ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

UDC 517.927:517.956.225 **doi: 10.32620/reks.2022.4.03**

Oleksii NIKOLAEV, Oleksandr HOLOVCHENKO, Nina SAVCHENKO

National Aerospace University "Kharkiv Aviation Institute", Kharkiv, Ukraine

GREEN'S FUNCTIONS OF THE FIRST AND SECOND BOUNDARY VALUE PROBLEMS FOR THE LAPLACE EQUATION IN THE NONCLASSICAL DOMAIN

The ***subject*** *of study is the Green's functions of the first and second boundary value problems for the Laplace equation. The study constructs the Green's functions of the first and second boundary value problems for the Laplace equation in space with a spherical segment in analytical form, as well as numerical analysis of these functions. Research* ***task****: to formalize the problem of determining Green's functions for the specified domain; using methods of Fourier, pair summation equations and potential theory to reduce mixed boundary value problems for auxiliary harmonic functions to a system of equations that has an analytical solution; investigate the compatibility of the algebraic system for determining constants of integration; formulate and prove a theorem about the jump of the normal derivative of the potential of a simple layer on the surface of a segment, with the help of which to present the Green's function in the form of the potential of a simple layer; conduct a numerical experiment and identify algorithms and areas of changing the parameters of effective calculations; analyze the behavior of Green's functions. Scientific* ***novelty****: for the first time, Green's functions of Dirichlet and Neumann boundary value problems for the Laplace equation in three-dimensional space with a spherical segment were constructed in analytical form, the obtained results were substantiated, and a comprehensive numerical experiment was conducted to analyze the behavior of these functions. The obtained* ***results****: mixed boundary value problems in the interior and exterior of the spherical surface to which the segment belongs are set for the auxiliary harmonic functions; using the Fourier method, the problem is reduced to systems of paired equations in series by Legendre functions, the solutions of which are found using discontinuous Mehler-Dirichlet sums. The specified functions are obtained in an explicit view in two forms: series based on the basic harmonic functions in spherical coordinates and the potential of a simple layer on the surface of the segment. To substantiate the results, the lemma on the compatibility of the algebraic system for determining the constants of integration and the theorem on the jump of the normal derivative of the potential of a simple layer on a segment are proved. A numerical experiment was conducted to analyze the behavior of the constructed functions.* ***Conclusions****: the analysis of numerical values of Green's functions obtained by different algorithms showed that the highest accuracy of results outside the surface of the segment was obtained when using images of Green's functions in the form of series. On the basis of the calculations, the lines of the level of the Green's functions of two boundary value problems in the plane of the singular point, as well as the graphs of the potential density of the simple layer for the Dirichlet problem and the potential jump for the Neumann problem on the segment at different locations of the singular point were constructed. In the partial case of the location of a singular point at the origin of the coordinates, the potential of the electrostatic field of a point charge near a conductive grounded thin shell in the form of a spherical segment is found. The main characteristics of such a field are found in closed form.*

Keywords: *Green's function; Dirichlet boundary value problem; Neumann boundary value problem; Laplace equation; harmonic function; spherical segment; simple layer potential; spherical functions; potential jump.*

Introduction

The mathematical apparatus of Green's functions has had a significant impact on the development of methods of mathematical physics over the past two centuries. It had been widely used to represent solutions of linear differential equations in space and in domains of different dimensions, reduce boundary value problems to integral equations, prove the existence of solutions to boundary value problems, construct efficient numerical methods, and analyze various types of physical fields with point and distributed sources.

The Green's function was first introduced by George Green in [1] for the Laplace equation. In the same essay, Green showed the use of this function to find the potentials of electrostatic and magnetostatic fields. Many mathematicians of the 19th century after Green set out to find solutions to the Dirichlet and Neumann problems for various differential equations, following Green's method. They obtained integral formulas, introduced functions similar to Green's function, studied their properties and applied them to solve problems in acoustics, hydrodynamics, heat conduction, magnetism, electrodynamics, elasticity theory for second-order linear differential equations of elliptic, hyperbolic and parabolic types. Among these scientists were H. Weber, L. Schläfli, C. Neumann, F. Neumann, E. Betti, R. Lipschitz, H. Helmholtz, B. Rieman, U. Dini and others. In commemoration of Green's contributions, all similar functions were also later called as Green's functions.

The problem of explicitly constructing the Green's function was rather simply solved for the Laplace equation in a plane domain. Even B. Rieman drew attention to the fact that the Green's function for a plane domain is closely related to the analytic function that performs a conformal mapping of the domain onto a unit circle. The existence of such a function for a simply connected plane domain with a piecewise smooth boundary follows from the well-known Riemann theorem, which he proved in 1851 [2]. The latter circumstance made it possible to construct Green's functions for many canonical domains on the plane [4]. The Green's function for some canonical spatial domains was constructed using the image method, the method of separation of variables in the Laplace equation in curvilinear coordinate systems, series in eigenfunctions of boundary value problems, and the method of integral transformations [4]. To apply the image method, a certain symmetry of the region is required, which, for example, takes place for a half-space and a sphere. Symmetry considerations were used by A. Greenhill in a rectangular parallelepiped, expressing the Green's function by series in elliptic theta functions [5]. The main difficulties in the implementation of the method of separation of variables were associated with the representation of the fundamental solution by a series or an integral over the basis harmonic functions in the corresponding area. In particular, they were overcome in [6] in cylindrical and spherical coordinate systems. The expansion of the Green's function for a rectangular parallelepiped in terms of the eigenfunctions of the Laplace operator is given by B. Riemann in his lectures [7]. Similar approaches were applied to the Helmholtz equation, the wave equation, the Klein-Gordon equation, the heat-conduction equation [3, 4, 8, 9]. An interesting version of the image method was proposed by A. Sommerfeld. In [10], he constructed the Green's function in a closed form for the problem of plane wave diffraction on a half-plane. In this case, the method of images on Riemann surfaces corresponding to multivalued solutions of the Helmholtz equation was used. Later, this method was used by E. W. Hobson, who constructed the Green's function for the Laplace equation for a disk [11]. The connection of the solutions of the equations of the static theory of elasticity with harmonic functions made it possible to solve the problem of concentrated forces for an unbounded and semi-limited homogeneous and isotropic elastic medium [12-14]. The explicit construction of the Green's tensor for the equilibrium equation in the displacements of an anisotropic elastic space is given in the article [15] using the integral Fourier transform.

The modern stage in the creation of the theory of Green's functions began in the second half of the 20th century. At this time, the ideas of the classics of natural science developed in several directions. In potential theory, one of the directions concerned multiply connected domains. For a plane multiply connected domain bounded by circles, the Green's function was found from the solution of the null-field integral equation by expanding the kernels of the equation and the function itself into Fourier series in angular variables [16]. In a particular case, a solution was obtained for an eccentric ring. The Green's function for a multiply connected domain of the complex plane bounded by polygons symmetric about the real axis was constructed in [17]. The method is based on a Schwarz–Christoffel conformal map of the part of the upper half-plane exterior to the problem domain onto a semi-infinite strip whose end contains some slits. For multiply connected domains with circular boundaries on the plane, Green's functions were constructed using the analytical Schottky–Klein function [18]. In a number of articles spatial multiconnected areas were considered. Thus, in the article [19], in fact, a generalization of the result of E. W. Hobson to the case of a system of coaxial circular disks is given. The potential is represented by a simple layer of charged disks. A system of integral equations of the Abel type is obtained for the charge densities of disks. For a space with a system of arbitrarily located spherical cavities, when constructing the Green's function, the addition theorems for spherical harmonics were used [20]. In this case, the coefficients of expansions of the desired functions into series in terms of spherical functions must satisfy infinite systems of linear algebraic equations.

The method of images also received a rebirth. In [21], Green's functions for the Laplace equation in the exterior and interior of a prolate spheroid, written as series in terms of harmonic basis functions in these domains, are used to implement the imaging method. The field created by the potential distribution in the form of a Green's function is represented by two point sources of a field in the exterior and interior of the spheroid and an additional field created by a surface charge on a confocal spheroidal surface located outside the main domain.

The techniques of integral Fourier transform and a Kontorovich–Lebedev transform were used to find the Green's function for two homogeneous wedges complementing each other up to space [21].

The above methods have also been widely used in problems of elasticity theory [23–30], acoustics [31], wave dynamics [32], stationary hydrodynamics [33, 34], and diffraction theory [35].

In [23], the Green's matrix was constructed using the Fourier transform for a homogeneous and orthotropic half-strip sandwiched between two absolutely rigid half-

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

planes. The elements of the Green's matrix, which physically represent the displacements of the points of the half-strip under the action of a concentrated force, are expressed in terms of elementary functions. Article [24] presents a method for calculating the static Green's functions in a multilayered transversally isotropic or isotropic half-space. To obtain Green's functions in the transformed domain, a cylindrical system of vector functions and the propagator matrix method are used. In the article [25], the Green's functions for a periodic system of concentrated forces and dislocations for a plane and a half-plane are constructed using the methods of the theory of analytic functions. Paper [26] presents an analysis of the problem of an external circular crack in a transversely isotropic piezoelectric solid subject to normal stresses and electric charges symmetrical about the plane of the crack. The obtained Green's functions for a point force and a point charge are recognized as exact and are expressed in terms of elementary functions. In article [27], Green's function is constructed, which characterizes the coupled electroelastic fields in a composite piezoceramic wedge under the action of concentrated shear forces or electric charges. Mellin and Fourier integral transformations are used. In [28] the three-dimensional Green's functions for anisotropic bimaterials are found using two-dimensional Fourier transforms. The Green's functions in the physical domain are expressed as the sum of the spatial Green's function and an additional part in the form of regular integrals. Paper [29] constructed Green's functions with selected features of axisymmetric and three-dimensional static elasticity problems for isotropic layered half-spaces and layers. The Hankel integral transform was used for axisymmetric cases and the Fourier transform in the three-dimensional case. In [30] Green's formulas for thermoelastic displacements and stresses in a half-plane were obtained for a point heat source.

It is important to note the directions of applications of the Green's function. In addition to the traditional ones described at the beginning of this section, there are works devoted to diagnosing cavities in plane bodies by measuring electrostatic or thermostatic characteristics at their boundaries [36], modeling seismic and postseismic deformations of faults in the Earth’s crust [37], layered nano- structures in semiconductors [38], stress and strain fields in composite materials [39], inverse problems of identifying sources of potential fields [40], etc.

In this article, the Green's functions of the first and second boundary value problems for the Laplace equation are constructed analytically in a nonclassical domain, which is considered to be a space with a spherical segment. The problems posed in it are reduced to solving mixed non-axisymmetric boundary value problems of potential theory. The class of mixed problems for the equations of mathematical physics includes problems in which there are lines of change of the type of boundary conditions on the boundary surfaces. Studies of these problems were widely developed in the second half of the 20th century. Some idea of the bibliography on this topic can be obtained in the books [41, 42]. In the very formulation of mixed problems, in contrast to the main problems, there are analytical difficulties associated with satisfying the boundary conditions, and computational difficulties with the slow convergence of computational algorithms. This may explain the small number of studies that deal with non-axisymmetric mixed problems. In the axisymmetric case, problems for a spherical segment in space were considered in [42–44]. In them, the exact satisfaction of mixed boundary conditions was based on the use of discontinuous Mehler – Dirichlet sums [45]

$$\sum_{n=0}^{\infty}\cos[(n+1/2)t]P_n(\cos\theta)= \begin{cases}[2(\cos t-\cos\theta)]^{-1/2}, & 0\le t<\theta<\pi,\\ 0, & 0<\theta<t\le\pi,\end{cases} \quad (1)$$

$$\sum_{n=0}^{\infty}\sin[(n+1/2)t]P_n(\cos\theta)= \begin{cases}0, & 0\le t<\theta<\pi,\\ [2(\cos\theta-\cos t)]^{-1/2}, & 0<\theta<t\le\pi.\end{cases} \quad (2)$$

In the book [42], a similar approach is applied formally to a particular non-axisymmetric problem for a segment, and the solution of the problem is only outlined, but not brought to real results. In [46, 47], axisymmetric and reducible problems were solved for a segment located inside some canonical bodies, and the addition of the segment surface to a complete spherical surface crossed the boundary of the external body, i.e. the generalized Fourier method was not suitable for solving such problems.

1. Problem Statement

In space $\mathbb{R}^3$ Cartesian and spherical coordinates of a point x will be denoted by (x_1, x_2, x_3) and (r,θ,φ) respectively. The connection between them is carried out by formulas

$$\begin{cases}x_1 = r\sin\theta\cos\varphi,\\ x_2 = r\sin\theta\sin\varphi,\\ x_3 = r\cos\theta.\end{cases} \quad (3)$$

Consider a spherical segment in space

$$\Gamma=\{x\in\mathbb{R}^3 : r=R,\ \theta\in[0,\beta],\ \varphi\in[0,2\pi]\}.$$

Let us pose the problem of finding the Green's function $G(x,x_0)$, $|x_0|<R$ of the Dirichlet problem for the Laplace equation in the domain $\mathbb{R}^3\setminus\Gamma$, which satisfies the following conditions:

$$\Delta G(x,x_0)=0 \ \ \forall x\in\mathbb{R}^3\setminus\Gamma: x\neq x_0, \tag{4}$$

$$G(x,x_0)=F(x-x_0)+u(x,x_0), \tag{5}$$

$$G(x,x_0)_{|x\in\Gamma}=0, \tag{6}$$

where Δ – the Laplace operator, x_0 – singular point of the Green's function, $F(x-x_0)=-(4\pi|x-x_0|)^{-1}$ is a fundamental solution of the Laplace equation, $u(x,x_0)$ is a harmonic function in the domain $\mathbb{R}^3\setminus\Gamma$.

Harmonic function $u(x,x_0)$ we will find in the form

$$u(x,x_0)=\begin{cases}u^+(x,x_0), |x|<R,\\ u^-(x,x_0), |x|>R,\end{cases} \tag{7}$$

where $u^{\pm}(x,x_0)$ – harmonic functions in the corresponding areas, and the function $u^-(x,x_0)$ is regular at infinity.

Regarding functions $u^{\pm}(x,x_0)$, on a spherical surface the following conjugation conditions follow from the potential properties:

$$\left(u^+(x,x_0)-u^-(x,x_0)\right)_{|r=R}=0, \tag{8}$$

$$u^{\pm}(x,x_0)_{|\Gamma}=(4\pi|x-x_0|)^{-1}{}_{|\Gamma}, \tag{9}$$

$$\left(\frac{\partial u^+}{\partial r}-\frac{\partial u^-}{\partial r}\right)_{|r=R,\theta\in(\beta,\pi]}=0. \tag{10}$$

Thus, to find the Green's function, it is necessary to find harmonic functions $u^{\pm}(x,x_0)$ in relevant areas that satisfy the conditions (8) – (10).

Now consider the formulation of the problem of determining the Green's function of the Laplace equation for the Neumann problem. It satisfies the following conditions:

$$\Delta G(x,x_0)=0 \ \ \forall x\in\mathbb{R}^3\setminus\Gamma: x\neq x_0, \tag{11}$$

$$G(x,x_0)=F(x-x_0)+u(x,x_0), \tag{12}$$

$$\frac{\partial G}{\partial n}(x,x_0)_{|x\in\Gamma}=0, \tag{13}$$

where n – normal to the surface of the segment.

As before, we will find the harmonic function $u(x,x_0)$ in the form (7). Then, with respect to the functions $u^{\pm}(x,x_0)$ on the spherical surface, we obtain the following conjugation conditions:

$$\left(\frac{\partial u^+}{\partial n}(x,x_0)-\frac{\partial u^-}{\partial n}(x,x_0)\right)_{|r=R}=0, \tag{14}$$

$$\frac{\partial u^{\pm}}{\partial n}(x,x_0)_{|\Gamma}=\frac{1}{4\pi}\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)_{|\Gamma}, \tag{15}$$

$$(u^+-u^-)_{|r=R,\theta\in(\beta,\pi]}=0. \tag{16}$$

Therefore, the solution of the mixed potential theory problem (14) - (16) leads to finding the Green's function of the Neumann problem.

2. Reduction of the first problem to a system of paired equations

Let's write the Laplace equation in spherical coordinates

$$\Delta\equiv\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0.$$

Next, we will use the expansion of the fundamental solution of the Laplace equation into a series by spherical functions [4]

$$F(x-x_0)==-\frac{1}{4\pi}\sum_{n=0}^{\infty}\begin{Bmatrix}r^n/r_0^{n+1}\\ r_0^n/r^{n+1}\end{Bmatrix}\times$$
$$\times\sum_{m=-n}^{n}(-1)^m P_n^{-m}(\cos\theta_0)P_n^m(\cos\theta)e^{im(\varphi-\varphi_0)}, \tag{17}$$

where the upper (lower) multiplier in curly brackets is selected at $r<r_0$ ($r>r_0$), (r_0,θ_0,φ_0) – spherical coordinates of a point x_0, $P_n^m(x)$ – attached Legendre function of the first kind.

Considering the formula (17) and the symmetry of the function $G(x,x_0)$ with respect to the plane $\{\varphi=\varphi_0\vee\varphi=\varphi_0+\pi\}$ we will find the functions $u^{\pm}(x,x_0)$ in the form of series

$$u^+=\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n\sum_{m=0}^{n}a_{n,m}P_n^m(\cos\theta)\cos[m(\varphi-\varphi_0)],$$
$$r<R; \tag{18}$$

$$u^-=\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=0}^{n}b_{n,m}P_n^m(\cos\theta)\cos[m(\varphi-\varphi_0)],$$
$$r>R. \tag{19}$$

The conjugation conditions and boundary conditions (8)–(10) for functions (18), (19) lead to the system of equalities

$$a_{n,m}=b_{n,m},\ n=0\div\infty, m=0\div n. \tag{20}$$

and the system of paired summative equations with respect to the coefficients of expansions (18), (19)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_{n,m}P_n^m(\cos\theta)\cos[m(\varphi-\varphi_0)]=$$
$$=-F(x-x_0)_{|r=R},\ \theta\in[0,\beta), \varphi\in[0,2\pi], \tag{21}$$

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

$$\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n} a_{n,m}P_n^m(\cos\theta)\cos[m(\varphi-\varphi_0)]=0,$$
$$\theta\in(\beta,\pi],\ \varphi\in[0,2\pi]. \qquad (22)$$

Substitute (17) into the right-hand side of equation (21), then write down equations (21), (22) with respect to the Fourier coefficients according to the system of functions $\{\cos[m(\varphi-\varphi_0)]\}_{m=0}^{\infty}$

$$\sum_{n=m}^{\infty} a_{n,m}P_n^m(\cos\theta)=$$
$$=\frac{2-\delta_{m,0}}{4\pi R}(-1)^m\sum_{n=m}^{\infty}\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)P_n^m(\cos\theta),$$
$$\theta\in[0,\beta),\ m\in\mathbb{Z}_+, \qquad (23)$$

$$\sum_{n=m}^{\infty}(2n+1)a_{n,m}P_n^m(\cos\theta)=0,$$
$$\theta\in(\beta,\pi],\ m\in\mathbb{Z}_+, \qquad (24)$$

where $\delta_{m,k}$ – the Kronecker symbol.

Let's transform the system (23), (24). Using the fact that

$$P_n^m(\cos\theta)=(-1)^m\sin^m\theta\left(\frac{d^mP_n(x)}{dx^m}\right)_{|x=\cos\theta},$$

let's divide both parts of equality (23) by $(-1)^m\sin^m\theta$, after which we integrate them m times by variable $\cos\theta$

$$\sum_{n=0}^{\infty} a_{n,m}P_n(\cos\theta)=f_m(\theta)+\sum_{n=0}^{m-1} c_{n,m}P_n(\cos\theta),\ \theta\in[0,\beta) \qquad (25)$$

$$(-1)^m\frac{d^m}{d\cos\theta^m}\sum_{n=0}^{\infty}(2n+1)a_{n,m}P_n(\cos\theta)=0,\ \theta\in(\beta,\pi], \qquad (26)$$

$$f_m(\theta)=\frac{2-\delta_{m,0}}{4\pi R}(-1)^m\sum_{n=m}^{\infty}\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)P_n(\cos\theta).$$

Here, the constants of integration are denoted by $c_{n,m}$ at $n=0,1,\ldots,m-1,\ m\ge1$. In formula (25), the second sum on the right-hand side is empty at $m=0$.

3. Solving the system of paired equations

To solve the system of paired equations (25), (26), we apply the method of Mehler-Dirichlet discontinuous sums (1), (2), known from works [42-44]. First of all, we note that since formulas (1), (2) are actually decompositions of functions from their right-hand sides into Fourier series by trigonometric systems of functions $\{\cos[(n+1/2)t]\}_{n=0}^{\infty}$, $\{\sin[(n+1/2)t]\}_{n=0}^{\infty}$ on the segment $[0,\pi]$, then for the Fourier coefficients we obtain the identities

$$P_n(\cos\theta)=\frac{2}{\pi}\int_0^{\theta}\frac{\cos[(n+1/2)t]}{\sqrt{2(\cos t-\cos\theta)}}dt, \qquad (27)$$

$$P_n(\cos\theta)=\frac{2}{\pi}\int_{\theta}^{\pi}\frac{\sin[(n+1/2)t]}{\sqrt{2(\cos\theta-\cos t)}}dt. \qquad (28)$$

We will look for a solution of system (25), (26) in the form

$$a_{nm}=\int_0^{\beta}\varphi_m(t)\cos[(n+1/2)t]dt, \qquad (29)$$

where $\varphi_m(t)$ – some unknown function.

Considering that $\theta\in(\beta,\pi]$, let's substitute the integral (29) in the series located on the left side of equation (26). After integration by parts, we get

$$\sum_{n=0}^{\infty}(2n+1)\int_0^{\beta}\varphi_m(t)\cos[(n+1/2)t]dtP_n(\cos\theta)=$$
$$=2\varphi_m(\beta)\sum_{n=0}^{\infty}\sin[(n+1/2)\beta]P_n(\cos\theta)-$$
$$-2\int_0^{\beta}\varphi'_m(t)\sum_{n=0}^{\infty}\sin[(n+1/2)t]P_n(\cos\theta)dt. \qquad (30)$$

Here it is assumed that the function $\varphi_m(t)$ is sufficiently smooth and the permutation of the infinite sum and the integral is admissible. Since $t\le\beta<\theta$, then, according to formula (2), the right-hand side of the last formula is equal to zero, that is, equation (26) is identically fulfilled. Substitute integral (29) into equation (25)

$$\int_0^{\beta}\varphi_m(t)\sum_{n=0}^{\infty}\cos[(n+1/2)t]P_n(\cos\theta)dt=$$
$$=f_m(\theta)+\sum_{n=0}^{m-1}c_{n,m}P_n(\cos\theta),\ \theta\in[0,\beta).$$

Substitute the sum (1) in the left part of this equality, and the integral (27) in the right part

$$\int_0^{\theta}\frac{dt}{\sqrt{2(\cos t-\cos\theta)}}\left\{\varphi_m(t)-\frac{2}{\pi}\sum_{n=0}^{m-1}c_{nm}\cos[(n+\frac{1}{2})t]-\right.$$
$$\left.-\frac{2-\delta_{m,0}}{2\pi^2R}(-1)^m\sum_{n=0}^{\infty}\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)\cos[(n+\frac{1}{2})t]\right\}=$$
$$=0,\ m\in\mathbb{Z}_+.$$

Using the formulas of the direct and inverse integral transformation of the Abel type, it is possible to solve the previous integral equation with respect to the function $\varphi_m(t)$ ($m \ge 0$)

$$\varphi_m(t) = \frac{2-\delta_{m,0}}{2\pi^2 R}(-1)^m \sum_{n=0}^{\infty}\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)\times$$
$$\times\cos[(n+1/2)t] + \frac{2}{\pi}\sum_{n=0}^{m-1} c_{n,m}\cos[(n+1/2)t]\,. \quad (31)$$

To determine integration constants $\{c_{nm}\}_{m,n=0}^{\infty,m-1}$ (when $m=0$ this set is empty) we will require that the function $\left(\frac{\partial u^+}{\partial r} - \frac{\partial u^-}{\partial r}\right)_{|(r,\theta,\varphi)\in\Gamma}$ be integrable on the surface Γ. It is not difficult to understand that a necessary condition for this is the integrability of the function

$$I_m(\cos\theta) \equiv \sum_{n=0}^{\infty}(2n+1)a_{n,m}P_n^m(\cos\theta)$$

with weight $\sin\theta$ on the segment $\theta\in[0,\beta]$ for any $m \ge 0$. Considering the formulas (30), (2), we write

$$I_m(\cos\theta) == \sqrt{2}(-1)^m \sin^m\theta \frac{d^m}{d\cos\theta^m}\left\{\frac{\varphi_m(\beta)}{\sqrt{\cos\theta-\cos\beta}} - \right.$$
$$\left. -\int_0^{\beta}\frac{\varphi_m'(t)dt}{\sqrt{\cos\theta-\cos t}}\right\}. \quad (32)$$

According to the method [42], let's transform (32), by replacing the variables, $\cos\theta = v$, $\cos t = u$ and the integrand function $\varphi_m(t) = \psi_m(u)$

$$I_m(v) == \sqrt{2}(-1)^m(1-v^2)^{m/2}\frac{d^m}{dv^m}\left\{\frac{\psi_m(v_0)}{\sqrt{v-v_0}} + \right.$$
$$\left. +\int_{v_0}^{v}\frac{\psi_m'(u)du}{\sqrt{v-u}}\right\}. \quad (33)$$

Let's integrate the integral in (33) by parts m times

$$I_m(v) = \sqrt{2}(-1)^m(1-v^2)^{m/2}\times$$
$$\times\frac{d^m}{dv^m}\left\{\sum_{k=0}^{m}\chi_k(v,v_0)\psi_m^{(k)}(v_0) + \right.$$
$$\left. +\int_{v_0}^{v}\chi_m(v,u)\psi_m^{(m+1)}(u)du\right\}. \quad (34)$$

where $\chi_k(v,v_0) = \frac{\sqrt{\pi}}{\Gamma(k+1/2)}(v-v_0)^{k-1/2}$, $\Gamma(x)$ – gamma function. Formula (34) shows that the function in (32) has the singularity of order $-m-1/2$ at a point $v=v_0$ and, in order to get rid of non-integrable singularities in (34), it is necessary to impose additional conditions on the functions $\psi_m(u)$:

$$\psi_m(v_0) = \psi_m'(v_0) = \cdots = \psi_m^{(m-1)}(v_0) = 0 \quad (35)$$

or, what exactly,

$$\varphi_m(\beta) = \varphi_m'(\beta) = \cdots = \varphi_m^{(m-1)}(\beta) = 0\,. \quad (36)$$

Thus, to determine the integration constants $\{c_{nm}\}_{m,n=0}^{\infty,m-1}$ in (31) for each m, we have m conditions (36). Let's find out the possibility of finding constants from these conditions.

Lemma. For each $m=1,2,\ldots$ and arbitrary $\beta\in(0,\pi)$ algebraic system

$$\sum_{n=0}^{m-1} c_{n,m}(n+1/2)^k\cos[(n+1/2)\beta + \pi k/2] = g_{m,k},$$
$$k = 0,1,\ldots,m-1 \quad (37)$$

has a unique solution for any set $\{g_{m,k}\}_{k=0}^{m-1}\in\mathbb{R}^m$.

Proof. To prove the lemma, we show that the homogeneous system (37) has only a trivial solution. For this, we will fix the integer $m>0$ and consider the function

$$P(t) = \sum_{n=0}^{m-1} c_n\cos[(n+1/2)t]\,.$$

Let's multiply both parts of the previous equality by $2\cos(t/2)$

$$Q(t) \equiv 2\cos\frac{t}{2}P(t) = \sum_{n=0}^{m-1} c_n 2\cos\frac{t}{2}\cos[(n+1/2)t] =$$
$$= \sum_{n=0}^{m-1} c_n\cos[(n+1)t] + \sum_{n=0}^{m-1} c_n\cos(nt) = \sum_{n=0}^{m}\tilde{c}_n\cos(nt)\,,$$

where $\tilde{c}_0 = c_0$, $\tilde{c}_n = c_n + c_{n-1}$ $(n = 1\div m-1)$, $\tilde{c}_m = c_{m-1}$.

The function $Q(t)$ can be rewritten as

$$Q(t) = \left\{e^{-imt}\tilde{c}_m e^{2mti} + \cdots + \tilde{c}_1 e^{(m+1)ti} + 2\tilde{c}_0 e^{mti} + \right.$$
$$\left. +\tilde{c}_1 e^{(m-1)ti} + \cdots + \tilde{c}_m\right\}.$$

The expression in the curly brackets is the inverse polynomial relative to e^{it}. If such a polynomial has a root e^{it} $t\in(0,\pi)$, then it has a root e^{-it} the same multiplicity as the first.

Assume that the homogeneous system (37) at $\beta\in(0,\pi)$ has a non-trivial solution $\{c_n\}_{n=0}^{m-1}$. Then the function $P(t)$ satisfies the conditions

$$P(\beta) = P'(\beta) = \cdots = P^{(m-1)}(\beta) = 0\,.$$

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

It is obvious that the same conditions are satisfied by the function $Q(t)e^{imt}$. Then the polynomial

$$\tilde{c}_m e^{2mti} + \dots + \tilde{c}_1 e^{(m+1)ti} + 2\tilde{c}_0 e^{mti} + \\ + \tilde{c}_1 e^{(m-1)ti} + \dots + \tilde{c}_m$$

has a pair of complex conjugate roots $e^{\pm i\beta}$, both multiples of m. However, this polynomial has one more root $e^{i\pi}$, different from those specified. If a polynomial of degree $2m$ has $2m+1$ roots, then it is identically equal to zero, that is, the coefficients $\{\tilde{c}_n\}_{n=0}^{m}$, and with them $\{c_n\}_{n=0}^{m-1}$, are zero. The resulting contradiction with the assumption proves the lemma statement.

It follows from the lemma, formula (46) and conditions (36) that the constants of integration $\{c_{n,m}\}_{m,n=0}^{\infty,m-1}$ are found uniquely from the system (37) when

$$g_{m,k} = \frac{(-1)^{m+1}(2-\delta_{m,0})}{4\pi R} \times \\ \times \sum_{n=m}^{\infty} \left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)\left(n+\frac{1}{2}\right)^k \cos\left[\left(n+\frac{1}{2}\right)\beta + \frac{\pi k}{2}\right]$$

that is, formula (31) uniquely defines the function $\varphi_m(t)$.

Note that the function $\psi_m(u)$ at $m \in \mathbb{Z}_+$ is an analytic function on the interval $(-1,1)$, and a derivative $\psi_m^{(k)}(u)$ for each $k \in \mathbb{Z}_+$ is bounded in the neighborhood of a point $u=1$. The latter follows from formula (31) and the representation

$$\cos[(k+1/2)\arccos u] = \\ = \sum_{m=0}^{k} (-1)^{k-m} C_{2k+1}^{2m+1} \left(\frac{1+u}{2}\right)^{m+1/2} \left(\frac{1-u}{2}\right)^{k-m}.$$

This means that the conditions of smoothness of the function $\psi_m(u)$, which were assumed above, are fulfilled.

4. Construction of the Green's function of the first boundary value problem

Substituting the function $\varphi_m(t)$ into formula (29), we obtain

$$a_{n,m} = \alpha_{n,m} + \gamma_{n,m}, \quad (38)$$

where

$$\beta_{n,k} = \begin{cases} s_{n+k+1}(\beta) + s_{n-k}(\beta), & n \neq k, \\ \beta + s_{2n+1}(\beta), & n = k, \end{cases} \quad s_k(\beta) = \frac{\sin(k\beta)}{k},$$

$$\gamma_{n,m} = \frac{2-\delta_{m0}}{4\pi^2 R}(-1)^m \sum_{k=0}^{\infty} \beta_{n,k} \left(\frac{r_0}{R}\right)^k P_k^{-m}(\cos\theta_0),$$

$$\alpha_{n,m} = \frac{1}{\pi} \sum_{k=0}^{m-1} \beta_{n,k} c_{k,m}.$$

Hence, the Green's formula of the first boundary value problem is restored by formulas (5), (7), (18) – (20), (38)

$$G(x,x_0) = -(4\pi|x-x_0|)^{-1} + \\ + \sum_{n=0}^{\infty} \begin{Bmatrix} (r/R)^n \\ (R/r)^{n+1} \end{Bmatrix} \sum_{m=0}^{n} a_{nm} P_n^m(\cos\theta)\cos[m(\varphi-\varphi_0)], \\ \begin{Bmatrix} r<R \\ r>R \end{Bmatrix}, r_0 < R. \quad (39)$$

The results of numerical calculations according to formula (39) are shown in Fig. 1 and in tables 1, 2. Calculations were carried out with the following parameter values: $\beta = \pi/3$, $r_0/R = 0.35$, $\theta_0 = 2\pi/5$, $\varphi_0 = 0$. In fig. 1 shows the level lines of the Green's function in the plane $x_2 = 0$ passing through a singular point.

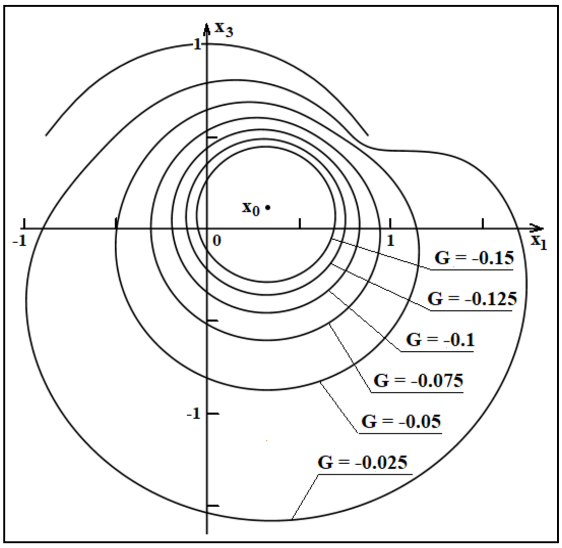

Fig. 1. Level lines of the Green's function of the first boundary value problem in the plane $x_2 = 0$

Table 1 shows the numerical values of the Green's function at some points of the plane $x_2 = 0$, $|x| < R$ (parameters are indicated above). Table 2 shows the relative error in percent of the calculation of the Green's function at the points of the plane $x_2 = 0$, $|x| < R$ at different ratios r/R and different angles θ. The largest error is observed at $\theta = 0$, but in range $\theta = \pi/5 \div \pi$ even with the ratio $r/R = 0.95$ at least two valid significant digits after the period are obtained.

Table 1

The value of the Green's function at the points of the plane $x_2 = 0$

$\theta \setminus r/R$	0.75	0.85	0.95
0	-0.03167	-0.01704	-0.004918
$\pi/5$	-0.06654	-0.03551	-0.01073
$2\pi/5$	-0.1303	-0.08931	-0.06358
$3\pi/5$	-0.1115	-0.08932	-0.07340
$4\pi/5$	-0.07462	-0.06422	-0.05586
π	-0.05422	-0.04793	-0.04299

Table 2

The relative error of calculating the Green's function (%)

$\theta \setminus r/R$	0.75	0.85	0.95
0	0.004	0.18	10.3
$\pi/5$	0.0005	0.02	1.07
$2\pi/5$	0.0002	0.007	0.16
$3\pi/5$	0.0002	0.007	0.13
$4\pi/5$	0.0004	0.01	0.2
π	0.003	0.07	1.18

5. Representation of a function u(x) in the form of a potentially simple layer

A fundamental fact of classical potential theory is the jump of the normal derivative of the potential of a simple layer when passing through a closed Lyapunov surface [48]. Let's prove a similar result for a surface with edge Γ.

Consider the potential of a simple layer for surfaces Γ with density $f(y)$

$$U(x) = \int_\Gamma \frac{f(y)}{|x-y|} d\sigma_y .$$

Let's fix the arbitrary $\varepsilon \in (0,\beta)$ and mark $\Gamma_\varepsilon = \{r = R, \theta \in [0, \beta - \varepsilon], \varphi \in [0, 2\pi]\}$. Let's take an arbitrary point $x = (r, \theta, \varphi) : r < R, \theta \in [0, \beta - \varepsilon]$ and the point $\xi = (R, \theta, \varphi)$ on the surface Γ_ε corresponding to it. Let $n_\xi = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ – unit vector of the normal to the surface Γ_ε in the point ξ, $\frac{\partial U}{\partial n_\xi}(x)$ – derivative of a function $U(x)$ at a point x in the direction of the normal n_ξ. We will assume that

$$f(y) \in C(\Gamma_\varepsilon) \ \forall \varepsilon \in (0,\beta) ; \ \int_\Gamma |f(y)| d\sigma_y < \infty . \quad (40)$$

Since the surface Γ is part of the Lyapunov surface, the following integrals exist according to the properties of the function $U(x)$

$$\frac{\partial U}{\partial n_\xi}(x) = \int_\Gamma \frac{(y-x, n_\xi)}{|x-y|^3} f(y) d\sigma_y = \int_\Gamma \frac{\cos\psi(x,y)}{|x-y|^2} f(y) d\sigma_y ,$$

$$\frac{\partial U}{\partial n_\xi}(\xi) = \int_\Gamma \frac{(y-\xi, n_\xi)}{|\xi-y|^3} f(y) d\sigma_y = \int_\Gamma \frac{\cos\psi(\xi,y)}{|\xi-y|^2} f(y) d\sigma_y ,$$

where $\psi(x,y)$ – the angle between the vectors x and y.

Theorem. For each $\varepsilon \in (0,\beta)$ and an arbitrary point $\xi \in \Gamma_\varepsilon$ if conditions (40) are fulfilled, there are limits

$$\frac{\partial U^\pm}{\partial n_\xi}(\xi) = \lim_{r \to R \mp 0} \frac{\partial U}{\partial n_\xi}(x) ,$$

moreover

$$\frac{\partial U^\pm}{\partial n_\xi}(\xi) = \frac{\partial U}{\partial n_\xi}(\xi) \pm 2\pi f(\xi) . \quad (41)$$

Proof. First, consider the analogue of the Gaussian integral for a surface Γ

$$G(x) = \int_\Gamma \frac{\cos\varphi(x,y)}{|x-y|^2} d\sigma_y ,$$

where $\varphi(x,y)$ – the angle between the vectors $y-x$ and y. In fig. 2 shows three different options for the location of the point x relative to the surface Γ ($|x_1| < R$, $|\xi| = R$, $|x_2| > R$). Let us denote the solid angle of a cone with the vertex at point x_1 and the direction AB - the edge of the surface Γ - by α_1. Then the solid angle at which the surface Γ is visible from the point x_1 is equal to $4\pi - \alpha_1$. If the point x_1 directs to the point $\xi_1 \in \Gamma$, then the solid angle is $(4\pi - \alpha_1) \to 4\pi - \alpha$, where α is the solid angle of the cone with vertex ξ_1 and direction AB.

Note that, unlike a closed surface, for which the solid angle relative to the internal point is equal 4π and does not depend on the position of the point, the solid angle $4\pi - \alpha_1$ depends on the position x_1.

Since $\cos\varphi(x,y)|x-y|^{-2} d\sigma_y$ with accuracy to the sign is equal to the magnitude of the solid angle under which the surface element is visible from the point x, then

$$\lim_{x_1 \to \xi_1} G(x_1) = 4\pi - \alpha .$$

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

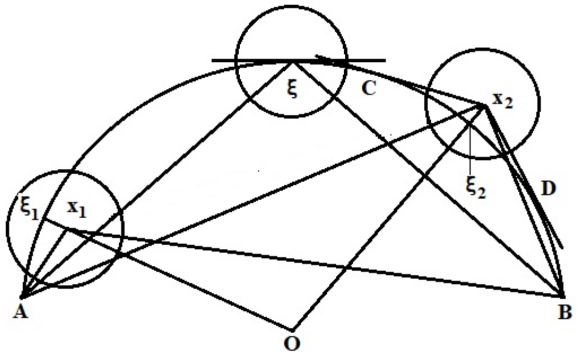

Fig. 2. Solid angles for calculating the potential

The same $G(\xi)=2\pi-\gamma$, where γ – the solid angle of the cone ξAB. To calculate $G(x_2)$, we draw a cone x_2CD that touches the surface Γ along a line CD. We denote by Γ_1 the part of the surface Γ that lies in the middle of the line CD, and the solid angle at which Γ_1 is visible from the point x_2 - by ω_1. Then, since the angle between $y-x$ and y is obtuse, then

$$\int_{\Gamma_1}\frac{\cos\varphi(x_2,y)}{|x_2-y|^2}d\sigma_y=-\omega_1 .$$

Similarly

$$\int_{\Gamma\backslash\Gamma_1}\frac{\cos\varphi(x_2,y)}{|x_2-y|^2}d\sigma_y=\omega_1-\omega_2 ,$$

where ω_2 – the solid angle of the cone x_2CD. Now

$$\lim_{x_2\to\xi_2}\int_{\Gamma}\frac{\cos\varphi(x_2,y)}{|x_2-y|^2}d\sigma_y=-\lim_{x_2\to\xi_2}\omega_2=-\omega ,$$

where ω – the solid angle of the cone ξ_2AB.

Let's return to the proof of the theorem. Let's choose a number $\delta\in(0,\varepsilon)$. Let's mark

$$W(x)=\int_{\Gamma}\frac{\cos\varphi(x,y)}{|x-y|^2}f(y)d\sigma_y ,\quad \lim_{x\to\xi}W(x)=W^{\pm}(\xi)\begin{cases}|x|<R,\\|x|>R.\end{cases}$$

Let's write the function $W(x)$ like this:

$$W(x)=\int_{\Gamma_\delta}\frac{\cos\varphi(x,y)}{|x-y|^2}f(y)d\sigma_y+\int_{\Gamma\backslash\Gamma_\delta}\frac{\cos\varphi(x,y)}{|x-y|^2}f(y)d\sigma_y=$$

$$=\int_{\Gamma_\delta}\frac{\cos\varphi(x,y)}{|x-y|^2}[f(y)-f(\xi)]d\sigma_y+f(\xi)\int_{\Gamma_\delta}\frac{\cos\varphi(x,y)}{|x-y|^2}d\sigma_y+$$

$$+\int_{\Gamma\backslash\Gamma_\delta}\frac{\cos\varphi(x,y)}{|x-y|^2}f(y)d\sigma_y . \qquad (42)$$

Since the function $f(y)$ is continuous on Γ_δ, the first integral on the right in formula (42) converges uniformly at point $x=\xi$ (proof, as in the classical theory [48]). The third integral on the right in (42) is also a continuous function of x due to the continuity of the kernel and the integrability of $f(y)$. Then the limit transition in (42) at $x\to\xi\in\Gamma_\varepsilon$ ($|x|<R$) taking into account the limit of the function $G(x)$ gives

$$W^{+}(\xi)=\int_{\Gamma_\delta}\frac{\cos\varphi(\xi,y)}{|\xi-y|^2}[f(y)-f(\xi)]d\sigma_y+f(\xi)(4\pi-\alpha_\delta)+$$

$$+\int_{\Gamma\backslash\Gamma_\delta}\frac{\cos\varphi(\xi,y)}{|\xi-y|^2}f(y)d\sigma_y=\int_{\Gamma}\frac{\cos\varphi(\xi,y)}{|\xi-y|^2}f(y)d\sigma_y-$$

$$-f(\xi)(2\pi-\alpha_\delta)+f(\xi)(4\pi-\alpha_\delta)=W(\xi)+2\pi f(\xi) , \quad (43)$$

where α_δ – the solid angle of a cone with vertex ξ, the direction of which is the boundary of the surface Γ_δ.

Now denote by $\theta(\xi,y)$ the angle between the vectors ξ and y. Let's take into account that $\psi(x,y)=\theta(x,y)+\varphi(x,y)$. Then the normal derivative of the potential can be written as:

$$\frac{\partial U}{\partial n_\xi}(x)=\int_{\Gamma}\frac{\cos\varphi(x,y)}{|x-y|^2}\cos\theta(\xi,y)f(y)d\sigma_y-$$

$$-\int_{\Gamma}\frac{\sin\varphi(x,y)}{|x-y|^2}\sin\theta(\xi,y)f(y)d\sigma_y . \qquad (44)$$

The second integral in the last formula is a continuous function of x, since $|x-y|=2R\sin(\theta(\xi,y)/2)$, and the first integral coincides with the integral $W(x)$ in which the function $f(y)$ was replaced by $\cos\theta(\xi,y)f(y)$. Therefore, the limit transition in formula (44) at $x\to\xi\in\Gamma_\varepsilon$ ($|x|<R$) gives

$$\frac{\partial U^{+}}{\partial n_\xi}(\xi)=\int_{\Gamma}\frac{\cos\varphi(\xi,y)}{|\xi-y|^2}\cos\theta(\xi,y)f(y)d\sigma_y+2\pi f(\xi)-$$

$$-\int_{\Gamma}\frac{\sin\varphi(\xi,y)}{|\xi-y|^2}\sin\theta(\xi,y)f(y)d\sigma_y=\frac{\partial U}{\partial n_\xi}(\xi)+2\pi f(\xi) .$$

The second formula (41) can be proved similarly.

Consequence. If the conditions of the theorem are satisfied, the following formula is correct

$$f(\xi)=\frac{1}{4\pi}\left[\frac{\partial U^{+}}{\partial n_\xi}(\xi)-\frac{\partial U^{-}}{\partial n_\xi}(\xi)\right]. \qquad (45)$$

Formula (45) makes it possible to represent function (7) by the potential of a simple layer. For this, obviously, it is necessary to restore the density of this potential. Taking into account formulas (18), (19), (45), we have

$$f(\theta,\varphi)=\frac{1}{4\pi}\left(\frac{\partial u^{+}}{\partial r}-\frac{\partial u^{-}}{\partial r}\right)_{|\Gamma}=$$

$$=\frac{1}{4\pi R}\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\sum_{n=0}^{\infty}(2n+1)a_{n,m}P_n^m(\cos\theta)=$$

$$=\frac{1}{4\pi R}\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]I_m(\cos\theta)\,, \qquad (46)$$

where $I_m(v)$ is determined by the formula (34).

Graphs of the potential density of a simple layer at $\varphi=\varphi_0$, constructed according to calculations obtained by formula (46), are shown in Fig. 3 Various options for the location of the singular point relative to the surface Γ are considered. A characteristic increase of density around the edge of the shell $v_0=\cos\beta=0.5$ is observed, which is associated with the root feature at this point.

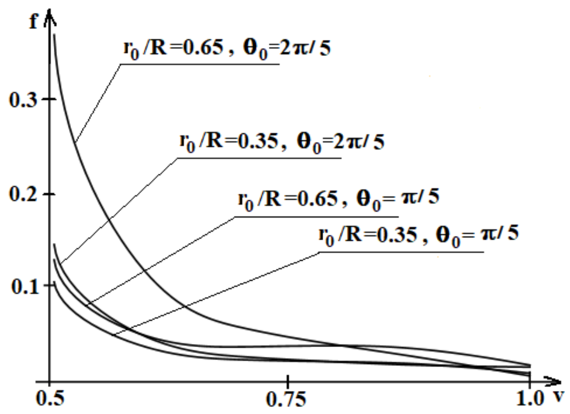

Fig. 3. Graphs of the potential density of a simple layer at $\varphi=\varphi_0$

6. The electrostatic field of a point charge near a grounded conducting spherical shell

As an example of the application of the results obtained above, we present the problem of determining the electrostatic field of a point unit charge near a grounded conducting spherical shell located in a vacuum. We will assume that the point charge is located at the point $x=0$, and the spherical shell is given by the conditions specified in Section 1. To comply with the above formulas, we consider that the charge is negative, and its numerical value is equal to the electric constant ε_0. This task is interesting because due to the axial symmetry of the electrostatic field, some of its characteristics are found in a closed form. It should be noted that the Green's function (5) at $x_0=0$ exactly determines the potential of the studied field. Here $F(x-x_0)$ is the potential of a point charge, and $u(x)$ is the potential of the field induced by the shell.

Since the partial case of problem (14)–(16) is considered, we will present only some final results.

Due to axial symmetry, we find the function $u^{\pm}(x)$ in the form

$$u^{\pm}(x)=\sum_{n=0}^{\infty}a_n\begin{Bmatrix}(r/R)^n\\(R/r)^{n+1}\end{Bmatrix}P_n(\cos\theta)\,, \qquad (47)$$

$$a_n=\frac{1}{4\pi^2R}\begin{cases}\beta+\sin\beta, & n=0,\\ \sin[(n+1)\beta]/(n+1)+\sin(n\beta)/n, & n\neq 0.\end{cases}$$

Further simplification of formula (47) is connected with the explicit summation of series

$$\sum_{n=0}^{\infty}\frac{\sin(n+1)\beta}{n+1}\mu^{n+1}P_n(\cos\theta)=$$

$$=\operatorname{Im}\ln\left(z-\cos\theta+\sqrt{1-2z\cos\theta+z^2}\right), \qquad (48)$$

$$\sum_{n=1}^{\infty}\frac{\sin n\beta}{n}\mu^{n}P_n(\cos\theta)=$$

$$=-\operatorname{Im}\ln\left(1-z\cos\theta+\sqrt{1-2z\cos\theta+z^2}\right), \qquad (49)$$

where $z=\mu e^{i\beta}$, $\mu=r/R$. The functions in the right-hand parts of formulas (48), (49) are multivalued functions of a complex variable z. In order to be able to extract single-valued branches of these functions, we will make cuts in the complex $z-$plane, as shown in Fig. 4.

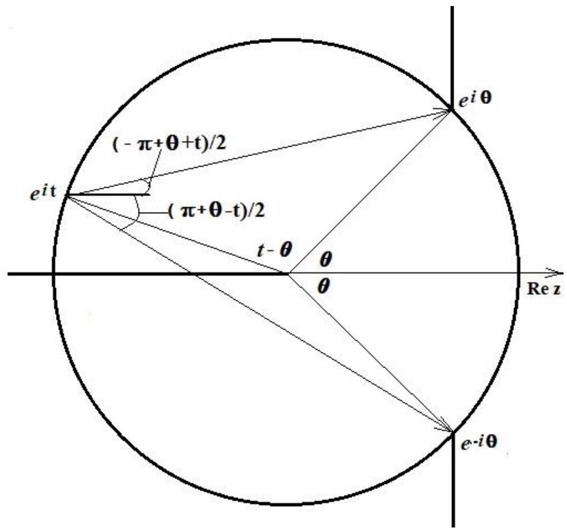

Fig. 4. Cuts of the complex $z-$plane for selection of single-valued branches of multivalued functions

We will choose the branches in such a way that the indicated functions take real values, and the roots take positive values at $z\in(0,1)$. Since the series in formulas (48), (49) converge at $\mu=1$, we will calculate their sums at this value of the parameter μ and $\theta\in[0,\beta)$. Bearing in mind the selection of single-valued branches of analytic functions, we can write

$$\arg(e^{\pm i\theta}-e^{i\beta})=-(\pi-\beta\mp\theta)/2\,,$$

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

$$|e^{\pm i\theta}-e^{i\beta}|=2\sin\left(\frac{\beta\mp\theta}{2}\right),$$

therefore

$$\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}=\sqrt{2(\cos\theta-\cos\beta)}e^{i(-\pi+\beta)/2},$$

$$e^{i\beta}-\cos\theta+\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}=$$
$$=\left(\sqrt{2}\sin\frac{\beta}{2}-\sqrt{\cos\theta-\cos\beta}\right)\left(\sqrt{\cos\theta-\cos\beta}+i\sqrt{2}\cos\frac{\beta}{2}\right).$$

Then

$$\arg\left(e^{i\beta}-\cos\theta+\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}\right)=$$
$$=\operatorname{arctg}\frac{\sqrt{2}\cos\frac{\beta}{2}}{\sqrt{\cos\theta-\cos\beta}}. \quad (50)$$

$$\arg\left(1-e^{i\beta}\cos\theta+\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}\right)=$$
$$=\beta-\pi+\operatorname{arctg}\frac{\sqrt{2}\cos\frac{\beta}{2}}{\sqrt{\cos\theta-\cos\beta}}. \quad (51)$$

Substitution of identities (50), (51) in formulas (47) – (49) at $\mu=1$ gives

$$u^+(R,\theta)=\frac{1}{4\pi R},\ \theta\in[0,\beta)$$

which corresponds to condition (9). Similarly at $\theta\in(\beta,\pi]$ we have

$$\arg\left(e^{i\beta}-\cos\theta+\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}\right)=$$
$$=\operatorname{arctg}\frac{\sqrt{2}\sin\frac{\beta}{2}}{\sqrt{\cos\beta-\cos\theta}}, \quad (52)$$

$$\arg\left(1-e^{i\beta}\cos\theta+\sqrt{1-2e^{i\beta}\cos\theta+e^{i2\beta}}\right)=$$
$$=\beta-\operatorname{arctg}\frac{\sqrt{2}\sin\frac{\beta}{2}}{\sqrt{\cos\beta-\cos\theta}} \quad (53)$$

and we get a formula for the function $u^+(R,\theta)$

$$u^+(R,\theta)=\frac{1}{2\pi^2 R}\operatorname{arctg}\frac{\sqrt{2}\sin\frac{\beta}{2}}{\sqrt{\cos\beta-\cos\theta}},\ \theta\in(\beta,\pi]. \quad (54)$$

Note that the limit of the function $u^+(x)$ at $\theta\downarrow\beta$ is equal to $\frac{1}{4\pi R}$, which is consistent with the continuity of the potential on the surface $r=R$.

In the closed form, it is still possible to find the charge density that is induced on the surface Γ

$$f(\theta)=\frac{\partial u^+}{\partial r}(R,\theta)-\frac{\partial u^-}{\partial r}(R,\theta)=$$
$$=\frac{1}{4\pi^2R^2}\left\{\pi+\frac{2\sqrt{2}\cos\frac{\beta}{2}}{\sqrt{\cos\theta-\cos\beta}}-2\operatorname{arctg}\frac{\sqrt{2}\cos\frac{\beta}{2}}{\sqrt{\cos\theta-\cos\beta}}\right\}. \quad (55)$$

and the magnitude of the total charge on this surface

$$q=\frac{1}{\pi}(\sin\beta+\beta). \quad (56)$$

The last formula corresponds to the result given in [49].

It is interesting to note that the density of surface charge depends only on the variable $t=\cos(\beta/2)/\cos(\theta/2)$, while the graph of the function $f(t)$, for example, at $\beta=\pi/3$, is the graph shown in Fig. 5.

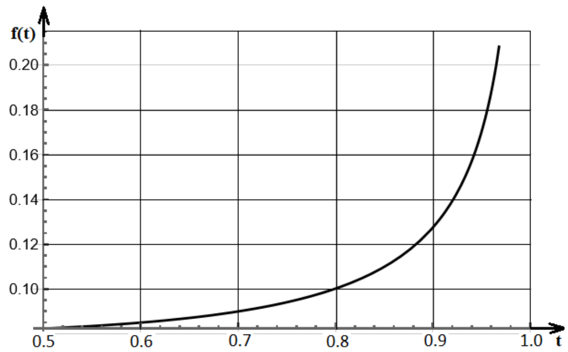

Fig. 5. Dependence of the charge density induced on the shell on a variable $t=\cos(\beta/2)/\cos(\theta/2)$ at $\beta=\pi/3$

7. Green's function of the Neumann problem

Now let's construct the Green's function of the Neumann problem (11) – (13). For it, auxiliary harmonic functions $u^{\pm}$ satisfy conditions (14) – (16). We will look for them again in the form (18), (19). After satisfying the conditions (14) – (16) with respect to the coefficients $a_{n,m}$, $b_{n,m}$, we obtain a system

$$na_{n,m}=-(n+1)b_{n,m},\ n=0\div\infty, m=0\div n;$$

$$\frac{1}{R}\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\sum_{n=m}^{\infty}n\,a_{n,m}P_n^m(\cos\theta)=$$
$$=-\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\frac{2-\delta_{m0}}{4\pi R^2}(-1)^m\times,$$
$$\times\sum_{n=m}^{\infty}(n+1)\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)P_n^m(\cos\theta)$$
$$\theta\in[0,\beta),\ \varphi\in[0,2\pi];$$

$$\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\sum_{n=m}^{\infty}\frac{2n+1}{n+1}a_{n,m}P_n^m(\cos\theta)=0,$$

$$\theta\in(\beta,\pi],\ \varphi\in[0,2\pi].$$

Let's apply the above method to solve this system. First, we reduce it to a system

$$\sum_{n=0}^{\infty}na_{n,m}P_n(\cos\theta)=\sum_{n=0}^{m-1}c_{n,m}P_n(\cos\theta)-$$

$$-\frac{2-\delta_{m,0}}{4\pi R}\sum_{n=m}^{\infty}(n+1)\left(\frac{r_0}{R}\right)^n(-1)^m P_n^{-m}(\cos\theta_0)P_n(\cos\theta),$$

$$\theta\in[0,\beta),\ m\in\mathbb{Z}_+; \quad (57)$$

$$(-1)^m\sin^m\theta\frac{d^m}{d\cos\theta^m}\sum_{n=0}^{\infty}\frac{2n+1}{n+1}a_{n,m}P_n(\cos\theta)=0,$$

$$\theta\in(\beta,\pi],\ m\in\mathbb{Z}_+, \quad (58)$$

where $c_{n,m}$ – constants of integration, which we will find from the formula

$$a_{n,m}=(n+1)\int_0^{\beta}\varphi_m(t)\cos[(n+1/2)t]dt. \quad (59)$$

From the formula (59) it follows that

$$\frac{a_{n,m}}{n+1}=\varphi_m(\beta)\frac{\sin[(n+1/2)\beta]}{n+1/2}-$$

$$-\frac{1}{n+1/2}\int_0^{\beta}\varphi_m'(t)\sin[(n+1/2)t]dt.$$

A necessary condition for the convergence of the series $\sum_{n=0}^{\infty}na_{n,m}P_n(\cos\theta)$ is the condition

$$\varphi_m(\beta)=0,\ m=0,1,\ldots. \quad (60)$$

Then

$$\frac{a_{n,m}}{n+1}=\varphi_m'(\beta)\frac{\cos[(n+1/2)\beta]}{(n+1/2)^2}-\varphi_m'(0)\frac{1}{(n+1/2)^2}-$$

$$-\frac{1}{(n+1/2)^2}\int_0^{\beta}\varphi_m''(t)\cos[(n+1/2)t]dt. \quad (61)$$

We impose one more condition on the function $\varphi_m(t)$

$$\varphi_m'(0)=0,\ m=0,1,\ldots. \quad (62)$$

As in Section 3, equation (58) is satisfied automatically using (61), and equation (57) leads to the differential equation ($m\ge 0$)

$$\varphi_m''(t)+\frac{1}{4}\varphi_m(t)=-\frac{2}{\pi}\sum_{k=0}^{m-1}c_{k,m}\cos[(k+1/2)t]+(-1)^m\times$$

$$\times\frac{2-\delta_{m0}}{2\pi^2R}\sum_{n=m}^{\infty}(n+1)\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)\cos\left[\left(n+\frac{1}{2}\right)t\right]. \quad (63)$$

Boundary conditions (60), (62) should be added to the equation (63).

The solutions of the obtained set of boundary value problems are functions

$$\varphi_0(t)=-\frac{\beta}{2\pi^2R}\mathrm{tg}(\beta/2)\cos(t/2)+\frac{t}{2\pi^2R}\sin(t/2)-$$

$$-\frac{1}{2\pi^2R}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r_0}{R}\right)^k P_k(\cos\theta_0)\tau_k(t), \quad (64)$$

$$\varphi_m(t)=\frac{2}{\pi}c_{0m}[\beta\,\mathrm{tg}(\beta/2)\cos(t/2)-t\sin(t/2)]-$$

$$-\frac{(-1)^m}{\pi^2R}\sum_{k=m}^{\infty}\frac{1}{k}\left(\frac{r_0}{R}\right)^k P_k^{-m}(\cos\theta_0)\tau_k(t)+\frac{2}{\pi}\sum_{k=1}^{m-1}\frac{c_{k,m}\tau_k(t)}{k(k+1)},$$

$$m\ge 1, \quad (65)$$

where

$$\tau_k(t)=\cos[(k+1/2)t]-\cos[(k+1/2)\beta]\frac{\cos(t/2)}{\cos(\beta/2)}.$$

To determine the integration constants $\{c_{n,m}\}_{m=1,n=0}^{\infty,m-1}$ we will require that the function $(u^+-u^-)_{|x\in\Gamma}$ may be bounded on the surface Γ. Since

$$(u^+-u^-)_{|x\in\Gamma}==\sqrt{2}\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\times$$

$$\times(-1)^m(1-v^2)^{\frac{m}{2}}\frac{d^m}{dv^m}\int_{v_0}^{v}\frac{\psi_m'(u)}{\sqrt{v-u}}du, \quad (66)$$

where $\cos t=u$, $\varphi_m(t)=\psi_m(u)$, $\cos\theta=v$, $\cos\beta=v_0$, then it is necessary to estimate the boundedness of the derivative in formula (66). The derivative of the integral (66) has a singularity at the point $v=v_0$. Its order can be determined if the integral is integrated by parts $m+1$ times.

$$\frac{d^m}{dv^m}\int_{v_0}^{v}\frac{\psi_m'(u)}{\sqrt{v-u}}du=\frac{d^m}{dv^m}\left\{\sum_{k=1}^{m+1}\chi_k(v,v_0)\psi_m^{(k)}(v_0)+\right.$$

$$\left.+\int_{v_0}^{v}\chi_{m+1}(v,u)\psi_m^{(m+2)}(u)du\right\}, \quad (67)$$

where the function $\chi_k(v,u)$ is defined in (34). Formula (67) shows that the function $(u^+-u^-)_{|(r,\theta,\varphi)\in\Gamma}$ at the point $v=v_0$ has a singularity of order $-m+1/2$, and in order to get rid of all the singularities in (67), it is necessary to impose additional conditions on the function $\psi_m(u)$

$$\psi_m'(v_0)=\psi_m''(v_0)=\cdots=\psi_m^{(m)}(v_0)=0 \quad (68)$$

or, what is the same,

$$\varphi_m'(\beta)=\varphi_m''(\beta)=\cdots=\varphi_m^{(m)}(\beta)=0. \quad (69)$$

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

Thus, to determine the integration constants $\{c_{n,m}\}_{m,n=0}^{\infty,m-1}$ in (65) for each $m \geq 1$ we have m conditions (69). Let's find out the possibility of finding constants from these conditions.

For $m=1$ we find the constant of integration c_{01} from the condition $\varphi_1'(\beta)=0$

$$c_{0,1}=\frac{1}{\pi R}\frac{\cos(\beta/2)}{\beta+\sin\beta}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r_0}{R}\right)^k P_k^{-1}(\cos\theta_0)\tau_k'(\beta). \quad (70)$$

For $m>1$ we find the constants of integration from the conditions

$$\varphi_m'(\beta)=0,\ \varphi_m''(\beta)+\frac{1}{4}\varphi_m(\beta)=0,\cdots,$$

$$\varphi_m^{(m)}(\beta)+\frac{1}{4}\varphi_m^{(m-2)}(\beta)=0, \quad (71)$$

which follow from equalities (60), (69). As a result, we obtain a linear algebraic system with respect to $\{c_{n,m}\}_{n=0}^{m-1}$

$$-c_{0m}\frac{\beta+\sin\beta}{\cos(\beta/2)}+2\sum_{n=1}^{m-1}\frac{c_{nm}}{n(n+1)}\tau_n'(\beta)=$$

$$=\frac{(-1)^m}{\pi R}\sum_{n=m}^{\infty}\frac{1}{n}\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)\tau_n'(\beta), \quad (72)$$

$$\sum_{n=0}^{m-1}c_{n,m}(n+1/2)^k\cos[(n+1/2)\beta+\pi k/2]=,$$

$$=\tilde{g}_{m,k},\ m=2,3,\ldots;\ k=0,1,\ldots,m-2, \quad (73)$$

where

$$\tilde{g}_{mk}=\frac{(-1)^m}{2\pi R}\sum_{n=m}^{\infty}(n+1)\left(\frac{r_0}{R}\right)^n P_n^{-m}(\cos\theta_0)(n+1/2)^k\times$$

$$\times\cos[(n+1/2)\beta+\pi k/2].$$

After solving the system (72), (73), the harmonic functions $u^{\pm}(x)$ can be restored by formulas (12), (18), (19), substituting

$$\frac{a_{n,0}}{n+1}=\frac{1}{2\pi^2 R}\left[-\beta\,\mathrm{tg}\left(\frac{\beta}{2}\right)\lambda_{0,n}+\nu_{0,n}-\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r_0}{R}\right)^k P_k(\cos\theta_0)\mu_{k,n}\right],\ n=0,1,\ldots; \quad (74)$$

$$\frac{a_{n,m}}{n+1}=\frac{2}{\pi}c_{0m}[\beta\,\mathrm{tg}(\beta/2)\lambda_{0,n}-\nu_n]+\frac{2}{\pi}\sum_{k=1}^{m-1}\frac{c_{k,m}\mu_{k,n}}{k(k+1)}-$$

$$-\frac{(-1)^m}{\pi^2 R}\sum_{k=m}^{\infty}\frac{1}{k}\left(\frac{r_0}{R}\right)^k P_k^{-m}(\cos\theta_0)\mu_{k,n}, \quad (75)$$

$$m=1\div\infty,\ n=m\div\infty$$

$$b_{n,m}=-na_{n,m}/(n+1),\ m=0\div\infty,\ n=m\div\infty, \quad (76)$$

where

$$\lambda_{k,n}=\int_0^{\beta}c_k(t)c_n(t)dt,\ \lambda_{k,n}=\int_0^{\beta}t\sin(t/2)c_n(t)dt,$$

$$\mu_{k,n}=\int_0^{\beta}\tau_k(t)c_n(t)dt,\ c_n(t)=\cos[(n+1/2)t].$$

The coefficients $\lambda_{k,n}$, ν_n, $\mu_{k,n}$ are calculated in a closed form, but are not given here due to their bulkiness.

The results of numerical calculations using formulas (12), (18), (19), (74) – (76) are shown in Fig. 6 and in tables 3, 4. Calculations were carried out with the following parameter values: $\beta=\pi/3$, $r_0/R=0.35$, $\theta_0=2\pi/5$, $\varphi_0=0$. The level lines of the Green's function in the plane $x_2=0$ are presented in Fig. 6. It shows a characteristic jump in the values of the Green's function when passing through the surface of the segment, so some of the level lines are not closed. Tables 3 and 4 show the values of the Green's function at individual points of the plane $x_2=0$ in which the singular point of the Green's function is located, and the relative error of the calculations in percent.

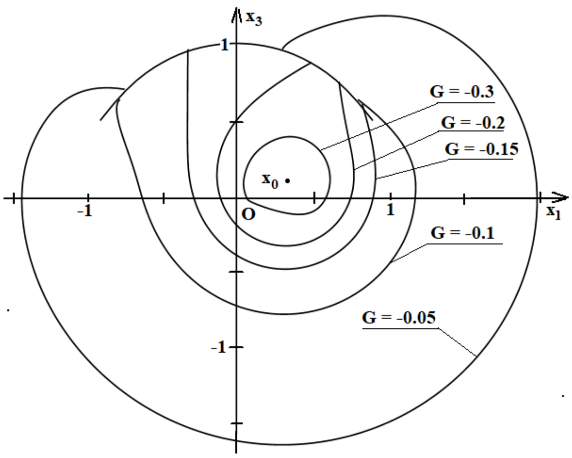

Fig. 6. Level lines of the Green's function of the Neumann problem in the plane $x_2=0$

Table 3

The value of the Green's function of the Neumann problem at the points of the plane $x_2=0$

$\theta \backslash r/R$	0.75	0.85	0.95
0	-0.06988	-0.07828	-0.08614
$\pi/5$	-0.06470	-0.07533	-0.08790
$2\pi/5$	-0.02328	-0.01966	-0.01482
$3\pi/5$	-0.01141	-0.00983	-0.008331
$4\pi/5$	-0.008661	-0.007659	-0.006845
π	-0.007918	-0.007038	-0.006099

The obtained results show a significantly higher accuracy of calculating the Green's function of the Neumann problem than the Green's function of the Dirichlet problem. Also, the decrease in accuracy is not in the vicinity of point 0, as in the Dirichlet problem, but in the vicinity of the point π.

Table 4

The relative error of calculating the value of the Green's function (%)

$\theta \backslash r/R$	0.75	0.85	0.95
0	0.00009	0.002	0.03
$\pi/5$	0.00014	0.003	0.04
$2\pi/5$	0.00016	0.004	0.09
$3\pi/5$	0.0002	0.005	0.1
$4\pi/5$	0.00058	0.015	0.3
π	0.00079	0.02	0.4

Note that the jump of the Green's function on the surface Γ according to formula (66) and conditions (68) can be expressed as follows:

$$J(v,\varphi) \equiv (u^{+} - u^{-})_{|x\in\Gamma} = = \sqrt{2}\sum_{m=0}^{\infty}\cos[m(\varphi-\varphi_0)]\times$$

$$\times(-1)^m(1-v^2)^{\frac{m}{2}}\int_{v_0}^{v}\frac{\psi_m^{(m+1)}(u)}{\sqrt{v-u}}du\,, \qquad (77)$$

where $v=\cos\theta$, $v_0=\cos\beta$.

The graphs of jumps of the Green's function of the Neumann problem on the surface Γ at $\beta=\pi/3$, $\varphi=\varphi_0=0$ are presented in Fig. 7.

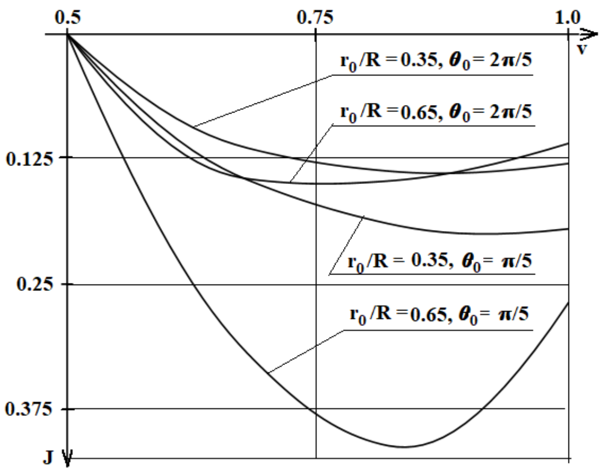

Fig. 7. Graphs of jumps of the Green's function on the surface Γ at $\varphi=\varphi_0$

The graphs show that the largest magnitude of jumps is observed when the singular point approaches the shell axis. At the edge of the shell (v=0.5), the values of the jumps are equal to zero, which corresponds to the continuity of the Green's function on the surface r=R outside Γ.

Discussions

The results of the study, given in chapters 1-7, showed that using the proposed methodology, the Green's functions of the Dirichlet and Neumann boundary value problems for the Laplace equation can be written in the explicit form of either series of basic harmonic functions for a sphere, or the potential of a simple layer for the surface of a segment. The resulting formulas include a finite set of m unknown constants of integration, which can be found by numerically solving a linear algebraic system for each $m=1,2,\ldots$. Calculations using various algorithms have shown that the highest accuracy of Green's function calculation is given by representations in the form of series. A natural problem associated with such an approach is a decrease in accuracy when approaching a spherical surface $r=R$. But the results of the calculations show that the values of the Green's functions are found with sufficient accuracy in almost the entire range $r/R=0\div0.95$ at $r_0/R\in[0,0.65]$ (see Tables 2, 4). The problem area in the Dirichlet problem with various numerical algorithms is the domain $\{r\in[R,R+\varepsilon], \theta\in[0,\beta), \varphi\in[0,2\pi]\}$ with a sufficiently small positive ε. In this area, the absolute error of calculating the Green's function exceeds the absolute value of the function itself, so the numerical results are not correct.

The Green's function of the Dirichlet problem (the potential of an electrostatic field created by a point charge near a grounded conducting shell) can be obtained in the closed form of the imaginary parts of some analytical functions only when the point source of the field is in the center of the spherical surface to which the segment Γ belongs. The characteristics of the field on the surface are expressed in terms of elementary functions.

The obtained results open up further opportunities for finding Green's functions in regions of more complex shape – doubly connected and multiply connected, one of the boundaries of which is a spherical segment. Another direction of the further development of the theory of Green's functions is related to the use of more complex boundary conditions in boundary value problems for the equations of mathematical physics. Impedance-type conditions are an example of such boundary conditions. An overview of the results of using such conditions is given in [50].

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

Conclusions

The article is devoted to the problem of constructing Green's functions of Dirichlet and Neumann boundary value problems for the Laplace equation in a non-classical domain – a three-dimensional space with a spherical segment. The statement of the problem leads to the need to determine auxiliary harmonic functions – solutions of mixed boundary value problems in the interior and exterior of the spherical surface to which the segment belongs. Boundary value problems using the Fourier method are reduced to systems of paired equations in series by Legendre functions, the solutions of which are found using discontinuous Mahler-Dirichlet sums. The specified functions are obtained in an explicit form in two forms: series based on the basic harmonic functions in spherical coordinates and the potential of a simple layer on the surface of the segment. Each representation of Green's functions contains a set of integration constants that can be found from additional conditions of integrability of the jump of the normal derivative of function on the segment surface (Dirichlet problem) or boundedness of the jump of the function itself on the segment surface (Neumann problem). The Lemma is proved for the Dirichlet problem, according to which the linear algebraic system for determining the constants of integration for each $m = 1, 2, \ldots$ has a unique solution. The representation of the Green's function of the Dirichlet problem in the form of the potential of a simple layer is based on the Theorem, which is proven in the article about the jump of the normal derivative of the potential of a simple layer on a segment. This result generalizes the classical theorem of the potential theory for closed Lyapunov surfaces to the case of a spherical segment.

The analysis of numerical values of Green's functions obtained by various algorithms showed that the highest accuracy of results outside the surface $r = R$ was obtained when using representations of Green's functions in the form of series. Numerical results and calculation errors are given in the tables. On the basis of the calculations, the level lines of the Green's functions of two boundary value problems were constructed in the plane of the singular point. Also the graphs of the potential density of the simple layer for the Dirichlet problem and the potential jump for the Neumann problem on the segment at different locations of the singular point were constructed.

In the partial case of the location of a singular point at the origin of the coordinates, the potential of the electrostatic field of a point charge near a grounded conducting thin shell in the form of a spherical segment is found. The main characteristics of such a field are found in closed form.

Contributions of authors: conceptualization – **Oleksii Nikolaev**; methodology – **Oleksii Nikolaev**; bibliography – **Oleksii Nikolaev**, **N. Savchenko**; formulation of tasks – **Oleksii Nikolaev**; analytic construction of Green's functions – **Oleksii Nikolaev, Oleksandr Holovchenko**; formulation and proof of lemma and theorem – **Oleksii Nikolaev,** software – **Oleksii Nikolaev, Oleksandr Holovchenko, Nina Savchenko**; verification – **Oleksii Nikolaev**; analysis of results – **Oleksii Nikolaev, Oleksandr Holovchenko, Nina Savchenko**; visualization – **Oleksii Nikolaev**; writing – original draft preparation – **Oleksii Nikolaev, Nina Savchenko**; writing – review and editing – **Oleksii Nikolaev, Nina Savchenko**.

All the authors have read and agreed to the published version of the manuscript.

References (GOST 7.1:2006)

1. Green, G. An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism [Text] / G. Grin. – Nottingham: Printed by T. Wheelhouse, 1828. – 72 p. Reprinted in three parts in Journal für die reine und angewandte Mathematik. – 1850. – vol. 39, no. 1. –P. 73-89; – 1852. – vol. 44, no. 4. –P. 356-374; – 1854. – vol. 47, no. 3. – P. 161-221. arXiv: 0807.0088v1 [physics.hist-ph] 1Jul 2008.

2. Bernhard Riemann's gesammelte mathematische Werke, wissenschaftlicher Nachlass [Text] / H. Weber, R. Dedekind. – Leipzig: Teubner. Reprint of the second edition of 1892, including the supplement prepared by M. Noether and W. Wirtinger in 1902, New York : Dover Publ., 1953. – 45 p.

3. Morse, P. M. Methods of Theoretical Physics. Part I [Text] / P. M. Morse, H. Feshbach. – New York : McGraw-Hill Publ., 1953. – 1060 p.

4. Morse, P. M. Methods of Theoretical Physics. Part II. [Text] / P. M. Morse, H. Feshbach. – New York : McGraw-Hill Publ., 1953. – 997 p.

5. Greenhill, A. G. On Green's function for a rectangular parallelepiped [Text] / A. G. Greenhill // Proc. Cambridge Philos. Soc. – 1879. – vol. 3. – P. 289-293.

6. Dougall, J. The determination of Green's function by means of cylindrical and spherical harmonics [Text] / J. Dougall // Proc. Edinburgh Math. Soc. – 1900. – vol. 18, ser. 1. – P. 33-83.

7. Riemann, B. Shwere, Elektrizität und Magnetismus [Text] / B. Riemann, K. Hattendorff. – Hannover: Publ. C. Rümpler, 1876. – 358 p.

8. Helmholtz, H. Theorie der Luftschwingunge in Röhren mit offenen [Text] / H. Helmholtz // J. Reine Angew. Math. – 1860. – vol. 57. – P. 1-72. http://eudml.org/doc/147771.

9. Carslaw, H. S. The use of Green's functions in the mathematical theory of the conduction of heat [Text]

/ H. S. Carslaw // Proc. Edinburgh Math. Soc. – 1902. – vol. 21, ser. 1. – P. 40-64. DOI: 10.1017/S0013091500034507.

10. Sommerfeld, A. Über verzweigte Potentiale im Raum [Text] / A. Sommerfeld // Proc. London Math. Soc. – 1897. – vol. 28, ser. 1. – P. 395-429.

11. Hobson, E. W. On Green's function for a circular disk with applications to electrostatic problems [Text] / E. W. Hobson // Trans. Cambridge Philos. Soc. – 1900. – vol. 18. – P. 277 – 291.

12. Thomson, W. Note the integration of the equations of equilibrium of an elastic solid [Text] / W. Thomson // The Cambridge and Dublin Mathematical Journal. – 1848. – vol. 3. – P. 87-89.

13. Boussinesq, J. Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques [Text] / J. Boussinesq. – Paris : Gauthier-Villars Publ., 1885. – 721 p. http://name.umd1.umich.edu/ABV5032.0001.001.

14. Mindlin, R. D. Force at a point in interior of a semiinfinite solid [Text] / R. D. Mindlin // J. Appl. Physics. – 1936. – vol. 7, no. 5. – P. 195-202. DOI: 10.1063/1.1745385.

15. Лифшиц, И. М. Построение тензора Грина для основного уравнения теории упругости в случае неограниченной упруго-анизотропной среды [Текст] / И. М. Лифшиц, Л. Н. Розенцвейг // ЖЭТФ. – 1947. – т. 17, № 9. – С. 783-791.

16. Chen, J.-T. Construction of Green's function using null-field integral approach for Laplace problems with circular boundaries [Text] / J.-T. Chen, J.-N. Ke, H.-Z. Liao // CMC. – 2009. – vol. 9, no. 2. – P. 93-109. DOI: 10.3970/cmc.2009.009.093.

17. Embree, M. Green's functions for multiply connected domains via conformal mapping [Text] / M. Embree, L. N. Trefethen // SIAM Rev. – 1999. – vol. 41, no. 4. – P. 745-761. PII. S0036144598349277.

18. Crowdy, D. Green's functions for Laplace's equation in multiply connected domains [Text] / D. Crowdy, J. Marshall // IMA Journal of Applied Mathematics. – 2007, no. 9. – P. 1-24. DOI: 10.1093/imamat/hxm007.

19. Fabricant, V. I. Electrostatic problem of several arbitrarily charged unequal coaxial disks [Text] / V. I. Fabricant // Journal of Computational and Applied Mathematics. – 1987. – vol. 18. – P. 129-147. PII:0377-0427(87)90012-4

20. Grebenkov, D. Semi-analitical computation of Laplacian Green functions in three-dimensional domains with disconnected spherical boundaries [Text] / D. Grebenkov, S. Traytak // Journal of Computational Physics. – 2019. – vol. 379. – P. 91-117. DOI: 10.1016/j.jcp.2018.10.033

21. Xue, C. Green's function and image system for the Laplace operator in the prolate spheroidal geometry [Text] / C. Xue, S. Deng // AIP Advances. – 2017. – vol. 7. – Article No. 015024. – P. 1-17. DOI: 10.1063/1.4974156.

22. Scharstein, R. W. Green's Function for the Harmonic Potential of the Three-Dimensional Wedge Transmission Problem [Text] / R. W. . Scharstein // Transactions on Antennas and Propagation. – 2004. – vol. 52, no. 2. – P. 452 – 460. DOI: 10.1109/TAP.2004.823949.

23. Dolgova, I. M. Green's matrix of the plane problem of elasticity theory for an orthotropic strip [Text] / I. M. Dolgova, Yu. A. Mel'nikov // Journal of Applied Mathematics and Mechanics. – 1989. – vol. 53, no. 1. – P. 81-84.

24. Pan, E. Static Green's functions in multilayered half space [Text] / E. Pan // Applied Mathematical Modeling. – 1997. – vol. 21, is. 8. – P. 509-521. DOI: 10.1016/S0307-904X(97)00053-X.

25. Grekov, M. A. Green's functions for periodic problems of elastic half-space [Text] / M. A. Grekov // Mechanics of Solids. – 1998. – vol. 33, no. 3. – P. 142-146. https://www.researchgate.net/publication/314606477.

26. Chen, W. Green's Functions of an External Circular Crack in a Transversely Isotropic Piezoelectric Medium [Text] / W. Chen, T. Shioya // JSME International Journal. Series A. – 1999. – vol. 42, no. 1. – P. 73-79. DOI: 10.1299/jsmea.42.73.

27. Fil'shtinskii, L. A. Investigation of coupled physical fields in a compound piezoceramic wedge [Text] / L. A. Fil'shtinskii, T. S. Matvienko // Mechanics of Composite Materials. – 1999. – vol. 35, no. 6. – P. 507-512. DOI: 10.1007/BF02259472.

28. Pan, E. Three-dimensional Green's functions in anisotropic biomaterials [Text] / E. Pan, F. G. Yuan // International Journal of Solid and Structures. – 2000. – vol. 37. – P. 5329-5351. DOI: 10.1016/S0020-7683(99)00216-4.

29. Процюк, Б. В. Функції Гріна задач статики для шаруватих тіл із плоскопаралельними межами поділу [Текст] / Б. В. Процюк // Мат. методи та фіз.-мех. поля. – 2005. – Т. 48, № 1. – С. 76-87.

30. Seremet, V. Static equilibrium of a thermoelastic half-plane: Green's functions and solutions in integrals [Text] / V. Seremet // Archive of Applied Mechanics. – 2014. – vol. 84, no. 4. – P. 553-570. DOI: 10.1007/s00419-013-0817-7.

31. Tadeu, A. Applications of Green functions in the study of acoustic problems in open and closed spaces [Text] / A. Tadeu, J. Antonio, L. Godinho // Journal of Sound and Vibration. – 2001. – vol. 247(1). – P. 117-130. DOI: 10.1006/jsvi.2001.3717.

32. Aleixo, E. Green's function for the lossy wave equation [Text] / E. Aleixo, E. Capelas de Oliveira // Revista Brasileira de Ensino de Fisica. – 2008. – vol. 30,

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

no. 1. – P. 1302.1-1302.5. DOI: 10.1590/S1806-11172008000100003.

33. Jun, S. Axial Green's functional method for steady Stokes flow in geometrically complex domains [Text] / S. Jun, D. W. Kim // Journal of Computational Physics. – 2011. – vol. 230(5). – P. 2095-2124. DOI: 10.1016/j.jcp.2010.12.007.

34. Newman, J. N. The Green function for potential flow in a rectangular channel [Text] / J. N. Newman // Journal of Numerical Analysis. – 1999. – vol. 19, no. 3. – P. 357-373. DOI: 10.1007/BF00043225.

35. Linton, C. M. The Green's Function for the Two-Dimensional Helmholtz Equation in Periodic Domains [Text] / C. M. . Linton // Journal of Engineering Mathematics. – 1998. – vol. 33, no. 4. – P. 377-401. DOI: 10.1023/A:1004377501747.

36. Melnikov, Yu. A. A Green's function method for detection of cavity from one boundary measurement [Text] / Yu. A. Melnikov, J. O. Powell // Transactions on Modeling and Simulation. – 1997. – vol. 15. – P. 121 – 130.

37. Barbot, S. Fourier-domain Green's function for an elastic semi-infinite solid under gravity, with applications to earthquake and volcano deformation [Text] / S. Barbot, Y. Fialko // Geophys. J. Int. – 2010. – vol. 182. – P. 568–582. DOI: 10.1111/j.1365-246X.2010.04655.x.

38. Kang, N. Green's function for Poisson Equation in Layered Nano-Structures including Graphene [Text] / N. Kang // A thesis presented to the University of Waterloo in fulfillment of the thesis requirement for the degree of Master of Mathematics in Applied Math. – Waterloo, Ontario, Canada, 2015. http://hdl.handle.net/10012/9312.

39. Tewary, V. K. Electrostatic Green's function for advanced materials subject to surface loading [Text] / V. K. Tewary // Journal of Engineering Mathematics. – 2004. – vol. 49, iss. 3. – P. 289-304. DOI: 10.1023/B:ENGI.0000031191.64358.a8.

40. Hon, Y. C. Inverse source identification by Green's function [Text] / Y. C. Hon, M. Li, Y. A. Melnikov // Engineering Analysis with Boundary Elements. – 2010. – vol. 34, no. 4. – P. 352-358. DOI: 10.1016/j.enganabound.2009.09.009.

41. Sneddon, I. N. Mixed boundary value problems in potential theory [Text] / I. N. Sneddon. – Amsterdam: North-Holland publ., New York : John Wily&Sons publ., 1966. – 283 p.

42. Уфлянд, Я. С. Метод парных уравнений в задачах математической физики [Текст] / Я. С. Уфлянд. – Ленинград : Наука, 1977. – 220 с.

43. Зюзин, В. А., В.И. Осесимметричное нагружение пространства со сферическим разрезом [Текст] / В. А. Зюзин, В. И. Моссаковский // Прикладная математика и механика – 1970. – Т. 34, вып. 1. – С. 179-183.

44. Улитко, А. Ф. Метод собственных векторных функций в пространственных задачах теории упругости [Текст] / А. Ф. Улитко. – Киев : Наукова думка, 1979. – 265 с.

45. Mehler, F. G. Notiz über die Dirichlet'schen Integralausdrücke für die Kugelfunction Pn(cosΘ) und über eine analoge Integralform für die Cylinderfunction Jm(x) [Text] / F. G. Mehler // Mathematische Annalen. – 1872. – vol. 5, iss. 1. – P. 141-144.

46. Ніколаєв, О. Г. Деякі типи фізичних полів у конусі з неоднорідністю у вигляді сферичного сегмента [Текст] / О. Г. Ніколаєв, К. П. Барахов // Мат. методи та фіз.-мех. поля. – 2005. – Т. 48, № 4. – С. 191-198.

47. Ніколаєв, О. Г. Задача теорії потенціалу для кругового циліндру зі сферичним сегментом [Текст] / О. Г. Ніколаєв, К. П. Барахов // Вісник Львівського ун-ту. Сер. прикл. математ. та інформ. – 2006. – Вип. 11. – С. 75-81.

48. Kellogg, O. D. Foundations of potential theory [Text] / O. D. Kellogg. – Berlin, Heidelberg, New York: Springer-Verlag, 1967. – 384 p.

49. Ландау, Л. Д. Теоретическая физика. Т. VIII. Электродинамика сплошных сред [Текст] / Л. Д. Ландау, Е. М. Лифшиц. – Москва: Наука, 1982. – 621 с.

50. Approximate boundary conditions for electromagnetic fields in electrodmagnetics [Text] / S. Berdnyk, A. Gomozov, D. Gretskih, V. Kartich, M. Nesterenko // Radioelectronic and Computer Systems. – 2022. – № 3. – P. 141-160. DOI: 10.32620/reks.2022.3.11.

References (BSI)

1. Green, G. *An Essay on the Application of mathematical Analysis to the theories of Electricity and Magnetism*. – Nottingham: Printed by T. Wheelhouse, 1828. *72 p.* Reprinted in three parts in *Journal für die reine und angewandte Mathematik*, 1850, vol. 39, no. 1, pp. 73-89; 1852, vol. 44, no. 4, pp. 356-74; 1854, vol. 47, no. 3, pp. 161-221. arXiv: 0807.0088v1 [physics.hist-ph] 1Jul 2008.

2. *Bernhard Riemann's gesammelte mathematische Werke, wissenschaftlicher Nachlass*. H. Weber and R. Dedekind, Eds. Leipzig: Teubner. Reprint of the second edition of 1892, including the supplement prepared by M. Noether and W. Wirtinger in 1902, NY., Dover Publ., 1953, pp. 3-45.

3. Morse, P. M., Feshbach, H. *Methods of Theoretical Physics. Part I.* NY.: McGraw-Hill Publ., 1953. 1060 p.

4. Morse, P. M., Feshbach, H. *Methods of Theoretical Physics. Part II.* NY.: McGraw-Hill Publ., 1953. 997 p.

5. Greenhill, A. G. On Green's function for a rectangular parallelepiped. *Proc. Cambridge Philos. Soc.*, 1879, vol. 3, pp. 289-293.

6. Dougall, J. The determination of Green's function by means of cylindrical and spherical harmonics. *Proc. Edinburgh Math. Soc.*, 1900, vol. 18, ser. 1, pp. 33-83.

7. Riemann, B., Hattendorff, K. *Shwere, Elektrizität und Magnetismus*. Hannover, Publ. C. Rümpler, 1876. 358 p.

8. Helmholtz, H. Theorie der Luftschwingunge in Röhren mit offenen. *J. Reine Angew. Math.*, 1860, vol. 57, pp. 1-72. http://eudml.org/doc/147771

9. Carslaw, H. S. The use of Green's functions in the mathematical theory of the conduction of heat. *Proc. Edinburgh Math. Soc.*, 1902, vol. 21, ser. 1, pp. 40-64. DOI: 10.1017/S0013091500034507.

10. Sommerfeld, A. Über verzweigte Potentiale im Raum. *Proc. London Math. Soc.*, 1897, vol. 28, ser. 1, pp. 395-429.

11. Hobson, E. W. On Green's function for a circular disk with applications to electrostatic problems. *Trans. Cambridge Philos. Soc.*, 1900, vol. 18, pp. 277 – 291.

12. Thomson, W. Note the integration of the equations of equilibrium of an elastic solid. *The Cambridge and Dublin Mathematical Journal*, 1848, vol. 3, pp. 87-89.

13. Boussinesq, J. *Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*. Paris, France: Gauthier-Villars, 1885. 721 p. http://name.umd1.umich.edu/ABV5032.0001.001

14. Mindlin, R.D. Force at a point in interior of a semiinfinite solid. *J. Appl. Physics*, 1936, vol. 7, no. 5, pp. 195-202. DOI: 10.1063/1.1745385.

15. Lifshits, I.M., Rozentsveyg, L.N. Postroenie tenzora Grina dlya osnovnogo uravneniya teorii uprugosti v sluchae neogranichennoy uprugo-anizotropnoy sredy [Construction of the Green's tensor for the basic equation of the theory of elasticity in the case of an unbounded elastically anisotropic medium]. *Zhurnal eksperimental'noy i teoreticheskoy fiziki*, 1947, vol. 17, no. 9, pp. 783-791.

16. Chen, J.-T., Ke, J.-N., Liao, H.-Z. Construction of Green's function using null-field integral approach for Laplace problems with circular boundaries. *CMC*, 2009, vol. 9, no. 2, pp. 93-109. DOI: 10.3970/cmc.2009.009.093.

17. Embree, M., Trefethen, L. N. Green's functions for multiply connected domains via conformal mapping. *SIAM Rev*., 1999, vol. 41, no. 4, pp. 745-761. PII. S0036144598349277.

18. Crowdy, D., Marshall, J., Green's functions for Laplace's equation in multiply connected domains. *IMA Journal of Applied Mathematics*, 2007, no. 9, pp. 1-24. DOI: 10.1093/imamat/hxm007.

19. Fabricant, V. I. Electrostatic problem of several arbitrarily charged unequal coaxial disks. *Journal of Computational and Applied Mathematics*, 1987, vol. 18, pp. 129-147. PII:0377-0427(87)90012-4.

20. Grebenkov, D., Traytak, S. Semi-analitical computation of Laplacian Green functions in three-dimensional domains with disconnected spherical boundaries. *Journal of Computational Physics*, 2019, vol. 379, pp. 91-117. DOI: 10.1016/j.jcp.2018.10.033.

21. Xue, C., Deng, S. Green's function and image system for the Laplace operator in the prolate spheroidal geometry. *AIP Advances*, 2017, vol. 7, article no.015024, pp. 1-17. DOI: 10.1063/1.4974156.

22. Scharstein, R.W. Green's Function for the Harmonic Potential of the Three-Dimensional Wedge Transmission Problem. *Transactions on Antennas and Propagation*, 2004, vol. 52, no. 2, pp. 452-460. DOI: 10.1109/TAP.2004.823949.

23. Dolgova, I. M., Mel'nikov, Yu. A. Green's matrix of the plane problem of elasticity theory for an orthotropic strip. *Journal of Applied Mathematics and Mechanics*, 1989, vol. 53, no. 1, pp. 81-84.

24. Pan, E. Static Green's functions in multilayered half space. *Applied Mathematical Modeling*, 1997, vol. 21, iss. 8, pp. 509-521. DOI: 10.1016/S0307-904X(97)00053-X.

25. Grekov, M. A. Green's functions for periodic problems of elastic half-space. *Mechanics of Solids*, 1998, vol. 33, no. 3, pp. 142-146. https://www.researchgate.net/publication/314606477.

26. Chen, W., Shioya, T. Green's Functions of an External Circular Crack in a Transversely Isotropic Piezoelectric Medium. *JSME International Journal*. *Series A*, 1999, vol. 42, no. 1, pp. 73-79. DOI: 10.1299/jsmea.42.73.

27. Fil'shtinskii, L. A., Matvienko, T. S. Investigation of coupled physical fields in a compound piezoceramic wedge. *Mechanics of Composite Materials*, 1999, vol. 35, no. 6, pp. 507-512. DOI: 10.1007/BF02259472.

28. Pan, E., Yuan, F.G. Tree-dimensional Green's functions in anisotropic bimaterials. *International Journal of Solid and Structures*, 2000, vol. 37, pp. 5329-5351. DOI: 10.1016/S0020-7683(99)00216-4.

29. Protsyuk, B. V. Funktsiyi Hrina zadach statyky dlya sharuvatykh til iz ploskoparalel'nymy mezhamy podilu [Green's functions of problems of statics for layered bodies with plane-parallel separation boundaries]. *Matematychni metody ta fizyko-mekhanichni polya*, 2005, vol. 48, iss. 1, pp. 76-87.

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

30. Seremet, V. Static equilibrium of a thermoelastic half-plane: Green's functions and solutions in integrals. *Archive of Applied Mechanics*, 2014, vol. 84, no. 4, pp. 553-570. DOI: 10.1007/s00419-013-0817-7.

31. Tadeu, A., Antonio, J., Godinho, L. Applications of Green functions in the study of acoustic problems in open and closed spaces. *Journal of Sound and Vibration*, 2001, vol. 247, no. 1, pp. 117-130. DOI: 10.1006/jsvi.2001.3717.

32. Aleixo, E., Capelas de Oliveira, E. Green's function for the lossy wave equation. *Revista Brasileira de Ensino de Fisica*, 2008, vol. 30, no. 1, pp. 1302.1-1302.5. DOI: 10.1590/S1806-11172008000100003.

33. Jun, S., Kim, D. W. Axial Green's functional method for steady Stokes flow in geometrically complex domains. *Journal of Computational Physics*, 2011, vol. 230, no. 5, pp. 2095-2124. DOI: 10.1016/j.jcp.2010.12.007.

34. Newman, J. N. The Green function for potential flow in a rectangular channel. *Journal of Numerical Analysis*, 1999, vol. 19, no. 3, pp. 357-373. DOI: 10.1007/BF00043225.

35. Linton, C. M. The Green's Function for the Two-Dimensional Helmholtz Equation in Periodic Domains. *Journal of Engineering Mathematics*, 1998, vol. 33, no. 4, pp. 377-401. DOI:10.1023/A:1004377501747.

36. Melnikov, Yu. A., Powell, J. O. A Green's function method for detection of cavity from one boundary measurement. *Transactions on Modeling and Simulation*, 1997, vol. 15, pp. 121-130.

37. Barbot, S., Fialko, Y. Fourier-domain Green's function for an elastic semi-infinite solid under gravity, with applications to earthquake and volcano deformation. *Geophys. J. Int*., 2010, vol. 182, pp. 568–582. DOI: 10.1111/j.1365-246X.2010.04655.x.

38. Kang, N. Green's function for Poisson Equation in Layered Nano-Structures including Graphene. A thesis presented to the University of Waterloo in fulfillment of the thesis requirement for the degree of Master of Mathematics in Applied Math. Waterloo, Ontario, Canada, 2015. http://hdl.handle.net/10012/9312.

39. Tewary, V. K. Electrostatic Green's function for advanced materials subject to surface loading. *Journal of Engineering Mathematics*, 2004, vol. 49, iss. 3, pp. 289-304. DOI: 10.1023/B:ENGI.0000031191.64358.a8.

40. Hon, Y. C., Li, M., Melnikov, Y. A. Inverse source identification by Green's function. *Engineering Analysis with Boundary Elements*, 2010, vol. 34, no. 4, pp. 352-358. DOI: 10.1016/j.enganabound.2009.09.009.

41. Sneddon, I. N. *Mixed boundary value problems in potential theory*. Amsterdam: North-Holland publ., New York, John Wily&Sons, 1966. 283 p.

42. Uflyand, Ya.S. *Metod parnykh uravneniy v zadachakh matematicheskoy fiziki* [*Method of pair equations in problems of mathematical physics*]. Leningrad, Nauka Publ., 1977. 220 p.

43. Zyuzin, V.A., Mossakovskiy, V.I. Osesim-metrichnoe nagruzhenie prostranstva so sfericheskim razrezom [Axisymmetric loading of space with a spherical cut]. *Prikladnaya matematika i mekhanika*, 1970, vol. 34, no. 1, pp. 179-183.

44. Ulitko, A.F. *Metod sobstvennyh vektornyh funkcij v prostranstvennyh zadachah teorii uprugosti* [*Eigenfunction Method in Spatial Problems of the Theory of Elasticity*]. Kiev, Naukova dumka Publ., 1979. 265 p.

45. Mehler, F.G. Notiz über die Dirichlet'schen Integralausdrücke für die Kugelfunction Pn(cosΘ) und über eine analoge Integralform für die Cylinderfunction Jm(x). *Mathematische Annalen*, 1872, vol. 5, iss. 1, pp. 141-144.

46. Nikolayev, O.H., Barakhov, K.P. Deyaki typy fizychnykh poliv u konusi z neodnoridnistyu u vyhlyadi sferychnoho sehmenta [Some types of physical fields in a cone with inhomogeneity in the form of a spherical segment]. *Matematychni metody ta fizyko-mekhanichni polya*, 2005, vol. 48, no. 4, pp. 191-198.

47. Nikolayev, O. H., Barakhov, K. P. Zadacha teoriyi potentsialu dlya kruhovoho tsylindru zi sferychnym sehmentom [Potential theory problem for a circular cylinder with a spherical segment]. *Visnyk L'vivs'koho universytetu. Seriya prykladna matematyka ta informatyka*, 2006, no. 11, pp. 75-81.

48. Kellogg, O. D. *Foundations of potential theory*. Berlin, Heidelberg, New York, Springer-Verlag Publ., 1967. 384 p.

49. Landau, L. D., Lifshic, E. M. Teoreticheskaja fizika. T. VIII. Jelektrodinamika sploshnyh sred [Theoretical physics. V. VIII. Electrodynamics of continuous media]. Moscow, Nauka Publ., 1982. 621 p.

50. Berdnyk, S. Gomozov, A., Gretskih, D., Kartich, V., Nesterenko, M. Approximate boundary conditions for electromagnetic fields in electrodmagnetics. *Radioelectronic and Computer Systems*. 2022, no. 3, pp. 141-160. DOI: 10.32620/reks.2022.3.11.

Надійшла до редакції 17.07.2022, розглянута на редколегії 20.11.2022

ФУНКЦІЇ ГРІНА ПЕРШОЇ ТА ДРУГОЇ КРАЙОВИХ ЗАДАЧ ДЛЯ РІВНЯННЯ ЛАПЛАСА В НЕКЛАСИЧНІЙ ОБЛАСТІ

Олексій Ніколаєв, Олександр Головченко, Ніна Савченко

Предметом вивчення є функції Гріна першої та другої крайових задач для рівняння Лапласа. **Метою** дослідження є побудова функцій Гріна першої та другої крайових задач для рівняння Лапласа в просторі зі сферичним сегментом в аналітичному вигляді, а також чисельний аналіз цих функцій. **Завдання:** формалізувати проблему визначення функцій Гріна для зазначеної області; **методами** Фур'є, парних суматорних рівнянь і теорії потенціалу звести мішані крайові задачі для допоміжних гармонічних функцій до системи рівнянь, яка має аналітичний розв'язок; дослідити на сумісність алгебраїчну систему для визначення сталих інтегрування; сформулювати і довести теорему про стрибок нормальної похідної потенціалу простого шару на поверхні сегмента, за допомогою якої подати функцію Гріна у вигляді потенціалу простого шару; провести числовий експеримент і виявити алгоритми і області зміни параметрів ефективних обчислень; проаналізувати характер поведінки функцій Гріна. **Наукова новизна**: вперше побудовано функції Гріна крайових задач Діріхле і Неймана для рівняння Лапласа в тривимірному просторі зі сферичним сегментом у аналітичному вигляді, обґрунтовано отримані результати та проведено всебічний числовий експеримент для аналізу поведінки цих функцій. **Отримані результати**: для допоміжних гармонічних функцій поставлено мішані крайові задачі у внутрішності та зовнішності сферичної поверхні, до якої належить сегмент; методом Фур'є задачі зведено до систем парних рівнянь у рядах за функціями Лежандра, розв'язки яких знаходяться за допомогою розривних сум Мелера – Діріхле. Указані функції отримано в явному вигляді в двох формах: рядів за базисними гармонічними функціями в сферичних координатах і потенціалу простого шару по поверхні сегмента. Для обґрунтування результатів доведено лему про сумісність алгебраїчної системи для визначення сталих інтегрування і теорему про стрибок нормальної похідної потенціалу простого шару на сегменті. Проведено числовий експеримент для аналізу поведінки побудованих функцій. **Висновки:** аналіз числових значень функцій Гріна, отриманих за різними алгоритмами, показав, що найбільшу точність результатів поза поверхнею сегмента отримано при використанні зображень функцій Гріна у формі рядів. На основі розрахунків побудовані лінії рівня функцій Гріна двох крайових задач в площині сингулярної точки, а також графіки щільності потенціалу простого шару для задачі Діріхле і стрибка потенціалу для задачі Неймана на сегменті при різному розташуванні сингулярної точки. У частинному випадку розташування сингулярної точки в початку координат знайдено потенціал електростатичного поля точкового заряду біля провідної заземленої тонкої оболонки у формі сферичного сегмента. Основні характеристики такого поля знайдено в замкненому вигляді.

Ключові слова: функція Гріна; крайова задача Діріхле; крайова задача Неймана; рівняння Лапласа; гармонічна функція; сферичний сегмент; потенціал простого шару; сферичні функції; стрибок потенціалу.

Ніколаєв Олексій Георгійович – д-р фіз.-мат. наук, проф., зав. каф. вищої математики та системного аналізу, Національний аерокосмічний університет ім. М. Є. Жуковського «Харківський авіаційний інститут», Харків, Україна.

Головченко Олександр Васильович – канд. фіз.-мат. наук, доц., проф. каф. вищої математики та системного аналізу, Національний аерокосмічний університет ім. М. Є. Жуковського «Харківський авіаційний інститут», Харків, Україна.

Савченко Ніна Валеріївна – канд. фіз.-мат. наук, доц. каф. вищої математики та системного аналізу, Національний аерокосмічний університет ім. М. Є. Жуковського «Харківський авіаційний інститут», Харків, Україна.

Oleksii Nikolaev – Doctor of Physical and Mathematical Sciences in Applied Mathematics, Professor, Head of Higher Mathematics and System Analysis Department, National Aerospace University "Kharkiv Aviation Institute", Kharkiv, Ukraine,
e-mail: a.nikolaev@khai.edu, ORCID: 0000-0002-7351-1392.

Oleksandr Holovchenko – PhD in Applied Mathematics, Associate Professor, Professor of Higher Mathematics and System Analysis Department, National Aerospace University "Kharkiv Aviation Institute", Kharkiv, Ukraine,
e-mail: o.holovchenko@khai.edu.

Nina Savchenko – PhD in Applied Mathematics, Associate Professor of Higher Mathematics and System Analysis Department, National Aerospace University "Kharkiv Aviation Institute", Kharkiv, Ukraine,
e-mail: n.savchenko@khai.edu, ORCID: 0000-0001-8144-9368.