ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

UDC 517.5:517.927 **doi: 10.32620/reks.2020.1.06**

V. O. RVACHOV, T. V. RVACHOVA, E. P. TOMILOVA

National Aerospace University "Kharkiv Aviation Institute", Ukraine

TOMIC FUNCTIONS AND LACUNARY INTERPOLATION SERIES IN BOUNDARY VALUE PROBLEMS FOR PARTIAL DERIVATIVES EQUATIONS AND IMAGE PROCESSING

In the paper we consider and solve the problem of construction of the so called tomic functions – the systems of infinitely differentiable functions which while retaining many important properties of the shifts of atomic function up(x) such as locality and representation of algebraic polynomials and being based on the atomic functions nevertheless have nonuniform character and therefore allow to take into account the inhomogeneous and changing character of the data encountered in real world problems in particular in boundary value problems for partial differential equations with variable coefficients and complex geometry of domains in which these boundary value problems must be solved. The same class of tomic functions can be applied to processing, denoising and sparse storage of signals and images by lacunary interpolation. The lacunary or Birkhoff interpolation of functions in which the function is being restored by the values of derivatives of order r *in points in which values of function and derivatives of order k<r are unknown is of great importance in many real world problems such as remote sensing. The lacunary interpolation methods using the tomic functions possesss important advantages over currently widely applied lacunary spline interpolation in view of infinite smoothness of tomic functions. The tomic functions can also be applied to connect (to stitch) atomic expansions with different steps on different intervals preserving smoothness and optimal approximation properties. The equations for of construction of tomic functions* $tofu_j(x)$ *– analogues of the basic functions of the generalized atomic Taylor expansions are obtained – which are needed for lacunary (Birkhoff) interpolation. For the applications in variational and collocation methods for solving bondary value problems for partial derivative and integral equations the tomic functions* $ftup_{r,j}(x)$ *are obtained that are analogues of B-splines and atomic functions* $fup_n(x)$*. Using similar methods, the tomic functions based on other atomic functions such as* $\Xi_n(x)$ *can be obtained.*

Keywords: *atomic functions; tomic functions; lacunary interpolation; Birkhoff interpolation; image processing and storage; variational methods; collocation method.*

Introduction

In this paper, we solve the problem of overcoming some limitations of atomic functions which found wide applications in solving the boundary value problems of electromagnetics and image processing and possess important properties of locality and representation of algebraic polynomials but does not possess sufficient flexibility to allow to take into account nonuniform and inhomogeneous character of the data of the objects of research – complex geometry and variable coefficients. The properties and parameters of the systems which are the objects of analysis and processing in electromagnetics, distant sensing, processing of multidimensional signals often undergo rapid changes and are described by differential equations with variable coefficients in the domains with complex geometry.

The aim of this paper is to introduce some generalization of atomic functions (AF) – so called tomic functions (TF) which while retaining most important advantages of AF take into account this inhomogeneity, variability of the behaviour of the solutions of the problems encountered in applications.

1. Formulation of the problem: Birkhoff or lacunary interpolation

The main task of this paper is starting from ideas and machinery of atomic functions offer the solution of the lacunary interpolation problem by construction the new class of function – **tomic functions**.

As one example we consider a solution with the help of this new apparatus of the constructive theory of functions of the problems of Birkhoff or lacunary interpolation. In the well- known Newton or Lagrange interpolation in order to reconstruct an unknown function we use its known values in some points. In Hermite interpolation in addition to the values of the function in some points the derivatives to some order in the same points are used. The Birkhoff (or lacunary) interpolation uses the known values of the derivatives of some order in points where the values of the function are unknown. Let us here make a remark that the opinion that it is more easy to determine the values of a variable than to find the values of the derivatives of this variable is not always true. It is in fact very often absolutely wrong.

Let us give some examples. Consider such frequently met functions as $\ln x$, $\operatorname{arctg} x$. Their derivatives are correspondingly

$$\frac{1}{x},\ \frac{1}{1+x^2}.$$

Its obvious that in this case the derivatives are computed much easier. One more frequently used in probability theory and its applications function is

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp(-t^2/2)dt.$$

The tabulated computed values of this function are contained in practically every probability and statistics textbook. Its derivative is

$$F'(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$$

– this is an elementary function. Let us recollect that the computation of integrals is often much harder task than computation of the derivatives. The derivative of an elementary function is always an elementary function while antiderivative (primitive) of an elementary function is not as a rule. The problems of solving the differential equations are problems of finding the function given some information about its derivatives and are called the integration of the differential equation. Now let us Consider such mechanical quantities as displacement (position), velocity, acceleration, jerk, snap etc. If displacement as function of time is denoted by $d(t)$, velocity as $v(t)$, acceleration $a(t)$, jerk $j(t)$, snap $sp(t)$, then by definition we have

$$\frac{d(a(t))}{dt} = j(t);\ \frac{d(d(t))}{dt} = v(t);$$
$$\frac{d(v(t))}{dt} = a(t);\ \frac{d(j(t))}{dt} = sp(t).$$

Determining of the displacement as function of time at given moment of time is considerably more difficult than determining of the velocity – first derivative of the displacement because for determining the velocity we need only local frame of reference while to determine displacement relative to some distant initial point of space we need global frame of reference, determining of the velocity is more difficult than determining of the acceleration in view of the Newton law $F = ma$ and we can determine the acceleration even without local reference frame.

Determining of the acceleration is more complex than determining of the jerk – the derivative of acceleration. What is measured by the sensors of plants and animals (including human beings) -temperature or its change, loudness of sound or its change, brightness of light or its change? Turning to computer science - what is more memory efficient for storage of the information on a function – 1) to store the values directly or 2) to store only some values with large steps and differences or both differences and second differences and third differences with diminishing small steps? The books containing very precise tables of logarithms and trigonometric functions with 10 digit accuracy of the pre-computer era used the second approach – values with large steps on the left side of a page, to the right to the column of values the column of the first differences with smaller step, and next to the right the column of second differences with the smallest step. But the differences by the Lagrange theorem are derivatives multiplied by the powers of the steps. And when the steps are small and constant the powers of steps are even more small, so in order to provide the needed accuracy of the values of the function, we need less digits to store the values of derivatives than to store the values of the function itself. Those are arguments in favour of lacunary or Birkhoff interpolation and one observes a lot of papers on lacunary spline interpolation which were published recently. Lacunary interpolation was invented by George Birkhoff (the same Birkhoff who proved the ergodic hypothesis and created the dynamic system theory) at the beginning of the twentieth century but was not very successful because mathematical tools he used were algebraic polynomials and an algebraic polynomial $P_n(x)$ of degree n has not more than n real roots, its derivative – not more than $n-1$ roots, its second derivative not more than $n-2$ roots and so on, and the derivative of the order n is constant not equal to zero, so has no roots at all. As well- known saying goes "new wine needs new skin bags". To develop lacunary interpolation new tools were needed – the splines [12, 16-18]. Numerous examples of applications of polynomial splines to lacunary interpolation are in [25-36]. But splines of degree n are functions of finite smoothness – only first $n-1$ derivatives are continuous. To construct the lacunary interpolation series we need the atomic functions. The generalized atomic Taylor expansion proposed in 1991 by V. A. Rvachev (V. O. Rvachov in Ukrainian) is an example of such lacunary interpolation of infinite order. Atomic and tomic functions are the tools for such kind of interpolation. Examples in medicine, geophysics, image processing, remote sensing [19-21, 23-25]. Now we will state reasons in favor of lacunary (Birkhoff) interpolation. There are many applications where robot motion with abrupt changes of jerk is not wanted, such as in transportation of people and goods where dropouts and breakages may easily occur. Limiting jerk in robot trajectories also contributes to

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

extended life of robot joints and thus to more precise trajectory tracking. A technique for time-jerk optimal planning of robot trajectories. The trajectory planning problem is a fundamental one in Robotics. It may be formulated thus: define a temporal motion law along a given geometric path, such as certain requirements set on the trajectory properties are fulfilled. Hence, the aim of trajectory planning is to generate the reference inputs for the control system of the manipulator, in order to be able to execute the motion. The inputs of any trajectory planning algorithm are: the geometric path, the kinematic and dynamic constraints; and the output is the trajectory of the joints (or of the end effector), expressed as a time sequence of position, velocity and acceleration values. Usually, the geometric path is specified in the operating space, i.e. with reference to the end effector of the robot. Standard generalized Taylor expansions on the basis of $up(x)$ function which were introduced by V. A. Rvachev in 1981 use values of the function and its first derivative in integer points, the values of derivative of the order 2 in half integer points. The values of the derivatives of the order in points of form $k2^{-n+1}$. In places where the unknown functions are varying rapidly the discretization step should be made smaller and alternately in regions where those functions are slow varying the step could be made larger. So, generalised atomic Tailor expansion (GATE) expansions should be made with different steps in different regions. But then the problem of smooth transition from one region to the neighbour must be solved. To solve this problem we have to introduce new smooth compactly supported functions similar to the atomic functions but which have zeroes of the derivatives placed non-uniformly or in other words the widths of the intervals between two neighbouring zeroes of the derivatives (parts of the derivative on this intervals which could be named hills and holes) should be different in different regions. Atomic function which satisfy functional differential equation (FDE) of pantograph type with some fixed compression coefficient? The hills and holes of the derivative of a given order have equal widths. Let us remind here that atomic functions by definition are compactly supported solutions of the equations of the form

$$Ly(x) = \sum_{k=1}^{n} c_k y(ax - b_k),$$

where L is linier differential operator of order m with constant coeffitients. Their Fourier transforms are of the form

$$F_{AF}(t) = \prod_{k=0}^{\infty} \frac{P_n(e^{it/a^k})}{Q_m(t/a^k)},$$

where $P_n(t), Q_m(t)$ are algebraic polynomials. If we denote

$$Snc(x) = \frac{\sin x}{x},$$

then Fourier transform of the up(x) function will be

$$Ftup(t) = \prod_{k=0}^{\infty} Snc(t/2^k),$$

$$Ly(x) = \sum_{k=1}^{m} c_k y(ax + b_k).$$

Convolution of $up(x)$ with itself is needed for application in variational and projection methods. To compute the convolution $upp = up(x) * up(x)$, we expand it into Fourier series on interval $[-2, 2]$. Coefficients of this series are values of the square of the Fourier transform of the function $upp(x)$ in points $k\pi/2$, where

$$upp(x) = \int_{-\infty}^{\infty} up(x-t)up(t)dt,$$

$$upp(x) = \int_{-\infty}^{\infty} e^{ixt} Fup^2(t)dt,$$

where

$$F_{up}(t) = \prod_{k=1}^{\infty} \frac{\sin t2^{-k}}{t2^{-k}}.$$

Support of the function $upp(x)$ – is the interval $[-2, 2]$. It is an even function so its Fourier expansion will contain only cosins. Function $upp(x)$ is used for computations when building orthogonal bases of the spaces of spans of shifts of functions $fup_n(x)$. It is obvious that

$$upp(0) = \gamma = \int_{-1}^{1} up^2(x)dx.$$

To solve multidimensional problems – on the plane or in the space, we use the functions

$$up(n, x) = \prod_{k=1}^{n} up(x_k).$$

Besides there are other atomic function which are much promising for applications. For example, the function $\Xi_n(x)$ is the solution with support $[-1,1]$ of FDE

$$y^{(n)}(x) = a\sum_{k=0}^{n}(-1)^k C_n^k y((n+1)x-2k+n).$$

In particular case $n=3$ we have

$$\Xi_3(x)^{(3)} = \lambda(\Xi_3(4x-3)-3\Xi_3(4x-1)+$$
$$+3\Xi_3(4x+1)-\Xi_3(4x+3)).$$

Application of this function instead of the function $up(x)$ has the advantage that the derivatives of it grow more slow than the derivatives of $up(x)$ function, and its derivatives possess fewer zeroes and in GATE on the basis of this function will contain fewer terms with derivatives of the given order. The orthogonal systems on the basis of $\Xi_n(x)$ functions have similar advantage. The negative feature of these functions when compared with the function $up(x)$ is in the fact that the derivatives of it from the first to the derivative of order $n-1$ are not linear combinations of shifts of this function with linearly transformed argument as is with $up(x)$ and to compute them we need additional formulas. The function of two variables

$$u(x_1,x_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(t_1x_1+t_2x_2)} F(t_1,t_2)dt_1dt_2,$$

where

$$F(t_1,t_2) = \prod_{k=1}^{\infty}\{\frac{\sin^2(t_1 3^{-k})+\sin^2(t_2 3^{-k})}{(t_1 3^{-k})^2+(t_2 3^{-k})^2}\}$$

is a fast decreasing solution of the equation

$$\Delta(u(x_1,x_2) = \frac{9}{4}(u(3x_1-2,3x_2)+u(3x_1+2,3x_2)+$$
$$+u(3x_1,3x_2-2)+u(3x_1,3x_2+2)-4u(3x_1-2,3x_2)),$$

where

$$\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2.$$

It can be shown that

$$up^{(n)}(x) = \sum_{k=1}^{2^n}\delta_k 2^{C_{n+1}^k} up(2^n x + 2^n + 1 - 2k),$$

where $\delta_1 = 1$, $\delta_{2k} = -\delta_k$, $\delta_{2k-1} = \delta_k$.

In addition, the construction of tomic functions from atomic functions, such as $up(x)$ requires knowledge of their moments.

2. Pantograph-type equations

Atomic functions are solutions with a compact support of linear functional differential equations with a linearly transformed argument.

Equations of this type are often called pantograph-type equations.

The generalized pantograph equation is

$$y'(t) = a(t)y(t)+b(t)y(\alpha t),$$

where $\alpha > 1$, has numerous applications. Such equations describe the absorption of light by an interstellar medium; they are found in the theory of electrical materials, mathematical cell biology uses similar equations to describe the number of cells in the process of division. Similar equations in physics are called Fokker-Planck equations. Fokker-Planck equation is

$$y''(x)+by'(x)+py(x)+qy(\alpha x) = 0.$$

Here, the coefficient shows how many parts the cell divides during division (mitosis), i.e. almost always equal to 2. But in nuclear physics, the coefficient can be greater than 2, since the nucleus can be divided into a larger number of particles - neutrons and protons.

3. Linear spaces generated by shifts of function up(x)

In approximation theory and computational mathematics, spaces of linear combinations of function $up(x)$ shifts are used [1-3, 7, 8]

$$UP_n = \{\sum_{k=1}^{m} c_k up(x-k2^{-n})\}.$$

It makes sense to consider a wider set of linear spaces

$$UP_n(m) = \{\sum_{k=1}^{s} c_k up(2^m x - k2^{m-n})\},\ n,m \in Z$$

since the operations of differentiation and integration of functions from spaces transform them into functions from. Note that here the limits in the sums can also be taken as infinite, since for each only a finite number of terms are different from 0 due to the compactness of the support of the function. Important elements of space are functions $fup_n(x)$. These functions among the elements of UP_n have minimal support (the support of a function is the set – the closure of the set of points where the function is not equal to 0).

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

Namely, the length of the function $\mathrm{fup}_n(x)$ support is equal to $(n+2)2^{-n}$. Function $\mathrm{fup}_0(x)$ is up(x) this. The shifts of these functions form a basis in space UP_n, consisting of functions with the smallest support. It is obvious that

$$\lim_{n\to\infty}(n+2)2^{-n}=0.$$

For comparison the length of the support of Shoenberg basic spline $B_n(x)$ степени n with step $h=2^{-n}$ is $(n+1)2^{-n}$, that is somewhat less, but the form of it is different for different n and it is of finite smoothness.

The letter f in notation of this function is taken from the first letter of the word fundamental (basic). There are two variants $\mathrm{Fup}_n(x)$ and $\mathrm{fup}_n(x)$. The initial variant had capital F and means the function normed by condition

$$\int_{-\infty}^{\infty}\mathrm{Fup}_n(x)dx=1.$$

The variant with f small means the function the maximum value of which is 1. This is convenient in applications of atomic functions in collocation methods (E. A. Fedotova, Gotovac (Chroatia (Split))). It is easily seen that

$$UP_n\subset UP_{n+1}.$$

Denote the orthogonal complement to UP_n in UP_{n+1} by OUP_n. In the linear space UP_n there is a base of shifts of the function $\mathrm{fup}_n(x)$ of the form $\mathrm{fup}_n(x-k2^{-n})$, in the linear space UP_{n+1} there is a base of shifts $\mathrm{fup}_{n+1}(x-k2^{-n-1})$. The supports of these functions are of lengths $(n+1)2^{-n}$ and $(n+2)2^{-n-1}$ respectively. In the space OUP_n a base of shifts of a function $\mathrm{ofup}_n(x)$ the length of the support of which is $(2n+1)2^{-n}$. As the space UP_n contains subspace of polynomials of degree n the function $\mathrm{ofup}_n(x)$ has zero moments of degree less or equal to n. This is an important fact for applications in border element methods (Boundary integral equations) because the solutions of Laplace equation and similar equations in half plane or circle which have zero moments of degree less or equal to given integer are small far from the boundary and the coefficients of the linear algebraic systems are accordingly small.

4. Achievements of atomic function theory and applications

Atomic functions were successfully applied to various problems of mathematical analysis including approximation theory, for solving the problem of representation of infinitely differentiable function by Taylor-Birkhoff expansions, in numerical methods for ordinary differential equation (ODE), FDE, partial differential (PDE), for signal and image processing [1-11]. Sometimes someone asks – why atomic functions are needed? Are not already used function classes sufficient for all purposes? The answer is following. The analytic functions are not sufficient and spline functions (which are piecewise analytic) are now widely applied. Splines have the advantage of being local – we can change a spline on small interval not changing elsewhere. But splines are only piecewise smooth and at the joining points of the pieces are differentiable only several times so that approximation rate is not very high. The approximation by atomic functions which are infinitely differentiable but nonanalytic allows to change the approximating aggregate on small intervals nevertheless providing high rate of approximation if the function that is approximated is very smooth. But approximation spaces generated by atomic functions have some limitations due to insufficient flexibility and refinement. As noted above the main task of this paper is starting from ideas and machinery of atomic functions offer the solution of the lacunary interpolation problem by construction the new class of function – tomic functions.

5. Necessity and indispensability of tomic function - motivations and definitions

Shortcoming of AF Disadvantage of using the atomic functions consists in the fact that that the zeroes od derivatives of an atomic function are spaced uniformly, equidistantly and hills and holes of the derivatives have equal widths. If the behaviour of the analyzed object – be it solution of a boundary value problem for a partial derivatives equation or the signal or image to be processed in different locations differs considerably – varies slow or fast – to take into account these variations the size of hills and holes of the derivatives of the function which describes this object ought to be accordingly variable. We certainly can try to divide the domain under exploration into more or less homogeneous parts and choose for each part the atomic functions of suitable size of the support and then sew together the different atomic expansions but we will need nevertheless functions with compact support and heterogeneous location of zeroes of the derivatives(the size of hills and holes of the derivatives) for area of transition. Introduction of tomic functions targets exactly this problem – the construction of expansions of the solutions of the problem which possess different behaviour in different parts of the domain of definition. The word atom means indivisible and the term tomic function means divisible function but to avoid mixing with other uses of the words such as division, divisor in different branches of mathematics we introduce the term tomic for our purpose.

6. Definition and construction

Tomic function of order 1 is a function $f(x)$ equal 0 outside the interval $[a,b]$, one time continuously differentiable, positive inside $[a,b]$ and which has unique point of strict maximum $c, a < c < b$ (where the derivative is zero $f'(c) = 0$). If $f(x)$ is twice continuously differentiable, i.e. $f(x) \in C^2[R]$ and let

$$l(x) = \begin{cases} f'(x), & x \in [a,c]; \\ 0, & x \notin [a,c]; \end{cases}$$

and

$$r(x) = \begin{cases} f'(x), & x \in [c,b]; \\ 0, & x \notin [c,b]. \end{cases}$$

Both $l(x))$ and $-r(x)$ are tomic functions of order 1, then the function $f(x)$ is called tomic function of order 2. If functions $l(x), -r(x)$ are tomic functions of order 2, then $f(x)$ is called tomic function of order 3. Similarly tomic function of any positive integer order is defined. Tomic function which for every positive integer k is tomic function of order k is called tomic function of infinite order. The point c is called the middle point of order 1. Middle points c_1, c_2 of functions $l(x), -r(x)$ are called middle points of order 2 and so on. The number of middle points of the order n of a tomic function of order n will be 2^{n-1}. Naturally two problems concerning tomic functions of infinite order - the problem of existence and the problem of uniqueness arise. It is obvious that function $up(x)$ is tomic function of infinite order. So tomic functions of infinite order exist. From the theorem of uniqueness of restoration of function $up(x)$ by zeroes of its derivatives which was proved in paper by V. A. Rvachev (V. O. Rvachov) in the paper [2] (Compactly supported solutions of functional-differential equations and their applications) follows that when middle points are in the middle. That is when ratio of lengths of supports of functions $l(x), r(x)$ at each step and in each part is 1 the existence and uniqueness of the tomic function takes place. One can prove the existence and uniqueness of tomic function of infinite order if the ratio of lengths of supports of functions $l(x), r(x)$ at each step and at each part takes place only starting with some order of derivatives. We will consider only such functions now because they are sufficient to take care of heterogeneousness for most problems. There are such two now unproven hypotheses. Hypothesis A. The existence and uniqueness of tomic function with given zeroes of derivatives and being normalized by some condition, such as its maximal value equals 1 holds if the ratio of the lengths of supports $l(x)/r(x) \to 1$ at each step and in each part. Hypothesis B. The existence and uniqueness of tomic function with given zeroes of derivatives and being normalized by some condition, such as its maximal value equals 1tholds if the ratio of the lengths of supports $l(x)/r(x)$ at each step and in each part is bounded $0 < a < l(x)/r(x) < b < +\infty$ where a, b do not depend on the order of a derivative and number of a part on which the support of the function is partitioned by the zeroes of the derivative.

7. Coordinated (coherent) systems of tomic functions

Spaces of atomic functions or Atomic spaces are by definition linear spaces of linear combinations of shifts (translates) of the function $up(x)$ of the form $up(x - k2^{-n})$ (in other words generated by the shifts of function $up(x)$ with constant step 2^{-n}) . The contain algebraic polynomials of degree n ,These spaces can be considered to be spaces of smoothed polynomial splines. In order to build the spaces of linear combinations of tomic functions (which will be called the tomic spaces), the zeroes of derivatives of which are not uniformly spaced we cannot use shifts (or translates) of a single tomic function. Here some construction of coherent systems of tomic functions is needed to satisfy the condition that the linear combinations of the functions belonging to such system must contain all algebraic polynomials of degrees not greater than some n. This condition provides for good approximation properties of tomic spaces. This can be done in the following way: subdivide the interval $[a,b]$, on which we want to build a coherent set of tomic functions with given steps between the nodes at which the senior derivatives of required order r are prescribed, by the required quantity of nodes from $x_1, ..., x_{2M}$. Then we add at the left and at the right sides additional 2^r nodes so that total num-

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

ber of nodes will be $2^M + 2^{r+1}$ rename all the nodes from left to right as

$$z_1 < z_2 < ... < z_N,$$

where $N = 2^M + 2^{r+1}$. Now we start construction of the sought coherent system of tomic functions $tofu_j(x)$ in the following way: on each interval $[z_k, z_{k+1}]$ we place the function of the form $up(a_k x - b_k)$ in such a way that its support coincided with the interval and denote it $g_k(x)$. Simple calculations give

$$a_k = \frac{2}{z_{k+1} - z_k} \qquad b_k = \frac{z_{k+1} + z_k}{z_{k+1} - z_k}.$$

For each j consider the function $\varphi_j(x)$ defined on the interval $[z_j, ... z_{j+2^r}]$ $\varphi_j(x) = \sum_{k=j}^{j+2^r} c_k g_k(x)$. Notice that unknown coefficients in fact depend on j $c_k = c_k(j)$, i.e. for different basic function $tofu_j(x)$ the coefficients $c_k = c_k(j)$ are different unlike the case of generalized Taylor series on the basis of the function $up(x)$. We assume that the function $\varphi_j(x)$ is to be the derivative of the order r of the function $tofu_j(x)$, that is $tofu_j^{(r)}(x) = \varphi_j(x)$. To find the unknown coefficients $c_k = c_k(j)$ we have some linear algebraic system. How do equations of this system for determining a basic tomic function $tofu_j(x)$ look like? The sought function $tofu_j(x)$ the zeroes of derivatives must be in prescribed points. For the derivative of the order r zeroes are all points z_k.As we assumed that the function $\varphi_j(x)$ is the derivative of order r of the function $tofu_j(x)$,so the conditions on the zeroes of the order r are satisfied by its definition. Unknown coefficients c_k we find from the conditions of vanishing of the derivatives of orders from 1 до $r-1$ in prescribed points and the condition that the function itself either equals to 1 in the middle pointв средней точке or the condition that its integral equals 1 (2 different normalizations which are convenient in different applications). For the derive of the order $r-1$ the zeroes are to be in points z_k with $k = j + 2s$. For the derive of the order $r-2$ the zeroes are to be in points z_k with $k = j + 4s$ and so on. The derivatives of the order less than r are found by successive integration (finding the primitive) of the derivative of the order r which is $\varphi_j(x) = \sum_{k=j}^{j+2^r} c_k g_k(x)$ in limits from z_j до to variable x. Therefore

$$(tofu_j(x))^{(r-1)} = \int_{z_j}^{x} \sum_{k=j}^{j+2^r} c_k g_k(t)dt = \sum_{k=j}^{j+2^r} c_k \int_{z_j}^{x} g_k(t)dt.$$

Integrals contained in these equations are successive primitives of compressed and shifted function $up(x)$ and as a consequence of the functional differential equation for this function satisfies , also are functions of the form $up(\beta x + \gamma)$ on left side of the support and further to the right an algebraic polynomial of the degree l for derivative $r - l - 1$). Here it is convenient to make use the Cauchy formula

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (t-x)^{n-1} f(t)dt.$$

Derivative of a negative order (antiderivative) is just what we need here -the operation of successive multiple integration to obtaining the primitives. With the help of this formula the equations for finding the coefficients c_k obtain a simple look. Now we see that expanding expressions $(t-x)^{n-1}$ in this integral it remains to compute the integrals of the form $\int_{z_j}^{z_{j+1}} t^s g_j(t)dt$. And as $g_j(x) = up(a_j x - b_j)$, such integrals are expressed via the known moments of the function $up(x)$. Moments of the $up(x)$ functions are computed by the recursive formula

$$a_n = \int_{-1}^{1} x^n up(x)dx, \; a_{2n+1} = 0, \; a_0 = 1,$$

$$a_{2n} = \frac{(2n)!}{2^{2n} - 1} \sum_{k=1}^{n} \frac{a_{2n-2k}}{(2n-2k)!(2k+1)!}.$$

Integrals of the form (half-moments)

$$b_{2n+1} = \int_0^1 x^{2n+1} up(x)dx$$

are computed by the recursive formula

$$b_{2n+1} = \frac{1}{(n+1)2^{2n+3}} \sum_{k=0}^{n+1} a_{2n+2-2k} C_{2n+2}^{2k}.$$

In particular $a_2 = \frac{1}{9}$, $a_4 = \frac{19}{3^3 5^2}$, $a_6 = \frac{583}{3^5 5 \cdot 7^2}$,

$b_1 = \frac{5}{36}$, $b_3 = \frac{143}{8 \cdot 27 \cdot 25}$, $b_5 = \frac{1153}{64 \cdot 3^6 \cdot 49}$.

The number of conditions (the numbers of equations) is equal to the number of unknowns and the structure of the matrix of this linear algebraic system is block diagonal. If the order of system we denote by $N = 2^n + 2^{r+1}$ then we have $N/2$ equations with two different unknowns in each, $N/4$ equations with 4 different unknowns in each, $N/8$ equations with 8 different unknowns in each and so on,… ending with 2 equations with half different unknowns in each corresponding to the 2 zeroes of first derivative and 1 equation for one zero at the right end for all unknowns and 1 equation with all unknowns to satisfy normalization condition. It is obvious that the matrix of the system is invertible. So the solution exists and is unique.From the construction of this coherent system of tomic follows that the linear combinations of the elements of it contain algebraic polynomials of the order $r-1$, but the proof of it being straightforward is rather lengthy and is omitted here. It can be calculated that due to the special block-diagonal structure of the matrix of the linear algebraic system for finding the 2^r coefficients c_k we need only $Cr2^r$ arithmetic operations (ao) (per one function). For $r = 5$ to find 33 c_k coefficients for one function $\varphi_j(x)$ takes 160 ao and for coherent system of 1000 functions – 200000 ao. To store 33 thousand of coefficient – adequate amount of memory. For $r = 10$ to compute 1025 coefficients per one tomic function we need 10 thousand ao and for computation of such coefficients for the coherent system of 10 thousand functions -100 million ao and storage of 10 million coefficients – the adequate amount of memory. If the norm C^r is insufficient we can break each interval $[z_k, z_{k+1}]$ by half and add to already built coherent system of tomic functions additional tomic functions. The tomic functions are analogues of shifts (translations) of the function up(x) – the functions $up(x - k2^{-n})$. But in applications to the methods of solution of the boundary value problems for PDE of the finite element type or boundary element type we use not the shifts of the function $up(x)$, but shifts of the functions $fup_n(x)$) because the supports of shifts of $up(x)$ are too wide and supports of $fup_n(x)$ are minimal possible. For GATE – Birkhoff interpolation series, where we do not integrate but collocate the shifts of $up(x)$ are optimal, but for methods where we integrate we need to build analogues of shifts of $fup_n(x)$) in the space of tomic functions. They $ftup_{r,j}(x)$ are constructed in the form of the antiderivative of the order r of the sums of the form $\psi_j(x) = \sum_{k=j}^{j+r+2} c_k g_k(x)$. With conditions only at the right end z_{j+r+3} antiderivatives of $\psi_j(x)$ of the order from 1 to r being 0, and possess minimal possible supports. They are analogues of the function $fup_n(x)$ for tomic expansions. Glueing (matching) two homogeneous atomic expansions with different steps depends on the kind of the problem we solve. In case of Birkhoff interpolation we need to construct tomic functions for interval of $2^r + r$ steps to the left and to the right of the transition point because we need wide tomic function which possess zeroes of derivatives in needed places. But in variational problems where we use orthogonal systems and in collocation we use the function $ftup_{r,j}(x)$ - analogues of B-splines and functions $fup_n(x)$ the transition interval (where we need tomic functions) is only $2r+3$ steps wide.

Conclusions

In this paper we introduce the tomic functions in order to transfer the application of atomic functions to solving problems with sharp geometric inhomogeneities and rapidly variable medium properties of the objects under study. The tomic functions are constructed on the basis of atomic function $up(x)$ by iterative procedure and are of two kinds, namely, designed for lacunary interpolation – analogues of the basic functions of atomic generalized Taylor series $bafu_{n,k}(x)$ and designed for variational and collocation methods in boundary value problems for partial differential equations – analogues of atomic functions $fup_n(x)$.

References (GOST 7.1:2006)

1. *Рвачёв, В. Л. Неклассические методы теории приближений в краевых задачах [Текст] / В. Л. Рвачёв, В. А. Рвачёв. – К. : Наукова думка, 1979. – 196 с.*

2. *Rvachev, V. A. Compactly supported solutions of functional-differential equations and their applications [Text] / V. A. Rvachev // Russian Math. Surveys. – 1990. – Vol. 45, No. 1. – P. 87 – 120.*

3. *Lemarié-Rieusset, P. G. Interpolating scaling functions, Bernstein polynomials and nonstationary wavelets [Text] / P. G. Lemarié-Rieusset // Revista Matemática Iberoamericana. – 1997. – Vol. 13, Iss. 1. – P. 91-188.*

4. Application of the Generalized Taylor – Birkhoff Series for Solving of the Initial Value Problem for Ordinary Differential Equations [Text] / V. O. Rvachov, T. V. Rvachova, Ye. P. Tomilova // Открытые информационные и компьютерные интегрированные технологии : сб. науч. тр. Нац. аэрокосм. ун-та «ХАИ». – 2018. – No. 79. – С. 153-161.

5. Rvachova, T. V. Finding Antiderivatives with the Help of the Generalized Taylor Series [Text] / T. V. Rvachova, Ye. P. Tomilova // Открытые информационные и компьютерные интегрированные технологии : сб. науч. тр. Нац. аэрокосм. ун-та «ХАИ». – 2016. – № 73. – С. 52-58.

6. Rvachov, V. O. On the construction of multimodal multiparameter exponential families probability laws [Text] / V. O. Rvachov, T. V. Rvachova // Радіоелектронні і комп'ютерні системи. – 2011. – № 4 (52). – С. 72-76.

7. Rvachova, T. V. On a relation between the coefficients and the sum of the generalized Taylor series [Text] / T. V. Rvachova // Matematicheskaya fizika, analiz, geometriya. – 2003. – Vol. 10, No 2. – P. 262–268.

8. Рвачев, В. А. Об эрмитовой интерполяции с помощью атомарных функций [Текст] / В. А. Рвачев, Т. В. Рвачева // Радіоелектронні і комп'ютерні системи. – 2010. – № 4(45). – С. 100–104.

9. Karlin, S. On Hermite-Birkhoff Interpolation [Text] / S. Karlin, J. Karon // J. of Approximation Theory. – 1972. – No. 6. – P. 90-114.

10. Рвачева, Т. В. Об асимптотике базисных функций обобщенного ряда Тейлора [Текст] / Т. В. Рвачева // Вісник ХНУ, сер. «Математика, прикладна математика і механіка». – 2003. – № 602. – С. 94–104.

11. Jwamer, K. Lacunary Interpolation Using Quartic B-Spline [Text] / K. Jwamer, B. Jamal // General Letters in Mathematic. – 2017. – Vol. 2, No. 3. – P. 129-137.

12. Al Bayati, Abbas Y. Construction of Lacunary Sextic spline function Interpolation and their Applications [Text] / Abbas Y. Al Bayati, Rostam K. Saeed, Faraidum K. Hama-Salh // J. Edu. & Sci. – 2010. – Vol. 23, No. 3. – P. 108-115. DOI: 10.33899/edusj.2010.58392.

13. Al Bayati, A. Y. Lacunary Interpolation by Quartic Splines with Application to Quadratures [Text] / Abbas Y. Al Bayati, Rostam K. Saeed, Faraidum K. Hama-Salh // Int. J. Open Problems Compt. Math. – 2010. – Vol. 3, No. 3. – P. 315-328.

14. Ponomaryov, V. Super-Resolution Procedures in Image and Video Sequences based on Wavelet Atomic Functions [Text] / V. Ponomaryov, F. Gomeztagle // Modelling, Simulation and Identification. – Sciyo, 2010. – P. 101-125. DOI: 10.5772/10016.

15. Ponomaryov, V. Optimal Wavelet Filters Selection for Ultrasound and Mammography Compression [Text] / V. Ponomaryov, J. L. Sanchez-Ramirez, C. Juarez-Landin // In Progress in Pattern Recognition, Image Analysis and Applications, Proceedings of 13th Iberoamerican Congress on Pattern Recognition, CIARP 2008, Havana, Cuba, September 9-12, 2008. – P. 62-69.

16. Kolodyazhny, V. M. Application of atomic functions to numeric simulation in electromagnetic theory [Text] / V. M. Kolodyazhny, V. A. Rvachev // MSMW'O4 Symposium Proceedings, Kharkov, Ukraine, June 21-26, 2004. – P. 916-918.

17. Кузниченко, В. М. Обобщенные ряды Тейлора для класса функций H(ρ, m, r) [Текст] / В. М. Кузниченко // Матем. заметки. – 1989. – Том 46, Вып. 4. – С. 120–122.

18. Kozulić, V. Computational Modeling of Structural Problems Using Atomic Basis Functions [Text] / V. Kozulić, B. Blaž Gotovac // In Mechanical and Materials Engineering of Modern Structure and Component Design. – Springer, 2015. – P. 207-229.

19. Brysina, I. V. Generalized atomic wavelets [Text] / I. V. Brysina, V. O. Makarichev // Radioelectronic and Computer Systems. – 2018. – No. 1 (85). – P. 23-31. DOI: 10.32620/reks.2018.1.03.

20. Теория R-функций и актуальные проблемы прикладной математики [Текст] / Ю.Г. Стоян, В. С. Проценко, Г. П. Манько, И. В. Гончарюк, Л. В. Курпа, В. А. Рвачев, Н. С. Синекоп, И. Б. Сироджа, А. Н. Шевченко, Т. И. Шейко. – К. : Наукова думка, 1986. – 264 с.

21. Makarichev, V. A. Approximation of periodic functions by $mup_s(x)$ [Text] / V. A. Makarichev // Math. Notes. – 2013. – Vol. 93, No. 6. – P. 858-880.

22. Brysina, I. V. Atomic wavelets [Text]/ I. V. Brysina, V. A. Makarichev // Radioelectronic and Computer Systems. – 2012. – № 1(53). – P. 37-45.

23. Tsay, R. S. Analysis of financial time series [Text] / R. S. Tsay. – John Wiley and Sons, 2010. – 714 p.

24. Dung, D. Hyperbolic Cross Approximation [Text] / D. Dung, V. Temlyakov, T. Ulrich. – Springer Nature, 2018. – 218 p.

25. Jwamer, Karwan H. F. (0,1,3) Lacunary Interpolation with Splines of Degree Six [Text] / Karwan H. F. Jwamer, Rostam K. Saeed // Journal of Applied and Industrial Sciences. – 2013. – Vol. 1(1). – P. 21- 24.

26. Jwamer, Karwan H. F. New Construction Seven Degree Spline Function to Solve Second Order Initial Value Problem [Text] / Karwan H. F. Jwamer, Abdullah I. Najim // American Journal of Numerical. – 2016. – Vol. 4, No. 1. – P. 11-20.

27. Jwamer, Karwan H. F. Generalization of (0, 4) Lacunary Interpolation by Quantic Spline [Text] / Karwan H. F. Jwamer, Karem G. Ridha // Journal of Mathematics and Statistics. – 2010. – Vol. 6, No. 1. – P. 72-78.

28. Viswanathan, P. Lacunary Interpolation by Fractal Splines with Variable Scaling Parameters [Text] / P. Viswanathan, A. K. B. Chand, K. R. Tyada // Numer. Math. Theor. Meth. Appl. – 2017. – Vol. 10, No. 1. – P. 65-83. DOI: 10.4208/nmtma.2017.m1514.

29. Jwamer, Karwan H. F. New Construction and New Error Bounds for (0, 2, 4) Lacunary Interpolation By Six Degree Spline [Text] / Karwan H. F. Jwamer, Ridha G. Karem // Raf. J. of Comp. & Math's. – 2011. – Vol. 8, No 1. – P. 37-46. DOI: 10.33899/csmj.2011.163606.

30. Feng-Gong Lang. Error Analysis for a Noisy Lacunary Cubic Spline Interpolation and a Simple Noisy Cubic Spline Quasi Interpolation [Text] / Feng-

Gong Lang, Xiao-Ping Xu // Hindawi Publishing Corporation Advances in Numerical Analysis. – 2014. Article ID 353194. – P. 1-8.

31. Singh, Kulbhushan. A Special Quintic Spline for (0,1,4) Lacunary Interpolation and Cauchy Initial Value Problem [Text] / Kulbhushan Singh // Journal of Mechanics of Continua and Mathematical Sciences. Mech. – 2019. – Vol. 14, No 4. – P. 533-537.

32. Hamasalh, Faraidun K. Inhomogeneous Lacunary Interpolation and Optimization Errors Bound of Seventh Spline [Text] / Faraidun K. Hamasalh, Karwan H. F. Jwamer // American Journal of Applied Mathematics and Statistics. – 2013. – Vol. 1, No. 3. – P. 46-51. DOI: 10.12691/ajams-1-3-3.

33. Karaballi, A. A. Lacunary interpolation by quartic splines on uniform meshes [Text] / A. A. Karaballi, S. Sallam // Journal of Computational and Applied Mathematics. – 1997. – Vol. 80, Iss. 1. – P. 97-104. DOI: 10.1016/S0377-0427(97)00015-0.

34. Srivastava, R. A New Kind of Lacunary Interpolation through g-Splines [Text] / R. Srivastava // International Journal of Innovative Research in Science, Engineering and Technology (An ISO 3297: 2007 Certified Organization). – August 2015. – Vol. 4, Iss. 8. – P. 7783-7786.

35. Kozulić, Vedrana. Application of the Solution Structure Method in Numerically Solving Poisson's Equation on the Basis of Atomic Functions [Text] / Vedrana Kozulić, Blaž Gotovac // International Journal of Computational Methods. – 2018. – Vol. 15, No. 05. Art. 1850033. – P. 1850033-1 1850033-25.

36. Lattice-based integration algorithms: Kronecker sequences and rank-1 lattices [Text] / Josef Dick,· Friedrich Pillichshammer, Kosuke Suzuki, Mario Ullrich, Takehito Yoshiki // Annali di Matematica. – 2018. – No. 197. – P. 109–126. DOI: 10.1007/s10231-017-0670-3.

References (BSI)

1. Rvachev, V. L., Rvachev, V. A. *Neklassicheskie metody teorii priblizhenii v kraevykh zadachakh* [Non-classical methods of approximation theory in boundary value problems]. Kyiv, "Naukova dumka" Publ., 1979. 196 p.

2. Rvachev, V. A. Compactly supported solutions of functional-differential equations and their applications. *Russian Math. Surveys*, 1990, vol. 45, no. 1, pp. 87-120.

3. Lemarié-Rieusset, P.G Interpolating scaling functions, Bernstein polynomials and nonstationary wavelets. *Revista Matemática Iberoamericana*, 1997, vol. 13, Iss. 1, pp. 91-188.

4. Rvachov, V. O., Rvachova, T. V., Tomilova, Ye. P. Application of the Generalized Taylor – Birkhoff Series for Solving of the Initial Value Problem for Ordinary Differential Equations. *Otkrytye informatsionnye i komp'yuternye integrirovannye tekhnologii : sb. nauch. tr. KhAI – Open information and computer integrated technologies KhAI*, 2018, no. 79, pp. 153-161.

5. Rvachova, T. V., Tomilova, Ye. P. Finding Antiderivatives with the Help of the Generalized Taylor Series. *Otkrytye informatsionnye i komp'yuternye integrirovannye tekhnologii : sb. nauch. tr. KhAI – Open information and computer integrated technologies KhAI*, 2016, no. 73, pp. 52-58.

6. Rvachov, V. O., Rvachova, T. V. On the construction of multimodal multiparameter exponential families probability laws. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems,* 2011, no. 4 (52), pp.72-76.

7. Rvachova, T. V. On a relation between the coefficients and the sum of the generalized Taylor series. *Matematicheskaya fizika, analiz, geometriya,* 2003, vol. 10, no. 2, pp. 262-268.

8. Rvachev, V. A., Rvacheva, T. V. Ob ermitovoi interpolyatsii s pomoshch'yu atomarnykh funktsii [On the hermite interpolation with the help of the atomic functions]. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2010, no. 4(45), pp. 100-104.

9. Karlin, S., Karon, J. On Hermite-Birkhoff Interpolation. *J. of Approximation Theory,* 1972, no. 6, pp. 90-114.

10. Rvachova, T. V. Ob asimptotike bazisnykh funktsii obobshchennogo ryada Teilora [On the asymptotics of the basis functions of a generalized Taylor series]. *Visnyk KhNU, ser. «Matematyka, prykladna matematyka i mekhanika» – KhNU Bulletin, ser. "Mathematics, Applied Mathematics and Mechanics"*, 2003, no. 602, pp. 94–104.

11. Jwamer, K., Jamal, B. Lacunary Interpolation Using Quartic B-Spline. *General Letters in Mathematic,* vol. 2, no. 3, June 2017, pp. 129-137.

12. Al Bayati, Abbas Y., Saeed, Rostam K., Hama-Salh, Faraidum K. Construction of Lacunary Sextic spline function Interpolation and their Applications. *J. Edu. & Sci.*, 2010, vol. 23, no. 3, pp. 108-115. DOI: 10.33899/edusj.2010.58392.

13. Al Bayati, Abbas Y., Saeed, Rostam K., Hama-Salh, Faraidum K. Lacunary Interpolation by Quartic Splines with Application to Quadratures. *Int. J. Open Problems Compt. Math.*, 2010, vol. 3, no. 3, pp. 315-328.

14. Ponomaryov, V., Gomeztagle, F. Super-Resolution Procedures in Image and Video Sequences based on Wavelet Atomic Functions. *Modelling, Simulation and Identification,* Sciyo, 2010, pp. 101-125. DOI: 10.5772/10016.

15. Ponomaryov, V., Sanchez-Ramirez, J. L, Juarez-Landin, C. Optimal Wavelet Filters Selection for Ultrasound and Mammography Compression. *In Progress in Pattern Recognition, Image Analysis and Applications, Proceedings of 13th Iberoamerican Congress on Pattern Recognition, CIARP 2008,* Havana, Cuba, September 9-12, 2008, pp. 62-69.

16. Kolodyazhny, V. M., Rvachev, V. A. Application of atomic functions to numeric simulation in electromagnetic theory. *MSMW'O4 Symposium Proceedings, Kharkov*, Ukraine, June 21-26, 2004, pp. 916-918.

17. Kuznichenko V. M. Obobshchennye ryady Teilora dlya klassa funktsii H(ρ,m,r) [Taylor generalized series for the class of functions H(ρ,m,r)]. *Matematicheskie zametki – Mathematical Notes*, 1989, vol. 46, no. 4, pp. 120-122.

18. Kozulić, V., Gotovac, Blaž B. Computational Modeling of Structural Problems Using Atomic Basis

Functions. *In Mechanical and Materials Engineering of Modern Structure and Component Design*, Springer, 2015, pp. 207-229.

19. Brysina, I. V., Makarichev, V. A. Generalized atomic wavelets. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2018, no. 1(85), pp. 23-31. DOI: 10.32620/reks.2018.1.03.

20. Stoyan, Yu. G., Protsenko, V. S., Man'ko, G. P., Goncharyuk, I. V., Kurpa, L. V., Rvachev, V. A., Sinekop, N. S., Sirodzha, I. B., Shevchenko, A. N., Sheiko, T. I. Teorija R-funkcij i aktual'nye problemy prikladnoj matematiki [Theory of R-functions and current problems of applied mathematics]. Kyiv, "Naukova dumka" Publ., 1986. 264 p.

21. Makarichev, V. A. Approximation of periodic functions by $mup_s(x)$. *Math. Notes*, 2013, vol. 93, no. 6, pp. 858-880.

22. Brysina, I. V., Makarichev, V. A. Atomic wavelets. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2012, no. 1(53), pp. 37-45.

23. Tsay, R. S. *Analysis of financial time series*. "John Wiley and Sons" Publ., 2010. 714 p.

24. Dung, D., Temlyakov, V., Ulrich, T. Hyperbolic Cross Approximation. *Springer Nature*, 2018. 218 p.

25. Jwamer, Karwan, H. F., Saeed, Rostam K. (0,1,3) Lacunary Interpolation with Splines of Degree Six. *Journal of Applied and Industrial Sciences*, 2013, vol. 1(1), pp. 21- 24.

26. Jwamer, Karwan H. F., Najim, Abdullah I. New Construction Seven Degree Spline Function to Solve Second Order Initial Value Problem. *American Journal of Numerical*, 2016, vol. 4, no. 1, pp. 11-20.

27. Jwamer, Karwan H. F., Ridha, G. Karem. Generalization of (0, 4) Lacunary Interpolation by Quantic Spline. *Journal of Mathematics and Statistics*, 2010, vol. 6, no. 1, pp. 72-78.

28. Viswanathan, P., Chand, A. K. B., Tyada, K. R. Lacunary Interpolation by Fractal Splines with Variable Scaling Parameters. *Numer. Math. Theor. Meth. Appl.*, 2017, vol. 10, no. 1, pp. 65-83. DOI: 10.4208/nmtma.2017.m1514.

29. Jwamer, Karwan H. F., Karem, Ridha G. New Construction and New Error Bounds for (0, 2, 4) Lacunary Interpolation By Six Degree Spline. *Raf. J. of Comp. & Math's.*, 2011, vol. 8, no. 1, pp. 37-46. DOI: 10.33899/csmj.2011.163606.

30. Lang, Feng-Gong, Xu, Xiao-Ping. Error Analysis for a Noisy Lacunary Cubic Spline Interpolation and a Simple Noisy Cubic Spline Quasi Interpolation. *Hindawi Publishing Corporation Advances in Numerical Analysis*, 2014, Article ID 353194, pp. 1-8.

31. Singh, Kulbhushan. A Special Quintic Spline for (0,1,4) Lacunary Interpolation and Cauchy Initial Value Problem. *Journal of Mechanics of Continua and Mathematical Sciences. Mech.*, 2019, vol. 14, no. 4, pp. 533-537.

32. Hamasalh, Faraidun K., Jwamer, Karwan H. F. Inhomogeneous Lacunary Interpolation and Optimization Errors Bound of Seventh Spline. *American Journal of Applied Mathematics and Statistics*, 2013, vol. 1, no. 3, pp. 46-51. DOI: 10.12691/ajams-1-3-3.

33. Karaballi, A. A., Sallam, S. Lacunary interpolation by quartic splines on uniform meshes. *Journal of Computational and Applied Mathematics*, 1997, no. 80, Iss. 1, pp. 97-104. DOI: 10.1016/S0377-0427(97)00015-0.

34. Srivastava, R. A New Kind of Lacunary Interpolation through g-Splines. *International Journal of Innovative Research in Science, Engineering and Technology (An ISO 3297: 2007) Certified Organization*, 2015, vol. 4, Iss. 8, pp. 7783-7786.

35. Kozulić, Vedrana., Gotovac, Blaž. Application of the Solution Structure Method in Numerically Solving Poisson's Equation on the Basis of Atomic Functions. *International Journal of Computational Methods*, 2018, vol. 15, no. 05, Art. 1850033, pp. 1850033-1 1850033-25.

36. Dick, J., Pillichshammer, F., Suzuki, K., Ullrich, M., Yoshiki. T. Lattice-based integration algorithms: Kronecker sequences and rank-1 lattices. *Annali di Matematica*, 2018, no. 197, pp. 109–126. DOI: 10.1007/s10231-017-0670-3.

Поступила в редакцию 3.01.2020, рассмотрена на редколлегии 20.01.2020

ТОМІК ФУНКЦІЇ ТА ЛАКУНАРНІ ІНТЕРПОЛЯЦІЙНІ РЯДИ У КРАЙОВИХ ЗАДАЧАХ ДЛЯ РІВНЯНЬ З ЧАСТИННИМИ ПОХІДНИМИ ТА ОБРОБЦІ ЗОБРАЖЕНЬ

В. О. Рвачов, Т. В. Рвачова, Є. П. Томілова

У цій статті ми розглядаємо і вирішуємо завдання побудови так званих томік функцій – систем нескінченно диференційовних функцій, які, зберігаючи багато важливих властивостей зсувів атомарної функції up(x) таких як локальність і зображення алгебраїчних многочленів і засновані на атомарних функціях, проте мають неоднорідний характер і отже дозволяють враховувати неоднорідний і мінливий характер даних, що зустрічаються в задачах реального світу, зокрема в крайових задачах для рівнянь з частинними похідними з змінними коефіцієнтами і складною геометрією областей, в яких ці крайові задачі вирішуються. Той же клас томік функцій може застосовуватися для обробки, усунення шумів і економного зберігання сигналів і зображень за допомогою лакунарної інтерполяції. Лакунарна або Біркгоффова інтерполяція функцій, в якій функція відновлюється за значеннями похідних порядку r в точках, в яких значення функції і її похідних порядку k<r невідомі, має велике значення в багатьох реальних задачах, таких, наприклад, як дистанційне зондування. Методи лакунарної інтерполяції, що використовують томік функції, мають важливі переваги у порівнянні з широко використовуваною лакунарною сплайн-інтерполяцією через нескінченну гладкість томик функцій. Томік функції також можуть застосовуватися для з'єднання (зшивання) атомарних розкладів з різним кроком на різних інтервалах, зберігаючи гладкість і оптимальні апроксимаційні властивості. Отримані рівняння для побудови томік функцій $tofu_j(x)$ – аналог

базисних функцій узагальнених атомарних рядів Тейлора, які потрібні для лакунарної (Біркгоффової) інтерполяції. Матриці лінійних алгебраїчних систем для обчислення коефіцієнтів томік функцій мають спеціальну блок-діагональну структуру і легко обернені. Для застосувань у варіаційних і коллокаційних методах розв'язання крайових задач для рівнянь з частинними похідними і інтегральних рівнянь отримані томік функції $ftup_{r,j}(x)$, які є аналогами В-сплайнів і атомарних функцій $fup_n(x)$. Використовуючи подібні методи, можна побудувати томік функції, засновані на інших атомарних функціях, таких, як $\Xi_n(x)$.

Ключові слова: атомарні функції; томік функції; лакунарна інтерполяція; Біркгоффова інтерполяція; обробка і зберігання зображень; варіаційний метод; метод коллокації.

ТОМИК ФУНКЦИИ И ЛАКУНАРНЫЕ ИНТЕРПОЛЯЦИОННЫЕ РЯДЫ В КРАЕВЫХ ЗАДАЧАХ ДЛЯ УРАВНЕНИЙ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ И ОБРАБОТКЕ ИЗОБРАЖЕНИЙ

В. А. Рвачёв, Т. В. Рвачёва, Е. П. Томилова

В этой статье мы рассматриваем и решаем задачу построения так называемых томик функций – систем бесконечно дифференцируемых функций, которые, сохраняя много важных свойств сдвигов атомарной функции $up(x)$ таких как локальность и представление алгебраических многочленов и основанные на атомарных функциях, тем не менее имеют неоднородный характер и следовательно позволяют учитывать неоднородный и изменчивый характер данных встречаемых в задачах реального мира, в частности в краевых задачах для уравнений с частными производными с переменными коэффициентами и сложной геометрией областей, в которых эти краевые задачи решаются. Тот же класс томик функций может применяться для обработки, устранения шумов и экономного хранения сигналов и изображений с помощью лакунарной интерполяции. Лакунарная или Биркгоффова интерполяция функций, в которой функция восстанавливается по значениям производных порядка r в точках, в которых значения функции и ее производных порядка $k<r$ неизвестны, имеет большое значение во многих реальных задачах, таких как дистанционное зондирование. Методы лакунарной интерполяции, использующие томик функции, обладают важными преимуществами по сравнению с широко используемой лакунарной сплайн-интерполяцией ввиду бесконечной гладкости томик функций. Томик функции также могут применяться для соединения (сшивания) атомарных разложений с различным шагом на разных интервалах, сохраняя гладкость и оптимальные аппроксимационные свойства. Получены уравнения для построения томик функций $tofu_j(x)$ – аналогов базисных функций обобщенных атомарных рядов Тейлора, которые нужны для лакунарной (Биркгоффовой) интерполяции. Матрицы линейных алгебраических систем для вычисления коэффициентов томик функций имеют специальную блок-диагональную структуру и легко обращаются. Для приложений в вариационных и коллокационных методах решения краевых задач для уравнений с частными производными и интегральных уравнений получены томик функции $ftup_{r,j}(x)$, которые являются аналогами В-сплайнов и атомарных функций $fup_n(x)$. Используя подобные методы, можно построить томик функции, основанные на других атомарных функциях, таких, как $\Xi_n(x)$.

Ключевые слова: атомарные функции; томик функции; лакунарная интерполяция; Биркгоффова интерполяция; обработка и хранение изображений; вариационный метод; метод коллокации.

Рвачев Владимир Алексеевич – д-р физ.-мат. наук, проф., проф. каф. высшей математики и системного анализа, Национальный аэрокосмический университет им. Н. Е. Жуковского «Харьковский авиационный институт», Харьков, Украина.

Рвачева Татьяна Владимировна – канд. физ.-мат. наук, доцент кафедры высшей математики и системного анализа, Национальный аэрокосмический университет им. Н. Е. Жуковского «Харьковский авиационный институт», Харьков, Украина.

Томилова Евгения Павловна – старший преподаватель кафедры высшей математики и системного анализа, Национальный аэрокосмический университет им. Н. Е. Жуковского «Харьковский авиационный институт», Харьков, Украина.

Rvachov Volodimir Olexijovych – DSc in Physics and Mathematics, PhD in Physics and Mathematics, Professor of Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkov Aviation Institute", Kharkov, Ukraine, e-mail: rvachov@gmail.com, ORCID Author ID: 0000-0002-5732-7405.

Rvachova Tatiana Volodimirivna – PhD in Physics and Mathematics, Associate Professor of Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkov Aviation Institute", Kharkov, Ukraine, e-mail: rvachova@gmail.com, ORCID Author ID: 0000-0003-0674-234X

Tomilova Evgenia Pavlovna – Senior teacher of Department of Higher Mathematics and System Analysis, National Aerospace University "Kharkov Aviation Institute", Kharkov, Ukraine, e-mail: tomilova.evgenia @gmail.com.