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I. V. BRYSINA, V. O. MAKARICHEV

National Aerospace University «Kharkov Aviation Institute», Kharkov, Ukraine

GENERALIZED ATOMIC WAVELETS

The problem of big data sets processing is considered. Efficiency of algorithms depends mainly on the appropriate mathematical tools. Now there exists a wide variety of different constructive tools for information analysis. Atomic functions are one of them. Theory of atomic functions was developed by V. A. Rvachev and members of his scientific school. A number of results, which prove that application of atomic functions is reasonable, were obtained. In particular, atomic functions are infinitely differentiable. This property is quite useful for smooth data processing (for example, color photos). Also, these functions have a local support, which allows to decrease complexity of numerical algorithms. Besides, it was shown that spaces of atomic functions have good approximation properties, which can reduce the error of computations. Hence, application of atomic functions is perspective. There are different ways to use atomic functions and their generalizations in practice. One such approach is a construction and application of wavelet-like structures. In this paper, generalized atomic wavelets are constructed using generalized Fup-functions and formulas for their evaluation are obtained. Also, the main properties of generalized atomic wavelets are presented. In addition, it is shown that these wavelets are smooth functions with a local support and have good approximation properties. Furthermore, the set of generalized atomic wavelets is a wide class of functions with flexible parameters that can be chosen according to specific needs. This means that the constructive analysis tool, which is introduced in this paper, gives researches and developers of algorithms flexible possibilities of adapting to the specifics of various problems. In addition, the problem of representation of data using generalized atomic wavelets is considered. Generalized atomic wavelets expansion of data is introduced. Such an expansion is a sum of trend or principal value function and several functions that describe the corresponding frequencies. The remainder term, which is an error of approximation of data by generalized atomic wavelets, is small. To estimate its value the inequalities from the previous papers of V. A. Rvachev, V. O. Makarichev and I. V. Brysina can be used.

Key words: *data processing, wavelets, atomic functions, V. A. Rvachev up-function, atomic wavelets, generalized atomic wavelet expansion.*

Introduction

The past few decades have been marked by the rapidly accelerating development of information technology. A huge number of opportunities, which seemed to be completely inaccessible earlier, has been appeared. At the same time new problems have been arisen. The volume of information has been increased and total expenses for its processing has been increased significantly. It is obvious that development of efficient algorithms is the basis for the successful application of new technology. We stress that the complexity of the algorithm and the accuracy of the results are key indicators of quality. These indicators are often highly dependent on the used mathematical tools.

In the last half of the twentieth century some new approximation tools such as wavelets and atomic functions were constructed. The reason was the inability to solve different engineering problems using classic mathematical approaches.

There are many different requirements that can be imposed on systems of functions. However, the most important are the order of smoothness, compactness of the function support (we say that the set

$$\operatorname{supp} f(x) = \overline{\{x :\ f(x) \neq 0\}}$$

is called a support of the function $f(x)$) and good approximation properties. The first one is important for the case of smooth data processing (for example, color photos). Further, if we use the system of locally supported functions, then it is possible to reduce time and memory complexity of the numerical algorithm. Finally, precision of the data representation and correctness of the results mainly depends on approximation properties. This implies that combination of the above features is necessary for the efficient algorithms of big data sets processing. In this paper we construct the new system of wavelets that have all these convenient properties.

There are many different definitions of the term “wavelet”. In general, wavelet is a function of zero mean that is defined on the real line and decreases sufficiently rapidly at infinity [1]. Various systems of wavelet functions are used in computer graphics [2 - 4], digital data processing and analysis [5 - 9], economic [10, 11] and so on.

Wavelets can be constructed in different ways. One of the methods is the application of solutions of so-called refinement equations

$$y(x) = \sum_k c_k \cdot y(a \cdot x - k) .$$

Note that the equation of this form is a partial case of the linear functional differential equation with a constant coefficients and linear transformations of the argument

$$y^{(n)} + a_1 \cdot y^{(n-1)} + \ldots + a_n \cdot y = \sum_k c_k \cdot y(a \cdot x + b_k) . \quad (1)$$

Solutions with a compact support of this equation are called atomic function [12, 13]. Necessary and sufficient conditions of existence of compactly supported solutions of the equation (1) were obtained by V.A. Rvachev in [12]. For this reason, the authors of the current paper consider it necessary to note that some fundamental principles of wavelet theory were introduced by V.A. Rvachev (see also [14]).

One of the most famous atomic functions is well-known V.A. Rvachev function

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin\left(t \cdot 2^{-k}\right)}{t \cdot 2^{-k}} dt .$$

This function is a solution with a support [-1,1] of the equation

$$y'(x) = 2(y(2x+1) - y(2x-1)) .$$

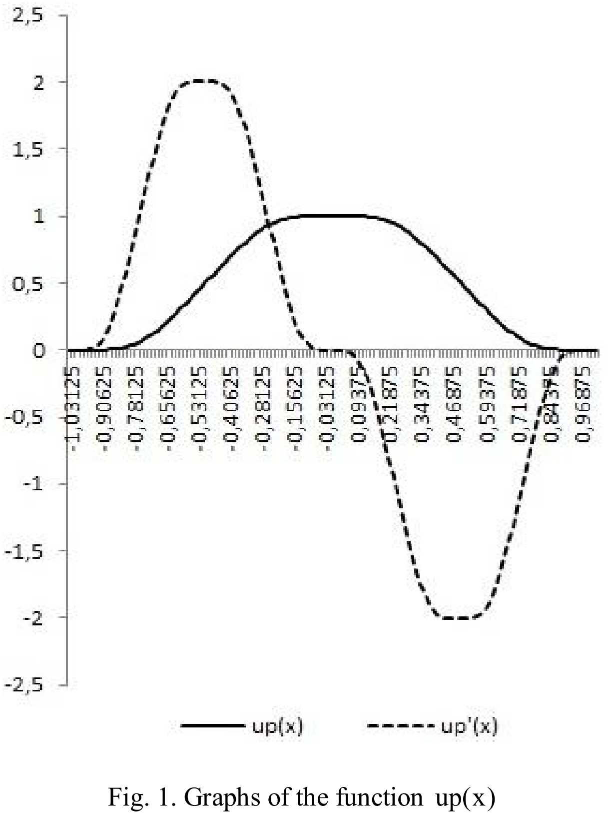

Fig. 1. Graphs of the function $up(x)$ and its derivative

Also, $up(x)$ is infinitely differentiable. Moreover, it has good approximation properties [12, 13, 15]. Besides, there is a basis of spaces

$$UP_n = \left\{ f(x): \; f(x) = \sum_k c_k \cdot up\left(x - \frac{k}{2^n} \right) \right\}$$

that consists of shifts of the locally supported atomic function

$$Fup_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \left(\frac{\sin\left(t2^{-n-1}\right)}{t2^{-n-1}} \right)^n F\left(\frac{t}{2^n}\right) dt ,$$

where $F(t)$ is the Fourier transform of $up(x)$.

Atomic functions $up(x)$ and $Fup_n(x)$ have a combination of convenient properties. Hence, they have a variety of applications to solution of real world problems [16 - 19]. Also, these functions were used in wavelet theory [20 - 22].

Some of the results of V.A. Rvachev on the approximation properties of the function $up(x)$ were generalized for the case of atomic function $mup_s(x)$, which is a solution of the equation

$$y'(x) = 2\sum_{k=1}^{s} \left(y(2sx + 2s - 2k + 1) - y(2sx - 2k + 1) \right),$$

where $s = 2,3,4,\ldots$.

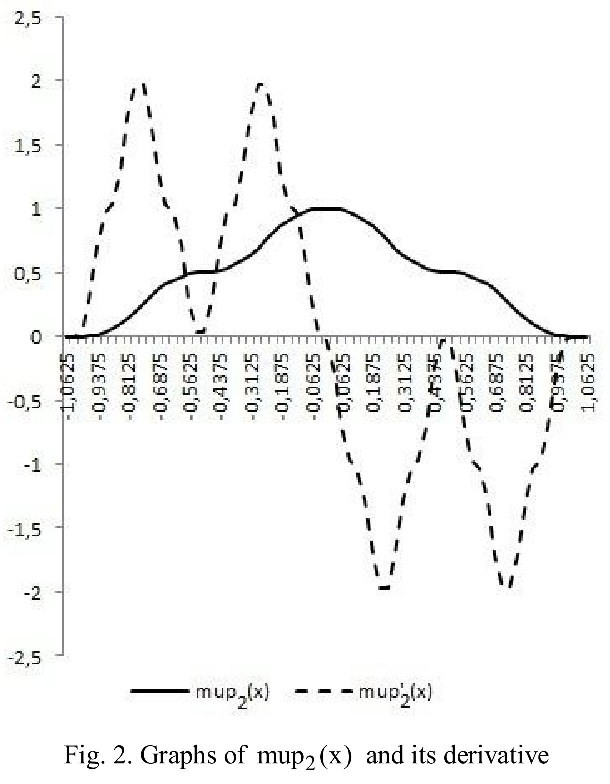

Fig. 2. Graphs of $mup_2(x)$ and its derivative

It was shown in [23] that spaces of linear combinations of mup_s-function shifts are asymptotically extremal for approximation of some classes of differentiable functions. Furthermore, the locally supported basis, which consists of shifts of the atomic function

$$\mathrm{Fup}_{s,n}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{itx}\left(\frac{\sin\frac{t}{2(2s)^n}}{\frac{t}{2(2s)^n}}\right)^n F_s\left(\frac{t}{(2s)^n}\right)dt,$$

where $F_s(t)$ is the Fourier transform of the function $\mathrm{mup}_s(x)$, was constructed.

There are different ways to use atomic functions in practice. Application of wavelet systems, which are constructed using atomic functions, is one of the approaches to solve real world problems. For this purpose atomic wavelets were introduced in [24 - 25]. In addition, it was shown in [27] that these wavelet systems can be effectively used in lossy image compression.

In [28, 29], a generalized Fup-functions were introduced. The function

$$f_{N,m}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{itx}\left(\frac{\sin(t/N)}{t/N}\right)^{m+1}F(t/N)dt,$$

where F(t) is the Fourier transform of the mother function $f(x)\in L_2(\mathbb{R})$ such that $\mathrm{supp}\, f(x)=[-1,1]$, $f(-x)=f(x)$, $f(x)\geq 0$ for any $x\in[-1,1]$ and $\int_{-\infty}^{\infty}f(x)dx=1$, $N\neq 0$ and $m\in\mathbb{N}$ is called a generalized Fup-function. Approximation properties of generalized Fup-functions were investigated in [28]. It was shown that spaces of shifts of these functions are asymptotically extremal for approximation of some classes of smooth functions. So, generalized Fup-functions have main advantages of the atomic functions $\mathrm{up}(x)$, $\mathrm{Fup}_n(x)$, $\mathrm{mup}_s(x)$ and $\mathrm{Fup}_{s,n}(x)$. Furthermore, by choosing of the mother function we can get convenient analysis tool with the required properties.

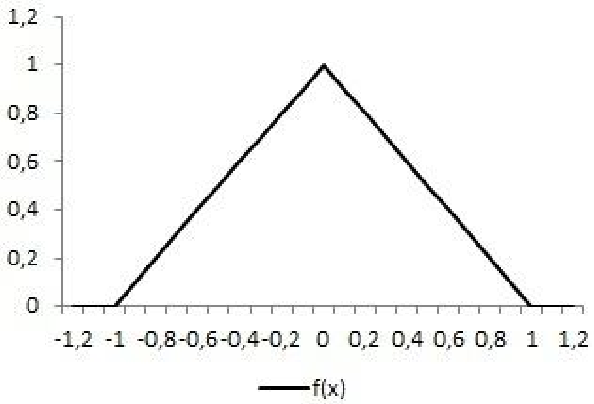

Fig. 3. Example of the mother function

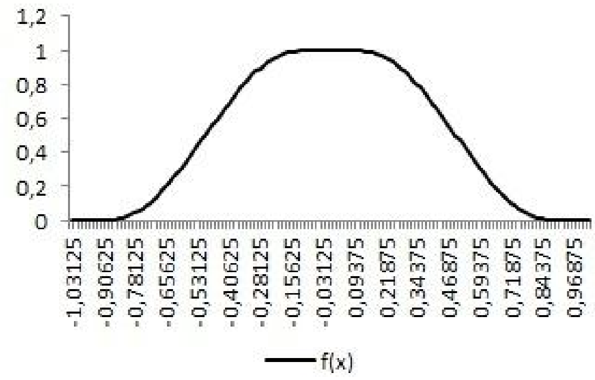

Fig. 4. Example of the mother function

Hence, we stress that generalized Fup-functions can be used in various applied problems.

In this paper we construct wavelets using generalized Fup-functions.

Formulation of the problem

Consider the following functions

$$v_k(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{itx}\cdot V_k(t)dt,\ k=0,1,2,...,$$

where

$$V_k(t)=\frac{\sin\frac{2^k t}{N}}{\frac{2^k t}{N}}\cdot\prod_{j=0}^{k-1}\cos\frac{2^j t}{N}\cdot F\left(\frac{t}{N}\right)$$

and F(t) is the Fourier transform of the mother function f(x) such that $\mathrm{supp}\, f(x)=[-1,1]$, $f(-x)=f(x)$, $f(x)>0$ for any $x\in(-1,1)$ and $\int_{-1}^{1}f(x)dx=1$. It can easily be checked that $v_k(x)$ is a generalized Fup-function

The aim of this paper is to construct wavelets using $v_k(x)$ and obtain formulas for their calculation.

Spaces of generalized atomic wavelets

First let us introduce some properties of the functions $v_k(x)$.

For any $k\in\mathbb{N}$ the following equality holds:

$$v_k(x)=\frac{1}{4}\cdot\left(v_{k-1}\left(x+\frac{2^k}{N}\right)+\right.$$
$$\left.+2\cdot v_{k-1}(x)+v_{k-1}\left(x-\frac{2^k}{N}\right)\right). \qquad (2)$$

In other words, the function $v_k(x)$ is a linear combination of shifts of $v_{k-1}(x)$. Indeed, it is clear that

$$V_k(t)=\frac{\sin\frac{2^{k-1}t}{N}\cdot\cos\frac{2^{k-1}t}{N}}{\frac{2^{k-1}t}{N}}\cdot\prod_{j=0}^{k-1}\cos\frac{2^j t}{N}\cdot F\left(\frac{t}{N}\right)=$$

$$=\frac{\sin\frac{2^{k-1}t}{N}}{\frac{2^{k-1}t}{N}}\cdot\prod_{j=0}^{k-2}\cos\frac{2^j t}{N}\cdot F\left(\frac{t}{N}\right)\cdot\cos^2\left(\frac{2^{k-1}t}{N}\right)=$$

$$=V_{k-1}(t)\cdot\cos^2\left(\frac{2^{k-1}t}{N}\right)=$$

$$=V_{k-1}(t)\cdot\left(\frac{e^{i\cdot\frac{2^{k-1}t}{N}}+e^{-i\cdot\frac{2^{k-1}t}{N}}}{2}\right)^2=$$

$$=V_{k-1}(t)\cdot\frac{1}{4}\cdot\left(e^{i\cdot\frac{2^k t}{N}}+2+e^{-i\cdot\frac{2^k t}{N}}\right).$$

It follows from the properties of Fourier transform that the equality (2) is satisfied.

Also, it is not hard to prove that

$$v_k(x)=0 \text{ for any } x\notin\left(-\frac{2^{k+1}}{N};\frac{2^{k+1}}{N}\right), \quad (3)$$

$$v_k(x)>0 \text{ for any } x\in\left(-\frac{2^{k+1}}{N};\frac{2^{k+1}}{N}\right), \quad (4)$$

where $k=0,1,2,\dots$.

Denote by L_k the space of the functions

$$f(x)=\sum_{j\in I(f)}c_j\cdot v_k\left(x-\frac{2^{k+1}j}{N}\right),$$

where $I(f)$ is a finite subset of integers. This means that L_k is a space of finite linear combinations of the function $v_k(x)$ shifts.

From (2) it follows that $L_k\supset L_{k+1}$ for any k.

Define the inner product of two functions as the integral

$$(f,g)=\int_{-\infty}^{\infty}f(x)\cdot g(x)dx.$$

Let W_k be the orthogonal complement to L_k in the space L_{k-1}:

$$W_k=\{f\in L_{k-1}:\ (f,g)=0 \text{ for any } g\in L_k\}.$$

This implies that

$$L_0=W_1\oplus W_2\oplus\ldots\oplus W_n\oplus L_n. \quad (5)$$

The construction of the generalized atomic wavelets is based on the special basis of the spaces W_k.

Theorem 1. For any natural k there exists the function $w_k(x)$ such that

1) the system of functions $\left\{w_k\left(x-\frac{2^{k+1}j}{N}\right)\right\}_{j\in\mathbb{Z}}$ is a basis of the space W_k;

2) $w_k(x)=0$ for any $x\notin\left(0;\frac{6\cdot 2^k}{N}\right)$;

3) $\int_{-\infty}^{\infty}w_k(x)dx=0$.

This statement is a generalization of theorem 2 from [25] and theorem 1 from [24] on the existence of atomic wavelets. Since $v_k(x)$ generalizes the function $\mathrm{Fup}_{s,n}(x)$, which was used for construction of atomic wavelets, we say that $w_k(x)$ is **a generalized atomic wavelet** and the linear space W_k is **a space of generalized atomic wavelets**. Theorem 1 can be proved in the same way as theorem 1 from [24].

In the next section we obtain formulas for evaluation of the generalized atomic wavelets.

Construction of generalized atomic wavelets

Consider the function

$$w(x)=\sum_{j=1}^{5}c_j\cdot v_{k-1}\left(x-\frac{2^k j}{N}\right)$$

such that $w(x)$ is orthogonal to the space L_k. Equivalently, $w(x)\perp v_k\left(x-\frac{2^{k+1}i}{N}\right)$ for any integer i.

It follows from (3) that $w(x)=0$ for any $x\notin\left(0;\frac{6\cdot 2^k}{N}\right)$ and $v_k\left(x-\frac{2^{k+1}i}{N}\right)=0$ for any $x\notin\left(\frac{2^{k+1}(i-1)}{N};\frac{2^{k+1}(i+1)}{N}\right)$. Hence, $w\perp L_k$ if and only if $w(x)\perp v_k\left(x-\frac{2^{k+1}i}{N}\right)$ for $i=0,1,2,3$. It means that

$$\int_{-\infty}^{\infty}w(x)\cdot v_k\left(x-\frac{2^{k+1}i}{N}\right)dx=0 \text{ for any } i=0,1,2,3.$$

Hence, coefficients $\{c_j\}_{j=1}^{5}$ satisfy the system of linear algebraic equations

$$A\cdot c=0, \quad (6)$$

where $c^T=(c_1\ \ c_2\ \ c_3\ \ c_4\ \ c_5)$, $A=(a_{ij})_{\substack{i=0,\dots,3\\ j=1,\dots,5}}$.

Besides,

$$a_{ij}=\int_{-\infty}^{\infty}v_k\left(x-\frac{2^{k+1}i}{N}\right)\cdot v_{k-1}\left(x-\frac{2^k j}{N}\right)dx.$$

If we combine this with (3), we get

$$A = \begin{pmatrix} \beta_k & \gamma_k & 0 & 0 & 0 \\ \beta_k & \alpha_k & \beta_k & \gamma_k & 0 \\ 0 & \gamma_k & \beta_k & \alpha_k & \beta_k \\ 0 & 0 & 0 & \gamma_k & \beta_k \end{pmatrix},$$

where

$$\alpha_k = \int_{-\infty}^{\infty} v_k(x) \cdot v_{k-1}(x) dx ,$$

$$\beta_k = \int_{-\infty}^{\infty} v_k(x) \cdot v_{k-1}\left(x - \frac{2^k}{N}\right) dx ,$$

$$\gamma_k = \int_{-\infty}^{\infty} v_k(x) \cdot v_{k-1}\left(x - \frac{2^{k+1}}{N}\right) dx .$$

It follows from (2) that $\alpha_k = (a_{k-1} + b_{k-1})/2$, $\beta_k = (a_{k-1} + 2 \cdot b_{k-1})/4$ and $\gamma_k = b_{k-1}/4$, where

$$a_{k-1} = \int_{-\infty}^{\infty} v_{k-1}^2(x) dx ,$$

$$b_{k-1} = \int_{-\infty}^{\infty} v_{k-1}(x) \cdot v_{k-1}\left(x - \frac{2^k}{N}\right) dx .$$

We obtain that the general solution of the system (6) is $c_1 = -b_{k-1} \cdot \delta$, $c_2 = (a_{k-1} + 2b_{k-1}) \cdot \delta$, $c_3 = -2(a_{k-1} + b_{k-1}) \cdot \delta$, $c_4 = (a_{k-1} + 2b_{k-1}) \cdot \delta$ and $c_5 = -b_{k-1} \cdot \delta$, where $\delta \in \mathbb{R}$. Therefore, the function

$$\begin{aligned} w_k(x) = &-b_{k-1} \cdot v_{k-1}\left(x - \frac{2^k}{N}\right) + \\ &+(a_{k-1} + 2b_{k-1}) \cdot v_{k-1}\left(x - \frac{2^k \cdot 2}{N}\right) - \\ &-2 \cdot (a_{k-1} + b_{k-1}) \cdot v_{k-1}\left(x - \frac{2^k \cdot 3}{N}\right) + \\ &+(a_{k-1} + 2b_{k-1}) \cdot v_{k-1}\left(x - \frac{2^k \cdot 4}{N}\right) - \\ &-b_{k-1} \cdot v_{k-1}\left(x - \frac{2^k \cdot 5}{N}\right) \end{aligned} \qquad (7)$$

is a generalized atomic wavelet.

Properties of generalized atomic wavelets

In this section we discuss the main properties of generalized atomic wavelets.

1. $\operatorname{supp} w_k(x) = \left[0; \frac{6 \cdot 2^k}{N}\right]$. This means that the function $w_k(x)$ has a local support.

2. $\int_{-\infty}^{\infty} w_k(x) dx = 0$. In other words, generalized atomic wavelets have zero mean value.

3. The function $w_k(x)$ is a smooth function. Depending on the choice of the mother function, we can get a generalized atomic wavelet with the desired order of smoothness. For example, if atomic function $up(x)$ is a mother function, then $w_k(x)$ is infinitely differentiable.

4. The system of generalized atomic wavelets has good approximation properties. It was shown in terms of the Kolmogorov width that spaces of linear combinations of generalized Fup-functions have almost the same approximation properties as trigonometric polynomials [28].

We see that by choosing such parameters as N, m and $f(x)$ we can obtain generalized atomic wavelets with the desired properties.

Practical approach to the application of generalized atomic wavelets

There are different ways to use wavelets in practice. In this section we consider the approach that is related to the construction of special bases in functional spaces.

Suppose some data are presented by the function $d(x)$. Denote by $p(x)$ an orthogonal projection of this function on the linear space L_0. And let $r(x) = d(x) - p(x)$. In this notation, $f(x) = p(x) + r(x)$. It follows from (5) and theorem 1 that

$$p(x) = \sum_{k=1}^{n} \sum_{j \in I_k(d)} \omega_{kj} \cdot w_k\left(x - \frac{2^{k+1} j}{N}\right) + \sum_{j \in J(d)} \upsilon_j \cdot v_n\left(x - \frac{2^{n+1} j}{N}\right), \qquad (8)$$

where $I_k(d)$ and $J(d)$ are subsets of integers. We say that $p(x)$ is a generalized atomic wavelet expansion of the function $d(x)$. Such an expansion can be used for the detection of the seasonal fluctuations and trend. Let $p_k(x) = \sum_{j \in I_k(d)} \omega_{kj} \cdot w_k\left(x - 2^{k+1} j / N\right)$ for $k = 1,...,n$ and $q(x) = \sum_{j \in J(d)} \upsilon_j \cdot v_n\left(x - \frac{2^{n+1} j}{N}\right)$. In this terms, $p(x) = p_1(x) + ... + p_n(x) + q(x)$. Each function $p_k(x)$ corresponds to the certain frequencies and $q(x)$ describes the principal value of the data function $d(x)$ (see fig. 5 – 10, $n = 3$).

The function $r(x)$ is a remainder term. Also, it represents an error of approximation of the function $d(x)$ by its generalized atomic wavelet expansion. It follows that such an error depends on approximation properties of the space L_0 and the mother function. An upper estimate of the best approximation of some classes of differentiable functions, which was obtained in [28], can be used in the general case to ensure the accuracy of the results. Let us remark that if $\mathrm{up}(x)$ or $\mathrm{mup}_s(x)$ is chosen as the mother function, then we can use results of V. A. Rvachev and V. A. Makarichev [12, 13, 15, 23] for this purpose.

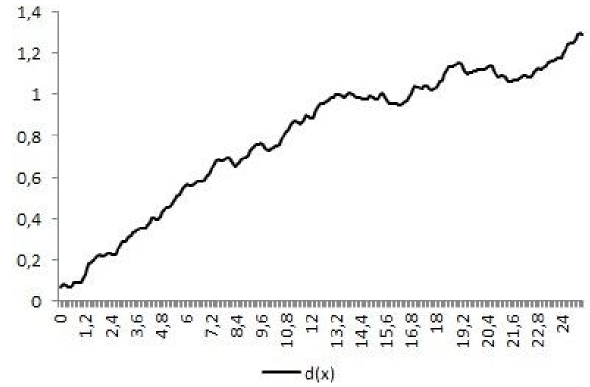

Fig. 5. Graph of the data function $d(x)$

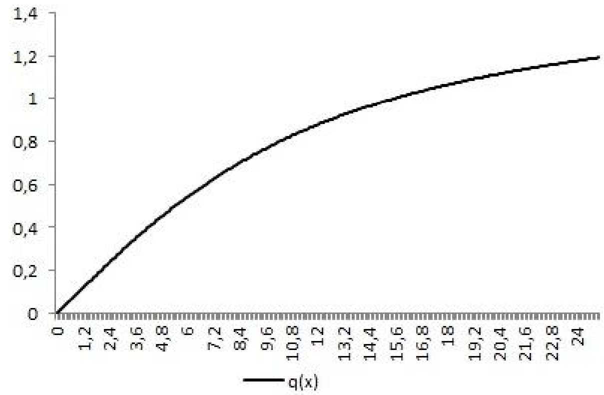

Fig. 6. Graph of $q(x)$ that describes trend or the principal value of $d(x)$

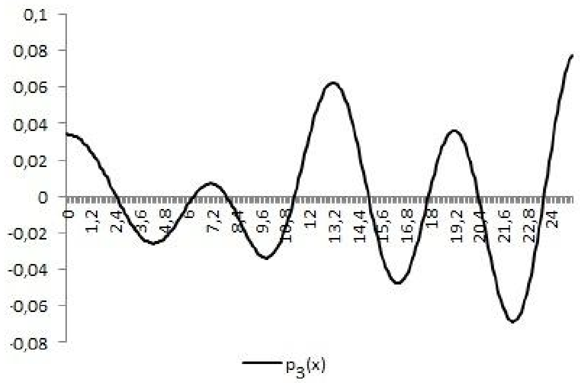

Fig. 7. Graph of $p_3(x)$ that corresponds to low-level frequencies

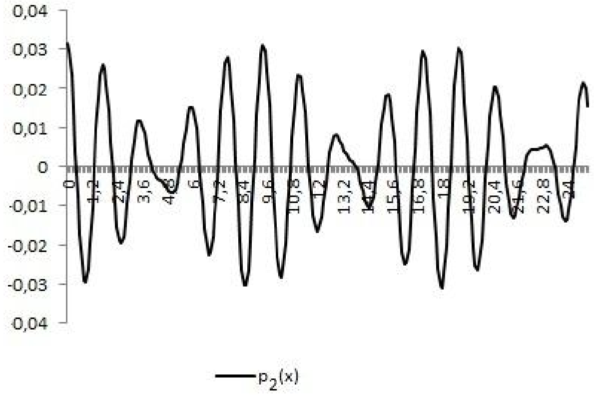

Fig. 8. Graph of $p_2(x)$ that corresponds to medium-level frequencies

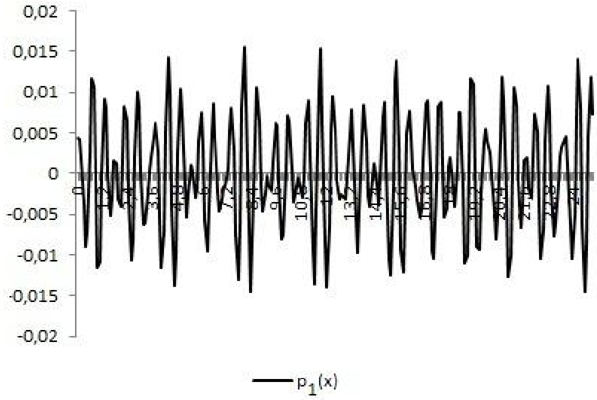

Fig. 9. Graph of $p_1(x)$ that corresponds to high-level frequencies

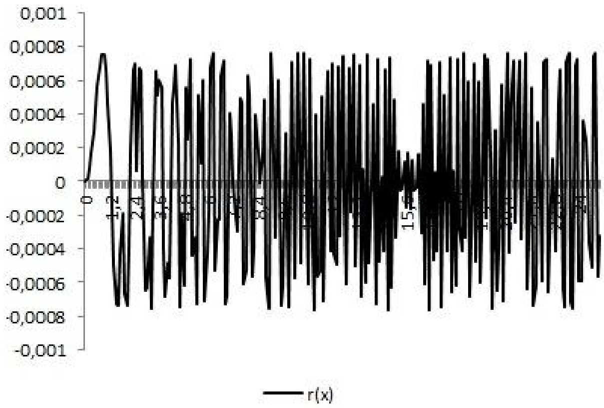

Fig. 10. Graph of the remainder term $r(x)$ that describes an error of approximation of the data function $d(x)$ by its generalized atomic wavelet expansion

Further, to obtain a generalized atomic wavelet expansion of $d(x)$ we get an orthogonal projection of this function on $W_1,...,W_n$ and L_n. It should be mentioned that the systems of functions

$$\left\{w_k\left(x-\frac{2^{k+1}j}{N}\right)\right\}_{j\in\mathbb{Z}} \text{ and } \left\{v_n\left(x-\frac{2^{n+1}j}{N}\right)\right\}_{j\in\mathbb{Z}}$$

are not orthogonal. It is clear that we can use some classic procedure to get orthogonal basis. But in practice it is more convenient to construct the corresponding biorthogonal system of functions. Construction of such a system will be the object of another paper.

Conclusions

In this paper we have constructed generalized atomic wavelets, which are locally supported smooth functions and have good approximation properties, and obtained convenient formulas for their evaluation. Also, we have introduced a generalized atomic wavelet expansion that can be used for data analysis.

Certainly, there are several unsolved problems relating generalized atomic wavelets. For instance, the problem of biorthogonal system construction is of interest.

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УЗАГАЛЬНЕНІ АТОМАРНІ ВЕЙВЛЕТИ

І. В. Брисіна, В. О. Макарічев

Розглянуто проблему обробки великих об'ємів даних. Ключову роль у розробці ефективних алгоритмів відіграє застосування відповідного математичного апарату. На сьогодні існує багато конструктивних засобів аналізу, серед яких можна виділити атомарні функції. Теорію атомарних функцій було розроблено у роботах В. О. Рвачова та представників його наукової школи. Було отримано низку результатів, що надають фундаментальне обґрунтування доцільності їх практичного застосування. Зокрема, атомарні функції нескінченно диференційовані, що є суттєвим при обробці даних з ефектом гладких переходів (наприклад, кольорові фотографії). Також ці функції мають локальний носій, що дозволяє значно скоротити витрати чисельних ресурсів. Окрім того, доведено наявність у просторів атомарних функцій гарних апроксимаційних властивостей, завдяки яким можна зменшити похибку обчислень. Тому застосування цього математичного апарату в алгоритмах обробки даних є достатньо перспективним. Існує декілька основних підходів до практичного використання атомарних функцій та їх узагальнень, одним із яких є побудова на їх основі вейвлетоподібних структур. У даній роботі за допомогою узагальнених Fup-функцій побудовано узагальнені атомарні вейвлети та отримано формули для їх обчислення. Також наведено їх основні властивості. Зокрема, встановлено, що узагальнені атомарні вейвлети поєднують у собі такі якості, як гладкість, локальність носія та гарні апроксимаційні властивості. Крім того, узагальнені атомарні вейвлети – це широкий клас функцій, параметри яких можна змінювати з урахуванням конкретних потреб. Це означає, що запропонований математичний апарат надає дослідникам та розробникам алгоритмів гнучкі можливості пристосування до специфіки різноманітних проблем. Також у статті розглянуто питання подання даних за допомогою узагальнених атомарних вейвлетів. Для цього у роботі запропоновано узагальнене атомарне розвинення даних, яке полягає у поданні інформації у вигляді суми тренд-функції та декількох доданків, що описують відповідні частоти. При цьому похибка цього розвинення описується залишковим членом, який, згідно з результатами попередніх досліджень, є незначним і можна оцінити за допомогою нерівностей, що були отримані у роботах В. О. Рвачова, В. О. Макарічева та І. В. Брисіної.

Ключові слова: обробка даних, вейвлети, атомарні функції, up-функція В. О. Рвачова, атомарні вейвлети, узагальнене атомарне розвинення даних.

ОБОБЩЕННЫЕ АТОМАРНЫЕ ВЕЙВЛЕТЫ

И. В. Брысина, В. А. Макаричев

Рассмотрена проблема обработки больших объемов данных. Ключевую роль при разработке эффективных алгоритмов играет применение подходящего математического аппарата. В работах В. А. Рвачева и его учеников была развита теория атомарных функций. В частности, был получен ряд результатов, дающих фундаментальное обоснование целесообразности их практического применения. В данной статье построены обобщенные атомарные вейвлеты, которые обладают рядом преимуществ по сравнению с другими аналогичными инструментами анализа. Также в работе предложен подход к практическому применению предложенных функций.

Ключевые слова: обработка данных, вейвлеты, атомарные функции, up-функция В. А. Рвачева, атомарные вейвлеты, обобщенное атомарное разложение данных.

Брысина Ирина Викторовна – канд. физ.-мат. наук, доцент, доцент кафедры высшей математики и системного анализа, Национальный аэрокосмический университет им. Н. Е. Жуковского «Харьковский авиационный институт», Харьков, Украина, e-mail: iryna.brysina@gmail.com.

Макаричев Виктор Александрович – канд. физ.-мат. наук, доцент кафедры высшей математики и системного анализа, Национальный аэрокосмический университет им. Н. Е. Жуковского «Харьковский авиационный институт», Харьков, Украина, e-mail: victor.makarichev@gmail.com.

Brysina Iryna Victorivna – Ph. D. in Physics and Mathematics, Associate Professor of Department of Higher Mathematics and System Analysis, National Aerospace University «Kharkov Aviation Institute», Kharkov, Ukraine, e-mail: iryna.brysina@gmail.com.

Makarichev Victor Olexandrovych – Ph. D. in Physics and Mathematics, Associate Professor of Department of Higher Mathematics and System Analysis, National Aerospace University «Kharkov Aviation Institute», Kharkov, Ukraine, e-mail: iryna.brysina@gmail.com.