end{aligned}$$

Having solved these equations as a system we'll find:

$$X_A = 0,375P \text{ N}, Y_A = 0,79P \text{ N}, N_D = 1,8 \text{ N}.$$

Thus pressure of rod on the pin at point A is  $X'_A = -0,375P \text{ N}$ ,  $Y'_A = -0,79P \text{ N}$  and pressure of the cylinder on the floor is  $N'_D = -1,8 \text{ N}$ .

$$\text{Answer: } X'_A = -0,375P \text{ N}, Y'_A = -0,79P \text{ N}, N'_D = -1,8 \text{ N}.$$

### Sample problem 3.9

Smooth rods AB and CD (Fig. 3.10, a) are fixed with pins at points A and E correspondingly and adjoin at point C. The distances are known  $AB=CD=2l$ ,  $AE=CE=DE=l$ . Weight of homogeneous bar AB is equal to  $P$ , weight of load  $Q$  is  $2P$ . Determine angle  $\gamma$  in a position of equilibrium.

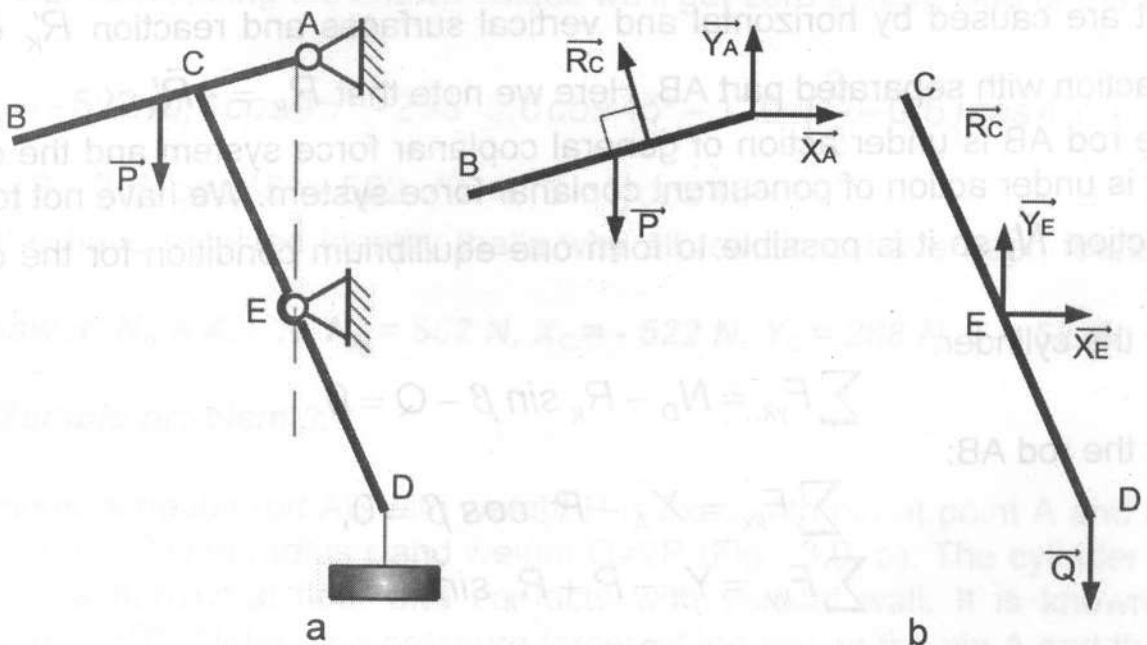


Fig. 3.10

### Solution

- Using method of section dismember the structure at point C onto two parts AB and CD.
- Form FBD for each part of the system (Fig. 3.10, b). The rod AB is under action of weight  $P$ , reaction  $\vec{R}_C$  and reactions  $\vec{X}_A, \vec{Y}_A$  that are caused by pin joint at the point A. The rod CD is under action of applied force  $Q$  as weight of a load, reactions  $\vec{X}_E, \vec{Y}_E$  that are caused by pin joint and reaction  $\vec{R}'_C$  of interaction with separated part AB. Here we note that  $\vec{R}_C = -\vec{R}'_C$ .
- The rods AB and CD are under action of general coplanar force system. To find angle  $\gamma$  it is sufficient to write only equations of moments for the rods. It is evident that ACE is isosceles triangle so angle AEC is equal to  $180^\circ - 2\gamma$ .

$$\begin{cases} \sum M_A = P \cdot l \cdot \sin \gamma - 2 \cdot l \cdot R_C \cdot \cos \gamma = 0; \\ \sum M_E = R_C \cdot l \cdot \cos \gamma - Q \cdot l \cdot \sin(180 - 2\gamma) = 0. \end{cases}$$

- We have got two independent equations of equilibrium. There are two unknown values in previous system. So the problem is statically determinate.
- From the 1<sup>st</sup> equation:

$$R_C = \frac{P \sin \gamma}{2 \cos \gamma}.$$

Substituting it to the second one we'll get:

$$\sin \gamma - 4 \cdot 2 \sin \gamma \cos \gamma = 0,$$

$$\sin \gamma (1 - 8 \cos \gamma) = 0.$$

This expression has two solutions:

$$\sin \gamma = 0, \quad \gamma_1 = 0;$$

$$\cos \gamma = \frac{1}{8}, \quad \gamma_2 = \arccos \frac{1}{8} \approx 83^\circ.$$

**Answer:**  $\gamma = 83^\circ$ .

### 3.3. Solving problems on your own

#### Problem 3.1

A cylinder is supported by a bar as shown in Fig. 3.11. The weight of the cylinder is 500 N and the weight of the bar is 100 N. If all surfaces are smooth, determine the reactions at supports A and B of the bar.



### Problem 3.2

A cylinder is supported by a bar and cable as shown in Fig. 3.12. The weight of the cylinder is 750 N and the weight of the bar is 100 N. If all surfaces are smooth, determine the reaction at support C of the bar and tension T in the cable.

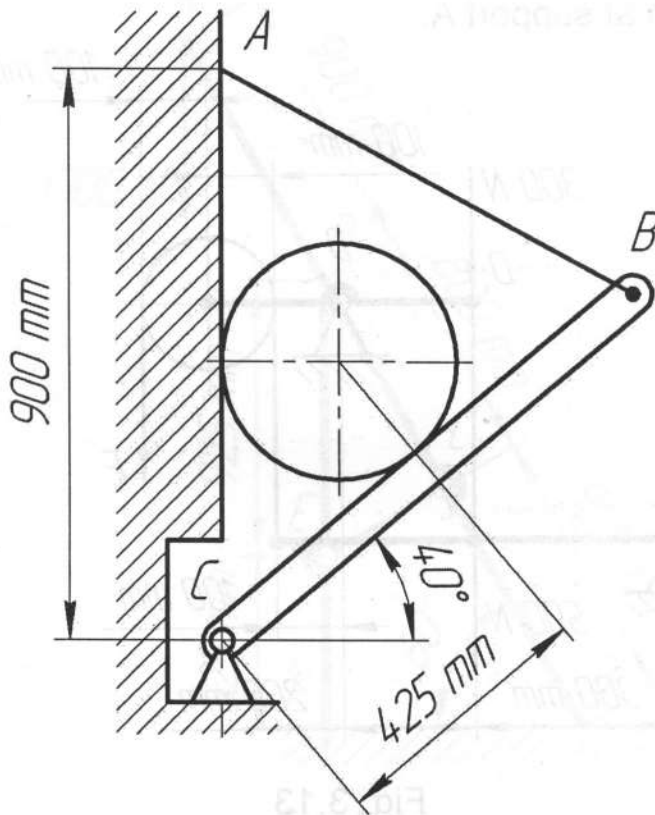
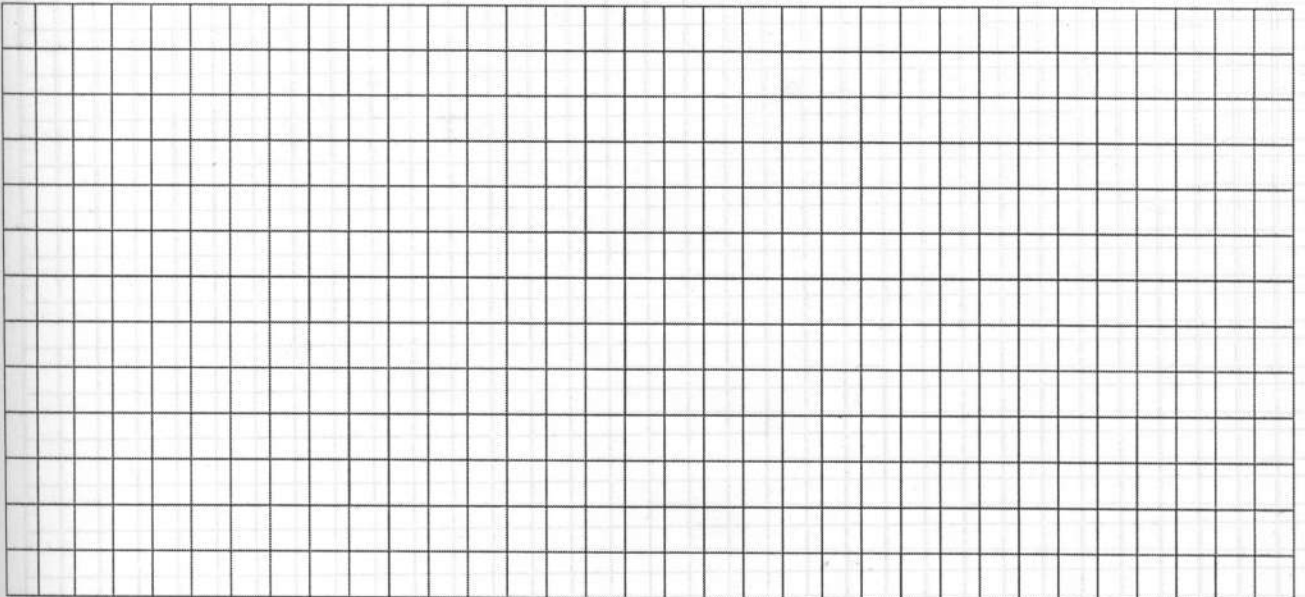


Fig. 3.12

### Solution



### Problem 3.3

A two-beam mechanism is loaded and supported as shown in Fig. 3.13. Determine:

- The force  $F$  required to hold the system in equilibrium.
- The tension in cable  $CD$ .
- The reaction at support  $B$ .
- The reaction at support  $A$ .

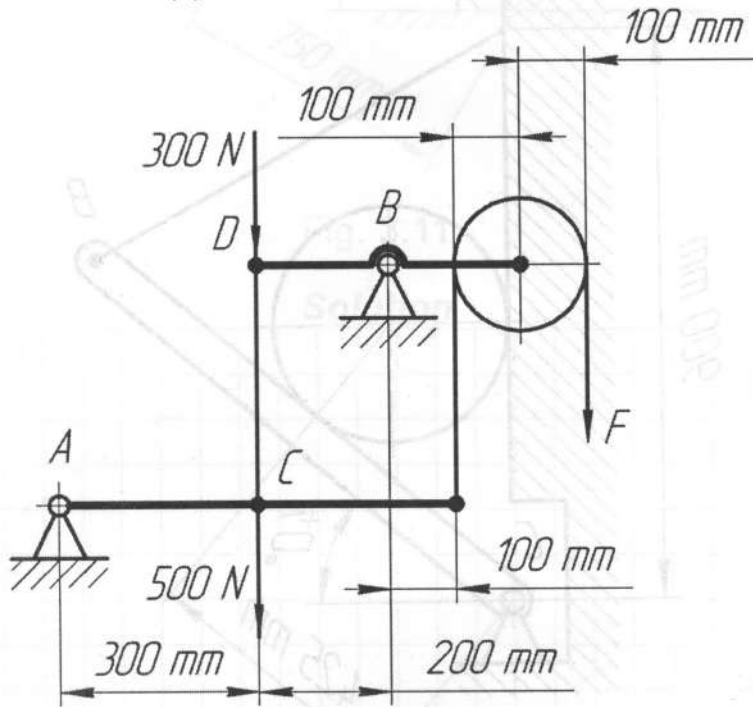
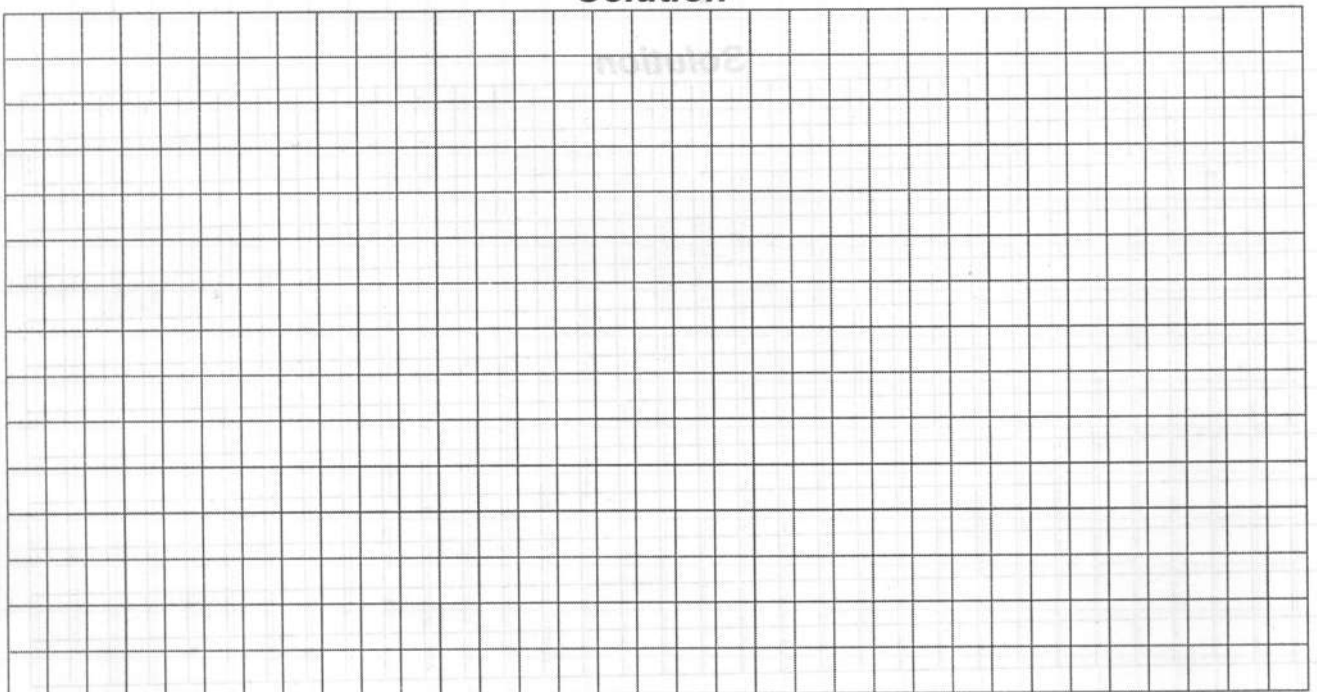


Fig. 3.13

### Solution





**Problem 3.4**

Bar AD shown in Fig. 3.14 weighs 600 N. Determine:

- The force exerted on the bar by link CE and the forces exerted on the bar at contact points B and D. All surfaces are smooth.
- The reaction at support F of the post.

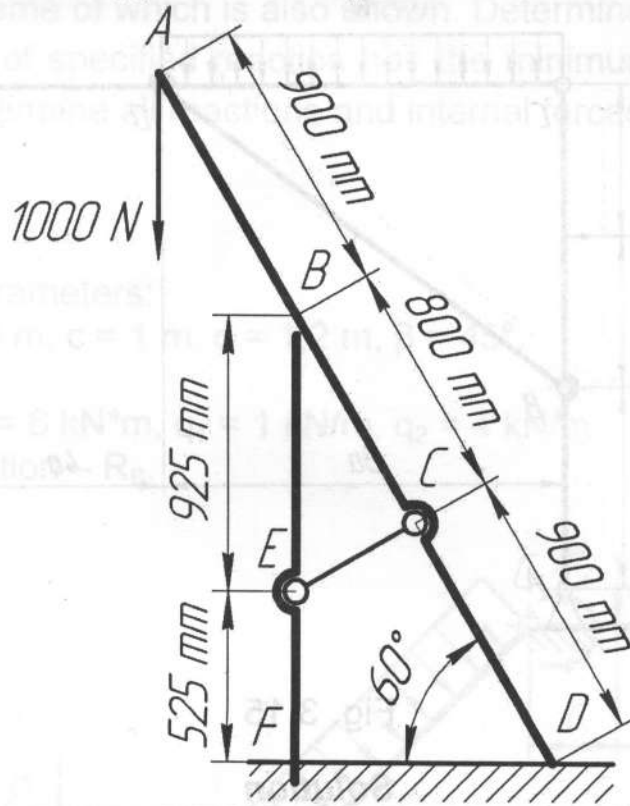


Fig. 3.14

**Solution**

**Problem 3.5**

The frame of Fig. 3.15 has a distributed load of  $w=200\text{ N/m}$  applied to member CDE and a concentrated force  $P=200\text{ N}$  applied to member ABC. If  $a=100\text{ mm}$ , determine all forces acting on member ABC.

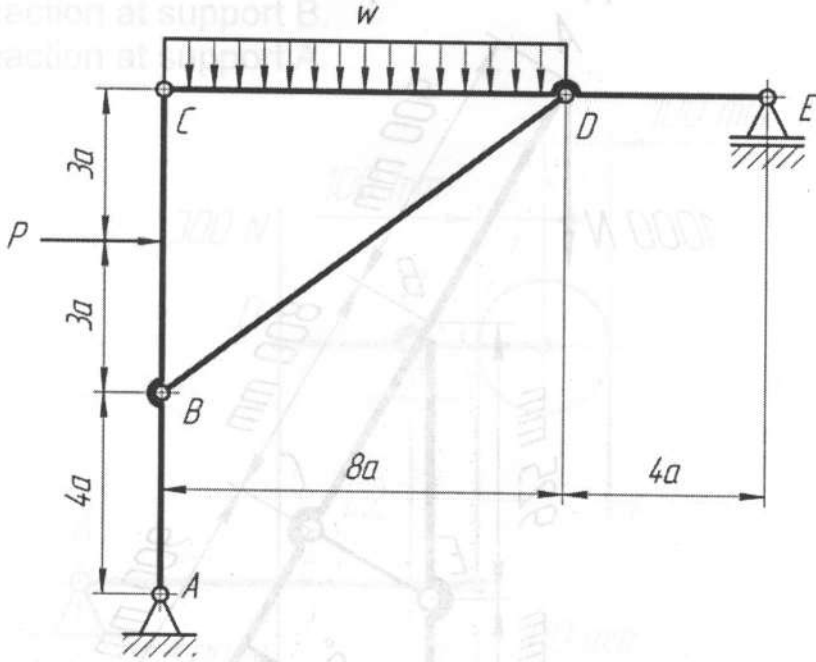
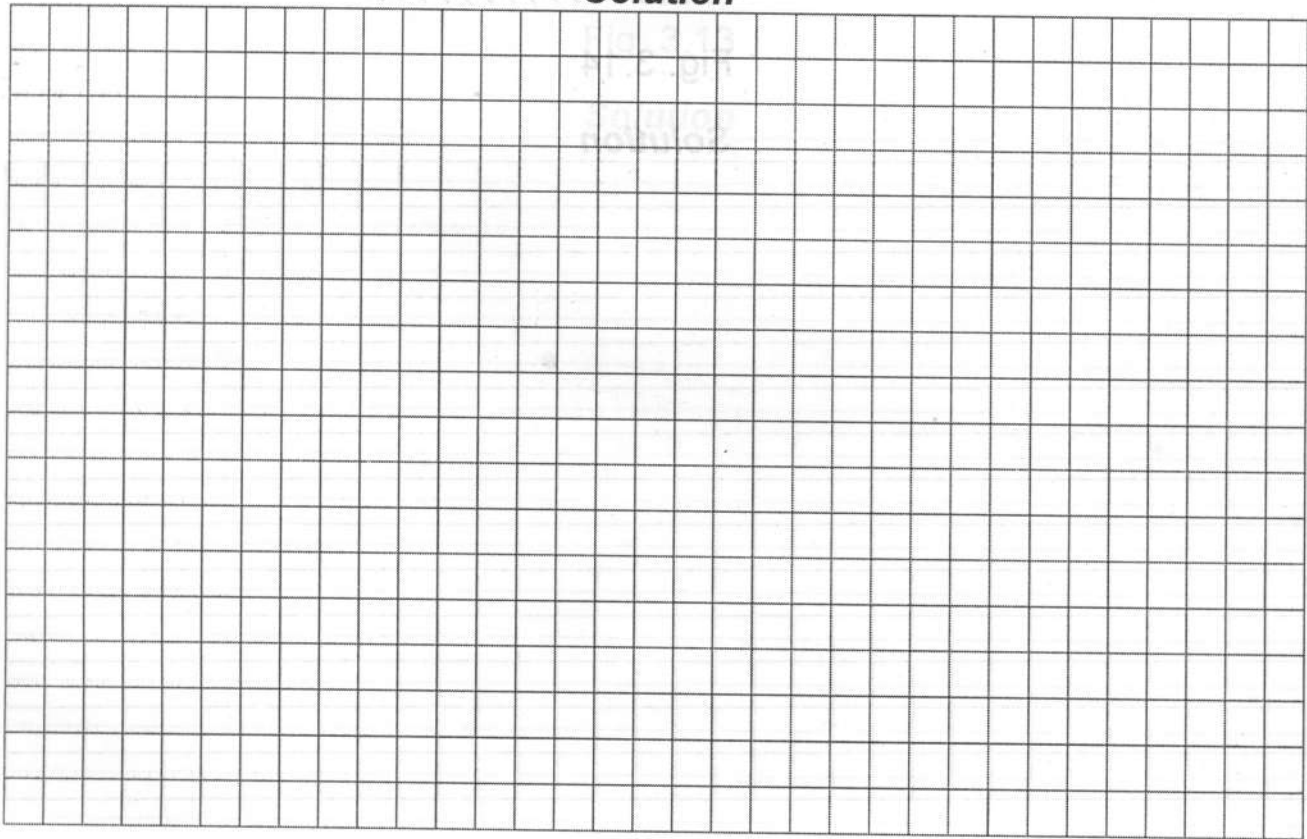


Fig. 3.15

**Solution**





in the problems (3.6 - 3.10) in Figures 3.16 - 3.20 compound structures are shown. They consist of two rods loaded by concentrated forces, distributed loads and couples with moments. Values of loads and geometrical parameters are specified in each problem. Consider two methods of finding internal forces at point C using the method of sections, one of which is also shown. Determine in which method of finding the magnitude of specified section has the minimum value and for that method of finding determine all sections and internal forces at point C.

Problem 3.6

Geometrical parameters:  
 $a = 3 \text{ m}$ ,  $b = 0.8 \text{ m}$ ,  $c = 1 \text{ m}$ ,  $d = 1.2 \text{ m}$ ,  $\beta = 45^\circ$   
 Loads:  
 $F_1 = 20 \text{ kN}$ ,  $M_1 = 0 \text{ kNm}$ ,  $F_2 = 1 \text{ kNm}$ ,  $q_1 = 4 \text{ kNm}$   
 Examining section - K

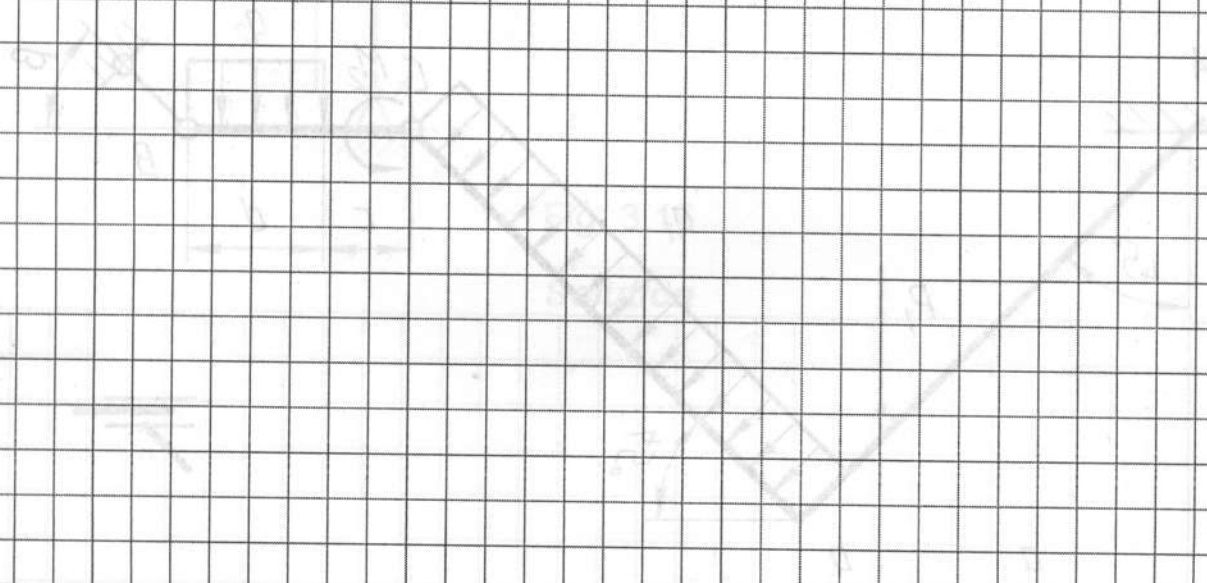


Fig. 3.16

Solution

### Problem 3.7

Geometrical parameters:

$a = 1,6 \text{ m}$ ,  $b = 1,4 \text{ m}$ ,  $c = 3 \text{ m}$ ,  $d = 1 \text{ m}$ ,  $\alpha = 180^\circ$ ,  $\beta = 90^\circ$ .

Loads:

$P_1 = 2 \text{ kN}$ ,  $P_2 = 6 \text{ kN}$ ,  $M_1 = 4 \text{ kN}\cdot\text{m}$ ,  $M_2 = 10 \text{ kN}\cdot\text{m}$ ,  $q_1 = 6 \text{ kN/m}$ .

Examining reaction –  $M_A$ .

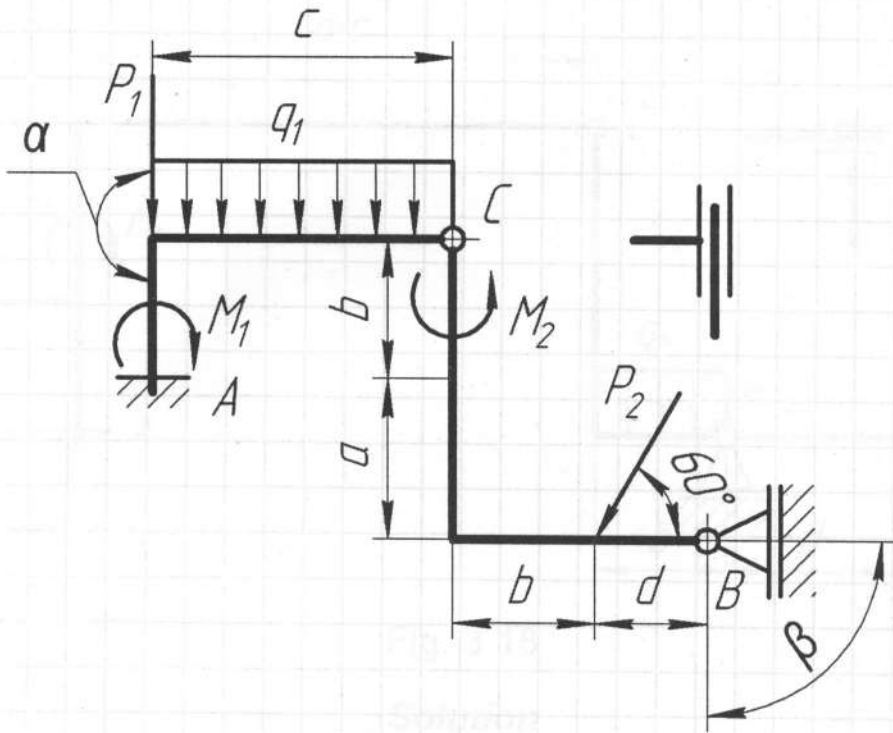


Fig. 3.17

### Solution



Geometrical parameters:

$$a = 7.8 \text{ m}, b = 14 \text{ m}, c = 8 \text{ m}, d = 17 \text{ m}, \alpha = 180^\circ, \beta = 90^\circ$$

Loads:

$$P_1 = 2 \text{ kN}, P_2 = 6 \text{ kN}, M_1 = 4 \text{ kNm}, M_2 = 10 \text{ kNm}, q = 5 \text{ kNm/m}$$

Examining section - M

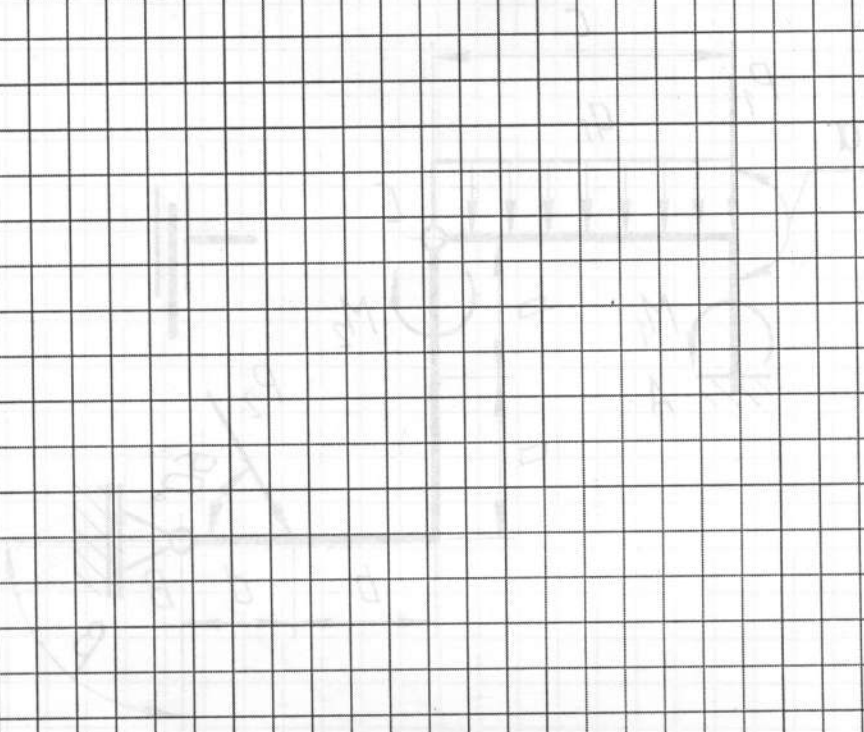


Fig. 3.17

Solution

### Problem 3.8

Geometrical parameters:

$$a = 2 \text{ m}, b = 1,5 \text{ m}, c = 4 \text{ m}, d = 2 \text{ m}, \alpha = 30^\circ.$$

Loads:

$$P_1 = 6 \text{ kN}, P_2 = 20 \text{ kN}, M_1 = 12 \text{ kN}\cdot\text{m}, q_2 = 3 \text{ kN/m}.$$

Examining reaction –  $R_B$ .

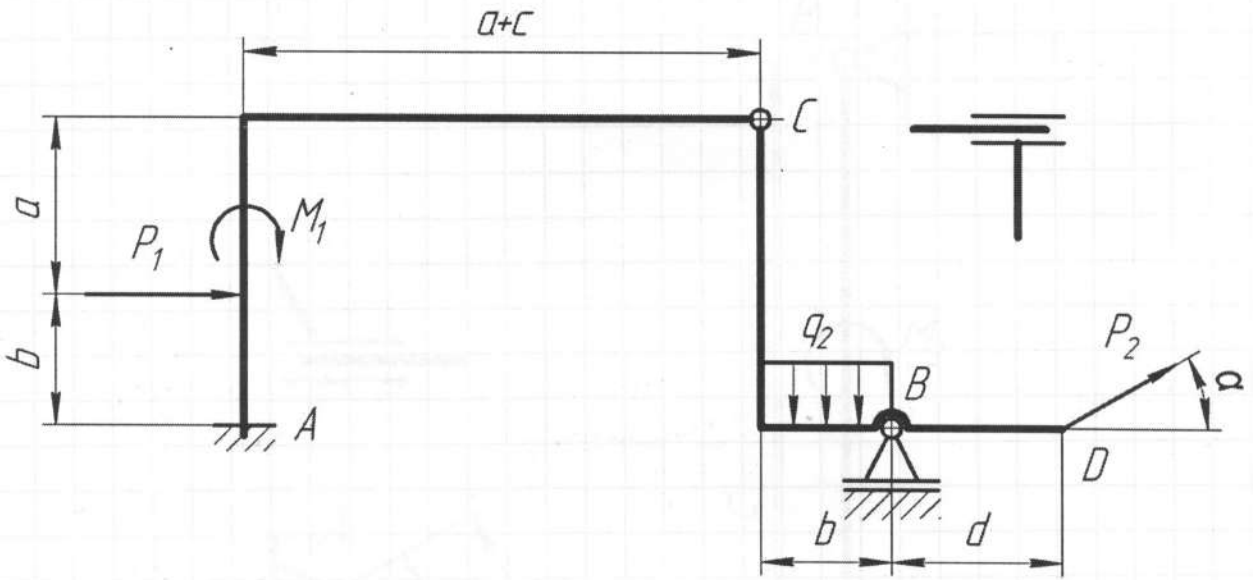
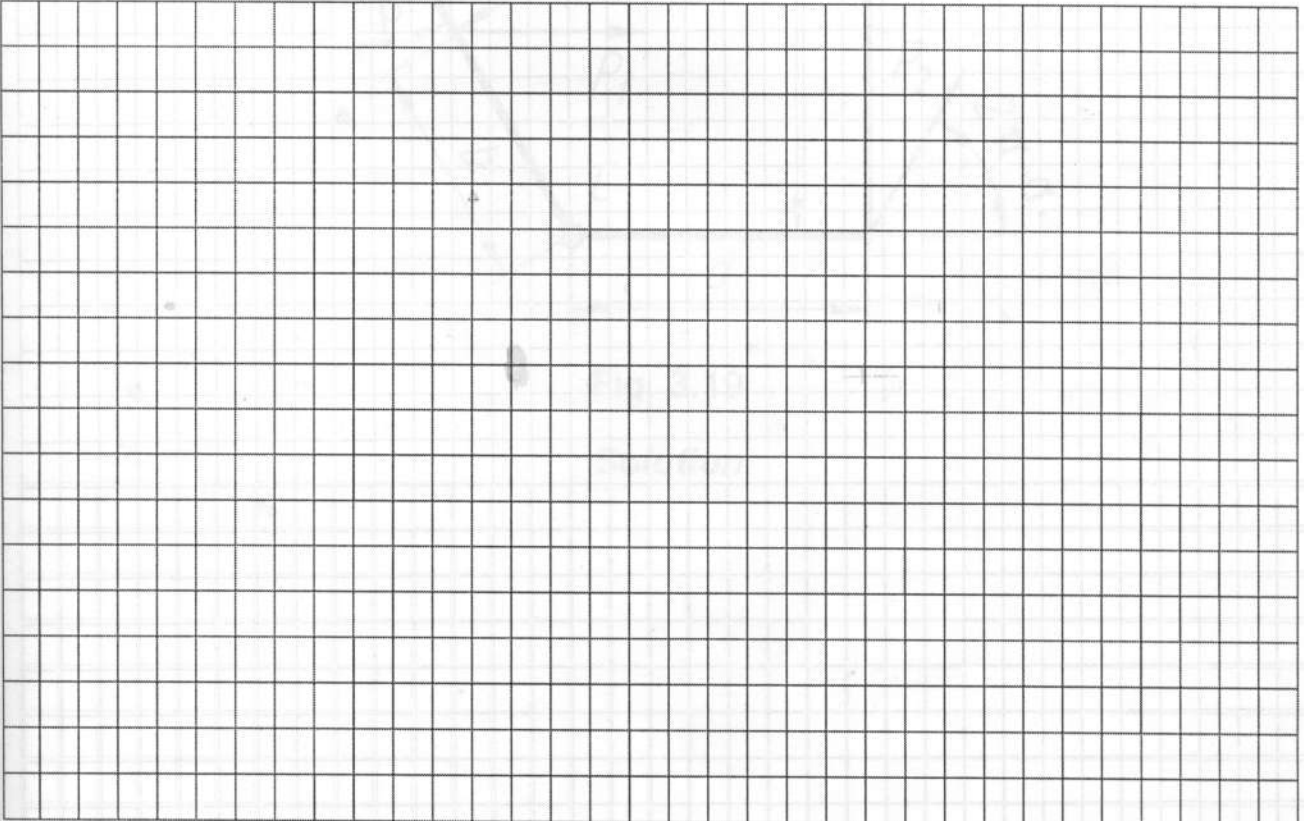


Fig. 3.18

### Solution



Problem 3.8

Geometrical parameters:

$$a = 2 \text{ m}, b = 1.5 \text{ m}, c = 4 \text{ m}, d = 2 \text{ m}, \alpha = 30^\circ$$

Loads:

$$P_1 = 6 \text{ kN}, P_2 = 20 \text{ kN}, M_1 = 12 \text{ kNm}, c_2 = 3 \text{ kNm}$$

Examining reaction -  $R_B$

Def.

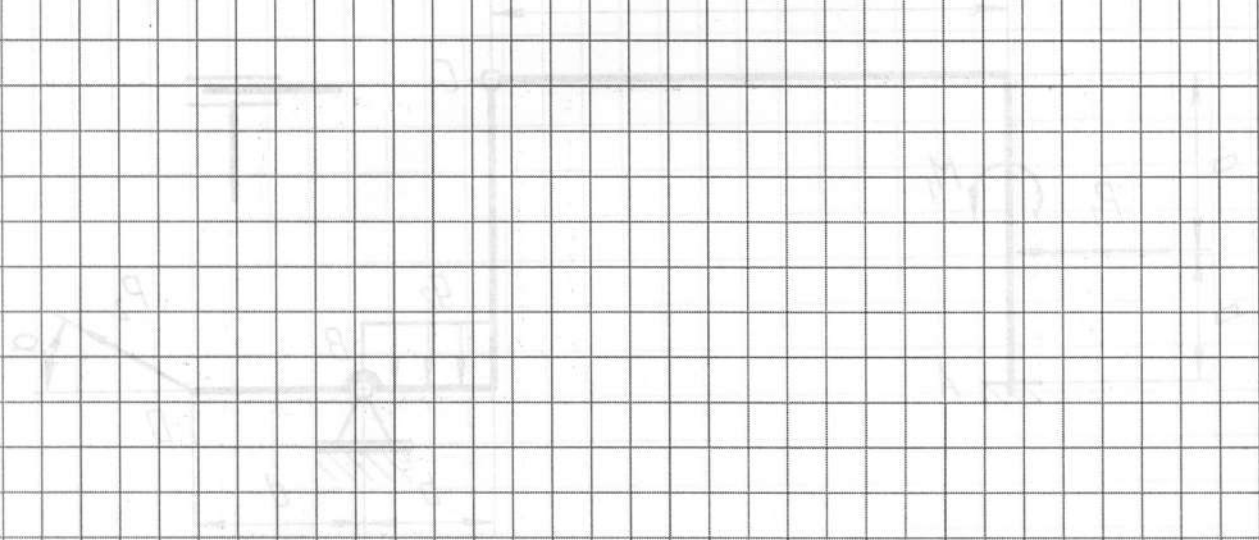


Fig. 3.18  
Solution



Problem 3.9

Geometrical parameters:

$$a = 1.5 \text{ m}, b = 1.5 \text{ m}, c = 6 \text{ m}, d = 2 \text{ m}, \alpha = 30^\circ, \beta = 30^\circ$$

Loads:

$$P_1 = 10 \text{ kN}, P_2 = 8 \text{ kN}, M_2 = 6 \text{ kNm}, q = 4 \text{ kNm/m}$$

Examining reaction -  $X_A$

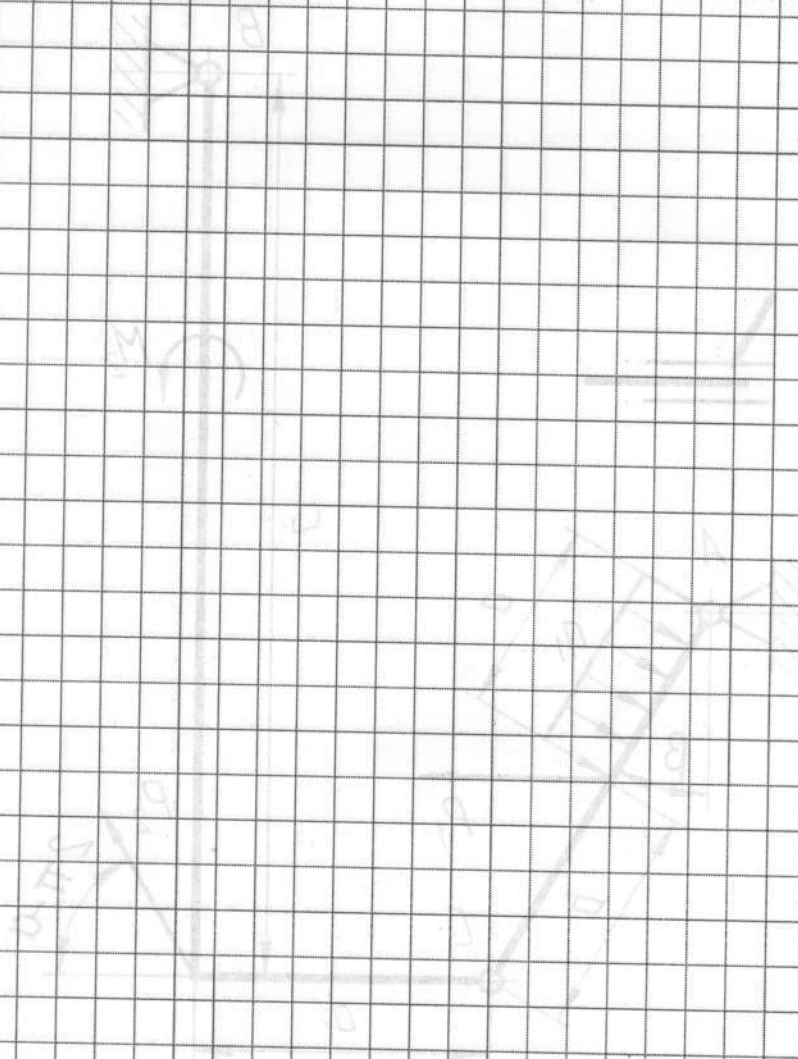


Fig. 3.19

Solution



### Problem 3.10

Geometrical parameters:

$a = 1\text{ m}$ ,  $b = 1,5\text{ m}$ ,  $c = 3\text{ m}$ ,  $d = 2\text{ m}$ ,  $\beta = 30^\circ$ .

Loads:

$P_1 = 6\text{ kN}$ ,  $P_2 = 10\text{ kN}$ ,  $M_1 = 4\text{ kN}\cdot\text{m}$ ,  $q_2 = 3\text{ kN/m}$ .

Examining reaction –  $R_A$ .

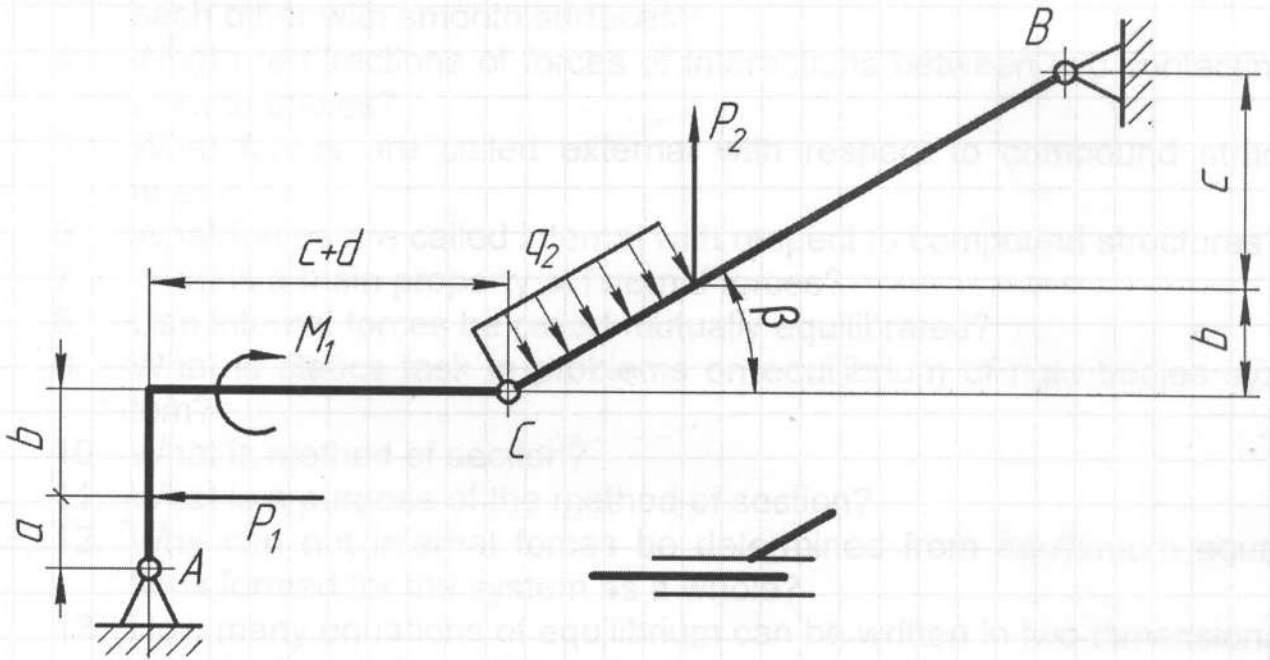
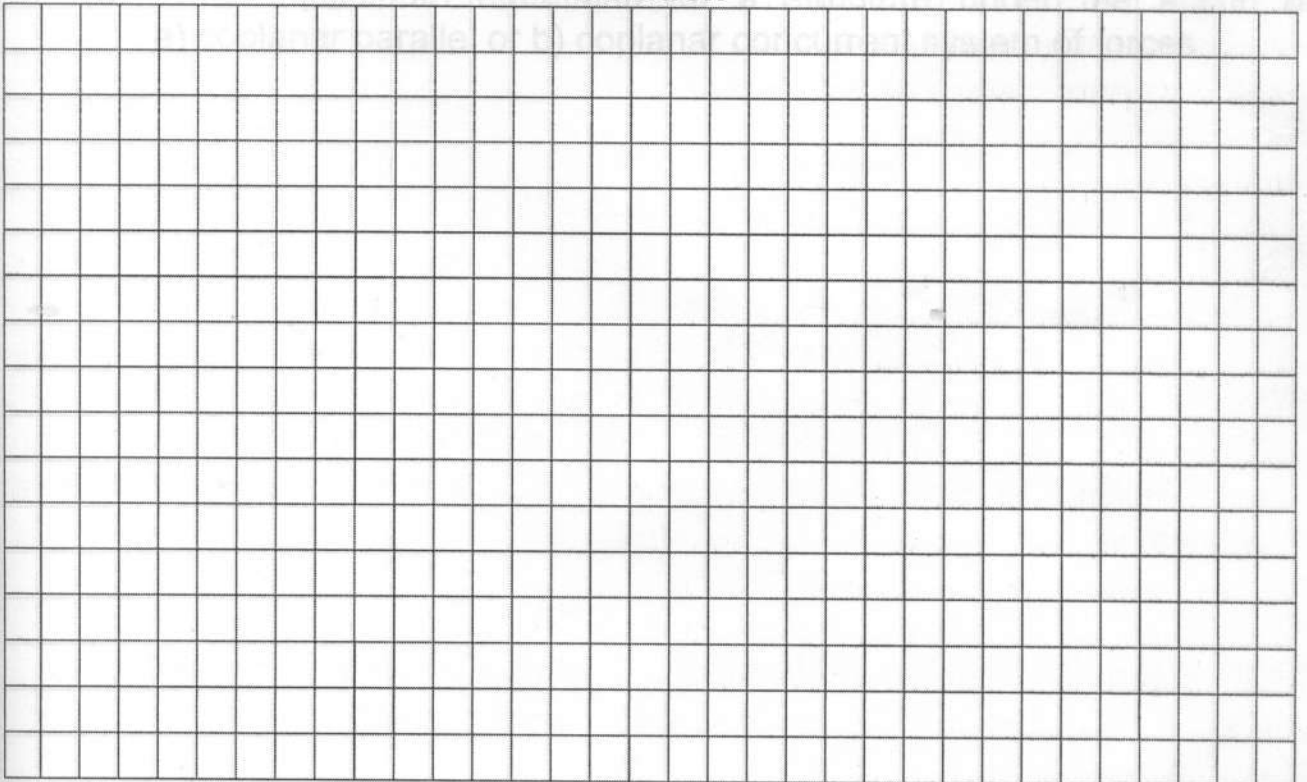


Fig. 3.20

### Solution



Problem 310

Geometrical parameters:

$$a = 1.5 \text{ m}, c = 3 \text{ m}, d = 2 \text{ m}, \beta = 30^\circ$$

Loads:

$$P_1 = 6 \text{ kN}, P_2 = 10 \text{ kN}, M_1 = 4 \text{ kN}\cdot\text{m}, q_2 = 2 \text{ kN/m}$$

Examining section - R.

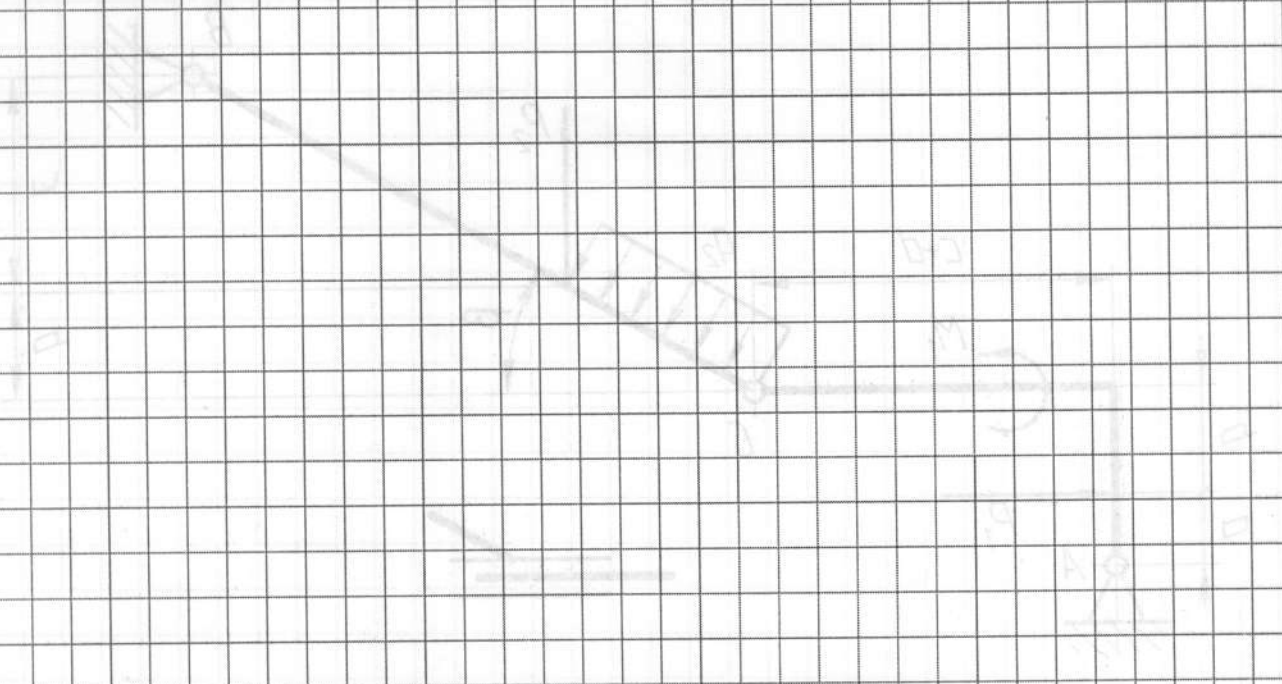


Fig. 3.30  
Solution

### 3.4. Questions for self-testing

1. What systems are called compound structures?
2. Give examples of compound structures in which bodies are joined by pins and show forces of interaction between parts of these structures.
3. What directions do forces of interaction have if bodies in the system each other with smooth surfaces?
4. What are directions of forces of interactions between two contacting smooth bodies?
5. What forces are called external with respect to compound structures?
6. What forces are called internal with respect to compound structures?
7. What is a main property of internal forces?
8. Can internal forces be called mutually equilibrated?
9. What is statics task in problems on equilibrium of rigid bodies system?
10. What is method of section?
11. What is a purpose of the method of section?
12. Why can not internal forces be determined from equilibrium equations formed for the system as a whole?
13. How many equations of equilibrium can be written in two-dimensional case for the system of  $N$  bodies?
14. How can coordinate axes and moment points be rationally chosen for forming equilibrium equations?
15. Write equilibrium equations for a structure under the action of  
a) coplanar parallel or b) coplanar concurrent system of forces.

## 4. GENERAL THREE DIMENSIONAL FORCE SYSTEM EQUILIBRIUM

### 4.1. Main information from the theoretical course

**General three-dimensional force system.** The sufficient and necessary conditions for the equilibrium of a rigid body under the action of general three-dimensional force system are expressed by the following six scalar equations

$$\sum_{k=1}^n F_{xk} = 0, \quad \sum_{k=1}^n F_{yk} = 0, \quad \sum_{k=1}^n F_{zk} = 0;$$
$$\sum_{k=1}^n M_x(\vec{F}_k) = 0, \quad \sum_{k=1}^n M_y(\vec{F}_k) = 0, \quad \sum_{k=1}^n M_z(\vec{F}_k) = 0.$$

It means that for rigid body complete equilibrium the algebraic sums of the forces projections on three noncoplanar axes must be equal zero and the algebraic sums of the moments of forces about the axes must be equal zero.

**Concurrent three-dimensional force system.** A rigid body under the action of three-dimensional forces all concurrent at the point is at the state of equilibrium if the algebraic sums of the forces projections on three noncoplanar axes are zero

$$\sum_{k=1}^n F_{xk} = 0, \quad \sum_{k=1}^n F_{yk} = 0, \quad \sum_{k=1}^n F_{zk} = 0.$$

**Parallel three-dimensional force system.** A rigid body under the action of parallel three-dimensional forces is at the state of equilibrium if the algebraic sum of the forces projections on the axis that is parallel to the forces is zero and the algebraic sums of the moments of forces about other two axes are zero.

If forces are parallel to the axis  $Ox$  ( $\vec{F}_k \parallel Ox$ ) then equilibrium conditions are

$$\sum_{k=1}^n F_{xk} = 0, \quad \sum_{k=1}^n M_y(\vec{F}_k) = 0, \quad \sum_{k=1}^n M_z(\vec{F}_k) = 0.$$

There are two ways of determination of force  $\vec{F}$  moment about an axis.

**Projection of force  $\vec{F}$**  on a plane (Fig. 4.1, a) is **vector** quantity that can be defined as:

$$|\vec{F}_{xy}| = |\vec{F}| \cdot \cos \alpha,$$

where  $\alpha$  is angle between the force and its projection on the plane  $XOY$ .

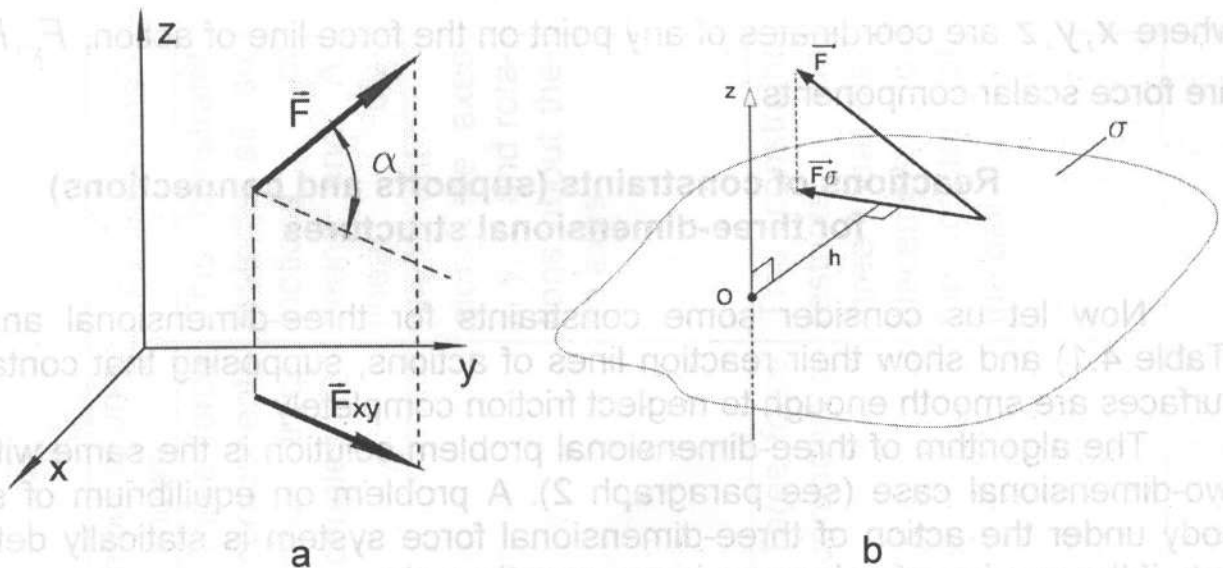


Fig. 4.1

The first. The moment of a force  $\vec{F}$  about the given axis may be obtained in the following three steps:

1. project the force into a plane that is perpendicular to the axis, as result we get the vector  $\vec{F}_\sigma$  (Fig. 4.1, b);
2. determine the arm  $h$  (see Fig. 4.1, b) of the projection  $\vec{F}_\sigma$  with respect to the point of axis and plane intersection, the arm  $h$  is the shortest distance between the projection line of action and the point;
3. calculate the moment of the force about the axis as positive or negative product of the magnitude of force projection and arm

$$M_u(\vec{F}) = \pm F_\sigma h.$$

The cases when axial moment of force is equal to zero may be determined with the help of formula:  $M_u(\vec{F}) = \pm F_\sigma h = 0$ :

- $F_\sigma = 0$  in the case when the force and the axis are parallel;
- $h = 0$  in the case when the force line of action crosses the point O, i.e. the line intersects the axis.

Evidently, both cases may be combined: an axial moment of a force is equal to zero if the force and the axis are in the same plane.

The second. The scalar magnitudes of a force moments about the coordinate axes may be determined by the following formulas

$$M_{Ox}(\vec{F}) = y \cdot F_z - z \cdot F_y,$$

$$M_{Oy}(\vec{F}) = z \cdot F_x - x \cdot F_z,$$

$$M_{Oz}(\vec{F}) = x \cdot F_y - y \cdot F_x,$$



where  $x, y, z$  are coordinates of any point on the force line of action,  $F_x, F_y, F_z$  are force scalar components.

### Reactions of constraints (supports and connections) for three-dimensional structures

Now let us consider some constraints for three-dimensional analysis (Table 4.1) and show their reaction lines of actions, supposing that contacting surfaces are smooth enough to neglect friction completely.

The algorithm of three-dimensional problem solution is the same with the two-dimensional case (see paragraph 2). A problem on equilibrium of single body under the action of three-dimensional force system is statically determinate if the number of unknowns is no more than six.

#### 4.2. Solution of problems

##### Sample problem 4.1

A force  $F_1 = 16 \text{ N}$  acts along the diagonal of the cube upper face, a force  $F_2 = 10 \text{ N}$  acts parallel to axis  $Ox$  along the edge of the cube lower face (Fig. 4.2). Cube edge size is  $a = 0,75 \text{ m}$ .

Determine the forces moments about coordinate axes.

##### Solution

Let's consider moments of the first force. To determine the moment of the first force about the axis  $Ox$  it is necessary to project the force onto the plane that is perpendicular to the axis  $Ox$  (plane  $yOz$ )

$$F_{1yOz} = F_1 \sin 45^\circ.$$

The vector of projection  $\vec{F}_{1yOz}$  is acted along the cube edge  $DE$  in positive direction of the axis  $Oy$ , so the shortest distance between the projection  $\vec{F}_{1yOz}$  and the point  $O$  is  $a$ . The moment of the projection is

$$M_X(\vec{F}_{1yOz}) = -F_{1yOz} a = -F_1 \sin 45^\circ a = -16 \cdot 0,7 \cdot 0,75 = -8,5 (\text{N} \cdot \text{m}).$$

Fig. 4.2

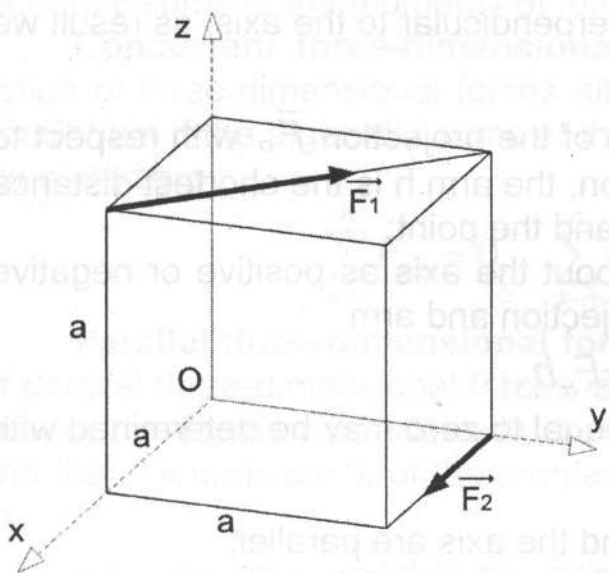
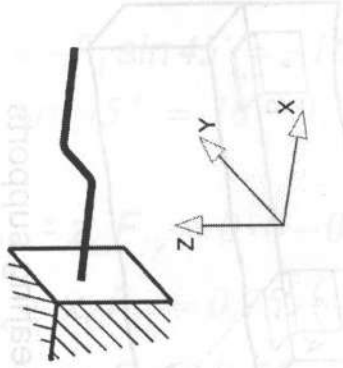
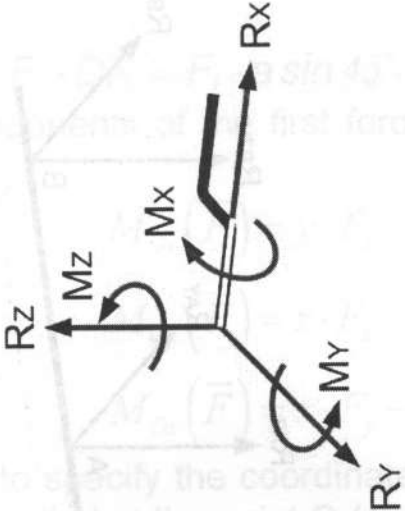
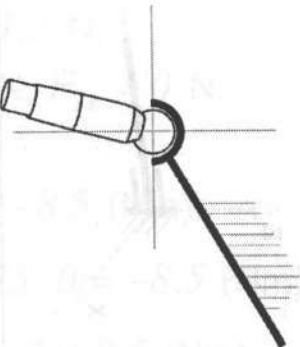
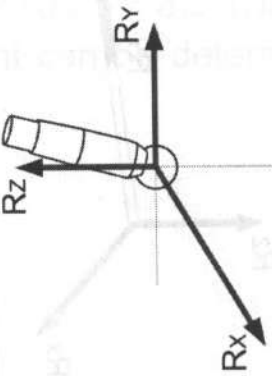
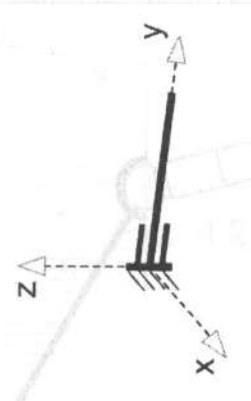
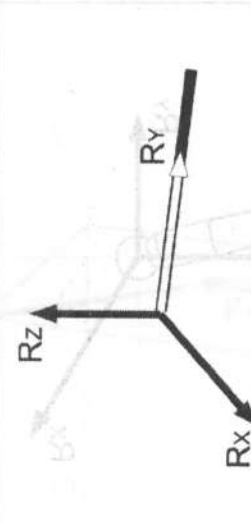
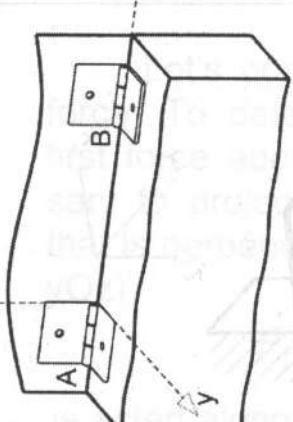
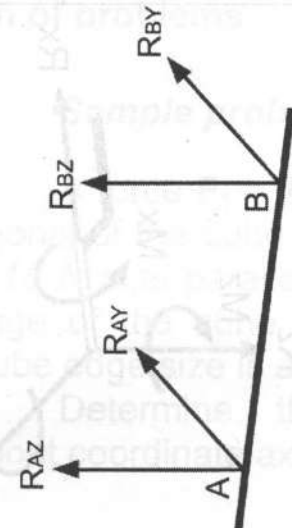


Table 4.1

Reactions at supports and connections for three-dimensional structure

Support or connections	Reactions	Number of unknown	Explanations
 <p>Fixed support</p>		<p>6</p> <p>Three force components and three couples</p>	<p>This constraint restricts all six motions of beam end A: linear placement along the axes x, y, z, and rotations about the axes</p>
 <p>Ball-and-socket joint and footstep bearing</p>		<p>3</p> <p>Three force components</p>	<p>This constraint restricts the three linear displacement of the center of the ball</p>

<p>Thrust bearing support</p>			<p>3</p> <p>Three force components</p>	<p>Properly aligned pairs of bearings restrict all displacements of the points on the axis of the shaft</p>
<p>Hinge and bearing support</p>	<p>Hinges</p>  <p>Bearing supports</p> 	<p>2</p> <p>Two force components</p>	<p>Properly aligned pairs of bearings (hinges) restrict the displacements of the points on the axis of the shaft in the directions perpendicular to the axis of the bearings (hinges)</p>	

To determine the moment of the first force about the axis Oy it is necessary to project the force onto the plane that is perpendicular to the axis Oy (plane xOz)

$$F_{1xOz} = F_1 \sin 45^\circ.$$

The vector of projection  $\vec{F}_{1xOz}$  is acted along the cube edge DG in negative direction of the axis Ox, so the shortest distance between the projection  $\vec{F}_{1yOz}$  and the point O is a. The moment of the projection is

$$M_y(\vec{F}_{1xOz}) = -F_{1xOz} a = F_1 \sin 45^\circ a = -16 \cdot 0,7 \cdot 0,75 = -8,5 \text{ (N} \cdot \text{m)}.$$

To determine the moment of the first force about the axis Oz it is necessary to project the force onto the plane that is perpendicular to the axis Oz, but the force is situated in the cube face that is perpendicular to the axis Oz, so we need to determine the shortest distance between the force line of action and point D only (DK)

$$M_z(\vec{F}_1) = F_1 \cdot DK = F_1 \cdot a \sin 45^\circ = 16 \cdot 0,7 \cdot 0,75 = 8,5 \text{ (Nm)}.$$

The scalar components of the first force moment can be determined by equations

$$M_{Ox}(\vec{F}) = y \cdot F_z - z \cdot F_y,$$

$$M_{Oy}(\vec{F}) = z \cdot F_x - x \cdot F_z,$$

$$M_{Oz}(\vec{F}) = x \cdot F_y - y \cdot F_x.$$

It is necessary to specify the coordinates of any point on the force line of action. The force is applied at the point G (a, 0, a), the force scalar components are the following:

$$F_{1x} = -F_1 \sin 45^\circ = -16 \cdot 0,7 = -11,2 \text{ N},$$

$$F_{1y} = F_1 \sin 45^\circ = 16 \cdot 0,7 = 11,2 \text{ N}, F_{1z} = 0 \text{ N}.$$

Then

$$M_x(\vec{F}_1) = y_G F_{1z} - z_G F_{1y} = 0 \cdot 0 - 0,75 \cdot 11,2 = -8,5 \text{ (Nm)},$$

$$M_y(\vec{F}_1) = z_G F_{1x} - x_G F_{1z} = 0,75 \cdot (-11,2) - 0,75 \cdot 0 = -8,5 \text{ (Nm)},$$

$$M_z(\vec{F}_1) = x_G F_{1y} - y_G F_{1x} = 0,75 \cdot 11,2 - 0 \cdot (-11,2) = 8,5 \text{ (Nm)}.$$

Moments of the second force.

The second force is situated in the plane xOy, so the moments of the force about the axes Ox and Oy are zero.

The force is in the plane that is perpendicular to the axis Oz, the arm of the force with respect to the axis Oz is a

$$M_z(\vec{F}_2) = -F_2 a = -10 \cdot 0,75 = -7,5 \text{ (Nm)}.$$

Answer:  $\vec{M}_O(\vec{F}_1) = (-8.5, 8.5, 8.5) \text{ Nm}$ ,  $\vec{M}_O(\vec{F}_2) = (0, 0, -7.5) \text{ Nm}$ .

### Sample problem 4.2

A winch supported by bearings **A** and **B** holds the 1-kN load **Q** in the position shown in the Fig. 4.3, a. A cable is tangent to the drum  $r = 0.05 \text{ m}$  and forms the angle  $\alpha = 60^\circ$  with the horizontal plane. Length of the handle **KD** is  $0.4 \text{ m}$ ,  $AD = 0.3 \text{ m}$ ,  $AC = 0.4 \text{ m}$ ,  $CB = 0.6 \text{ m}$ .

Determine the reactions at **A** and **B** and force **P** that balances the system in the position shown ( the handle **KD** is in horizontal plane).

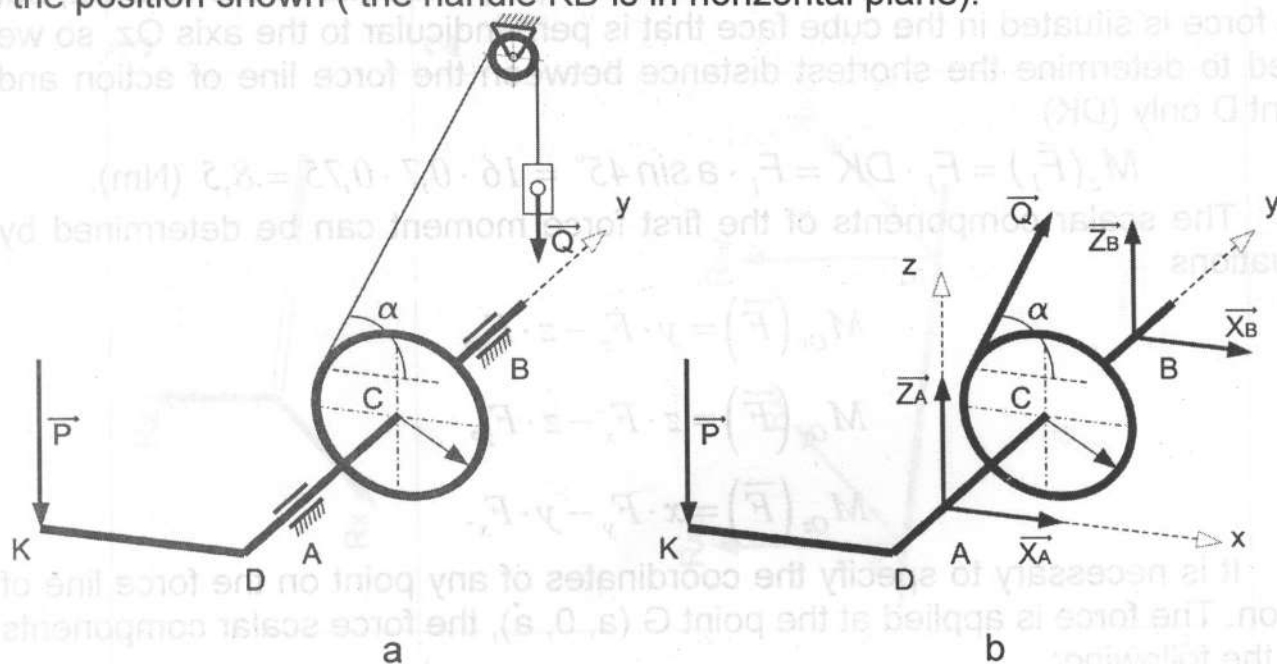


Fig. 4.2

### Solution

1. FBD (Fig. 4.3, b). The winch is at the state of equilibrium. Applied forces are unknown force **P** and tension **T** that is equal to the weight **Q** of the load if we neglect friction in the block. Motion of the winch is restricted by two bearings so reactions are the following: two forces at the bearing **A**  $\vec{X}_A, \vec{Z}_A$  and two forces  $\vec{X}_B, \vec{Z}_B$  at the bearing **B**. The force system acting on the winch is general three-dimensional. Let us choose coordinate system **Axyz**, axis **Ay** coincides with the axis of winch.

2. Problem is statically determinate because we have five unknown forces and we can write six independent equation of equilibrium for the winch.



3. Let us write the equilibrium equations. It is useful to project the forces onto the coordinate planes and reduce the three-dimensional problem to the set of two-dimensional problems.

From the  $xAy$  projection (Fig. 4.4, a) we get the equations  $R_x = \sum F_{xk} = 0$ ,  $R_y = \sum F_{yk} = 0$  and  $\sum M_z(\vec{F}_k) = 0$  (the axis  $Az$  is perpendicular to the  $xAy$  coordinate plane, so the projections of the forces onto this plane form the moments about axis  $Az$ ):

$$\begin{cases} R_x = \sum F_{xk} = X_A + X_B + Q \cdot \cos \alpha = 0; \\ R_y = \sum F_{yk} = 0; \\ \sum M_z(\vec{F}_k) = -X_B \cdot AB - Q \cdot \cos \alpha \cdot AC = 0. \end{cases}$$

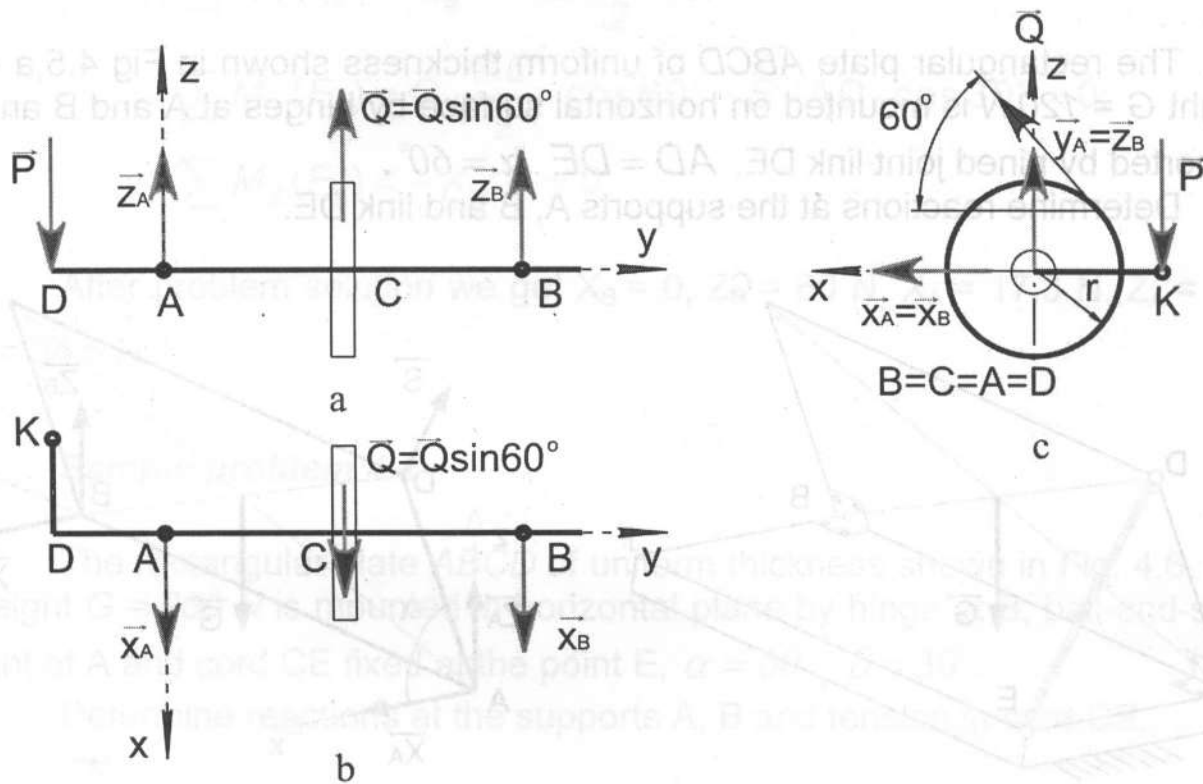


Fig. 4.3

Using the  $yAz$  projection (Fig. 4.4, b) we get the equations

$R_z = \sum F_{zk} = 0$  and  $\sum M_x(\vec{F}_k) = 0$  (the axis  $Ax$  is perpendicular to the  $yAz$  coordinate plane, so the projections of the forces onto this plane form the moments about axis  $Ax$ ):

$$\begin{cases} R_z = \sum F_{zk} = Z_A + Z_B - P + Q \cdot \sin \alpha = 0. \\ \sum M_x(\vec{F}_k) = Z_B \cdot AB + Q \cdot \sin \alpha \cdot AC + P \cdot AD = 0; \end{cases}$$

To write the last equation  $\sum M_y(\vec{F}_k) = 0$  we use the xAz projection (Fig. 4.4,c)

$$\sum M_y(\vec{F}_k) = Q \cdot r - P \cdot KD = 0.$$

Putting the known values and solving the system of the five equations we get

$$P = 125 H, X_A = -300 H, Z_A = -357 H, X_B = -200 H, Z_B = -384 H.$$

The negative signs associated with the A and B components indicate that they are in the opposite direction to those shown on the FBD.

**Sample problem 4.3**

The rectangular plate ABCD of uniform thickness shown in Fig 4.5,a has weight  $G = 120 N$  is mounted on horizontal surface by hinges at A and B and is supported by pinned joint link DE,  $AD = DE$ .  $\alpha = 60^\circ$ . Determine reactions at the supports A, B and link DE.

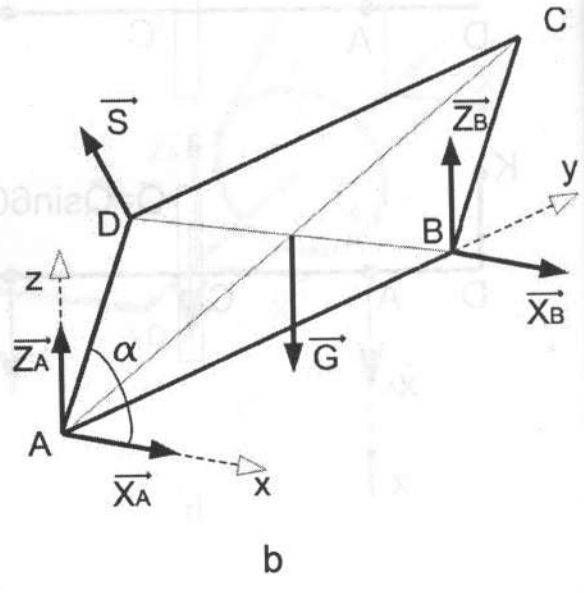
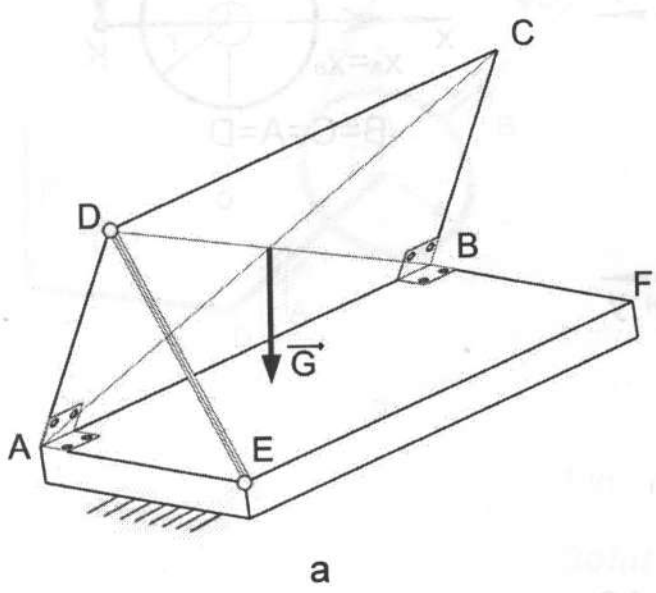


Fig. 4.4

**Solution**

1. FBD (Fig. 4.5, b). The plate ABCD is at the state of equilibrium. Applied force is weight  $\vec{G}$ . Motion of the plate is restricted by two hinges and by pinned joint link so reactions are the following: two forces at the point A  $\vec{X}_A, \vec{Z}_A$ , forces  $\vec{X}_B, \vec{Z}_B$  at the point B and reaction of the link DE  $\vec{S}$ . The force system acting on the plate is general tree-dimensional. Let us choose coordinate system Axyz, axis Ay coincides with the side AB.

2. Problem is statically determinate because we have five unknown forces and we can write six independent equation of equilibrium for the winch.
3. Let us write the equilibrium equations. It is useful to note that the triangle ADE is right-angled and situated in xAz plane, so moments of the force  $\vec{S}$  about axes Ax and Az are zero.

$$\begin{cases} \sum F_{xk} = X_A + X_B - S \cdot \cos 60^\circ = 0; \\ \sum F_{yk} = 0; \\ \sum F_{zk} = Z_A + Z_B - G + S \cdot \sin 60^\circ = 0. \end{cases}$$

$$\begin{cases} \sum M_x(F_k) = Z_B \cdot l - G \cdot \frac{l}{2} = 0; \\ \sum M_y(F_k) = G \cdot \frac{AD}{2} \cdot \cos 60^\circ - S \cdot AD \cdot \cos 30^\circ = 0; \\ \sum M_z(F_k) = -X_B \cdot l = 0. \end{cases}$$

After problem solution we get  $X_B = 0$ ,  $Z_B = 60 \text{ N}$ ,  $X_A = 17,3 \text{ N}$ ,  $Z_A = 30 \text{ N}$ ,  $S = 34,5 \text{ N}$ .

#### Sample problem 4.4

The rectangular plate ABCD of uniform thickness shown in Fig. 4.6, a has weight  $G = 200 \text{ N}$  is mounted in horizontal plane by hinge at B, ball-and-socket joint at A and cord CE fixed at the point E,  $\alpha = 60^\circ$ ,  $\beta = 30^\circ$ .

Determine reactions at the supports A, B and tension in cord CE.

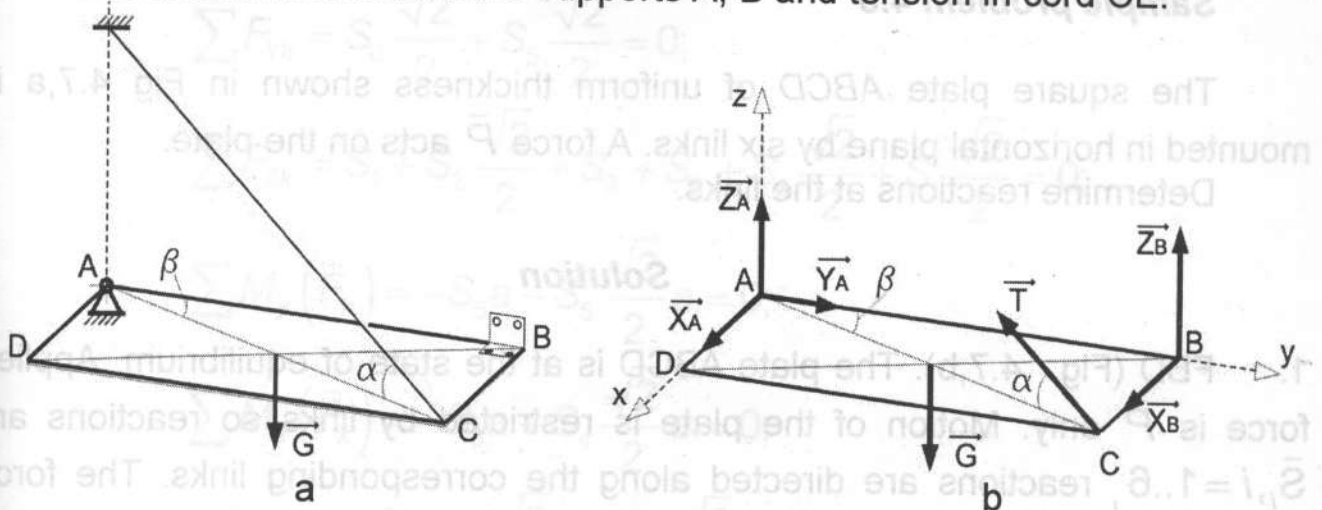


Fig. 4.6

### Solution

1. FBD (Fig. 4.6,b). The plate ABCD is at the state of equilibrium. Applied force is weight  $\vec{G}$  only. Motion of the plate is restricted by hinge at B, ball-and-socket joint at A and cord CE fixed at the point E so reactions are the following: two forces at the point B  $\vec{X}_B, \vec{Z}_B$ , three forces  $\vec{X}_A, \vec{Y}_A, \vec{Z}_A$  at the point A and reaction of the cord CE  $\vec{T}$ . The force system acting on the plate is general three-dimensional. Let us choose coordinate system Axyz, axis Ay coincides with the side AB.
2. Problem is statically determinate because we have six unknown forces and we can write six independent equations of equilibrium for the winch.
3. Let us write the equilibrium equations.

$$\begin{cases} \sum F_{xk} = X_A + X_B - T \cos \alpha \sin \beta = 0; \\ \sum F_{yk} = Y_A - T \cos \alpha \cos \beta = 0; \\ \sum F_{zk} = Z_A + Z_B - G + T \sin \alpha = 0. \end{cases}$$

$$\begin{cases} \sum M_x(F_k) = -Z_B AB - G \frac{AB}{2} + T \sin \alpha AB = 0; \\ \sum M_y(F_k) = G \frac{BC}{2} - T \sin \alpha \cdot BC = 0; \\ \sum M_z(F_k) = -X_B AB = 0. \end{cases}$$

After problem solution we get  $X_B = 0, Z_B = 0, Z_A = 100 \text{ N}, Y_A = 150 \text{ N}, X_A = 86,6 \text{ N}, T = 200 \text{ N}$ .

### Sample problem 4.5

The square plate ABCD of uniform thickness shown in Fig 4.7,a is mounted in horizontal plane by six links. A force  $\vec{P}$  acts on the plate. Determine reactions at the links.

### Solution

1. FBD (Fig. 4.7,b). The plate ABCD is at the state of equilibrium. Applied force is  $\vec{P}$  only. Motion of the plate is restricted by links so reactions are  $\vec{S}_i, i=1..6$ , reactions are directed along the corresponding links. The force system acting on the plate is general three-dimensional. Let us choose coordi-

nate system Dxyz, axis Dx coincides with the side DA, axis Dy coincides with the side DC.

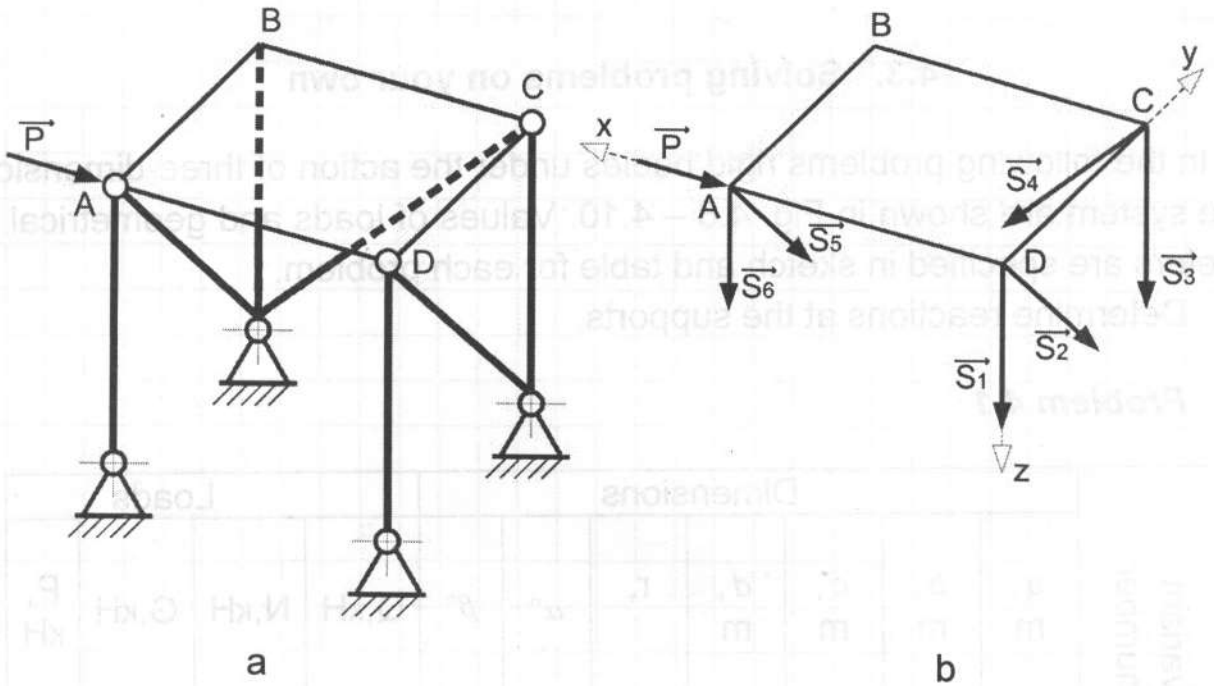


Fig. 4.7

2. Problem is statically determinate because we have six unknown forces and we can write six independent equations of equilibrium for the truss.
3. Let us write the equilibrium equations:

$$\sum F_{xk} = S_4 \frac{\sqrt{2}}{2} - P = 0;$$

$$\sum F_{yk} = S_2 \frac{\sqrt{2}}{2} + S_5 \frac{\sqrt{2}}{2} = 0;$$

$$\sum F_{zk} = S_1 + S_2 \frac{\sqrt{2}}{2} + S_3 + S_6 + S_5 \frac{\sqrt{2}}{2} + S_4 \frac{\sqrt{2}}{2} = 0;$$

$$\sum M_y(\vec{F}_k) = -S_6 a - S_5 \frac{\sqrt{2}}{2} a = 0;$$

$$\sum M_x(\vec{F}_k) = -S_3 a + S_4 \frac{\sqrt{2}}{2} a = 0;$$

$$\sum M_z(\vec{F}_k) = -S_4 \frac{\sqrt{2}}{2} a + S_5 \frac{\sqrt{2}}{2} a = 0.$$





Problem 4.2

Problem	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	
1	0.5	0.6	0.3	0.2	0.1	0.8	1.0	8	4	?																	
2	0.3	0.5	0.4	0.2	0.2	0.5	0.3	10	10	?																	
3	0.4	0.6	0.4	0.3	0.2	0.3	0.5	5	5	?																	

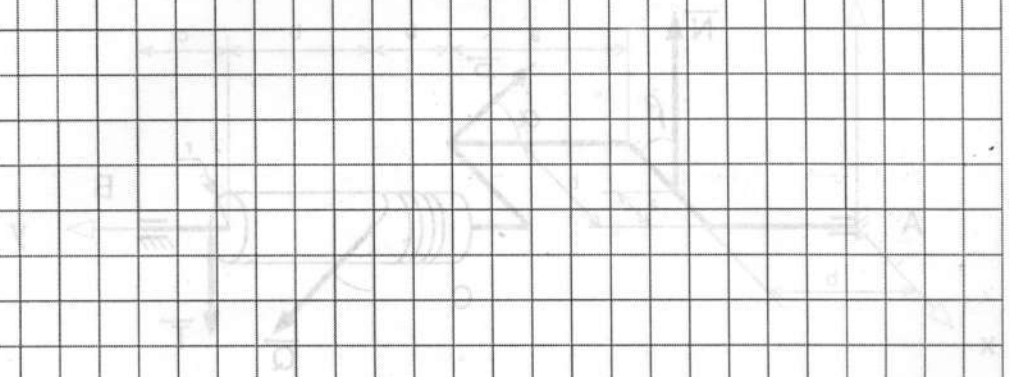


Fig. 4.2  
Solution

**Problem 4.2**

variant number	Dimensions							Loads			
	a, m	b, m	c, m	d, m	r, m	$\alpha^\circ$	$\beta^\circ$	Q, kH	N, kH	T, kH	P, kH
1	0,5	0,6	0,3	0,2	0,1	0	60	10	8	4	?
2	0,3	0,5	0,4	0,2	0,2	45	30	12	10	10	?
3	0,4	0,6	0,4	0,3	0,2	30	45	5	5	15	?

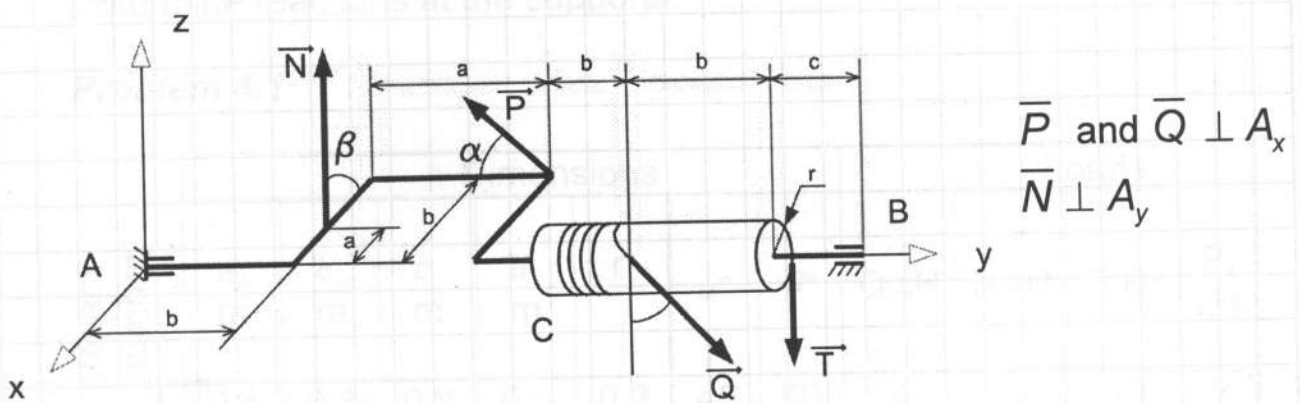
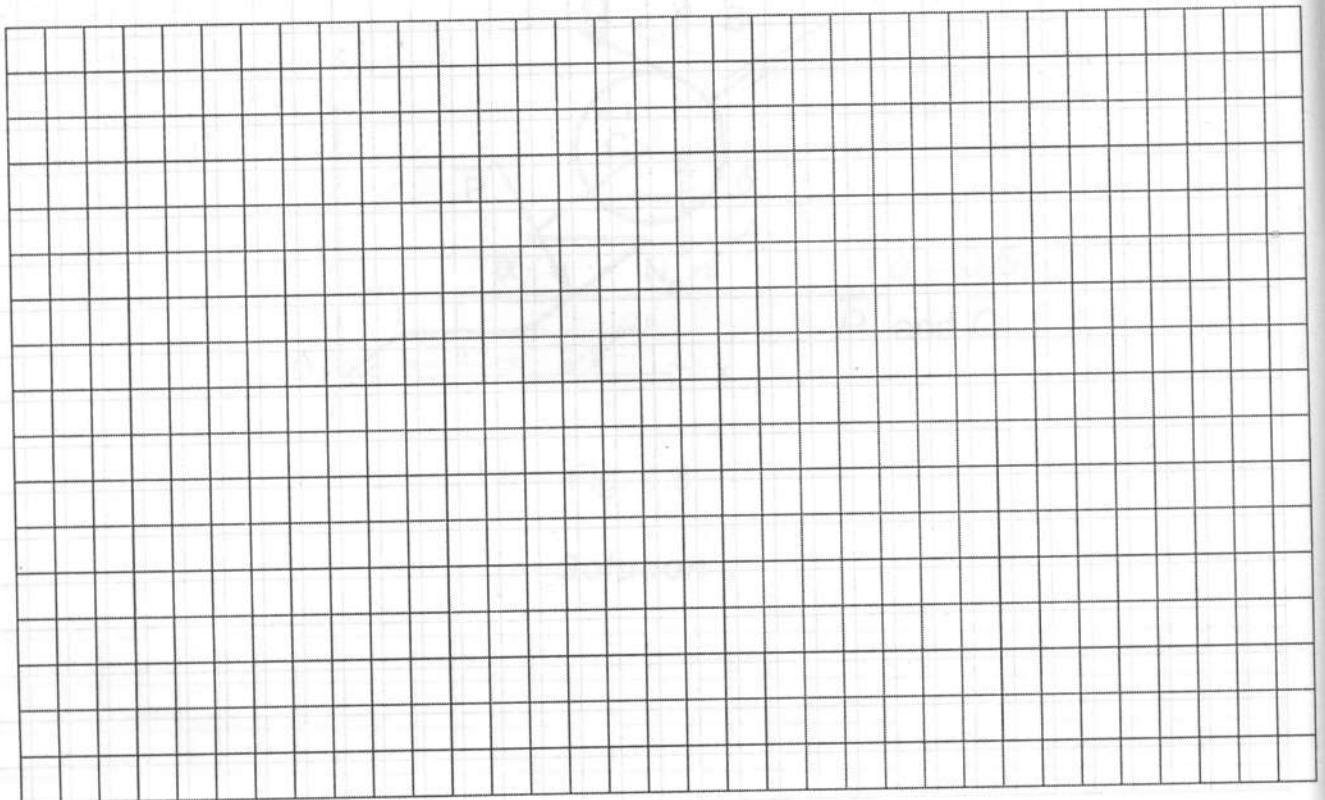


Fig. 4.9

**Solution**



Dimensions

Member	$a$ (m)	$b$ (m)	$c$ (m)	$d$ (m)	$e$ (m)	$f$ (m)	$g$ (m)	$h$ (m)	$i$ (m)	$j$ (m)	$k$ (m)	$l$ (m)
1	0.5	0.6	0.4	0.8	0.2	0.3	0	1.0	4	1.2	1	1
2	0.5	0.4	0.3	0.6	0.3	0.2	0.8	2.0	5	3	2	2
3	0.4	0.3	0.4	0.3	0.1	0.6	0.5	1.0	6	1	3	3

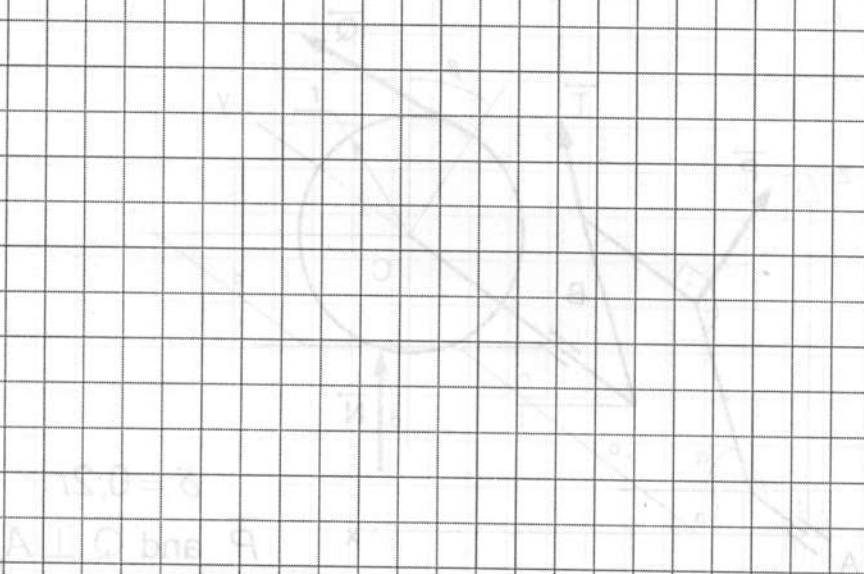


Fig. A.10  
Solution







#### 4.4. Questions for self-testing

1. Write the equilibrium equations of general three-dimensional force system in vector form and their scalar components.
2. How many independent equilibrium equations of three-dimensional concurrent force system can be written?
3. How many independent equilibrium equations of three-dimensional parallel force system can be written?
4. Definition of a force moment about an axis.
5. Describe the two ways to computing of a force moment about an axis?
6. Describe the conditions under which a force moment about an axis is zero?
7. Write the equations for computation of force moments about the coordinate axes.
8. Describe the algorithm for determination of supports reactions that act on a body under the three-dimensional force system?

## 5. REDUCING OF FORCE SYSTEM TO THE SIMPLEST FORM

### 5.1. Main information from the theoretical course

Let us consider force system  $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$ .

The vector sum of the system forces is called **total vector**

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = \sum_{k=1}^n \vec{F}_k. \quad (5.1)$$

Projecting left and right parts of the expressions (5.1) onto the axes of Cartesian rectangular coordinate system with origin in the center O we obtain the expressions for the components of the system total vector  $\vec{F}$

$$\left. \begin{aligned} F_x &= \sum_{k=1}^n F_{k_x}, \\ F_y &= \sum_{k=1}^n F_{k_y}, \\ F_z &= \sum_{k=1}^n F_{k_z}; \end{aligned} \right\} \quad (5.2)$$

The total vector magnitude is determined by next expression

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{\left(\sum_{k=1}^n F_{k_x}\right)^2 + \left(\sum_{k=1}^n F_{k_y}\right)^2 + \left(\sum_{k=1}^n F_{k_z}\right)^2}, \quad (5.3)$$

Its direction is determined with help of direction cosines

$$\cos(\vec{F}, \vec{i}) = \frac{F_x}{F}, \quad \cos(\vec{F}, \vec{j}) = \frac{F_y}{F}, \quad \cos(\vec{F}, \vec{k}) = \frac{F_z}{F}, \quad (5.4)$$

where  $(\vec{F}, \vec{i})$ ,  $(\vec{F}, \vec{j})$ ,  $(\vec{F}, \vec{k})$  are the angles between the total vector direction and positive x, y, z direction.

The **moment of the force  $\vec{F}$  about the point** is vector with magnitude determined as product of force magnitude and the shortest distance between the point and the force line of action. The vector of moment is directed along perpendicular to the plane formed by the point and force and points in such way that rotation of the force about the point is viewed anti-clockwise from the end of the vector.

The magnitude and the direction of vector moment are completely defined from the relation:

$$\vec{M}_0(\vec{F}) = \vec{r} \times \vec{F},$$

where  $\vec{r}$  is vector position from the center O on force point of application (Fig. 5.1).

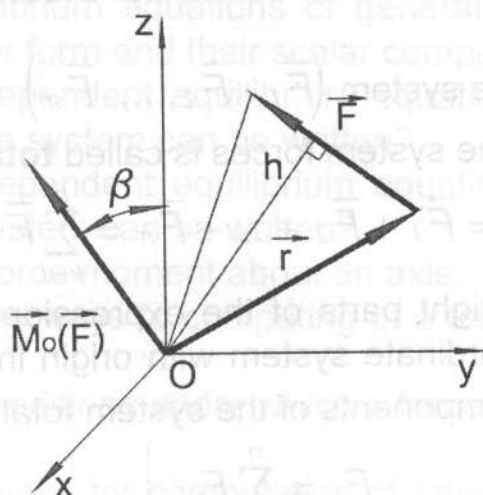


Fig. 5.1

The magnitude of moment by cross product definition is equal to

$$M_0(\vec{F}) = r \cdot F \cdot \sin \alpha = F \cdot h,$$

where  $h$  is the shortest distance between the point and the force line of action ("an arm"). Vector position, force and moment form "right-hand system".

The projection  $\vec{M}_0$  on axis through the point O is called **axial moment**.

For example,

$$M_z(\vec{F}) = np_z \vec{M}_0 = M_0 \cos \beta,$$

where  $\beta$  is angle between axis and vector moment (see Fig. 5.1).

If you know the forces projections on coordinate axes  $F_x, F_y, F_z$  and coordinates of any point on force line of action  $x, y, z$ , the moment of the force about the point can be computed using the law of vector product:

$$\vec{M}_0(\vec{F}) = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \vec{i}(yF_z - zF_y) + \vec{j}(zF_x - xF_z) + \vec{k}(xF_y - yF_x).$$

Then

$$M_x = y \cdot F_z - z \cdot F_y, M_y = z \cdot F_x - x \cdot F_z, M_z = x \cdot F_y - y \cdot F_x,$$

where  $M_x$  is the moment of force  $\vec{F}$  with respect to coordinate axes.

The vector sum of moments of forces about a given center is called **total moment about the center O**

$$\vec{M}_O = \sum_{k=1}^n \vec{M}_O(\vec{F}_k) = \sum_{k=1}^n \vec{r}_k \times \vec{F}_k, \quad (5.5)$$

where  $\vec{r}_k$  is vector position of the force  $\vec{F}_k$  with respect to the center O.

Projecting left and right parts of the expression (5.5) onto the axes of Cartesian rectangular coordinate system with origin in the center O we obtain the expressions for the components of the system total vector  $\vec{M}_O$ :

$$\left. \begin{aligned} M_{Ox} &= \sum_{k=1}^n M_{Ox}(\vec{F}_k), \\ M_{Oy} &= \sum_{k=1}^n M_{Oy}(\vec{F}_k), \\ M_{Oz} &= \sum_{k=1}^n M_{Oz}(\vec{F}_k). \end{aligned} \right\} \quad (5.6)$$

In the same manner magnitude and direction of the system  $(\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n)$  total moment  $\vec{M}_O$  about center O can be determined

$$M_O = \sqrt{M_{Ox}^2 + M_{Oy}^2 + M_{Oz}^2}, \quad (5.7)$$

$$\cos(\vec{M}_O, \vec{i}) = \frac{M_{Ox}}{M}, \quad \cos(\vec{M}_O, \vec{j}) = \frac{M_{Oy}}{M}, \quad \cos(\vec{M}_O, \vec{k}) = \frac{M_{Oz}}{M}, \quad (5.8)$$

wherein  $M_{Ox}$ ,  $M_{Oy}$ ,  $M_{Oz}$  are given by expressions (5.6).

Any general force system applied to a rigid body can be reduced to an equivalent force-couple system acting at a chosen center O. The force is equal to the total vector. The couple has vector moment equal to the total moment of original force system about chosen center O.

The magnitude and direction of the total vector are independent of center of reduction position. The magnitude and direction of the total moment vary with changing of center of reduction position.

There are several special opportunities of force system reducing depending on magnitude and mutual orientation of  $\vec{R}$  and  $\vec{M}_O$  (Table 5.1).

It is important to note that resultant of a force system and total vector are no equivalent notion although they are determined by the same equation.

Some value is called invariant with respect to the some parameter (argument) if the value does not vary when the parameter changes.

As was stated above the total vector does not change when the center of reduction changes its position. So the total vector is invariant with respect to the center of reduction position.



Total vector, total moment	The simplest equivalent force system
$\vec{F} = 0; \vec{M}_0 = 0$	Balanced force system
$\vec{F} \neq 0; \vec{M}_0 = 0$	The resultant $\vec{R} = \vec{F}$ passing through the chosen center of reduction O.
$\vec{F} \neq 0; \vec{M}_0 \neq 0; \vec{F} \perp \vec{M}_0$	The resultant $\vec{R} = \vec{F}$ with the line of action determined by the equation $\vec{M}_O(\vec{R}) = \vec{M}_O$
$\vec{F} \neq 0; \vec{M}_0 \neq 0; \vec{F} \perp \vec{M}_0$	The wrench
$\vec{F} = 0; \vec{M}_0 \neq 0$	The couple

The total vector magnitude is called the first statics invariant

$$I_1 = |\vec{F}_O| = \left| \sum_{k=1}^n \vec{F}_k \right| = \sqrt{F_{O_x}^2 + F_{O_y}^2 + F_{O_z}^2}. \quad (5.9)$$

The dot product of the total vector and the total moment is called the second statics invariant:

$$I_2 = \vec{F}_O \cdot \vec{M}_O. \quad (5.10)$$

## 5.2. Solution of problems

### Sample problem 5.1

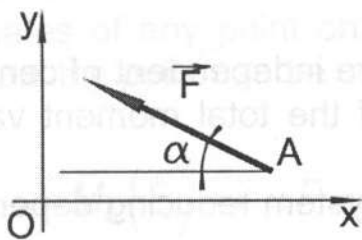


Fig. 5.2

The force  $F=420 \text{ N}$  applied at the point  $A$  is situated in  $XOY$  plane (Fig. 5.2).

Determine the moment of force about point  $O$ , if  $x_A = 0,2 \text{ m}, y_A = 0,3 \text{ m}, \alpha = 30^\circ$ .

### Solution

In a common case

$$\vec{M}_0 = \vec{i}(yF_z - zF_y) + \vec{j}(zF_x - xF_z) + \vec{k}(xF_y - yF_x).$$

Because the force lies in  $XOY$  plane its projection on  $z$ -axis and coordinate  $z$  are equal to zero ( $F_z = 0, z = 0$ ). That's why we can simplify the previous formula

$$\vec{M}_0 = \vec{k}(xF_y - yF_x),$$

where

$$F_x = -F \cos \alpha = -420 \cdot 0,85 = -357 \text{ (N)},$$

$$F_y = F \sin \alpha = 420 \cdot 0,5 = 210 \text{ (N)}.$$

Thus

$$|M_0| = |x \cdot F_y - y \cdot F_x| = |0,2 \cdot 210 - 0,3 \cdot (-357)| = 149,1 \text{ (N} \cdot \text{m)}.$$

### Sample problem 5.2

Total vector  $R$  (0; 3; 4) (N) and total moment  $M_o$  (2; -1; 2) (N·m) of a system applied to a rigid body are known. Determine the simplest system equivalent to the given one.

### Solution

Scalar components of the system total vector and total moment are:

$$F_x = 0, F_y = 3, F_z = 4;$$

$$M_{0x} = 2, M_{0y} = -1, M_{0z} = 2.$$

Magnitudes of total vector and total moment are:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{3^2 + 4^2} = 5 \text{ (N)};$$

$$M_0 = \sqrt{M_{0x}^2 + M_{0y}^2 + M_{0z}^2} = \sqrt{2^2 + 1^2 + 2^2} = 3 \text{ (N} \cdot \text{m)}.$$

Total vector and total moment are not equal to zero. Using the feature that dot product of two nonzero vectors is zero if they are mutually perpendicular we can check the orthogonality of the total vector and total moment (second statics invariant):

$$\vec{F} \cdot \vec{M}_0 = F_x M_{0x} + F_y M_{0y} + F_z M_{0z} = 0 \cdot 2 + 3 \cdot (-1) + 4 \cdot 2 = 5.$$

So the total vector is perpendicular to the total moment about the center.

As far as  $F \neq 0, M_0 \neq 0, \vec{F} \perp \vec{M}_0$  we can make a conclusion that system of forces is reduced to a wrench.

### Sample problem 5.3

Couples of forces with moments  $M_1=M_2=M_3=2 \text{ N} \cdot \text{m}$  are acted on a cube. Determine the magnitude of couple resultant.

### Solution

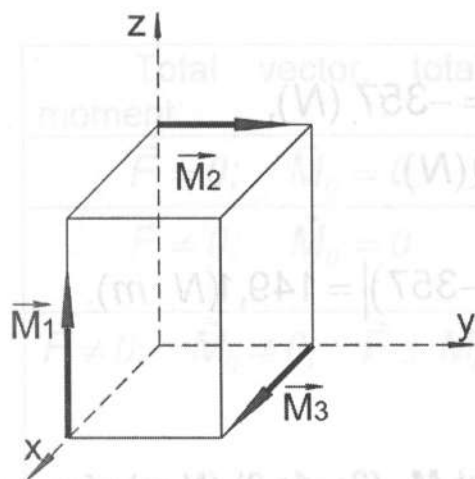


Fig. 5.3

As far as the magnitude of couple resultant is  $M_o = \sqrt{M_x^2 + M_y^2 + M_z^2}$ , we determine scalar components of the system total moment:

$$M_x = \sum M_{xk} = M_3 = 2 \text{ (N}\cdot\text{m)};$$

$$M_y = \sum M_{yk} = M_2 = 2 \text{ (N}\cdot\text{m)};$$

$$M_z = \sum M_{zk} = M_1 = 2 \text{ (N}\cdot\text{m)}.$$

$$\text{Then } M_o = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3} \text{ (N}\cdot\text{m)}.$$

### Sample problem 5.4

Forces  $F_1 = F_2 = F_3 = F_4 = F$  (N) are applied at the following points of a cube: **A** (along the diagonal **AC**), **H** (along the edge **HF**), **B** (along the diagonal **BE**), **D** (along the diagonal **DG**) respectively (Fig. 5.4).

Reduce the system to an equivalent force-couple system at the center **A**. Determine the simplest system equivalent to the given one.

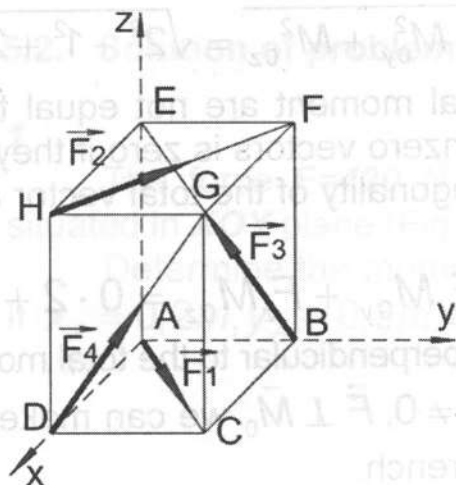


Fig. 5.4

### Solution

Determine the scalar components of the system total vector and the system total moment about the point **A**:

$$\vec{F} = \vec{i}F_x + \vec{j}F_y + \vec{k}F_z;$$

$$\vec{M}_A = \vec{i}M_x + \vec{j}M_y + \vec{k}M_z.$$

Scalar components of the system total vector are:

$$\begin{cases} F_x = \sum F_{xk} = -P_2 \frac{\sqrt{2}}{2} + P \frac{\sqrt{2}}{2} = 0; \\ F_y = \sum F_{yk} = P_2 \frac{\sqrt{2}}{2} + P_1 \frac{\sqrt{2}}{2} - P_3 \frac{\sqrt{2}}{2} + P_4 \frac{\sqrt{2}}{2} = P\sqrt{2} (N); \\ F_z = \sum F_{zk} = P_3 \frac{\sqrt{2}}{2} + P_4 \frac{\sqrt{2}}{2} = P\sqrt{2} (N). \end{cases}$$

The total vector is situated in the plane  $\mathbf{zAy}$ , total vector magnitude is

$$|\vec{F}| = \sqrt{(P\sqrt{2})^2 + (P\sqrt{2})^2} = 2P (N).$$

The scalar components of the system total moment about the point A are:

$$\begin{cases} M_x = \sum M_x (\vec{F}_k) = -P_2 \frac{\sqrt{2}}{2} a + P_3 \frac{\sqrt{2}}{2} a = 0; \\ M_y = \sum M_y (\vec{F}_k) = -P_2 \frac{\sqrt{2}}{2} a - P_4 \frac{\sqrt{2}}{2} a = -Pa\sqrt{2}; \\ M_z = \sum M_z (\vec{F}_k) = P_2 \frac{\sqrt{2}}{2} a + P_4 \frac{\sqrt{2}}{2} a = Pa\sqrt{2}. \end{cases}$$

The total moment magnitude is

$$|\vec{M}_A| = \sqrt{(Pa\sqrt{2})^2 + (Pa\sqrt{2})^2} = 2Pa(N \cdot m).$$

It is obviously that the total moment is situated in the plane  $\mathbf{zAy}$  too.

Using the feature that dot product of two nonzero vectors is zero if they are mutually perpendicular we can check the orthogonality of the total vector and total moment

$$\begin{aligned} \vec{F} \cdot \vec{M}_A &= F_x M_x + F_y M_y + F_z M_z = 0 \cdot 0 + (P\sqrt{2})(-Pa\sqrt{2}) + (P\sqrt{2}) \times \\ &\quad \times (Pa\sqrt{2}) = 0. \end{aligned}$$

So the total vector is perpendicular to the total moment about the center A.

We have thus found that  $\vec{F} \neq 0$ ,  $\vec{M}_A \neq 0$  и  $\vec{F} \perp \vec{M}_0$ , so the original force system can be reduced further to a resultant. The resultant is parallel to the total vector and is at a distance  $d$  of the total vector

$$d = \frac{M_A}{R} = \frac{Pa\sqrt{2}}{P\sqrt{2}} = a.$$

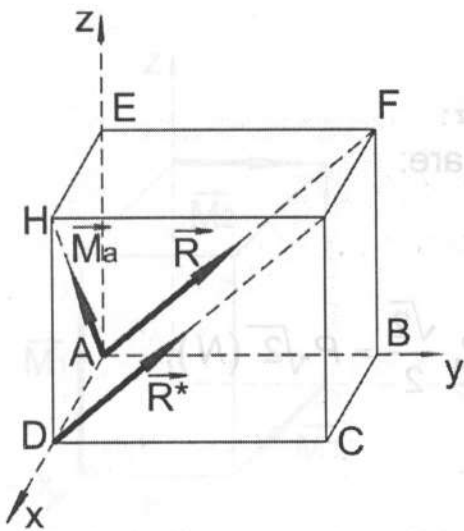


Fig. 5.5

The procedure of drawing of the resultant is the following: the plane  $\Pi$  perpendicular to the total moment must be drawn at the point A, in the plane  $\Pi$  total vector must be moved to another line of action that is away from the initial center A at the distance  $a$ , the direction of moving is such that the vector of the system total moment and the vector moment of resultant  $\vec{R}$  with respect to the initial center of reduction have the same sense. We obtain that the resultant is applied at the point D and directed along the diagonal  $DG$  (Fig. 5.5).

### Sample problem 5.5

Force system is reduced to total vector  $F(0; 3; 4)$  and total moment  $M_0(0; 4; 0)$  about an origin. Determine angle  $\gamma$  between these vectors.

### Solution

Dot product of vectors can be represented in the following view:

$$\vec{F} \cdot \vec{M}_0 = F \cdot M_0 \cdot \cos \gamma.$$

At the same time:

$$\vec{F} \cdot \vec{M}_0 = F_x M_x + F_y M_y + F_z M_z.$$

Then

$$\cos \gamma = \frac{F_x M_x + F_y M_y + F_z M_z}{F \cdot M_0}.$$

Let's find magnitudes of total vector and total moment:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{3^2 + 4^2} = 5 \text{ (N)};$$

$$M_0 = \sqrt{M_x^2 + M_y^2 + M_z^2} = \sqrt{4^2} = 4 \text{ (N} \cdot \text{m)}.$$

Finally we have got:

$$\cos \gamma = \frac{0 \cdot 0 + 3 \cdot 4 + 4 \cdot 0}{5 \cdot 4} = \frac{3}{5},$$

$$\gamma = \arccos(0,6) = 53,13^\circ.$$



### Sample problem 5.6

Forces  $P_1=P_2=P_3=P$  act along the edges of parallelepiped. Length of parallelepiped sides are  $a, b, c$ . Determine projections of total vector and total moment on axes.

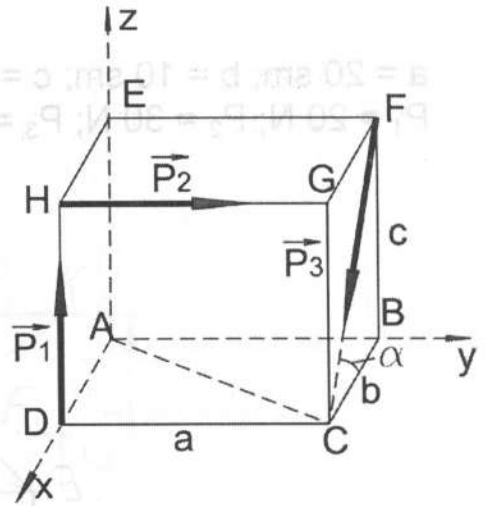


Fig. 5.6

### Solution

According to geometrical considerations:

$$\sin \alpha = \frac{c}{\sqrt{b^2 + c^2}}; \quad \cos \alpha = \frac{b}{\sqrt{b^2 + c^2}}.$$

Determine scalar components of the system total vector:

$$F_x = \sum P_{xk} = P_3 \cos \alpha = P \frac{b}{\sqrt{b^2 + c^2}};$$

$$F_y = \sum P_{yk} = P_2 = P;$$

$$F_z = \sum P_{zk} = P_1 - P_3 \sin \alpha = P \left( 1 - \frac{c}{\sqrt{b^2 + c^2}} \right).$$

Determine scalar components of the system total moment:

$$M_x = \sum M_x(P_k) = -P_2 c - P_3 \sin \alpha \cdot a = P(c - a \sin \alpha);$$

$$M_y = \sum M_y(P_k) = -P_1 b + P_3 \cos \alpha \cdot c = P(c \cdot \cos \alpha - b);$$

$$M_z = \sum M_z(P_k) = P_2 b - P_3 \cos \alpha \cdot a = P(b - a \cos \alpha).$$

### 5.3. Solving problems on your own

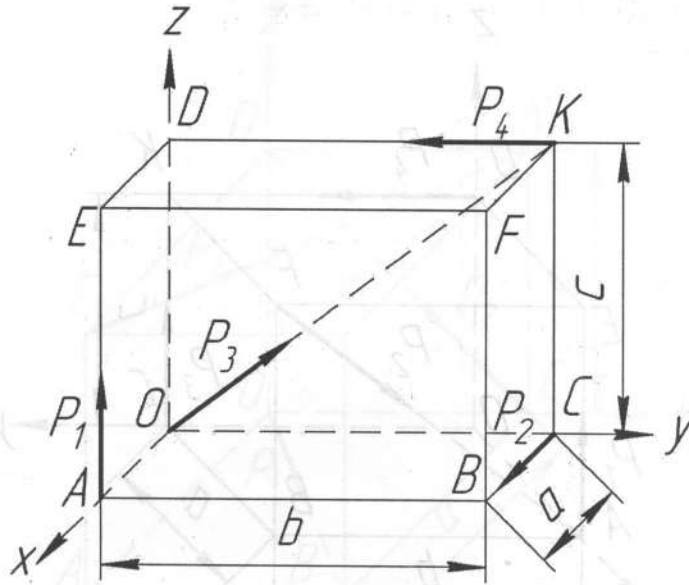
In the following problems determine total vector and total moment of given force system about center O. Ascertain the simplest system equivalent to the given one. Dimensions of parallelepiped and magnitudes of forces are given.



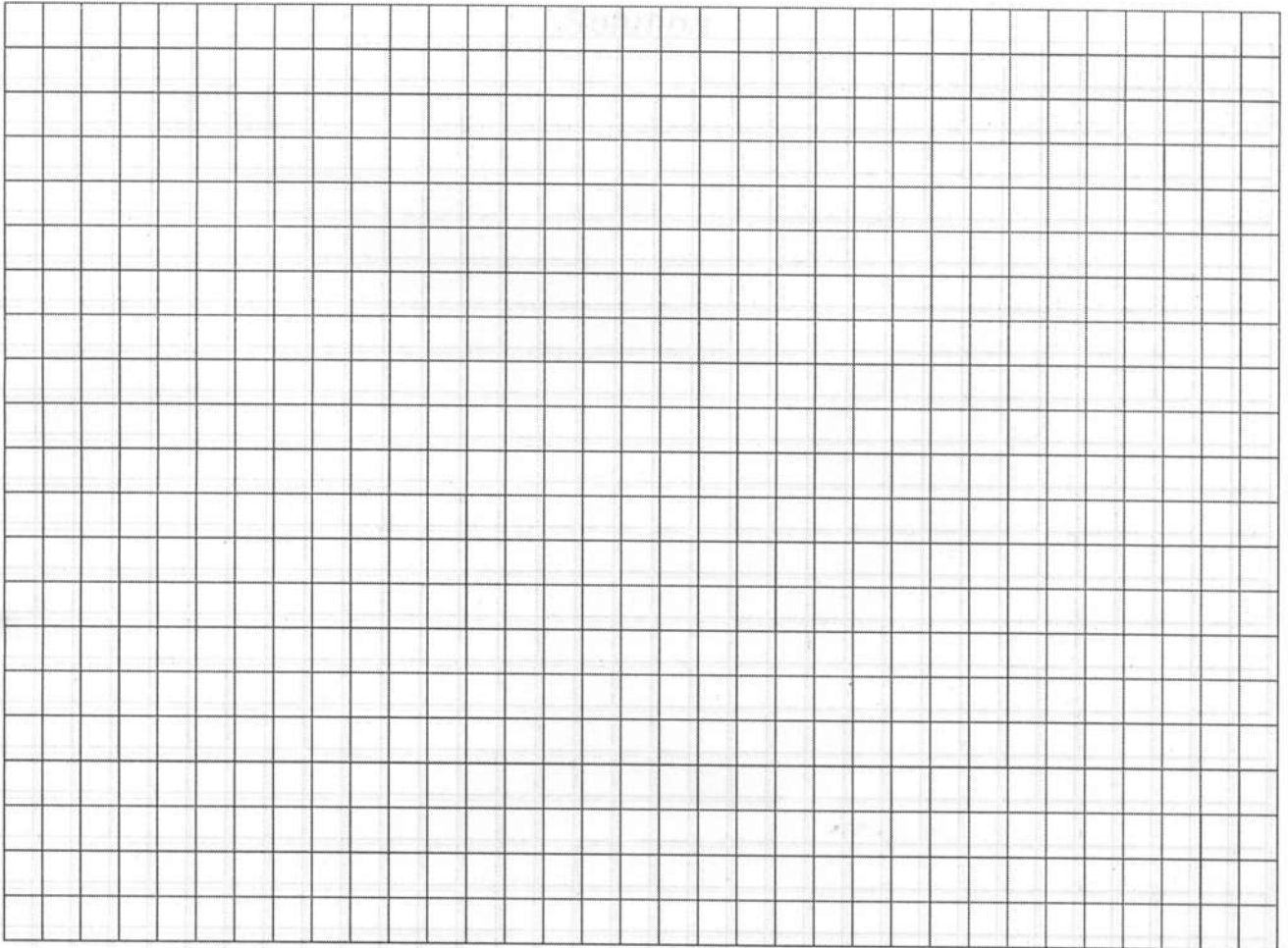
**Problem 5.2**

$a = 20 \text{ sm}; b = 40 \text{ sm}; c = 30 \text{ sm}.$

$P_1 = 8 \text{ N}; P_2 = 12 \text{ N}; P_3 = 20 \text{ N}; P_4 = 16 \text{ N}.$



**Solution**

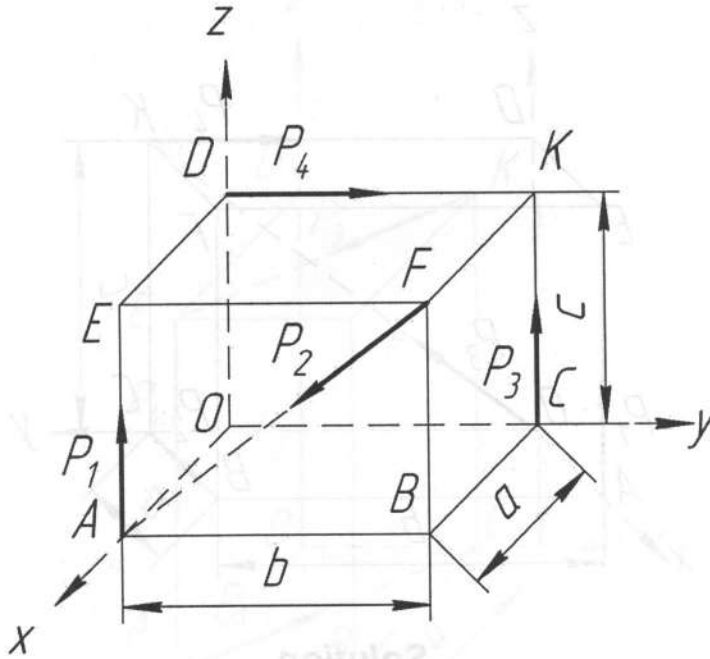


**Problem 5.3**

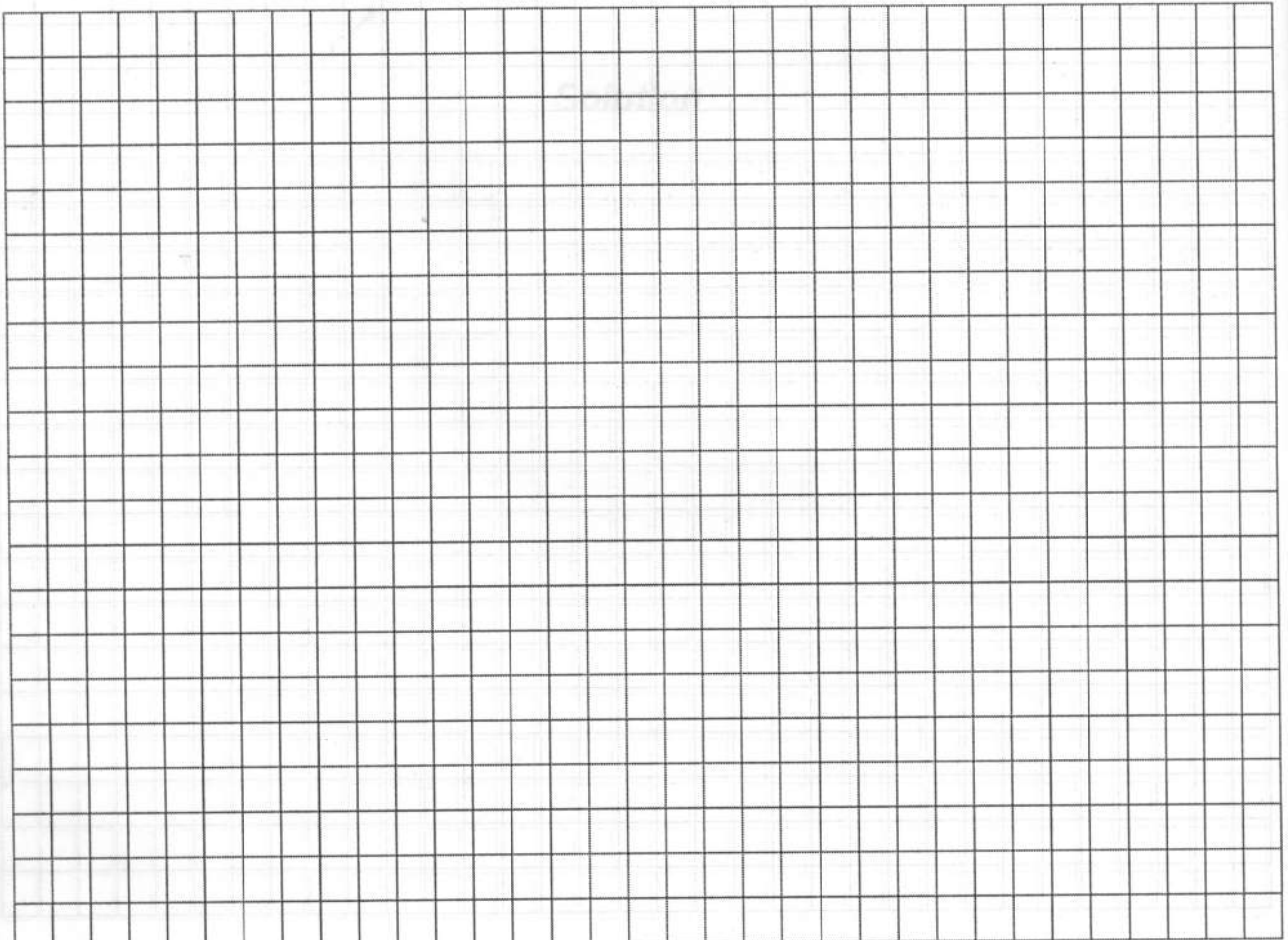
Problem 5.3

$a = 8 \text{ cm}$ ;  $b = 8 \text{ cm}$ ;  $c = 6 \text{ cm}$ .

$P_1 = 6 \text{ N}$ ;  $P_2 = 20 \text{ N}$ ;  $P_3 = 10 \text{ N}$ ;  $P_4 = 8 \text{ N}$ .

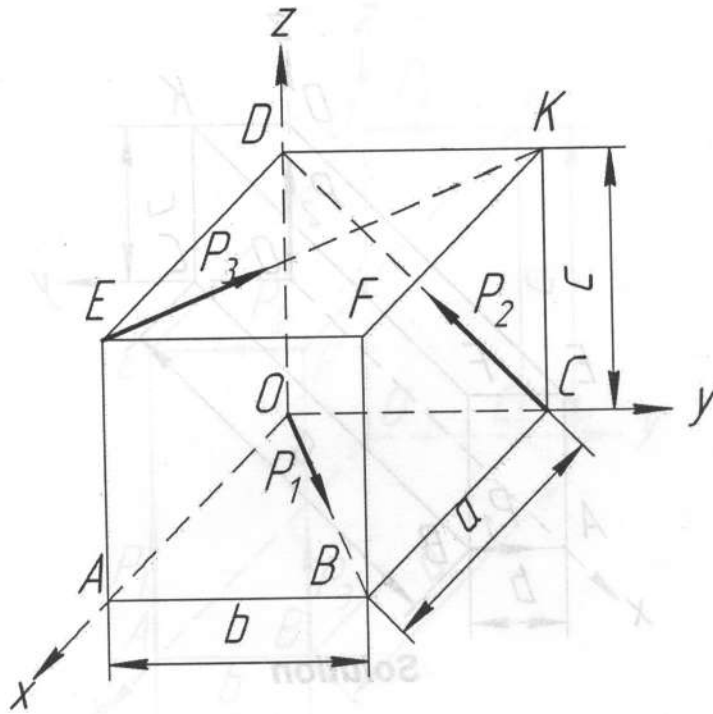


**Solution**

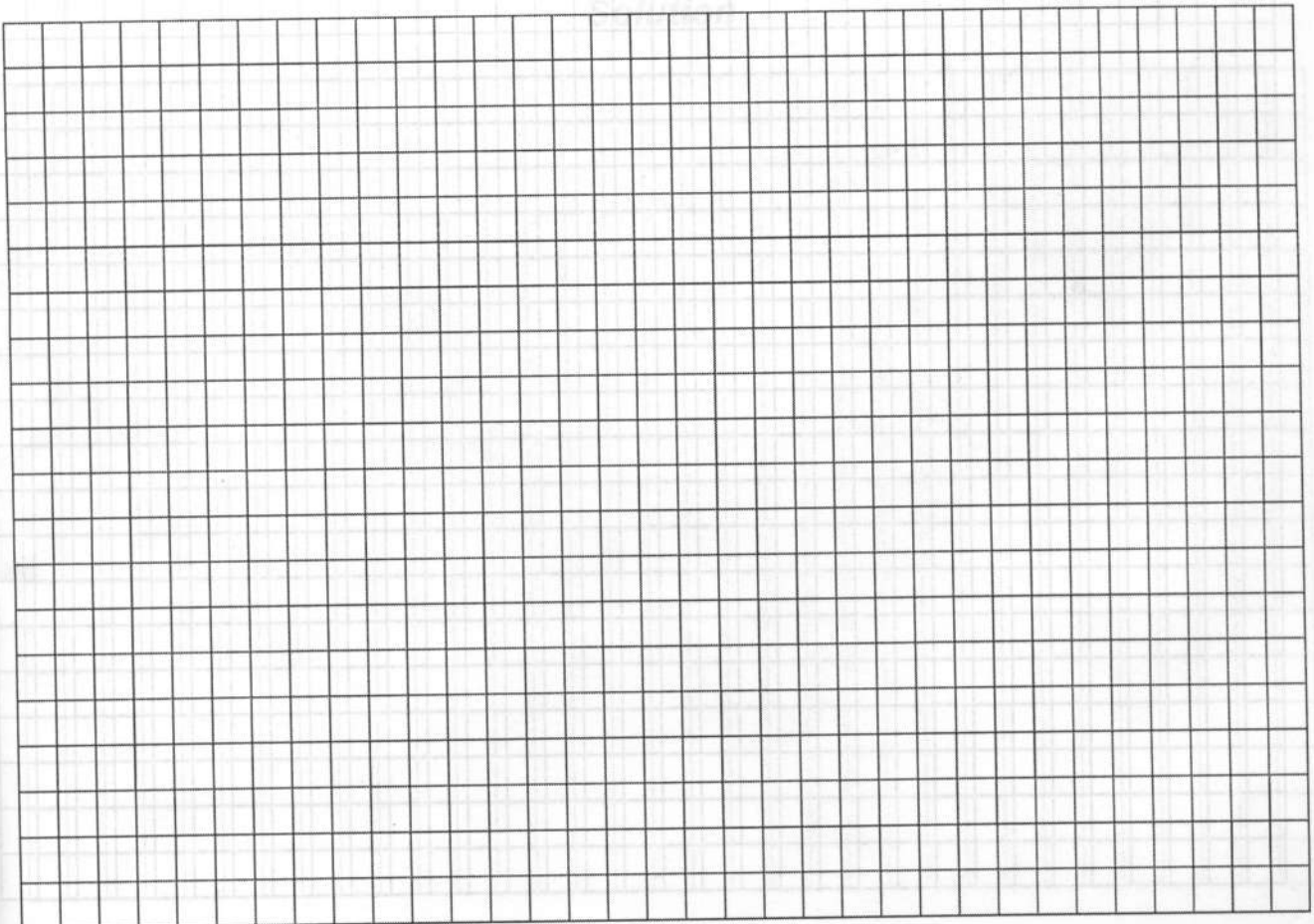


**Problem 5.4**

$a = 40 \text{ sm}$ ;  $b = 20 \text{ sm}$ ;  $c = 20 \text{ sm}$ .  
 $P_1 = 8 \text{ N}$ ;  $P_2 = 8 \text{ N}$ ;  $P_3 = 8 \text{ N}$ .



**Solution**

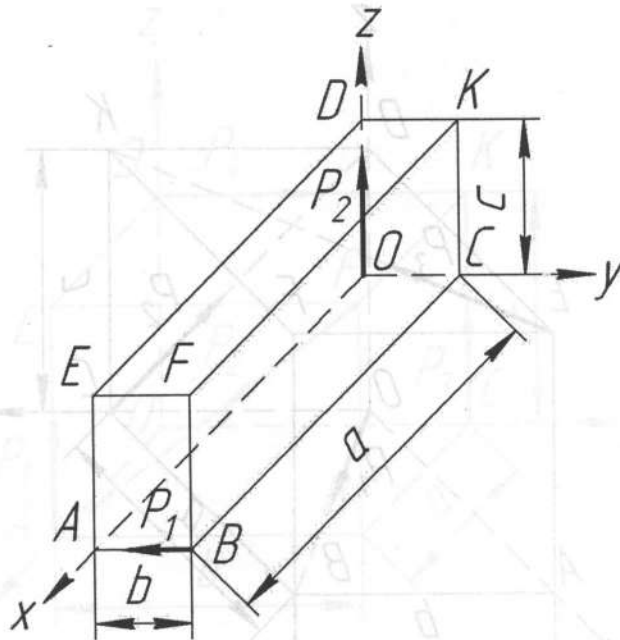




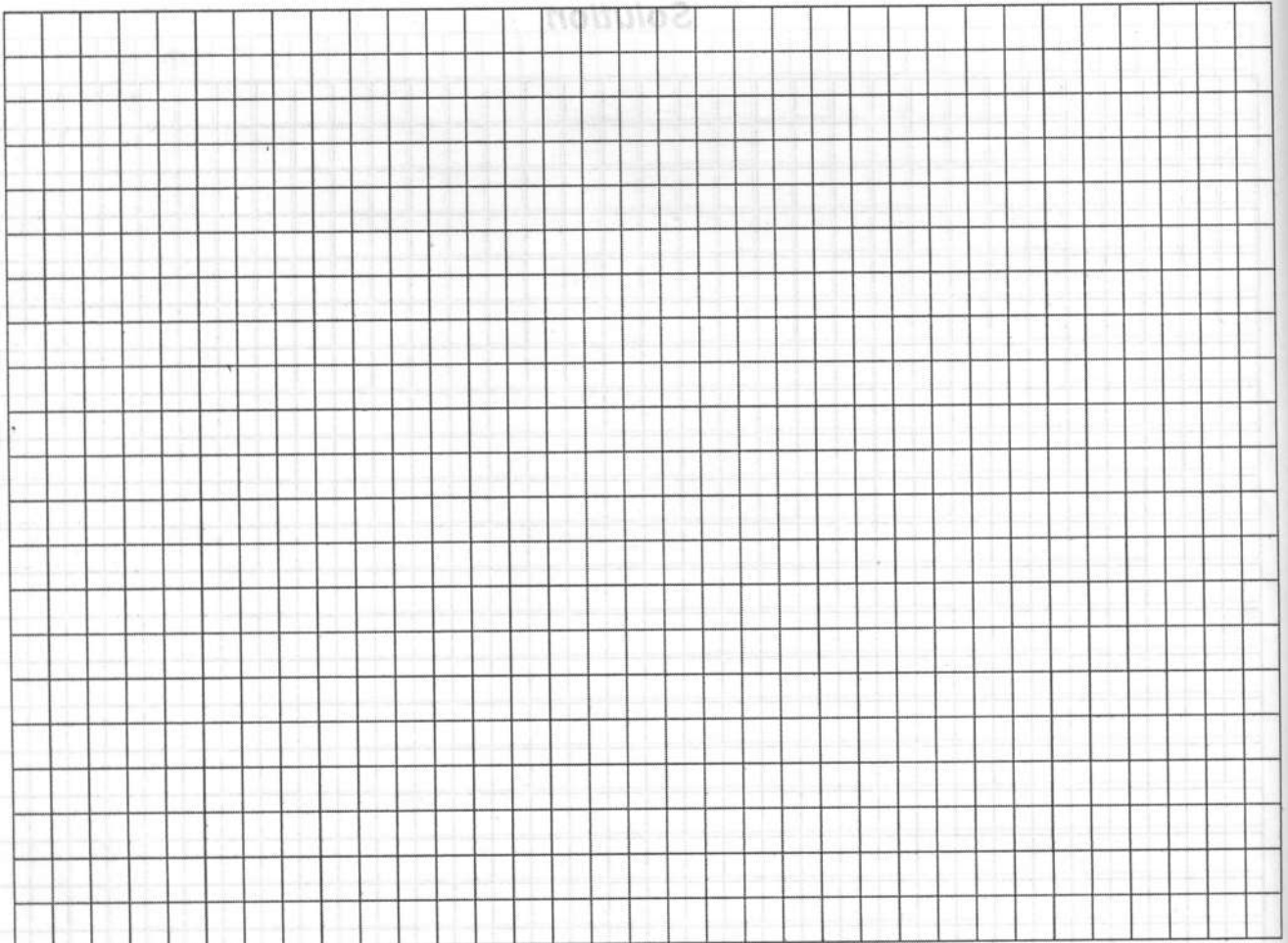
### Problem 5.5

$a = 40 \text{ cm}$ ;  $b = 5 \text{ cm}$ ;  $c = 8 \text{ cm}$ .

$P_1 = 40 \text{ N}$ ;  $P_2 = 30 \text{ N}$ .



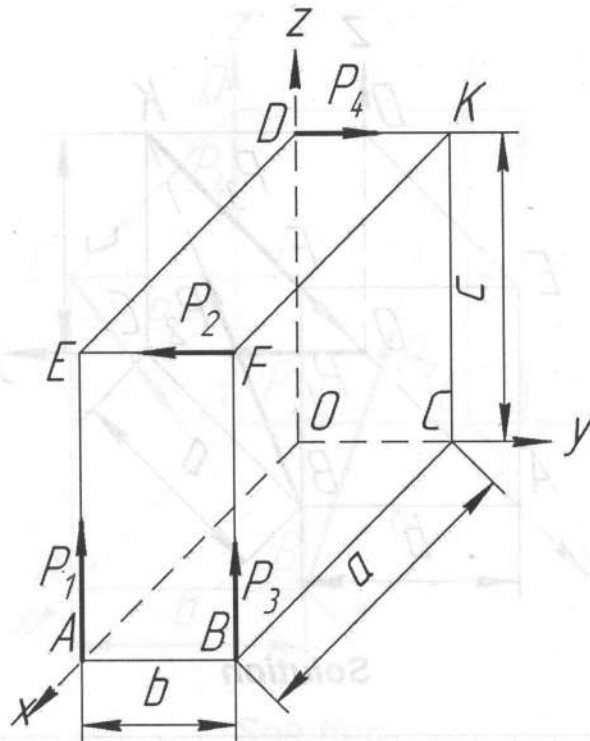
**Solution**



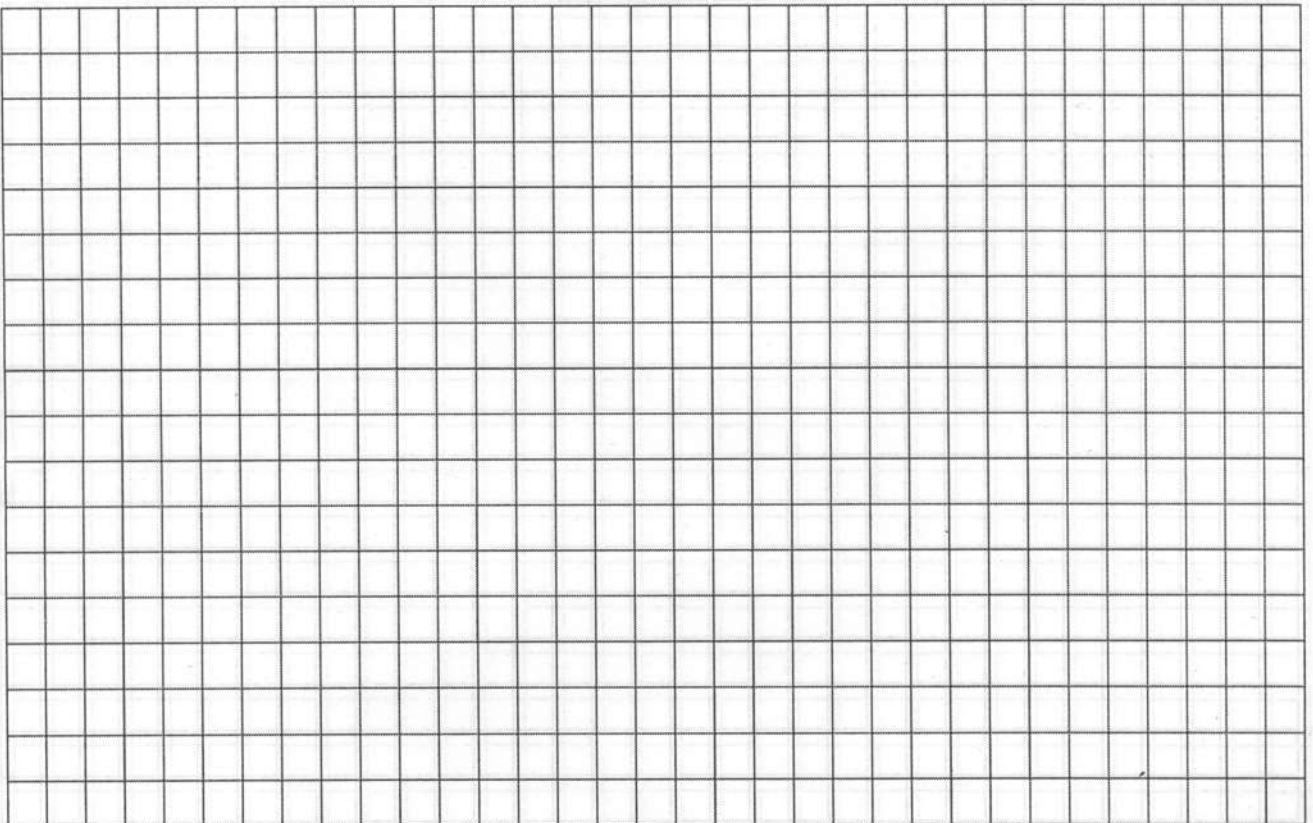
**Problem 5.6**

$a = 80 \text{ sm}$ ;  $b = 20 \text{ sm}$ ;  $c = 40 \text{ sm}$ .

$P_1 = 15 \text{ N}$ ;  $P_2 = 10 \text{ N}$ ;  $P_3 = 15 \text{ N}$ ;  $P_4 = 10 \text{ N}$ .



**Solution**

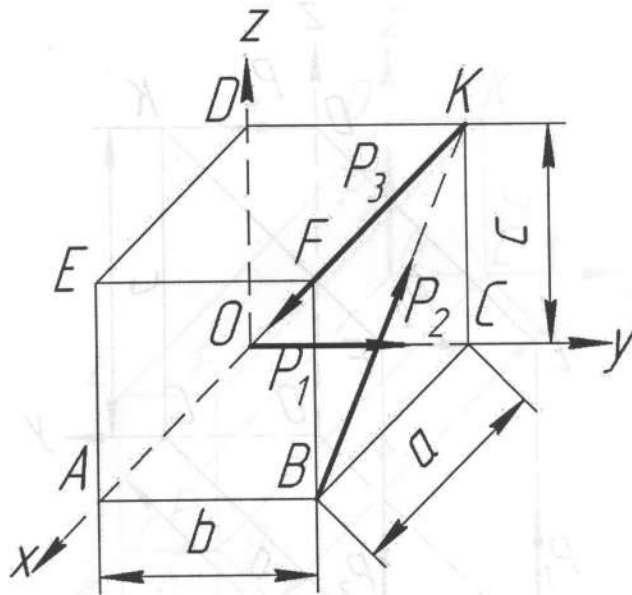


**Problem 5.7**

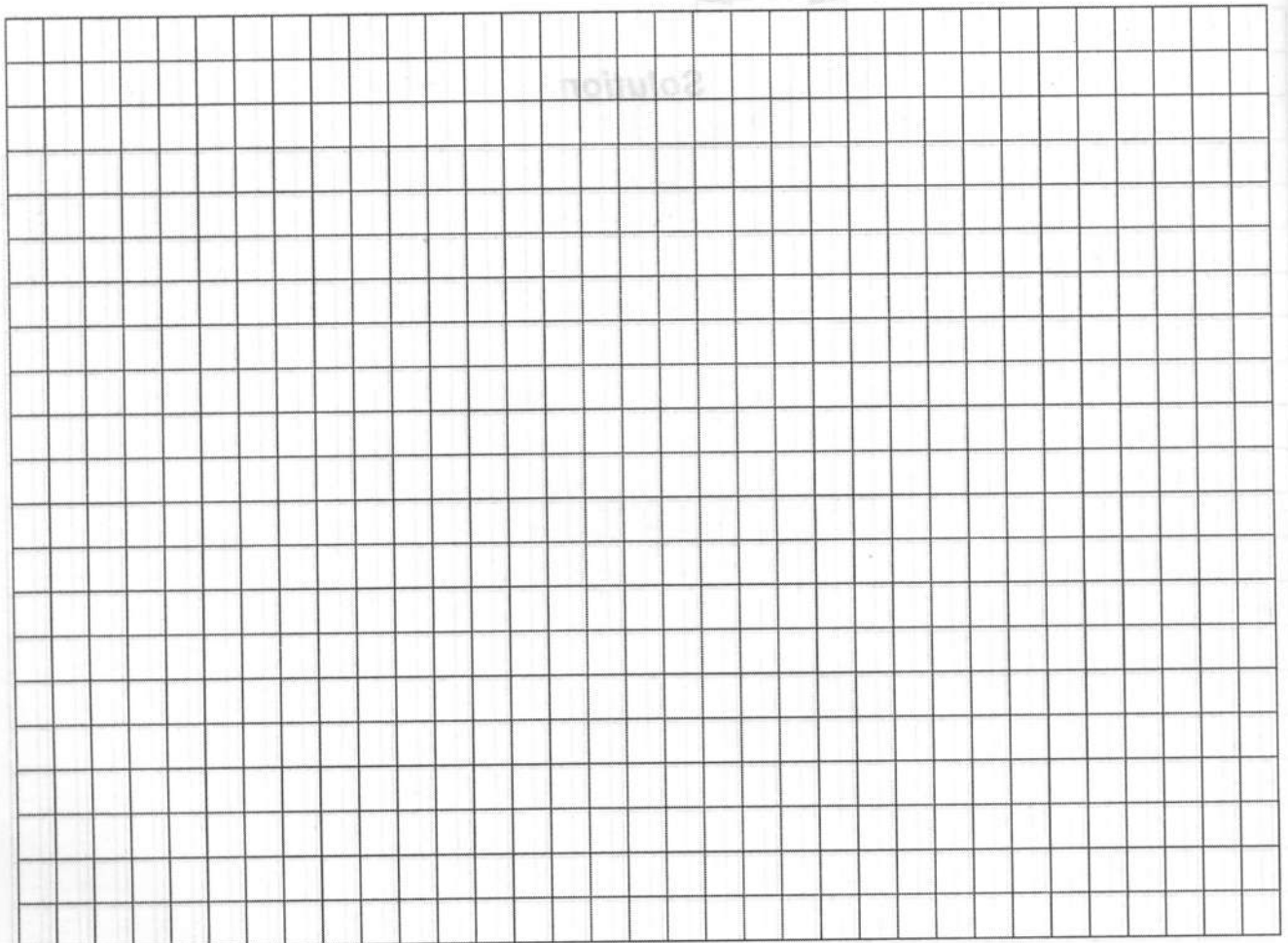
Problem 5.8

$a = 60 \text{ cm}; b = 30 \text{ cm}; c = 30 \text{ cm}.$

$P_1 = 6 \text{ N}; P_2 = 10 \text{ N}; P_3 = 20 \text{ N}.$



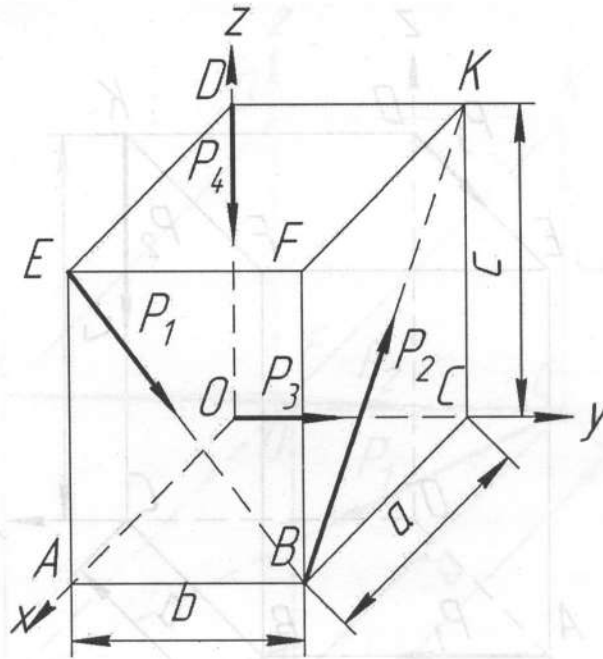
**Solution**



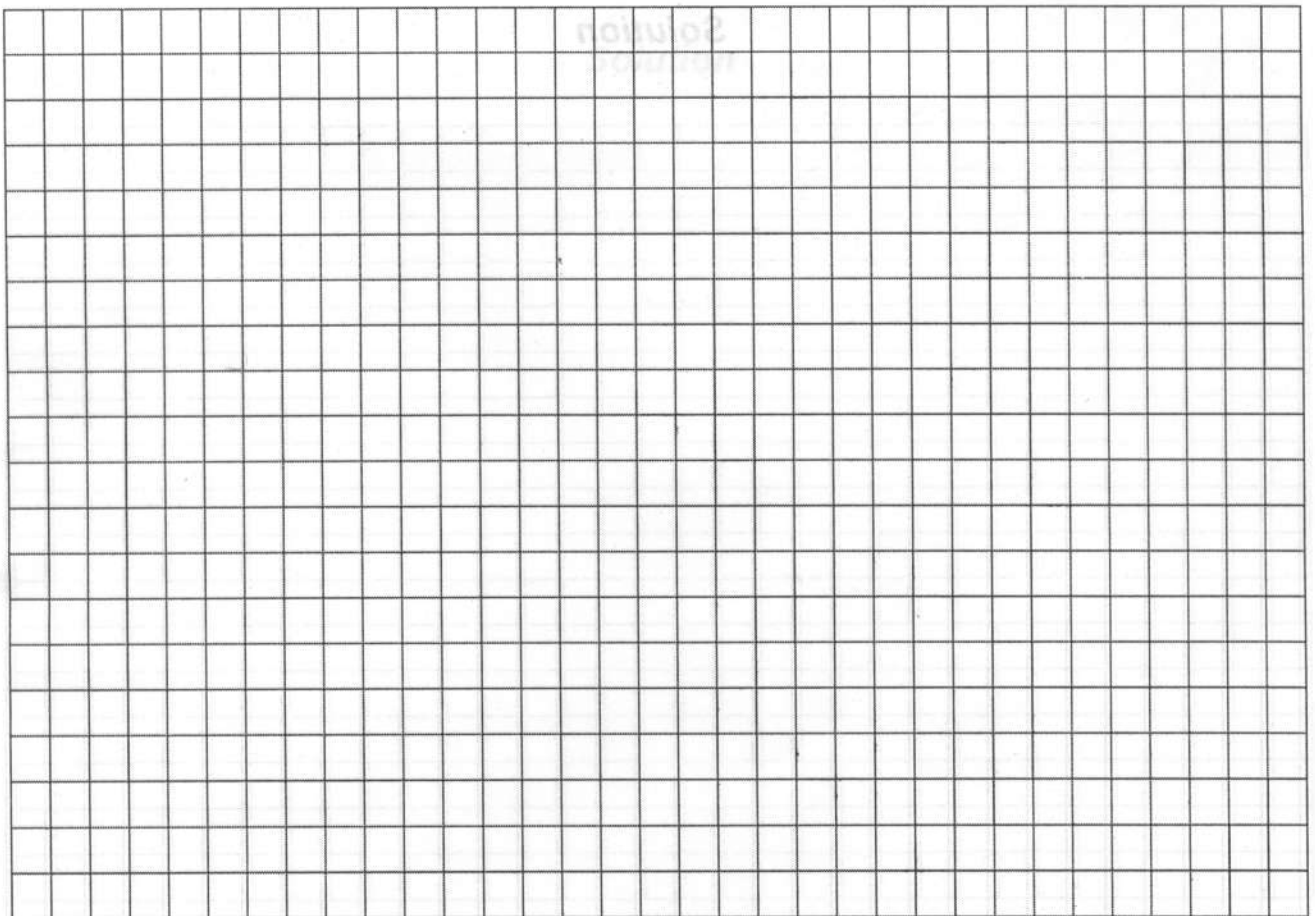
**Problem 5.8**

$a = 30 \text{ sm}; b = 15 \text{ sm}; c = 20 \text{ sm}.$

$P_1 = 30 \text{ N}; P_2 = 40 \text{ N}; P_3 = 10 \text{ N}; P_4 = 32 \text{ N}.$



**Solution**

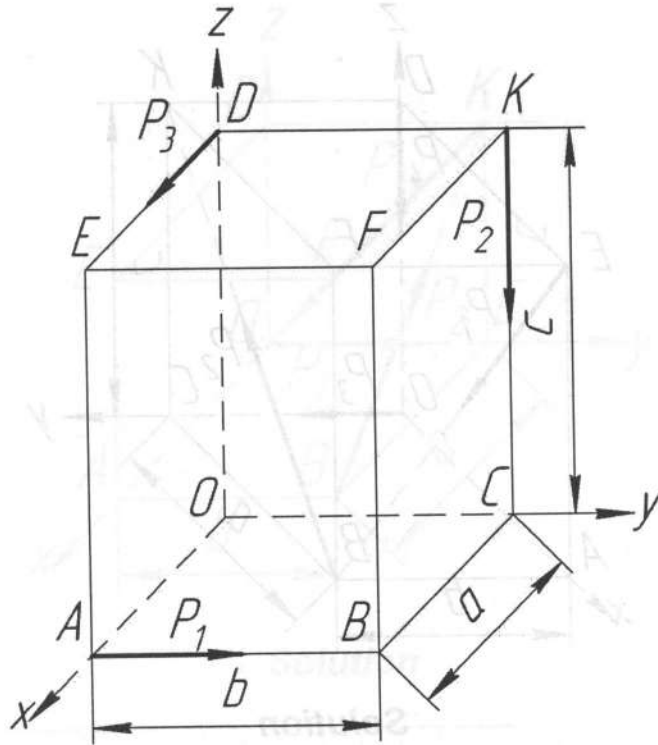


**Problem 5.9**

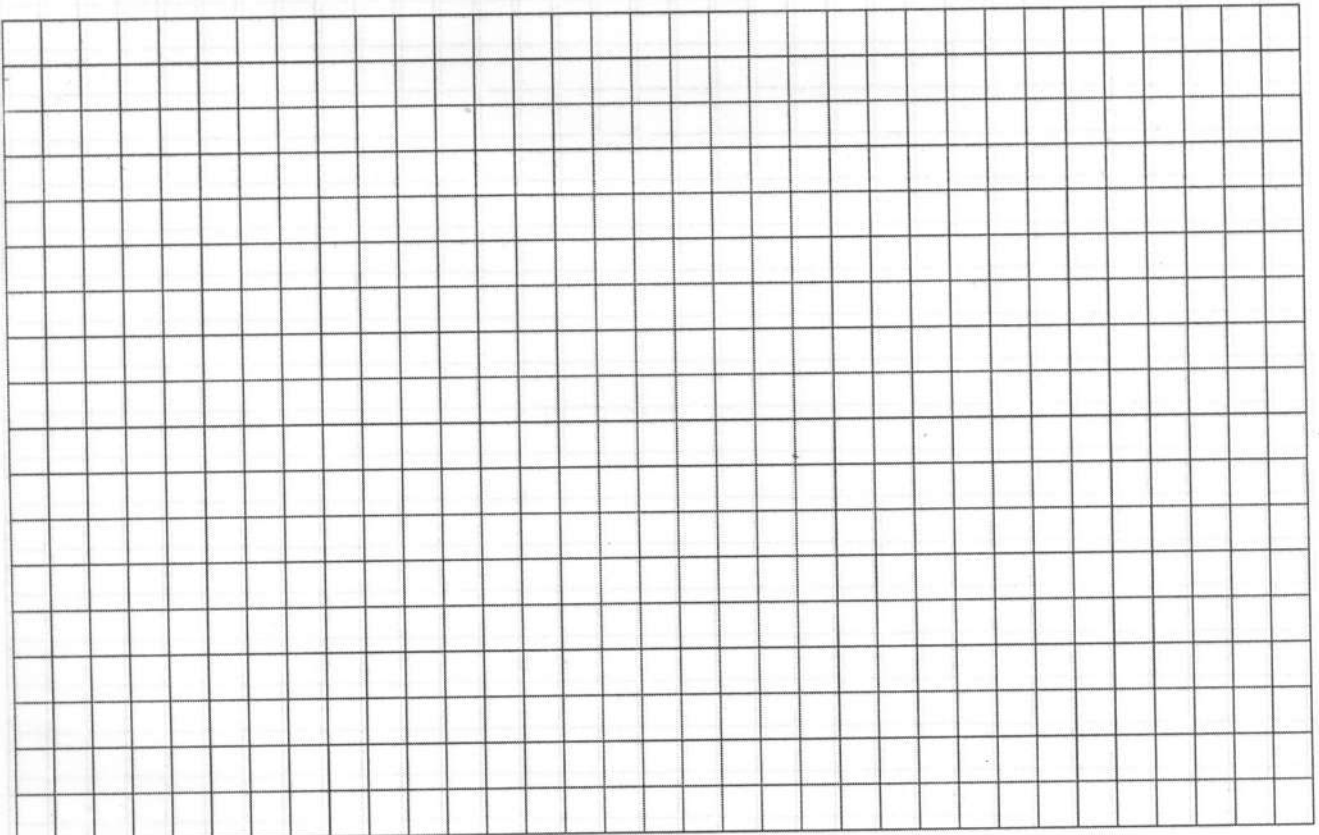
Problem 5.8

$a = 20 \text{ cm}$ ;  $b = 15 \text{ cm}$ ;  $c = 20 \text{ cm}$ .

$P_1 = 40 \text{ N}$ ;  $P_2 = 20 \text{ N}$ ;  $P_3 = 16 \text{ N}$ .



**Solution**

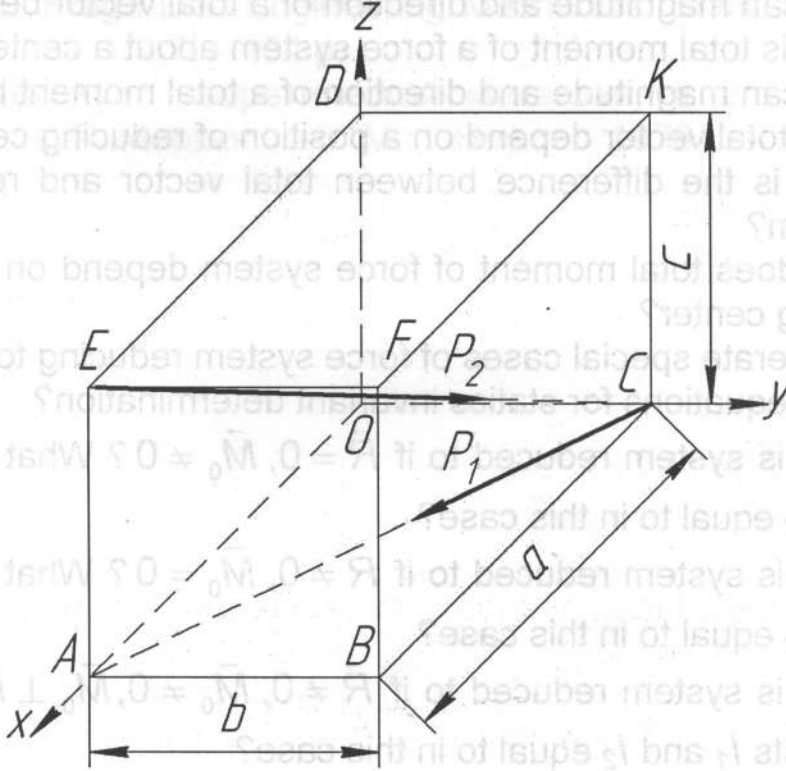




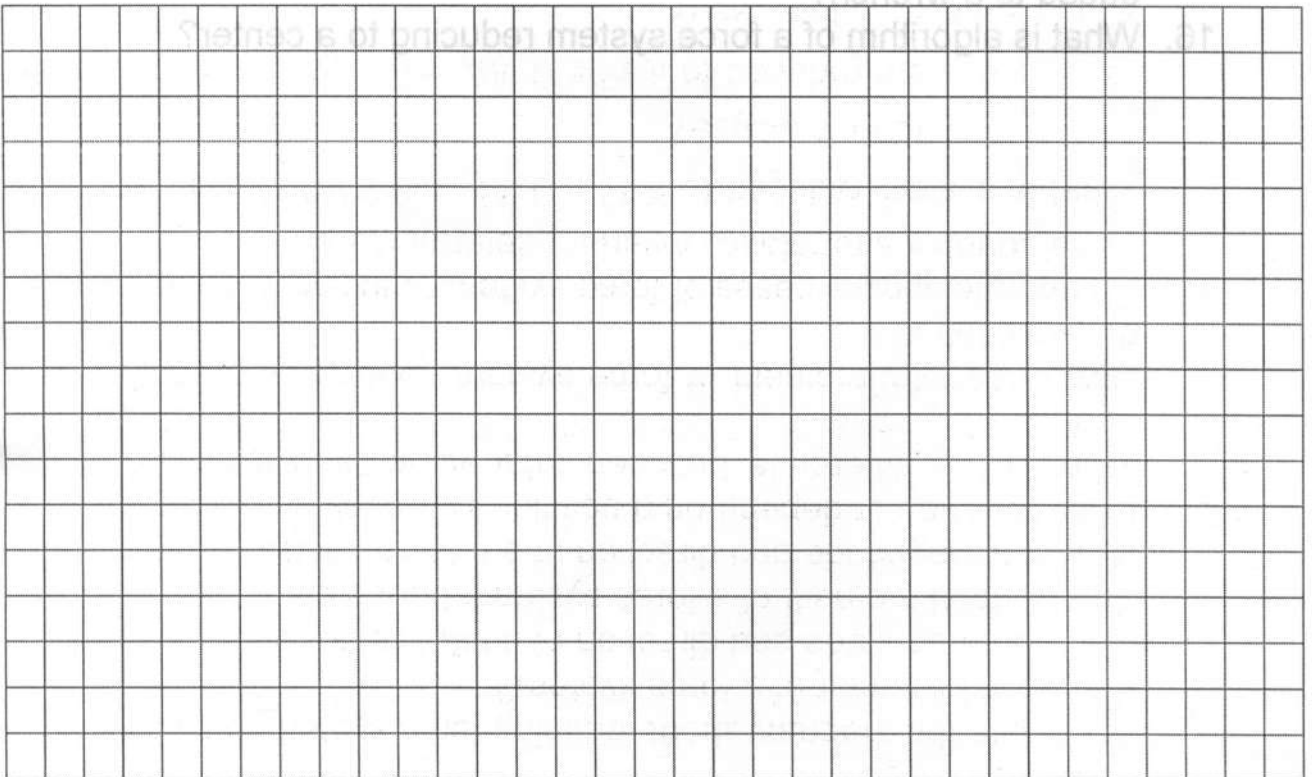
**Problem 5.10**

$a = 40 \text{ cm}$ ;  $b = 15 \text{ cm}$ ;  $c = 15 \text{ cm}$ .

$P_1 = 40 \text{ N}$ ;  $P_2 = 20 \text{ N}$ .



**Solution**



1. Formulate the theorem about reducing force system to a center.
2. What is total vector of a force system?
3. How can magnitude and direction of a total vector be determined?
4. What is total moment of a force system about a center?
5. How can magnitude and direction of a total moment be determined?
6. Does total vector depend on a position of reducing center?
7. What is the difference between total vector and resultant of force system?
8. How does total moment of force system depend on a position of reducing center?
9. Enumerate special cases of force system reducing to a center.
10. Write equations for statics invariant determination?
11. What is system reduced to if  $\vec{R} = 0, \vec{M}_0 \neq 0$ ? What are invariants  $I_1$  and  $I_2$  equal to in this case?
12. What is system reduced to if  $\vec{R} \neq 0, \vec{M}_0 = 0$ ? What are invariants  $I_1$  and  $I_2$  equal to in this case?
13. What is system reduced to if  $\vec{R} \neq 0, \vec{M}_0 \neq 0, \vec{M}_0 \perp \vec{R}$ ? What are invariants  $I_1$  and  $I_2$  equal to in this case?
14. In what case is a force system reduced to a wrench?
15. What are statics invariants  $I_1$  and  $I_2$  equal to if system can be reduced to a wrench?
16. What is algorithm of a force system reducing to a center?

## 6. BIBLIOGRAPHY

1. Beer F. P., Johnston E. R., Eisenberg E. R. and G. H. Staab, Vector Mechanics for Engineers: Statics. – New-York: McGraw-Hill, 2004. – 621 p.
2. Meriam J.L., Krage L.G. Engineering Mechanics: Statics. New-York: John Willey and sons, 1987. – 454 p.
3. Бутенин Н.В. Курс теоретической механики: в 2 т. / Н.В. Бутенин, Я.Л. Лунц, Д.Р. Меркин. – М.: Наука, 1976. – Т. 1. – 272 с.

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