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Finding Antiderivatives with the Help of the Generalized Taylor Series

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Finding antiderivatives of the given functions is an important component in a number of problems of applied mathematics. Difficulties occur in a case when the antiderivative in question cannot be expressed in terms of the elementary functions. In this paper we propose to use the atomic generalized Taylor series (AGTS), modified appropriately, for finding antiderivatives. More convenient formulas for the basic functions of AGTS are proposed.

Keywords: antiderivative, integral equation, atomic generalized Taylor series, basic functions of the atomic generalized Taylor series.

1. Statement of the problem and analysis of recent research and publications

In a number of problems of applied mathematics it is necessary to find antiderivatives of the given functions

$$F(x)=\int_0^x f(t)dt \qquad (1)$$

and these antiderivatives cannot be expressed in terms of elementary functions. One of the examples of such a problem is a problem of finding the distribution function for the normal law with probability density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Solving the linear non-homogeneous differential equations with constant coefficients, which often occur in applications, by the method of variation of constants, requires finding antiderivatives.

Solving of non-linear differential equations by the iteration method requires repeated finding antiderivatives of the given functions in a form convenient for the next iteration.

For example, the initial value problem for the differential equation of the first order

$$y(x)=F(x,y(x)),$$

$$y(x_0)=y_0$$

is equivalent to the Volterra integral equation

$$y(x)=y_0+\int_{x_0}^x F(t,y(t))dt\,.$$

If we solve this equation by iteration method, assuming $y_0(x)=y_0$ and

$$y_{n+1}(x)=y_0+\int_{x_0}^x F(t,y_n(t))dt\,,$$

then for every iteration step we should find the antiderivative of a function $F(x,y_n(x))$.

We cannot use usual quadrature formulas in this case, since the upper limit of the integral is variable. Classical Taylor series has some restrictions in application, first-

ly since its radius of convergence may be insufficient, and mainly because the substitution of the power series instead $y_n(t)$ into a function $F(t, y_n(t))$ requires further transformations for the obtaining the power series under the integral sign.

In this paper we propose to use for finding antiderivatives the atomic generalized Taylor series (AGTS) [1-4], modified appropriately. This series for the antiderivative $F(x)$ in question has a form

$$F(x) = \sum_{n=0}^{\infty} \sum_{k \in N_n} F^{(n)}(x_{n,k}) \varphi_{n,k}(x). \qquad (2)$$

Here the functions $\varphi_{n,k}(x)$ are so called basic functions of AGTS, which can be expressed in terms of the atomic function $\mathrm{up}(x)$ [5-8].

These functions have the following properties, which define them uniquely.

$$\varphi_{n,k}^{(m)}(x_{m,l}) = \delta_n^m \delta_l^k.$$

The points $x_{n,k}$ are defined as follows:

$$\text{for } n = 0 \quad x_{0,k} = k,$$

$$\text{for } n > 0 \quad x_{n,k} = k2^{n-1}.$$

So $x_{1,k} = k$, $x_{2,k} = \frac{k}{2}$, $x_{3,k} = \frac{k}{4}$ and so on.

2. Modification of the atomic generalized Taylor series and the more convenient formulas for the basic functions

In the series (2) in its standard form $N_0 = \mathrm{Z}$, $x_{0,k} = k$, so we should know the values of the integrals

$$F(k) = \int_0^k f(t)dt$$

that is not convenient in our case, because of the necessity to calculate the definite integrals.

We propose to modify this series in such a way that $N_0 = \{0\}$, knowing that $F(0) = 0$. In this process the basic functions of the generalized Taylor series $\varphi_{n,k}(x)$ are substituted by the modified basic functions of AGTS $\tilde{\varphi}_{n,k}(x)$. Namely, instead of defining the values of a function $F(x)$, represented by AGTS, at the points $k \neq 0$, we define the derivative of this function at the points $k - 1/2, k > 0, k + 1/2, k < 0$. Thus the modified $\tilde{x}_{1,k} = \frac{k}{2}$.

The corresponding modified basic functions for $k < 0$ are of the form

$$\tilde{\varphi}_{1,k-1/2}(x) = \begin{cases} 0.5up(x-k), x \le k, \\ 0.5, x > k. \end{cases}$$

Similarly we build the basic functions $\tilde{\varphi}_{1,k+1/2}(x)$ for $k>0$.

Further, all the other modified basic functions $\tilde{\varphi}_{n,k}(x)$ we obtain from the standard basic functions by subtraction of functions $\alpha\cdot\varphi_{1,k-1/2}(x)$, where α we choose to make the first derivatives of $\tilde{\varphi}_{n,k}(x)$ at the points $k-1/2,\ k>0,\ k+1/2,\ k<0$ equal to 0.

Consider in a bit more detail the case of the initial value problem for the differential equation of the first order with a smooth right-hand side

$$y'(x)=F(x,y(x)),$$
$$y(x_0)=y_0,$$

which is equivalent to the Volterra integral equation of the second kind

$$y(x)=y_0+\int_{x_0}^{x}F(t,y(t))dt.$$

Expanding both sides of this equation to atomic generalized Taylor series using its modification mentioned above in the right-hand side, we obtain the infinite system of non-linear equations with respect to the unknown coefficients of the AGTS. Let

$$f(t)=F(t,y(t)).$$

Then

$$y'(x)=f(x),$$
$$y''(x)=f'(x),$$
$$y^{(n)}(x)=f^{(n-1)}(x)\ n>1.$$

In its turn

$$f'(t)=\frac{\partial F}{\partial x}(t,y(t))+\frac{\partial F}{\partial y}(t,y(t))y'(t).$$

A representation obtained above one can also use for the calculation of Duhamel's integral

$$x(t)=\frac{1}{m\omega_d}\int_0^t p(\tau)e^{-\varsigma\omega_n(t-\tau)}\sin(\omega_d(t-\tau)d\tau,$$

which obviously is the indefinite integral.

When using constructions mentioned above for practical computations it is important to obtain convenient expressions for the basic functions of the atomic generalized Taylor series. In their turn the formulas for $\varphi_{n,k}(x)$ are based on the formulas for the atomic function $\mathrm{up}(x)$ and the atomic functions $\mathrm{fup}_n(x)$.

Let us introduce a function $\mathrm{up}^+(x)=\mathrm{up}(x-1)$, which is equal to 0 for $x\le 0$. Notice that this function is also used in [9]. It satisfies the functional-differential equation

$$y'(x)=2y(2x)-2y(2x-2). \qquad (3)$$

Now let us introduce a function $up^+(x,0)$, which is equal to $\mathrm{up}^+(x)$ on $[0,1]$ and is equal to 1 for $x\ge 1$.

This function is infinitely differentiable, as at a point 1 the function $\mathrm{up}^+(x)$ is equal to 1, and all of its derivatives at this point are equal to 0.

It is easily seen that

$$(up^+(x,0))' = 2up^+(2x)$$

Similarly, let's introduce the functions

$$up^+(x,n) = \begin{cases} up^+(x),\ x \in [0, 2^{-n}], \\ P_n(x),\ x > 2^{-n}. \end{cases}$$

where $P_n(x)$ is an algebraic polynomial of the degree n, which is the expansion of the function $\mathrm{up}^+(x)$ to a Taylor series at a point 2^{-n}.Coefficients of this polynomial are the rational numbers, which can be calculated by the recurrent formulas given in [5]. These functions are also infinitely differentiable by the construction. Thus

$$(up^+(x,n-1))^{(n)} = 2^{n(n+1)/2} up^+(2^n x).$$

Basic functions of the atomic generalized Taylor series are the linear combinations of shifts of the functions $up^+(x,n)$, and the coefficients of these linear combinations and the values of the shifts are given by the simple formulas. In its turn, calculation of the functions $up^+(x,n)$ reduces itself to the obtaining of the function $up^+(x) = up(x-1)$, and the formulas for $up(x)$ are given in [5].

In many of the important applications of the mechanics and electrodynamics the right-hand side of the ordinary differential equation of the first order or of a system of such equations is a function, which contains the integrals with respect to space variables, that is, the equation is integro-differential. In this case it's appropriate to use the atomic generalized Taylor series twice, that is with respect to both time and space variable. Thus it's necessary to calculate integrals of the products of basic functions of the atomic generalized Taylor series (bfAGTS) and from the product of basic function AGTS by the function not represented by AGTS. In this case it is convenient to change the representation in terms of bfAGTS with one in terms of the orthonormal system, obtained by orthogonalization of the sequence bfAGTS, which we denote by AOS (atomic orthogonal system). If we use Gram – Schmidt process for the orthogonalization, it's easy to obtain the formulas for bfAGTS in terms of AOS, since the corresponding transformation matrices are triangular. To orthogonalize bfAGTS we need to calculate the integrals of the products of functions $up^+(x,n)$

$$\int_a^b up^+(x,n) up^+(x,m) dx$$

with sufficient precision. All of these integrals can be expressed precisely in terms of the integral of $\mathrm{up}(x)$ squared, which in turn is a sum of a quickly convergent series, obtained by Parseval's identity.

Give the values of the function $\mathrm{up}(x)$, which are necessary for the calculation of the polynomials $P_{n-1}(x)$ using the definition of the functions $up^+(x,n)$

$$\mathrm{up}(-1/2) = up^{+}(1/2) = 1/2 = 0.5,$$

$$\mathrm{up}(-3/4) = up^{+}(1/4) = 5/72 = 0.069(4),$$

$$\mathrm{up}(-7/8) = up^{+}(1/8) = 1/288 = 0.00347(2),$$

$$\mathrm{up}(-15/16) = up^{+}(1/16) = 143/2073600 \approx 0.000068962,$$

$$up(-31/32) = up^{+}(1/32) = 19/33177600 \approx 5.72 \cdot 10^{-7},$$

$$up(-63/64) = up^{+}(1/64) = 1153/561842749440 \approx 2.05 \cdot 10^{-9},$$

$$up(-127/128) = up^{+}(1/128) = 583/179789679820800 \approx 3 \cdot 10^{-12},$$

$$up(-255/256) = up^{+}(1/256) = 1616353/704200217922109440000 \approx 2 \cdot 10^{-15},$$

$$up(-511/512) = up^{+}(1/512) = 132809/180275255788060016640000 \approx 7 \cdot 10^{-19},$$

$$up(-1023/1024) = up^{+}(1/1024) =$$

$$= 134926369/12463948513585393872383508480000 \approx 1 \cdot 10^{-22},$$

$$up(-2047/2048) = up^{+}(1/2048) =$$

$$= 46840699/6381541638955721662660356341760000 \approx 7 \cdot 10^{-27},$$

$$up(-4095/4096) = up^{+}(1/4096) =$$

$$= 67545496213157/2922147328878987139869165759252570709760000000 \approx$$

$$\approx 2 \cdot 10^{-31}.$$

Indeed,

$$P_{n-1}(x) = \sum_{k=0}^{n-1} \frac{up^{+}(2^{-n})^{(k)}}{k!}(x - 2^{-n})^{k},$$

and, as it was mentioned above

$$(up^{+}(x, n-1))^{(n)} = 2^{n(n+1)/2} up^{+}(2^{n} x).$$

Particularly

$$(up^{+}(2^{k}, n-1))^{(n)} = 2^{n(n+1)/2} up^{+}(2^{k+n}).$$

Give the recurrent formulas for the basic functions of the atomic generalized Taylor series. Formulas for the basic functions, corresponding to the values and the first derivatives are given above. Functions corresponding to n-th derivatives, we build as follows.

The function

$$\chi_{n,k}(x) = 2^{-n(n+1)/2} up^{+}(x - (2k-1)2^{-n}, n-1)$$

at a point $x_{n,k} = k2^{-n+1}$ has the n-th derivative equal to 1, and at the other points $x_{n,l}$, $l \neq k$ its n-th derivative is equal to 0.

Its derivatives of the m-th order, $m < n$ are of the form

$$\chi_{n,k}{}^{(m)}(x) = 2^{-n(n+1)/2+m(m+1)/2} up^{+}(2^{m}(x-(2k-1)2^{-n})), n-m-1)$$

and at the points $x_{m,s}$, $m < n$ are equal to the known numbers

$$a_{n,k,m,s} = \chi_{n,k}{}^{(m)}(x_{m,s}) =$$

$$= 2^{-n(n+1)/2+m(m+1)/2} up^{+}(2^{m}(x_{m,s}-(2k-1)2^{-n}), n-m-1).$$

Since the basic functions of AGTS

$$\varphi_{n,k}(x) = \chi_{n,k}(x) - \sum_{m=0}^{n-1} \sum_{s \in N_m} a_{n,k,m,s} \varphi_{m,s}(x).$$

This is the required recurrent formula for the basic functions of AGTS $\varphi_{n,k}(x)$.

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Обчислення первісних за допомогою узагальненого ряду Тейлора

Обчислення первісних заданих функцій є важливою складовою багатьох задач прикладної математики. Труднощі виникають у випадку, коли шукана первісна не може бути виражена через елементарні функції. У цій статті пропонується використовувати для обчислення первісних атомарний узагальнений ряд Тейлора (АУРТ), видозмінений відповідним чином. Запропоновано більш зручні формули для базисних функцій АУРТ.

Ключові слова: первісна, інтегральне рівняння, атомарний узагальнений ряд Тейлора, базисні функції атомарного узагальненого ряду Тейлора.

Вычисление первообразных с помощью обобщенного ряда Тейлора

Вычисление первообразных заданных функций является важной составляющей многих задач прикладной математики. Трудности возникают в случае, когда искомая первообразная не выражается через элементарные функции. В этой статье предлагается использовать для вычисления первообразных атомарный обобщенный ряд Тейлора (АОРТ), видоизмененный соответствующим образом. Предложены более удобные формулы для базисных функций АОРТ.

Ключевые слова: первообразная, интегральное уравнение, атомарный обобщенный ряд Тейлора, базисные функции атомарного обобщенного ряда Тейлора.

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