МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
Національний аерокосмічний університет ім. М.Є. Жуковського
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ТЕОРЕТИЧНА МЕХАНІКА.
СТАТИКА

Навчальний посібник

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THEORETICAL MECHANICS.
STATICS

Textbook

Харків «ХАІ» 2010

УДК [531+531.2] (075.8)
UDK [531+531.2] (075.8)

Теоретична механіка. Статика: навч. посіб. / В.О.Ойкін, О.Ю. Кладова, В.М. Павленко, Я.О.Стеценко. – Х.: Нац. аерокосм. ун-т «Харк. авіац. ін-т», 2010. – 84 с.
Theoretical mechanics. Statics: Textbook / V.A. Oikin, O.Y. Kladova, V.N. Pavlenko, Y.A. Stetsenko. — Kharkov: National Aerospace University 'Kharkov Aviation Institute', 2010. – 84 p.

Посібник призначено для забезпечення зрілого, поглибленого сприйняття механіки, а також для розвитку твердого розуміння основних принципів статики замість механічного вивчення специфічних методологій.

Розглянуто дві основні проблеми статики: еквівалентне перетворення систем сил, що діють на тверде тіло, та визначення умов, які мусить задовольняти система сил для забезпечення рівноваги твердого тіла. Наведено загальні рівняння для визначення положення центру ваги двовимірних і тривимірних твердих тіл.

Для студентів механічних та інших спеціальностей (з повною та скороченою програмою з теоретичної механіки).

Designed to provide a more mature, in-depth treatment of mechanics this textbook focuses on developing a solid understanding of basic principles of statics rather than rote learning of specific methodologies.

The textbook considers two basic problems of statics: the composition of forces and reduction of force system acting on rigid bodies to a form as simple as possible, and determination of the conditions for the equilibrium of force system acting on rigid bodies. The general equations for locating the center of gravity of two- and three-dimensional bodies are developed in the textbook.

For college students studying theoretical mechanics.

Іл. 38. Табл. 1. Бібліогр.: 6 назв

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1 Fundamental quantities and axioms of mechanics

1.1 Force. System of forces. Rigid body equilibrium

Statics is the branch of mechanics which treats of bodies that are at rest or in the state of uniform motion. Statics studies the laws of composition of forces and the conditions of equilibrium of engineering structures under the action of forces.

Force is a fundamental quantity of mechanics. In mechanics, a force can be defined as measure of mechanical interaction between bodies. As a result of this interaction bodies can be accelerated or deformed (i.e., bodies change their shape). So when we say “a force acts on the body” we know that there is another body acting as a source of the force.

The physical nature of forces is not studied in mechanics. We shall distinguish forces solely by the mode of their interaction. A force may act through direct contact like the elevating force acting on the airplane wing in incident flow (**close-range interaction**), or it may act from a distance like gravitational or magnetic attraction (**long-range interaction).**

In mechanics, force is determined by three characteristics: its magnitude, direction and point of application. So it can be represented by a vector whose length is equal to the force’s magnitude in scale. This vector is applied at a given point and directed along the force direction. Therefore, operations with forces obey the rules of vector algebra.

In the SI system the unit of force is Newton (1 N); in the MKS system it is one kilogram-force (1 kgf). The relation between these units can be defined as follows:

$$1\,N = 1\,\frac{kg \cdot m}{s^2}\ ;\ 1\,kgs \approx 9{,}81\,N\ ;\ 1\,N \approx 0{,}102\,kgs\ .$$

A set of forces applied to a material object (point, body, system of bodies) and treated as a group is called a **system of forces or force system**.

Consider a body under the action of force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$. If the physical state of the body does not change when we replace this force system with another one $\left(\vec{P}_1, \vec{P}_2, ..., \vec{P}_n\right)$, these force systems are said to be **equivalent**. Equivalence of the force systems is designated as follows:

$$\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right) \sim \left(\vec{P}_1, \vec{P}_2, ..., \vec{P}_k\right)\ . \qquad (1.1)$$

If system of force $\left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n\right)$ is equivalent to single force $\vec{R}$, i.e.

$$\left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n\right) \sim \vec{R} \ . \tag{1.2}$$

that force is the **resultant** of the force system. It means that the resultant force $\vec{R}$ has the same effect on the body as the given force system $\left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n\right)$.

A system of forces is said to be **equilibrated (balanced),** or **equivalent to zero,** if under the action of this system a body is at rest:

$$\left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n\right) \sim 0 \ . \tag{1.3}$$

Note, that if two forces are equivalent, then two vectors representing these forces are equal. However, the equality of two vectors ($\vec{F} = \vec{P}$) does not mean that the forces are equivalent ($\vec{F} \sim \vec{P}$).

There are several idealizations used in mechanics. One of these idealizations is a rigid body. A body is called **rigid** if the distance between any two points of the body does not change during its interaction with other solids. Consequently, the angle between any two straight lines in the body remains constant.

The following two reasons account for the introduction of the rigid body model:

in many cases the deformation of a body is negligibly small as compared with other results of force action, so we can treat the body as rigid;

the conditions for a rigid body to be at rest, which the forces acting on it must satisfy, are at the same time the necessary conditions of equilibrium for any deformable body. So, statements of rigid body statics can be used to study the equilibrium conditions of real physical bodies.

1.2 Statics axioms

As we know, an axiom is a statement that needs no proof. In statics, axioms are the simplest and the most general laws, which are valid for forces acting on the rigid body or applied to interacting bodies.

The first axiom states necessary and sufficient conditions under which two forces form a balanced system.

Axiom 1. A rigid body which is acted upon by two forces will be in equilibrium if and only if the two forces have the same magnitude and the same line of action but opposite sense. This case is shown in Fig.1.1.

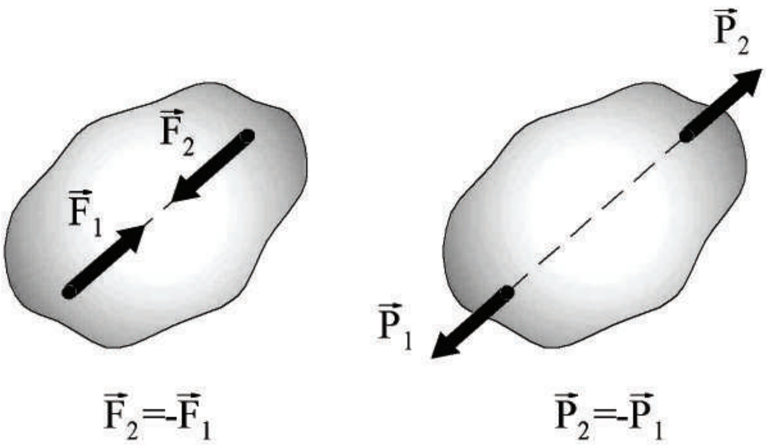


Fig. 1.1

The following two axioms formulate the simplest equivalence operations with forces.

Axiom 2. The action of a given force system on a rigid body remains unchanged if another balanced force system is added to, or subtracted from, the original system. In a special case, in accordance with Axiom 1, this balanced force system can consist of two equal and opposite forces acting along the common line (Fig. 1.2).

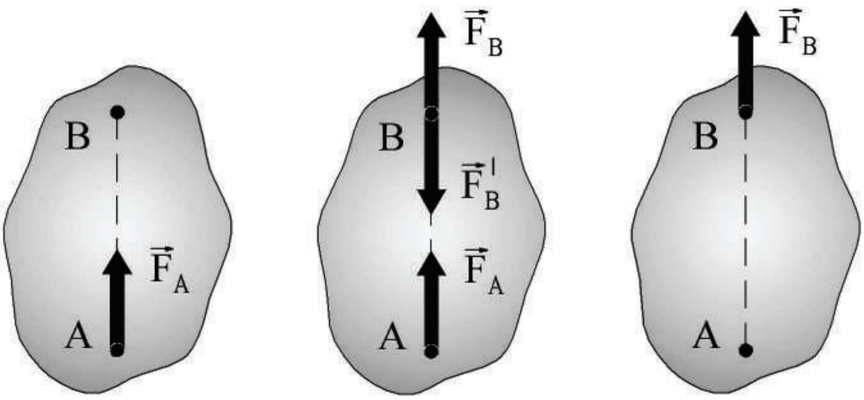


Fig. 1.2

It follows from Axiom 2 (corollary) that a force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts. The corollary is named the **principle of transmissibility**.

When only the resultant external effects of a force are to be investigated, the force may be treated as a **sliding vector**, and it is necessary and sufficient to specify the magnitude, direction, and line of action of the force.

This can be easily proved by the following relations

$$\vec{F}_B = \vec{F}_A;\ \left(\vec{F}'_B, \vec{F}_B\right) \sim 0\ ;\ \vec{F}_A \sim \left(\vec{F}_A, \vec{F}'_B, \vec{F}_B\right) \sim \vec{F}_B\ . \quad (1.4)$$

But in some cases it is necessary to take into consideration the position of the force application point. It will become evident when we study the doctrine of parallel forces center and center of gravity.

Axiom 3 (the parallelogram law). Two forces applied at one point of a body have as their resultant a force applied at the same point and represented by the diagonal of a parallelogram constructed with the two given forces as its sides (Fig. 1.3),

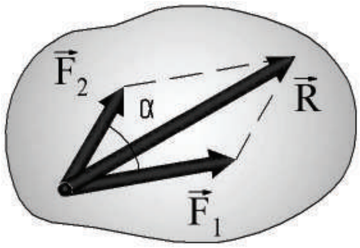


Fig. 1.3

i.e. a force system $\left(\vec{F}_1, \vec{F}_2\right)$ is equivalent to its resultant

$$\left(\vec{F}_1, \vec{F}_2\right) \sim \vec{R}\ . \quad (1.5)$$

This resultant is denoted according to the rule of vector addition for two vectors:

$$\vec{R} = \vec{F}_1 + \vec{F}_2\ , \quad (1.6)$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1 \cdot F_2 \cos\alpha}\ . \quad (1.7)$$

Note that the first two axioms and principle of transmissibility are valid only for rigid bodies, while the third axiom is also valid for any deformable body.

The following axiom determines the relation between the forces of interaction between two bodies.

Axiom 4 (principle of action and reaction). The forces of action and reaction existing between contacting bodies are equal in magnitude and act along the same line in opposite directions.

It means that if body 1 (Fig.1.4) acts on body 2 with the force $\vec{F}_{12}$ and body 2 acts on body 1 with the force $\vec{F}_{21}$, these forces satisfy the equation

$$\vec{F}_{21} = -\vec{F}_{12} \tag{1.8}$$

and act along the same line.

It is important to remember that the forces of action and reaction (e.g. $\vec{F}_{12}$ and $\vec{F}_{21}$)) do not form a balanced system of forces because they are applied to different bodies.

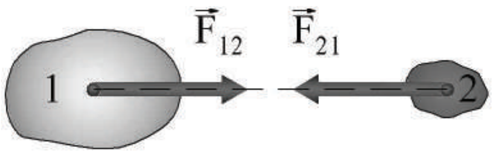


Fig. 1.4

Axiom 5 (principle of solidification). If a deformable body is in the state of static equilibrium, it would also be in static equilibrium if the body were rigid.

It means that if a deformable body (structure, system of bodies, etc.) is in the state of equilibrium, the rigid body produced from the deformable body by solidification is at equilibrium too.

It follows from Axiom 5 that for a system of deformable bodies the equilibrium conditions of a rigid body system are necessary, but not sufficient.

1.3 Vector and axial moment of force

When we deal with the problem of equilibrium of a lever as the simplest mechanism, we come to the notion of force moment. Let us consider the first sort lever AOB (Fig. 1.5) with O as fulcrum (pivot).

Forces $\vec{P}$ and $\vec{Q}$ are applied to the ends of the bar perpendicular to the AB. The equilibrium condition can be formulated in the following way: to be in the state of equilibrium, the lever's fulcrum must be situated at the point dividing the distance between the points of the force application into parts so that they are inversely proportional to the forces' magnitudes.

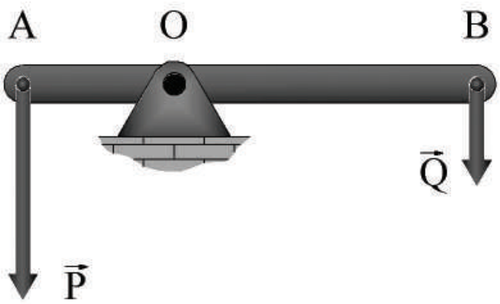


Fig. 1.5

So it must be true that

$$\frac{OA}{AB} = \frac{Q}{P} , \tag{1.9}$$

or

$$P \cdot OA = Q \cdot AB \ . \tag{1.10}$$

Thus, it follows from the theory of lever equilibrium that we should pay attention to the force-by-distance product. Now, we will introduce the notion of the moment of force about a point to generalize everything discussed above.

The **moment of a force about a center** (point) is a vector whose magnitude is determined as the product of the force magnitude and its arm. The **arm** is the shortest (perpendicular) distance between the center and the force's line of action. The vector of moment is perpendicular to the plane of the force and the center. The vector of moment is directed so that the rotation is counter-clockwise when viewed from the end of the vector.

The moment of a force about the center O is denoted as $\vec{M}_O\left(\vec{F}\right)$. The following cross product conforms fully to the definition of a vector of moment about the center

$$\vec{M}_O\left(\vec{F}\right) = \vec{r} \times \vec{F} , \tag{1.11}$$

where $\vec{r}$ is a position vector of the force application point relative to the center O.

According to the definition of the cross-product the magnitude of moment $\vec{M}_O\left(\vec{F}\right)$ is equal to

$$M_O\left(\vec{F}\right) = r \cdot F \cdot \sin\alpha = F \cdot h \ . \qquad (1.12)$$

Now consider a triangle with a vertex located at the point O and a base F (Fig. 1.6). The product $F \cdot h$ is equal to the doubled triangular area; so the magnitude of moment $\vec{M}_O\left(\vec{F}\right)$ can be expressed as

$$M_O\left(\vec{F}\right) = 2S \ , \qquad (1.12a)$$

where S is OAB triangle area.

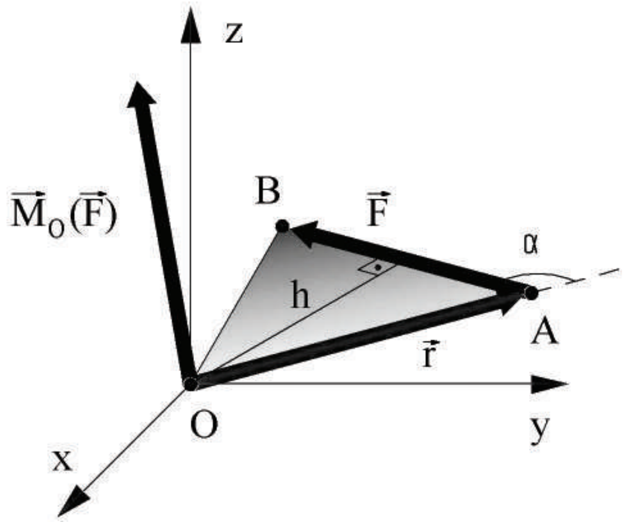


Fig. 1.6

Let O be the origin of the Cartesian rectangular coordinate system. Thus, the position vector and force can be expressed as

$$\vec{r} = \vec{i} \cdot x + \vec{j} \cdot y + \vec{k} \cdot z \ , \qquad (1.13)$$

$$\vec{F} = \vec{i} \cdot F_x + \vec{j} \cdot F_y + \vec{k} \cdot F_z \ , \qquad (1.14)$$

and the moment vector is

$$\overrightarrow{M}_O\left(\vec{F}\right)=\vec{r}\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ F_x & F_y & F_z\end{vmatrix}=\vec{i}\left(yF_z-zF_y\right)+$$

$$+\vec{j}\left(zF_x-xF_z\right)+\vec{k}\left(xF_y-yF_x\right). \quad (1.15)$$

On the other hand, a vector of moment can be resolved into components along the coordinate axes

$$\overrightarrow{M}_O\left(\overrightarrow{F}\right)=\overrightarrow{i}\cdot M_{Ox}\left(\overrightarrow{F}\right)+\overrightarrow{j}\cdot M_{Oy}\left(\overrightarrow{F}\right)+\vec{k}\cdot M_{Oz}\left(\overrightarrow{F}\right), \quad (1.16)$$

wherein $M_{Ox}\left(\overrightarrow{F}\right)$, $M_{Oy}\left(\overrightarrow{F}\right)$, $M_{Oz}\left(\overrightarrow{F}\right)$ are projections of the vector of moment on the axes Ox, Oy, Oz respectively.

Comparing the equalities (1.15) and (1.16), we have

$$M_{Ox}\left(\overrightarrow{F}\right)=y\cdot F_z-z\cdot F_y,$$

$$M_{Oy}\left(\overrightarrow{F}\right)=z\cdot F_x-z\cdot F_z, \quad (1.17)$$

$$M_{Oz}\left(\overrightarrow{F}\right)=x\cdot F_y-y\cdot F_x.$$

Now let us introduce the notion of a force moment about the axis. For this purpose *we* shall write the formulas that determine the moments of a force about the points O and O_1 situated on the axis Oz (Fig. 1.7):

$$\overrightarrow{M}_O\left(\overrightarrow{F}\right)=\overrightarrow{r}\times\overrightarrow{F}, \quad (1.18)$$

$$\overrightarrow{M}_{O_1}\left(\overrightarrow{F}\right)=\overrightarrow{r_1}\times\overrightarrow{F}. \quad (1.19)$$

Let us prove that the projections of force moments about the points O and O_1 onto the axis Oz are equal

$$M_{Oz}\left(\overrightarrow{F}\right)=M_{O_1z}\left(\overrightarrow{F}\right). \quad (1.20)$$

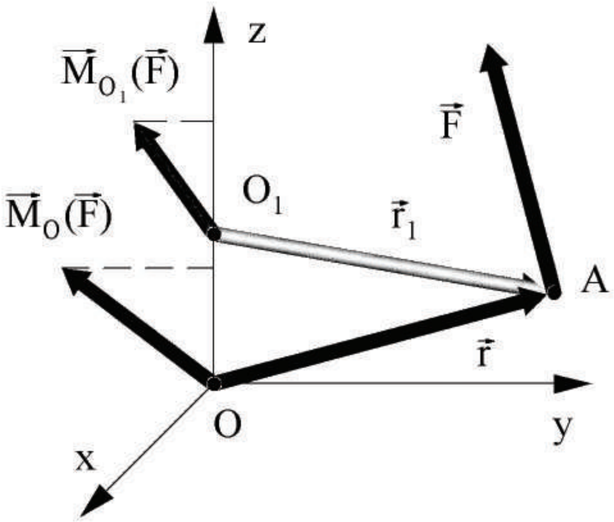


Fig. 1.7

Proceeding from the vector subtraction rule (see vector triangle OO_1A), we get

$$\vec{r}_1 = \vec{r} - \overrightarrow{OO_1} \quad . \tag{1.21}$$

$\overrightarrow{OO_1}$ does not have components along axes Ox, Oy. Therefore we have

$$\begin{aligned} r_{1x} &= r_x = x \ ; \\ r_{1y} &= r_y = y \ ; \\ r_{1z} &= r_z - OO_1 \ . \end{aligned} \tag{1.22}$$

By using formula (1.15), we can get the following expression for the vector moment $\vec{M}_{O_1}\left(\vec{F}\right)$

$$\vec{M}_{O_1}\left(\vec{F}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z - OO_1 \\ F_x & F_y & F_z \end{vmatrix} . \tag{1.23}$$

It follows that for the projection of vector $\vec{M}_{O_1}\left(\vec{F}\right)$ onto Oz

$$M_{O_1z}\left(\vec{F}\right) = x \cdot F_y - y \cdot F_x \ . \qquad (1.24)$$

Comparing this expression with the third formula from (1.17), we see that the equality (1.20) holds. As point O_1 is arbitrarily chosen on axis Oz, the following conclusion may be drawn: if we project the vector moment of the force about the center lying on the axis, the result is independent of the choice of the center on the axis. So, hereinafter we will use denotation $M_z\left(\vec{F}\right)$ instead of $M_{Oz}\left(\vec{F}\right)$, $M_{O_1z}\left(\vec{F}\right)$. Let us call this projection of vector moment $\vec{M}_O\left(\vec{F}\right)$ onto axis Oz the moment of force about the axis Oz, i.e.

$$M_z\left(\vec{F}\right) = x \cdot F_y - y \cdot F_x \ . \qquad (1.25)$$

Analyzing formula (1.25), we may state that the moment of force about axis Oz is determined by the following parameters: coordinates x and y of the force application point and the force $\vec{F}$ projections F_x and F_y onto axes Ox and Oy respectively. In its turn, the vector-position $\vec{r}$ and the force $\vec{F}$ projections onto plane xOy are equal to

$$\vec{r}_{xy} = \vec{i}\,x + \vec{j}\,y \ , \qquad (1.26)$$

$$\vec{F}_{xy} = \vec{i}\,F_x + \vec{j}\,F_y \ . \qquad (1.27)$$

Let us calculate the cross product

$$\vec{r}_{xy} \times \vec{F}_{xy} = \left(\vec{i}\,x + \vec{j}\,y\right) \times \left(\vec{i}\,F_x + \vec{j}\,F_y\right) =$$

$$= \vec{k}\left(xF_y - yF_x\right) = \vec{k} M_z\left(\vec{F}\right). \qquad (1.28)$$

By termwise multiplication, we take into account the equalities $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{i} = -\vec{k}$.

On the other hand (Fig. 1.7), we have

$$\vec{r}_{xy} \times \vec{F}_{xy} = \vec{k} \cdot \left| r_{xy} \right| \cdot \left| F_{xy} \right| \cdot \sin\alpha = \vec{k} \cdot \left| F_{xy} \right| \cdot h , \qquad (1.29)$$

as $\left| r_{xy} \right| \cdot \sin\alpha = h$. Here **h** is the length of the perpendicular dropped from point O onto the line of action of force $\vec{F}$, i.e. the arm of the projection $\vec{F}_{xy}$ with respect to point O.

Comparing formulas (1.28) and (1.29), we get another formula for the moment of force $\vec{F}$ about axis Oz:

$$M_z\left(\vec{F}\right) = \pm \left| F_{xy} \right| \cdot h . \qquad (1.30)$$

In accordance with this formula, the moment of force $\vec{F}$ about the given axis (for example, Oz), may be obtained in the following three steps:

1) project the force onto a plane that is perpendicular to axis Oz (plane xOy);
2) determine the arm h of the projection $\vec{F}_{xy}$ with respect to the point of axis and plane intersection;
3) calculate the moment of force about the axis as the positive or negative product of the magnitude of the force projection and its arm.

The rule of axial moment $M_z\left(\vec{F}\right)$ sign determination is as follows: if a force tends to rotate a body counterclockwise, then the moment of force $M_z\left(\vec{F}\right)$ is considered positive. Similarly, if a force tends to rotate a body clockwise, then the moment of force is negative.

This means that steps 1, 2 and 3 of the procedure stated above are valid for the axes of any direction irrespective of the frame of reference.

If the axial moment of force is equal to zero, in accordance with formula (1.30): $M_z\left(\vec{F}\right) = 0$. If $F_{xy} = 0$, the force and the axis are parallel; if $h = 0$, the force $\vec{F}_{xy}$ line of action passes through point O, i.e. the line intersects the axis.

Evidently, both cases may be combined: the axial moment of a force is equal to zero if the force and the axis are in the same plane.

In practice, it is convenient to determine a moment of force about one of the coordinate axes by decomposing the force into two or three components parallel to the coordinate axes. Then the moment of force about the chosen axis is equal to the sum of the component moments about this axis. The component arms are equal to the modulus of one of the coordinates of the force application point:

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z \; ; \tag{1.31}$$

$$M_x\left(\vec{F}\right) = M_x\left(\vec{F}_y\right) + M_x\left(\vec{F}_z\right) \; ; \tag{1.32}$$

$$M_y\left(\vec{F}\right) = M_y\left(\vec{F}_x\right) + M_y\left(\vec{F}_z\right) \; ; \tag{1.33}$$

$$M_z\left(\vec{F}\right) = M_z\left(\vec{F}_x\right) + M_z\left(\vec{F}_y\right) . \tag{1.34}$$

Example. Force $\vec{F}$ is applied to the vertex of the rectangular parallelepiped with edges a, b, c, where the force is directed along the upper cube face (Fig. 1.8). Determine the moments of the force with respect to coordinate axes Ox, Oy, Oz.

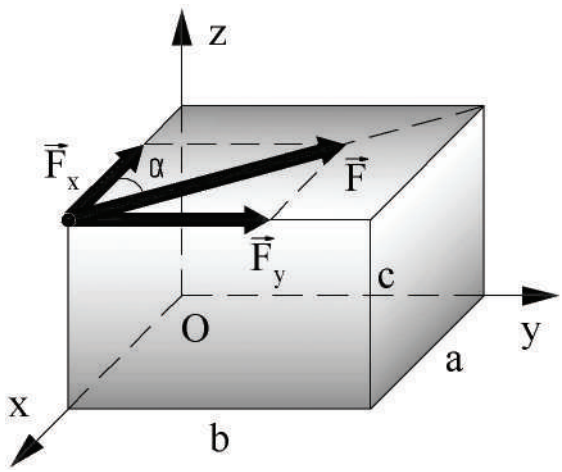


Fig. 1.8

Let us resolve force F into components along these axes

$$\vec{F} = \vec{F}_x + \vec{F}_y \; ; \; F_z = 0 .$$

At the same time,

$$\left|\vec{F}_x\right| = F \cdot \cos\alpha \; , \; \left|\vec{F}_y\right| = F \cdot \sin\alpha \; ,$$

wherein **F** is the magnitude of the force $\vec{F}$,

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \; ; \; \sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \; .$$

Taking into account the conditions for the axial moment to vanish and equalities (1.32) – (1.34), we obtain:

$$M_x\left(\vec{F}\right) = -\left|\vec{F}_y\right| \cdot c = -F\frac{b \cdot c}{\sqrt{a^2+b^2}};$$

$$M_y\left(\vec{F}\right) = -\left|\vec{F}_x\right| \cdot c = -F\frac{a \cdot c}{\sqrt{a^2+b^2}};$$

$$M_z\left(\vec{F}\right) = \left|\vec{F}_y\right| \cdot a = F\frac{a \cdot b}{\sqrt{a^2+b^2}}.$$

1.4 Couple. Couple vector moment

A set of two equal noncollinear parallel forces of opposite sense is called a couple. The plane containing these two forces is called a couple plane.

Let us consider couple $\left(\vec{P},\vec{P}'\right)$. Π is the couple plane.

Now we shall determine the sum of the vector moments of couple forces with respect to the arbitrarily chosen center O (Fig. 1.9).

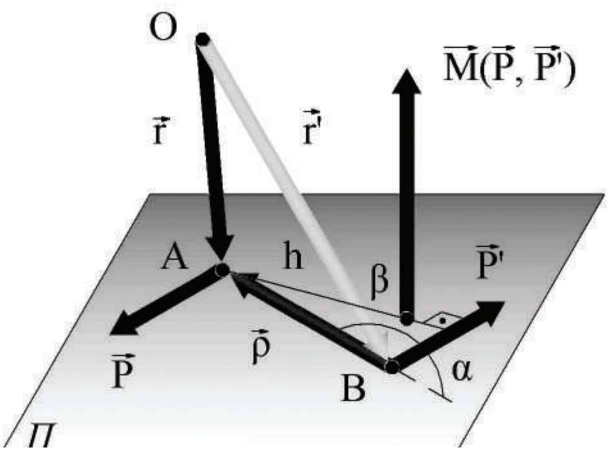


Fig. 1.9

In accordance with the definition of a couple, we have

$$\vec{P}' = -\vec{P} . \tag{1.35}$$

As we see from Fig. 1.9,

$$\vec{r}' = \vec{r} - \vec{\rho} . \tag{1.36}$$

Thus we obtain

$$\vec{M}_O\left(\vec{P}\right) + \vec{M}_O\left(\vec{P}'\right) = \vec{r} \times \vec{P} + \vec{r}' \times \vec{P}' =$$

$$= \vec{r} \times \vec{P} + \left(\vec{r} - \vec{\rho}\right) \times \left(-\vec{P}\right) =$$

$$= \vec{r} \times \vec{P} - \vec{r} \times \vec{P} + \vec{\rho} \times \vec{P} = \vec{\rho} \times \vec{P} . \tag{1.37}$$

The vector $\vec{\rho} \times \vec{P} = \overrightarrow{BA} \times \vec{P}$ does not depend on the location of center O. It depends on the mutual location of application points of forces $\vec{P}$, $\vec{P}'$ (i.e. it depends on vector $\overrightarrow{BA} = -\overrightarrow{AB}$). Vector $\vec{\rho} \times \vec{P} = \overrightarrow{BA} \times \vec{P}$ is called a **couple moment**

$$\vec{M}\left(\vec{P}, \vec{P}'\right) = \overrightarrow{BA} \times \vec{P} = \overrightarrow{AB} \times \vec{P}' . \tag{1.38}$$

The magnitude of the couple moment vector is

$$M\left(\vec{P}, \vec{P}'\right) = AB \cdot P' \cdot \sin\alpha = AB \cdot P' \cdot \sin\beta = P' \cdot h . \tag{1.39}$$

In formula (1.39) $\alpha = 180^\circ - \beta$ and $AB \cdot \sin\beta = h$.

The shortest distance between the lines of action of the couple forces h is called the **arm** of the couple.

The vector of a couple moment is directed perpendicularly to the plane of the couple so that, if seen from its head, the rotation of the plane is anticlockwise (the direction of the vector may be determined by the right-hand screw rule).

The features of couples and operations with them will be presented in Chapter 3.

1.5 Constrains and their reaction

A body is considered **free** if its displacements are not restricted by any other bodies; otherwise a body is **constrained**. The bodies that prevent the motion of the first body are called constraints imposed upon the body. Constraints imposed upon a rigid body restrict the body's freedom of motion. If we compare the motion of a free body under the action of a given force system and the motion of a constrained body under the action of the same force system, we can see that these motions are different. So, it may be stated that the mechanical effect of a constraint is the same as the action of a force. Therefore the action of a constraint in the body may be replaced by the forces that are called **reactions**.

In the Russian school of mechanics it is assumed that any constrained body can be considered free if constraints applied to the body are mentally eliminated and replaced with the corresponding reaction forces. In real conditions the magnitude and direction of the reaction are unknown. Moreover, the position of the distributed reaction force resultant application point is also unknown (the contact between bodies is not localized, but there is an area contact). All these unknown parameters of the reaction (magnitude, direction and point of application) depend on loads applied to the body. These parameters are determined by the conditions of equilibrium in the equations for the body. These conditions will be given later. Using some simplification and features of constraints in problems of statics for a rigid body which is not free, we may determine the reaction force line of action.

Constraints may be subdivided into two types. The **first type** includes constraints imposed on a body under the action of forces that are in the same plane (coplanar force system). Constrains are referred to the **second type** if they are imposed on a body under the action of forces that are not in the same plane (non-coplanar force system).

Now let us consider some constraints and show their reaction lines of actions, assuming that contacting surfaces are smooth enough to neglect friction completely.

By definition, a reaction force may be treated as counteraction to displacement in some direction. A body under the action of constraints of the first type can be subject to three types of displacement. These are two linear, mutually perpendicular displacements of a contact point of the body and constraint, and a rotational displacement about the axis that passes through the contact point and is perpendicular to the plane of the forces. Linear displacements are prevented by reaction forces. Rotational displacement is prevented by the reaction couple characterized by its moment.

Fig. 1.10, a shows **a smooth surface**. This constraint impedes the motion of the contact point along the common normal to the contacting surfaces. So, the smooth surface reaction is directed along the common normal. Fig. 1.10, g and 1.10, h show the supporting smooth surfaces as constraints. Reactions $\vec{R}_A$, $\vec{R}_B$, $\vec{N}$ are directed perpendicularly to the corresponding surfaces. In Fig. 1.10, e the body is suspended by means of two **ideal cords**, which are flexible, weightless, and inextensible. Each cord counteracts displacements of the attachment points. These counteractions are effected by forces $\vec{T}_A$, $\vec{T}_B$ directed along the cords.

A body is called **a two-force body** if the following four conditions are satisfied:

– the body has a negligible weight;
– the body is pinned at only two locations to other objects (in 2D);
– the body can be treated as rigid;
– no active forces are applied to the body's internal points.

In accordance with Axiom 1, the reaction of the body, as well as that of the constraint, is directed along the straight line crossing the pin centers (Fig. 1.10, d)).

The **sliding joint and pinned joint** are shown in Fig. 1.10, b and 1.10, c. The sliding joint impedes the motion of the beam end A in the direction that is perpendicular to the plane where the joint is installed; so the sliding joint reaction is oriented perpendicularly to the plane that supports the sliding joint. The pinned joint prevents motion in two directions: along axis Ax and along axis Ay, so the reaction of the pinned joint has two components: $\vec{X}_A$, $\vec{Y}_A$. The resulting reactive force is $\vec{R}_A = \vec{X}_A + \vec{Y}_A$.

The **clamped joint**, or fixed support, shown in Fig. 1.10, f, precludes motion in all directions (along axes Ax and Ay, rotation about axis Az). Therefore, there should exist two mutually perpendicular components of reaction forces $\vec{X}_A$, $\vec{Y}_A$ and a couple with moment M_{Az} to replace the clamped joint. **Sliding support** is shown in Fig. 1.10, i. This constraint impedes motion along axis Ay and rotation about axis Az. So, sliding support may be replaced by reaction force $\vec{Y}_A$ and couple M_{Az}. **Double stage sliding support** (Fig. 1.10, j) prevents rotation about axis Az, so in this case reaction couple M_{Az} occurs.

Ball and socket joint and footstep bearing (Fig. 1.11, a and 1.11, b**).** The contact of spherical surfaces (external and internal thrust bearing race) occurs in various points under different loads acting on the body. In any case, reaction $\vec{R}_O$ is directed along a common normal to the contacting surfaces, i.e. its line of action intersects the center of the spherical surfaces. In solving problems, it is useful to resolve the reaction force into three rectangular components $\vec{X}_O$, $\vec{Y}_O$, $\vec{Z}_O$, so that $\vec{R}_O = \vec{X}_O + \vec{Y}_O + \vec{Z}_O$. After a problem is solved and the magnitudes and direction of these components are known, the magnitude and direction of resulting reaction $\vec{R}_O$ may be determined by vector algebra rules. This reasoning holds for footstep bearing, too.

Fig. 1.11, c shows the **fixed support** of a body under the action of a non-coplanar force system. Such a constraint restricts all six motions of beam end A: linear displacements along axes x, y, z, and rotations about these axes. So, we have six unknowns: components $\vec{X}_A$, $\vec{Y}_A$, $\vec{Z}_A$ of a concentrated reaction $\vec{R}_A$ and projections of reaction couple vector moment $\vec{M}$ on the axes.

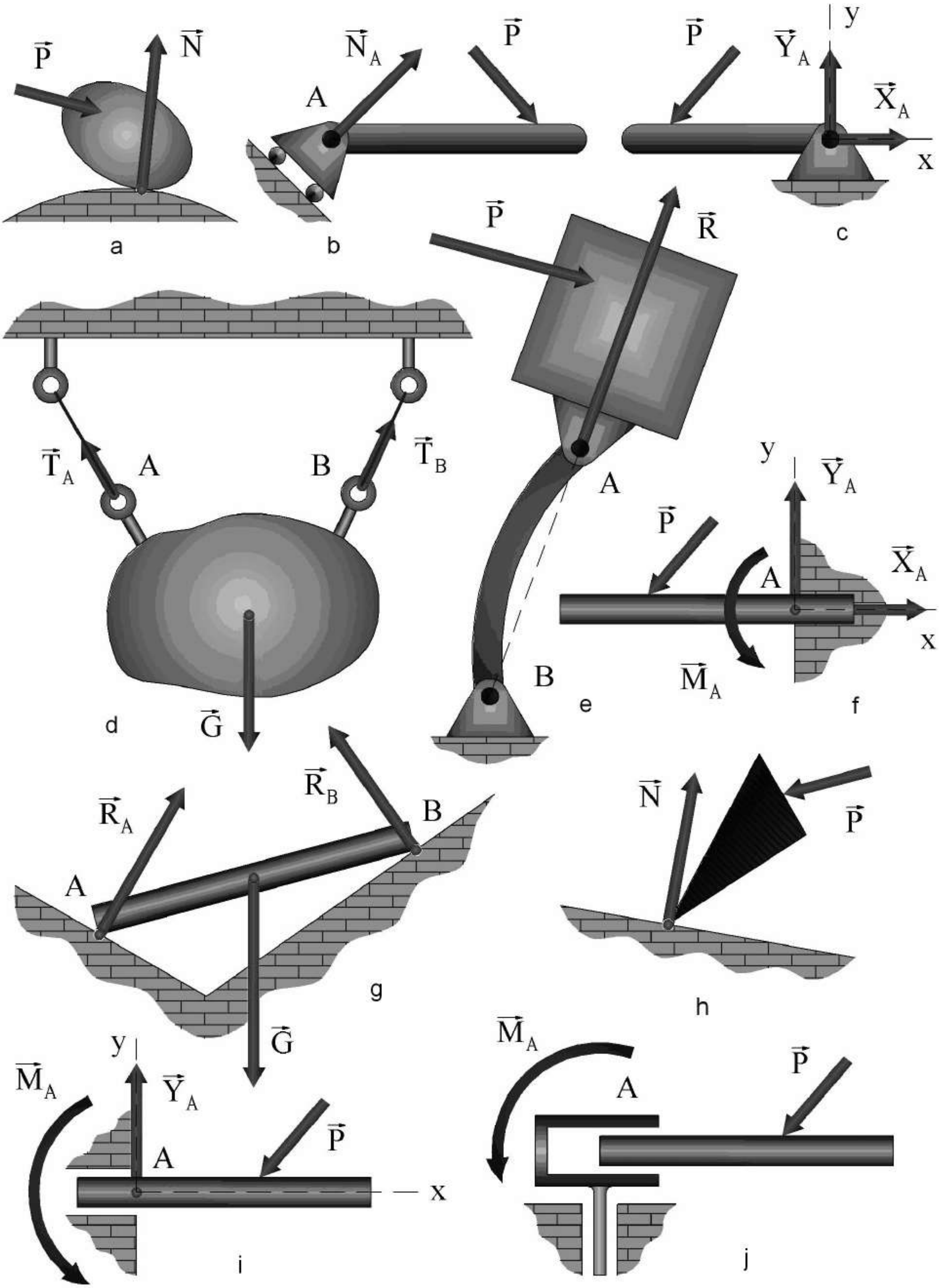
It is worth noting that an unknown force may be represented by its two (Fig. 1.10, в, f) or three (Fig. 1.11, a, b, c) components along coordinate axes. There is another way of presenting an unknown reaction. For example, it is possible to characterize the pin joint reaction force (Fig. 1.10 в) by the unknown magnitude and angle between the reaction and axis x. The ball joint reaction may be represented by a magnitude and angles between the reaction and any two coordinate axes. The third angle may be determined by the expression $cos^2 \alpha + cos^2 \beta + cos^2 \gamma = 1$.

1.6 Problems of statics. Force classification

There are two general problems of rigid body statics. They are:

- the composition of forces and the reduction of the force system acting on rigid bodies to the simplest possible form;
- the determination of conditions for the equilibrium of the force system acting on rigid bodies.

In addition to the problems formulated above, in some cases methods of statics make it possible to solve the following problem: if a body is partially fixed, it has several equilibrium positions; it is necessary to find these positions. However, the most general and effective way of solving this problem is given in analytical mechanics and is not considered in this book.


P
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a
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Fig. 1.10

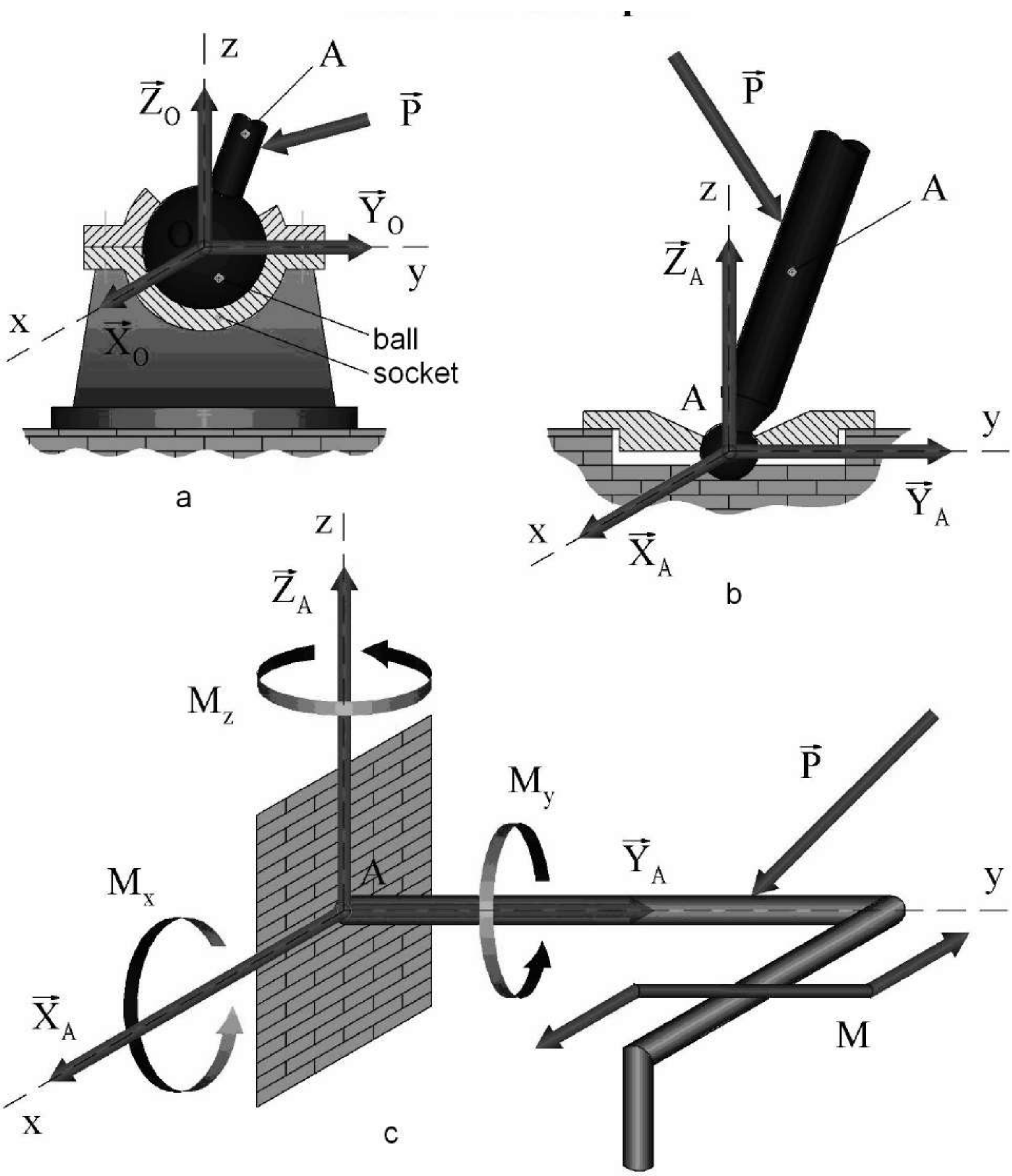


Fig. 1.11

The first problem deals with a force system applied to a rigid body; it is not known if the body is at rest or not. So the solution of this problem and the results obtained are of great importance not only for statics but for dynamics, too.

The solution of the second problem provides general balance conditions for a given force system. If affected by a force system, a body is fixed by constraints, and we should take into consideration all the forces applied as well as the reaction forces when calculating equilibrium conditions. In this case, equilibrium conditions assume a mathematical form of equations with unknown reaction forces. The determination of unknown reaction forces can be viewed as a major result of solving the second general problem of statics.

In mechanics, all forces acting on a body or mechanical system are divided into two classes in accordance with two principles.

First, forces are divided into active and reactive. **Active forces** are independent of constraints and are supposed to be given in the problem statement.

Secondly, forces are divided into external and internal ones.

Forces are called **external** with respect to a given object (body or mechanical system) if they are produced by particles and other bodies are not included into this object (i.e. external objects). Interaction forces between particles or parts of the given object are called internal.

As all forces, internal forces act in pairs and obey the action and reaction equality principle, which states that to every action, there is an equal and opposite reaction.

2 Concurrent force system

If lines of action of all forces of a system intersect at the same point, the force system is called concurrent. Research into concurrent force systems is of prime theoretical and practical importance. The results are used in designing trusses (railway bridges, tangent towers, etc.). In addition, data obtained will be used in the general force system analysis.

2.1 Concurrent force system resultant

The following theorem has to be proved to answer the question formulated in the first problem of statics (theorem about resultant).

A concurrent force system is equivalent to a single force (resultant). The resultant equals the vector sum of the system forces with its line of action passing through the point where all the lines of action of the system forces intersect.

Proof. Consider a concurrent force system applied to a rigid body $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$. Lines of action intersect at point O (Fig. 2.1, a).

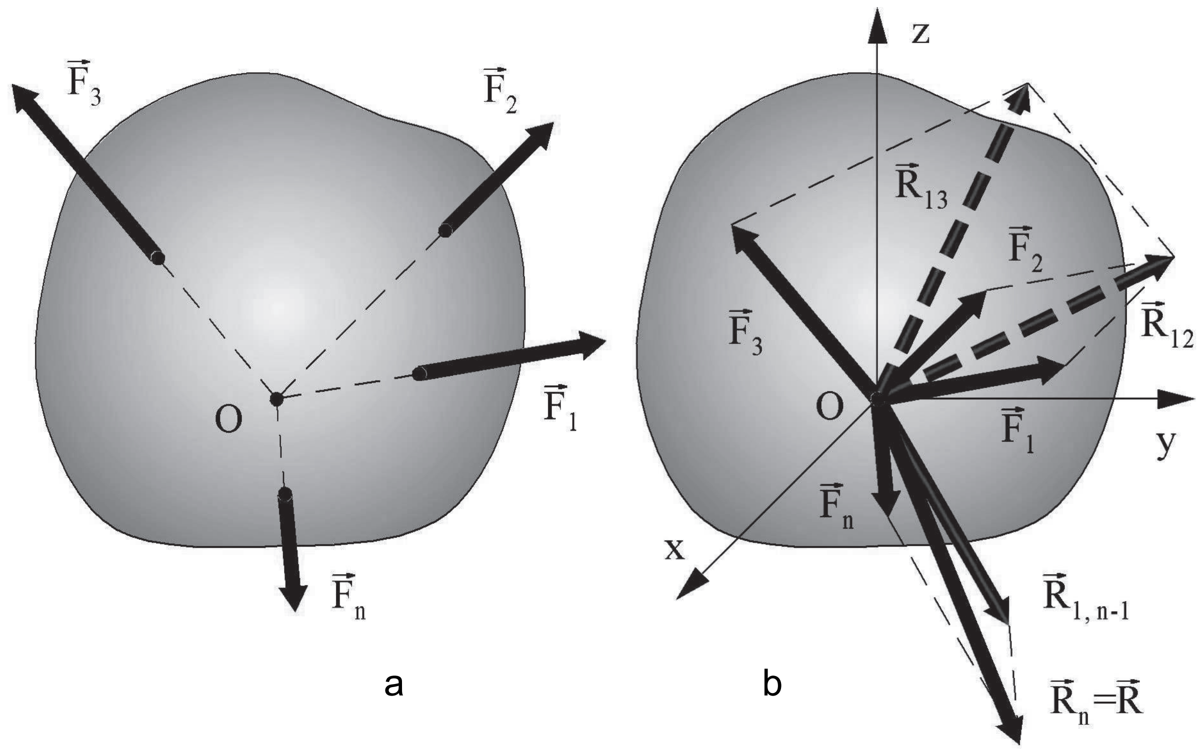


Fig. 2.1

In accordance with the corollary of Axiom 2, we can transpose each force along its line of action to point O. We obtain a force system acting at point O (Fig. 2.1 b). Applying Axiom 3 (parallelogram law) and summing forces $\vec{F}_1, \vec{F}_2, ..., \vec{F}_n$ sequentially, we obtain

$$\vec{R}_{12} = \vec{F}_1 + \vec{F}_2 ; \qquad (2.1)$$

$$\vec{R}_{13} = \vec{R}_{12} + \vec{F}_3 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 ; \qquad (2.2)$$

$$\vec{R}_{1n} = \vec{R} = \vec{R}_{1,n-1} + \vec{F}_n = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{k=1}^{n} \vec{F}_k . \qquad (2.3)$$

All previous operations were performed on the basis of statics axioms, so equalities (2.1) – (2.3) can be supplemented with the relations of equivalence:

$$\vec{R}_{12} \sim \left(\vec{F}_1, \vec{F}_2\right) ; \qquad (2.4)$$

$$\vec{R}_{13} \sim \left(\vec{F}_1, \vec{F}_2, \vec{F}_3\right) ; \qquad (2.5)$$

$$\vec{R} \sim \left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right) . \qquad (2.6)$$

Relations (2.3), (2.6) prove the theorem.

The sequence of operations executed to prove the theorem is called the geometrical method of calculating the concurrent force system.

In practice, the geometrical method of calculating the force system resultant can be used in some particular cases. The procedure for constructing a coplanar concurrent force system resultant consists of the following four steps:

Step 1. Draw forces $\vec{F}_1$ to a chosen scale at point O (the point where the lines of action intersect).

Step 2. Draw to scale force $\vec{F}_2$ from the tip of force $\vec{F}_1$ in a head-to-tail fashion.

Step 3. The process of adding forces $\vec{F}_i$, $i = 3, ..., n,$ continues until all the forces are joined head-to-tail.

Step 4. Draw a vector from the tail of the first vector $\vec{F}_1$ to the head of the last vector, $\vec{F}_n$. This vector drawn to a chosen scale is a force system resultant.

Thus, for example, we examine the method described above for a system consisting of five forces (Fig. 2.2) lying in the plane of Fig. 2.2: $F_1 = 20N$, $F_2 = 35N$, $F_3 = 40N$, $F_4 = 30N$, $F_5 = 25N$.

Choose a scale coefficient $\mu = 1\frac{N}{mm}$. Draw forces $\vec{F}_1,...,\vec{F}_5$ (Fig. 2.2, a) to a chosen scale. For convenience, further tracings are given separately (Fig. 2.2, b).

Drawn to a chosen scale, vector $\vec{R}$ represents the resultant of the force system $\left(\vec{F}_1,\vec{F}_2,...,\vec{F}_5\right)$. Multiplying the length of vector $\vec{R}$ measured in mm by scale coefficient μ, we obtain the resultant magnitude.

$$R = |OE| \cdot \mu = 11 \cdot 1 = 11(N) .$$

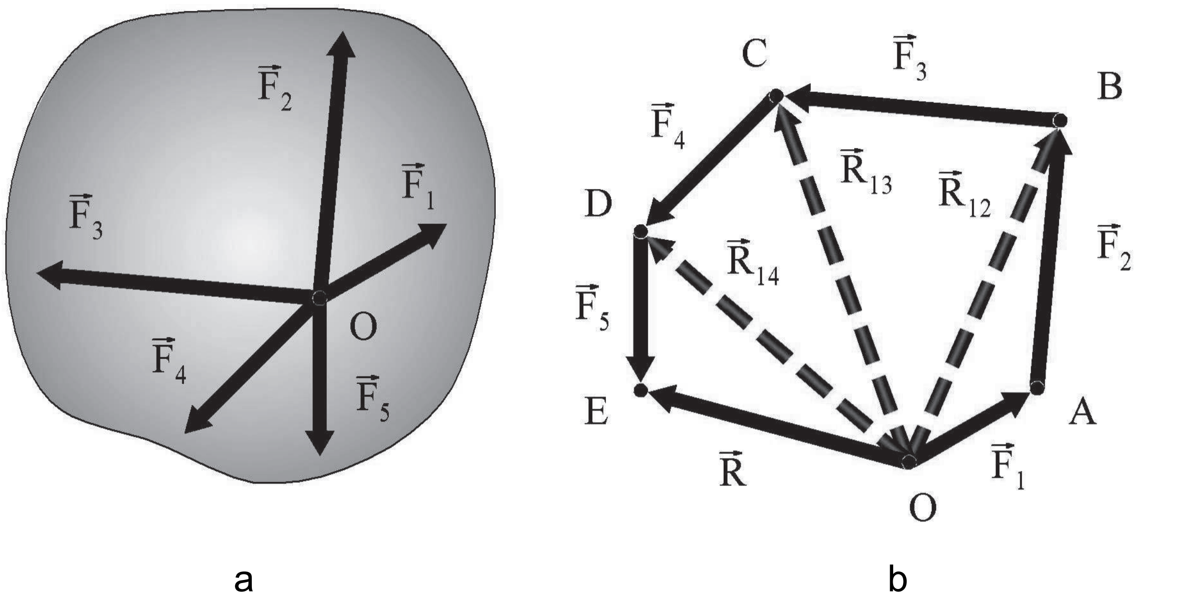


Fig. 2.2

The vector $\vec{R}$ direction is determined by tracing. The broken line OABCDE is called the polygon of forces or diagram of forces. If the head of the last force (point E) coincides with the tail of the first one (point O), the polygon of forces is said to be closed.

The graphical method of finding a force system resultant based on polygon construction and measuring the length of its segments gives only rough approximation, while the exact value of the resultant force can be obtained by the analytical method of vector addition. Consider a rectangular coordinate system with its origin at point O (see Fig. 2.1, b). By projecting the left and right sides of equality (2.3) onto axes x, y, z, we obtain expressions for the resultant projections onto these axes:

$$\left.\begin{aligned} R_x &= F_{1x} + F_{2x} + \ldots + F_{nx} = \sum_{k=1}^{n} F_{kx}; \\ R_y &= F_{1y} + F_{2y} + \ldots + F_{ny} = \sum_{k=1}^{n} F_{ky}; \\ R_z &= F_{1z} + F_{2z} + \ldots + F_{nz} = \sum_{k=1}^{n} F_{kz}. \end{aligned}\right\} \quad (2.7)$$

With the help of these projections we can find the resultant magnitude:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{\left(\sum_{k=1}^{n} F_{kx}\right)^2 + \left(\sum_{k=1}^{n} F_{ky}\right)^2 + \left(\sum_{k=1}^{n} F_{kz}\right)^2}\,. \quad (2.8)$$

The resultant $\vec{R}$ direction can be determined by using direction cosines

$$\left.\begin{aligned} \cos\left(\vec{R}, x\right) &= \frac{R_x}{R}; \\ \cos\left(\vec{R}, y\right) &= \frac{R_y}{R}; \\ \cos\left(\vec{R}, z\right) &= \frac{R_z}{R}. \end{aligned}\right\} \quad (2.9)$$

So the resultant of the given force system $\left(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n\right)$ can be obtained from expressions (2.7) – (2.9) by performing some analytical calculations.

The graphical and analytical methods described above can be used to solve the first problem of rigid body statics. The theorem of the resultant of a concurrent force system gives us an answer to this problem.

2.2 Equilibrium conditions for the concurrent force system

As we know, a rigid body under the action of a force system is in the state of equilibrium (at rest) if the force system is equivalent to zero. On the other hand, a concurrent force system is always equivalent to its resultant. So, to be balanced, any concurrent force system must have a zero resultant, i.e.

$$\vec{R} = \sum_{k=1}^{n} \vec{F}_k = 0 \ , \qquad (2.10)$$

where n is the number of forces the system consists of.

Equality (2.10) is a vector form of the equilibrium condition for a concurrent force system.

Condition (2.10) in its geometrical form (Fig. 2.1 b, 2.2 b) can be formulated as follows: the closure of the polygon of forces is a necessary and sufficient condition for the concurrent force system to be in balance.

Analytical equilibrium conditions for a concurrent force system can be obtained from equalities (2.8), (2.10)

$$\begin{cases} \sum_{k=1}^{n} F_{kx} = 0 \, ; \\ \sum_{k=1}^{n} F_{ky} = 0; \\ \sum_{k=1}^{n} F_{kz} = 0. \end{cases} \qquad (2.11)$$

The algebraic sum of the force components lying along any three axes that do not belong to the same plane or to parallel planes (non-coplanar axes) is equal to zero. Usually, three mutually perpendicular axes (the Cartesian rectangular coordinate system) are chosen.

2.3 The static indeterminacy of the problem of equilibrium

Both analytical and geometrical equilibrium conditions allow us to answer the question whether the force system under consideration is in equilibrium or whether the body situated in a given position is in equilibrium under the action of the given force system.

In practice we often use equality (2.11) in order to determine the unknown reaction forces.

When determining the equilibrium conditions of a rigid body, we should take into consideration the forces acting on it, both active (or given) and constraint reactions (for constrained bodies). If a body under consideration is a constrained one, we should replace the constraints acting upon it by their reactions according to the constraint eliminating axiom. If the equilibrium conditions are formulated only for a free body, equalities (2.11) can be treated as equilibrium equations in order to find the unknown reactions. To be soluble, the problem must have not more than three unknowns in three equations (for a noncoplanar concurrent force system) and not more than two unknowns in two equations (for a coplanar concurrent force system). If the number of unknowns is greater than the number of equilibrium equations the problem is called **statically indeterminate** and can not be solved solely by means of rigid body statics.

3 The system of couples

Notions concerning couples and their vector moments were introduced in Section 1.4. It was shown that the sum of vector moments of couple forces does not depend on the choice of the center about which the moments are calculated. Thus, a couple vector moment is a **free vector** which can be applied to any point without changing its magnitude and direction.

We know that any force acting upon a rigid body can be shifted along the line of action of the force (corollary from Axiom 2). So we can suppose that segment AB connecting the tails of the vectors of couple forces is perpendicular to their lines of action. For the further analysis of couple features, we need to find the sum of two parallel forces applied to a rigid body.

The following two statements are assumed without proof.

The arrangement of two parallel forces is equivalent to the resultant force, i.e. to the vector sum of the original forces and the line of action that divides the segment between the lines of action of original forces into parts inversely proportional to the force magnitudes (Fig. 3.1, a) and belongs to the segment.

The arrangement of two nonparallel unequal forces is equivalent to their resultant force, i.e. to the vector sum of the original forces and the line of action that divides the segment between the lines of action of original forces into parts inversely proportional to the force magnitudes (Fig. 3.1, b) and lies beyond the segment:

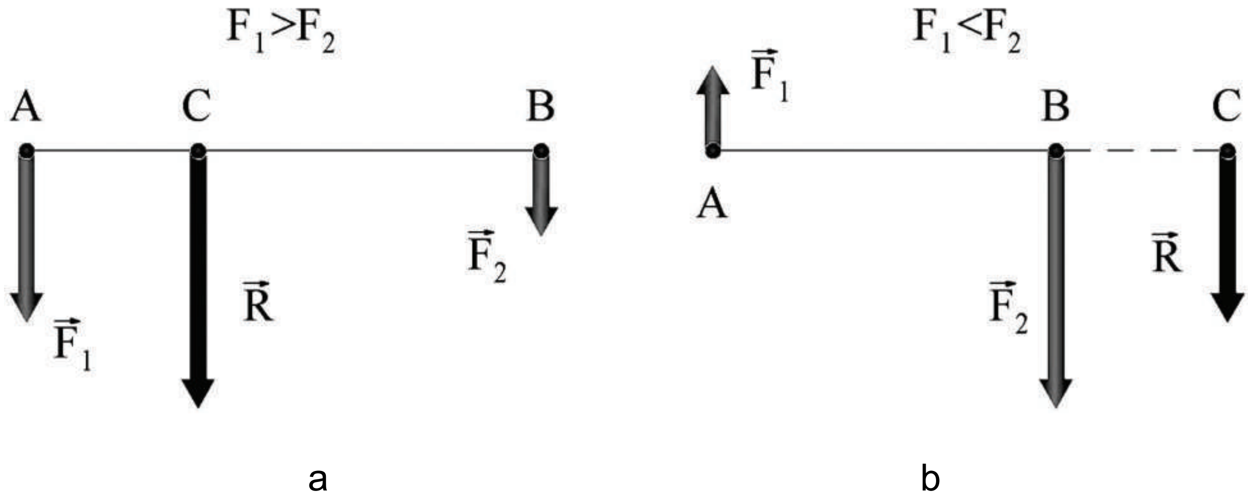


Fig. 3.1

$$\left(\vec{F}_1, \vec{F}_2\right) \sim \vec{R}\,, \tag{3.1}$$

$$\vec{R} = \vec{F}_1 + \vec{F}_2\,, \tag{3.2}$$

$$\frac{AC}{BC} = \frac{F_2}{F_1}\,. \tag{3.3}$$

Relations (3.1) – (3.3) are valid for both cases and unambiguously determine the resultant vector of magnitude $\boldsymbol{R}$, direction $\frac{\bar{R}}{R}$, and application point C.

3.1 Couple features

1) Let us consider equations (3.2), (3.3) and Fig. 3.1, b. Taking into consideration that $BC = AC - AB$, the distance AC can be found from equality (3.3):

$$AC = \frac{F_2}{F_1 - F_2} \cdot AB\,. \tag{3.4}$$

Executing the limiting process, provided that $F_2 \to F_1$, we get $R=0$ from equality (3.2). Simillarly, we get $AC = \infty$ from equality (3.4) (the limiting value of the resultant magnitude and distance AC). Thus, after limiting, the arrangement of forces $\left(F_1, F_2\right)$ is transformed into a couple.

This limiting process implies that a couple has no resultant and can not be reduced any more. Alongside with a force, a couple is a self-dependent element of statics.

2) Couple characteristics (the arm and magnitude of a force) can be changed provided the couple moment remains constant.

Consider two couples $\left(\vec{F}, \vec{F}'\right)$ and $\left(\vec{P}, \vec{P}'\right)$ with the arms AB and CD correspondingly (Fig. 3.2). Let us prove that the couple equivalence

$$\left(\vec{F}, \vec{F}'\right) \sim \left(\vec{P}, \vec{P}'\right), \tag{3.5}$$

meets the condition

$$F \cdot AB = P \cdot CD \ . \tag{3.6}$$

According to Axioms 1 and 2, we can apply two equal and opposite forces $\vec{Q}$, $\vec{Q}'$ at the middle point O of the arm AB (or CD). The magnitude of $\vec{Q}$

$$Q = P - F \ . \tag{3.7}$$

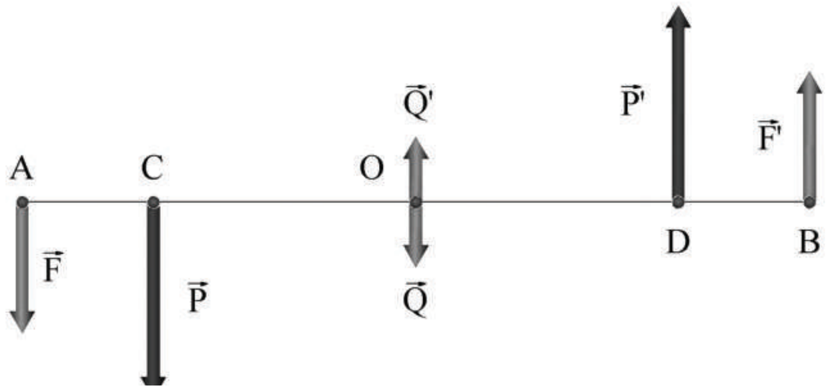


Fig. 3.2

It is obvious that $\vec{P}$ is the resultant force of $\vec{F}$ and $\vec{Q}$. In its turn, $\vec{P}'$ is the resultant force of $\vec{F}'$ и $\vec{Q}'$. Indeed, in accordance with (3.7), we have

$$\vec{P} = \vec{F} + \vec{Q} = \vec{F} + \vec{P} - \vec{F} = \vec{P} \ . \tag{3.8}$$

Condition (3.2) holds.

The same is valid for the forces $\vec{F}'$ и $\vec{Q}'$.

In Fig. 3.2 $AC = DB$, $CD = AB - 2 \cdot AC$.

Thus, we have

$$\left(\vec{F}, \vec{F}'\right) \sim \left(\vec{F}, \vec{F}', \vec{Q}, \vec{Q}'\right) \sim \left(\vec{F}, \vec{Q}; \vec{F}', \vec{Q}'\right) . \tag{3.9}$$

So, in accordance with equation (3.8), the sum of forces $\vec{F}$, $\vec{Q}$ is force $\vec{P}$; and the sum of $\vec{F}'$ и $\vec{Q}'$ is force $\vec{P}'$.

From equality (3.3) and Fig. 3.1, a we get (Fig. 3.2)

$$\frac{AC}{OC} = \frac{Q}{F} = \frac{P - F}{F} \ . \qquad (3.10)$$

Therefore, condition (3.3) for $\vec{F}$, $\vec{Q}$ holds. It means that force $\vec{P}$ is the resultant of $\vec{F}$, $\vec{Q}$.

$$\left(\vec{F},\vec{Q}\right) \sim \vec{P} \ . \qquad (3.11)$$

If we repeat the same reasoning for $\vec{F}'$ и $\vec{Q}'$, we get

$$\left(\vec{F}',\vec{Q}'\right) \sim \vec{P}' \ . \qquad (3.12)$$

Moreover, (3.9) and (3.7) conform to condition (3.5). Considering that $\left(\vec{Q},\vec{Q}'\right) \sim 0$ we finally come to the following relation:

$$\left(\vec{F},\vec{F}'\right) \sim \left(\vec{F},\vec{F}',\vec{Q},\vec{Q}'\right) \sim \left(\vec{F},\vec{Q};\vec{F}',\vec{Q}'\right) \sim \left(\vec{P},\vec{P}'\right) . \qquad (3.13)$$

3) The couple effect on a given rigid body remains unchanged if it is translated (is shifted by parallel translation).

Let us consider a couple, $\left(\vec{F},\vec{F}'\right)$, in the plane Π. We will now prove that a couple $\left(\vec{P},\vec{P}'\right)$ in the plane Π' parallel to the plane Π is equivalent to the original couple $\left(\vec{F},\vec{F}'\right)$ if $\vec{P} = \vec{F}$, $\vec{P}' = \vec{F}'$, and ***CD=AB*** (Fig. 3.3).

We add the couple $\left(\vec{Q},\vec{Q}'\right)$ to the couple, $\left(\vec{P},\vec{P}'\right)$ on the same arm ***CD*** where $\vec{Q} = -\vec{P}$, $\vec{Q}' = -\vec{P}'$. In the plane Π' we get the balanced force system because $\left(\vec{Q},\vec{P}\right) \sim 0$, $\left(\vec{Q}',\vec{P}'\right) \sim 0$ in accordance with Axiom 1.

Based on the construction, polygon ACDB is a parallelogram (AB and CD are equal in magnitude and parallel). O is the point where its diagonals intersect, so the point O divides the diagonals in halves. For the resultant of the forces $\vec{F}'$, $\vec{Q}$ we obtain

$$\vec{F}' + \vec{Q} = 2\vec{F}'. \tag{3.14}$$

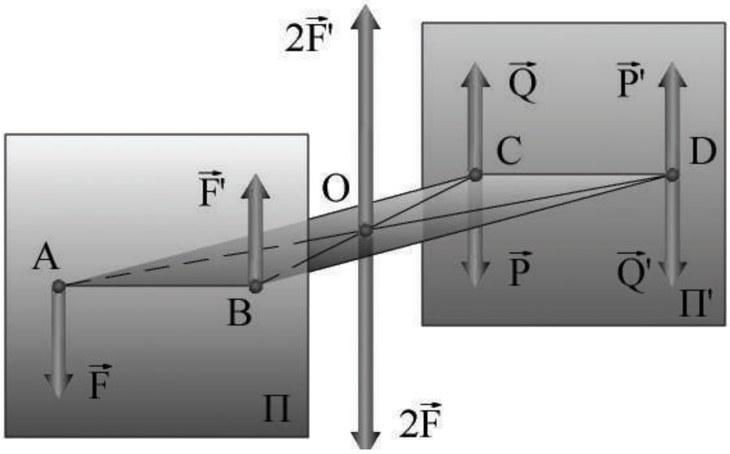


Fig. 3.3

The resultant goes through O (O is the middle point of segment BC so that as $\vec{F}' = \vec{Q}$). Then the resultant of $\vec{F}$, $\vec{Q}'$ is

$$\vec{F} + \vec{Q}' = 2\vec{F} \tag{3.15}$$

and passes through O, too.

It is evident that $\left(2\vec{F}, 2\vec{F}'\right) \sim 0$ because the magnitudes of couple forces are equal: $F' = F$.

Finally, we get

$$\left(\vec{F}, \vec{F}'\right) \sim \left(\vec{F}, \vec{F}'; \vec{P}, \vec{P}'; \vec{Q}, \vec{Q}'\right) \sim \left(\vec{F}', \vec{Q}; \vec{F}, \vec{Q}', \vec{P}, \vec{P}'\right) \sim$$

$$\sim \left(2\vec{F}', 2F; \vec{P}, \vec{P}'\right) \sim \left(\vec{P}, \vec{P}'\right). \tag{3.16}$$

The statement is proven.

4) The action of a couple on a given rigid body remains unchanged if the couple is turned through an orbital angle in the plane of the couple.

First, we rotate the couple $\left(\vec{F},\vec{F}'\right)$ about O (O is the middle point of AB) through an orbital angle. Now we will prove that couple $\left(\vec{P},\vec{P}'\right)$ is equivalent to the original couple $\left(\vec{F},\vec{F}'\right)$ (Fig. 3.4).

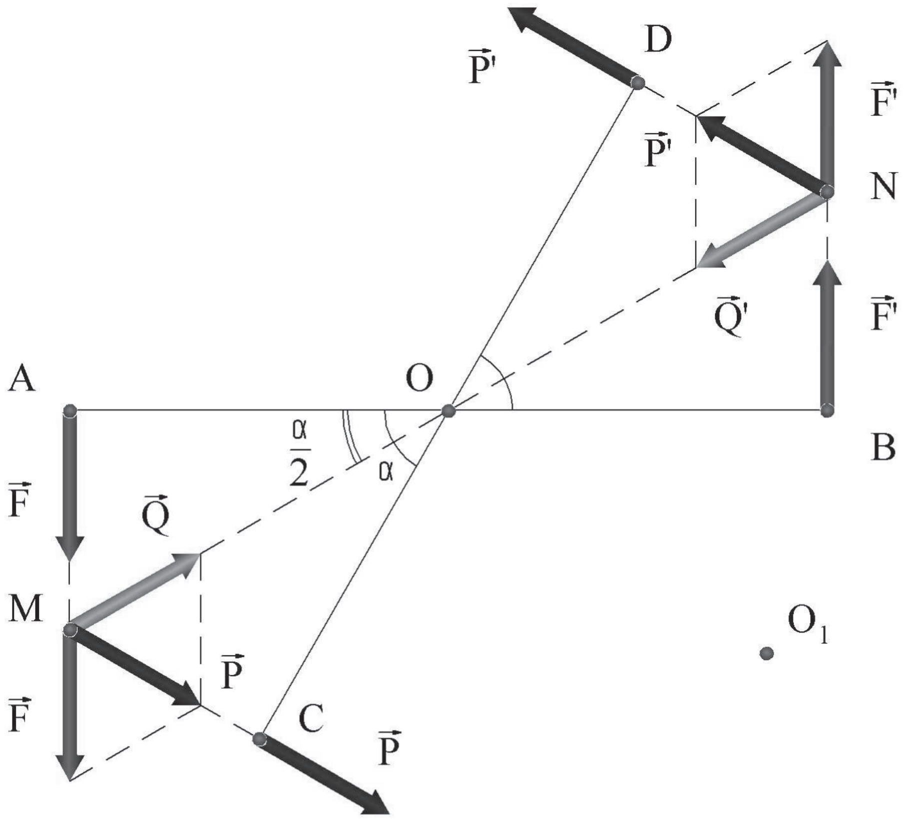


Fig. 3.4

Let us move forces $\vec{F}$ and $\vec{P}$ along their lines of action to the point of their intersection M. Similarly we move $\vec{F}'$ to point N. In accordance with the corollary from Axiom 2 the force system obtained is equivalent to the original force system.

Consider triangles OAM and OCM. They are equal because they are rectangular $\left(\angle A = \angle C = 90^\circ\right)$, have equal catheta (***OA=OC***) and a common hypotenuse, OM. The equality of angles $\angle AOM = \angle COM = \dfrac{\alpha}{2}$ follows from the equality of the triangles. Reasoning similarly for triangles

ODN и OBN, we get the equality of angles: $\angle DON = \angle BON = \frac{\alpha}{2}$.

Therefore, line MN goes through point O and divides $\angle\alpha$ in half.

Add forces $\vec{Q}$, $\vec{Q}'$ to forces $\vec{F}$, $\vec{P}$, $\vec{F}'$, $\vec{P}'$. They are equal in magnitude and opposite in direction along the line MN

$$\left(\vec{F},\vec{Q}\right) \sim \vec{P}\ ,\ \left(\vec{F}',\vec{Q}'\right) \sim \vec{P}'\ , \tag{3.17}$$

$$\vec{P} = \vec{F} + \vec{Q}\ ,\ \vec{P}' = \vec{F}' + \vec{Q}'\ , \tag{3.18}$$

$$\left(\vec{Q},\vec{Q}'\right) \sim 0\ . \tag{3.19}$$

The equivalence relations are:

$$\left(\vec{F},\vec{F}'\right) \sim \left(\vec{F},\vec{F}',\vec{Q},\vec{Q}'\right) \sim \left(\vec{F},\vec{Q};\vec{F}',\vec{Q}'\right) \sim \left(\vec{P},\vec{P}'\right)\ . \tag{3.20}$$

So, we proved the equivalence of the original couple and the couple rotated through an arbitrary angle about the middle point of the original couple arm.

To prove that a couple can be rotated in its plane about an arbitrary point, O_1 (Fig. 3.4) it is sufficient to rotate the couple about point O (the equivalence of this motion was proven above) and use the parallel translation $O \to O_1$ (the equivalence of this motion was proven in Chapter 3).

3.2 Couple equivalence and addition

As shown above, a couple moment is its comprehensive characteristic and takes full account of its static effect on a rigid body. It means that operations with couples may be substituted for the operation with the vectors of couple moments.

The following statements (corollaries) follow from the couple features 2, 3, and 4.

Corollary 1. Two couples applied to a rigid body and having the same vector moment are equivalent.

It means that if the following equality is valid for couples $\left(\vec{P},\vec{P}'\right)$ and $\left(\vec{Q},\vec{Q}'\right)$

$$\vec{M}\left(\vec{P},\vec{P}'\right)=\vec{M}\left(\vec{Q},\vec{Q}'\right) \qquad (3.21)$$

then

$$\left(\vec{P},\vec{P}'\right)\sim\left(\vec{Q},\vec{Q}'\right). \qquad (3.22)$$

Corollary 2. Two couples applied to a rigid body may be replaced by a single couple with a vector moment equal to the sum of vector moments of the original couples.

It means that if we add couples $\left(\vec{F}_1,\vec{F}_1'\right)$ and $\left(\vec{F}_2,\vec{F}_2'\right)$ to vector moments $\vec{M}_1$ and $\vec{M}_2$ correspondingly, we get couple $\left(\vec{Q},\vec{Q}'\right)$ with the moment that is the geometrical sum of $\vec{M}_1$ and $\vec{M}_2$:

$$\vec{M}\left(\vec{F}_1,\vec{F}_1'\right)=\vec{M}_1\ ;\ \vec{M}\left(\vec{F}_2,\vec{F}_2'\right)=\vec{M}_2\ ; \qquad (3.23)$$

$$\left(\vec{F}_1,\vec{F}_1';\vec{F}_2,\vec{F}_2'\right)\sim\left(\vec{Q},\vec{Q}'\right); \qquad (3.24)$$

$$\vec{M}\left(\vec{Q},\vec{Q}'\right)=\vec{M}=\vec{M}_1+\vec{M}_2\ . \qquad (3.25)$$

Since a couple moment is a free vector, it is possible to combine the tails of moments $\vec{M}_1$ and $\vec{M}_2$ by parallel translation and add vectors according to the parallelogram law.

3.3 Solution to the main problems of statics for a couple system

Reduction of a couple system to a simpler equivalent.

A couple system is applied to a rigid body $\left(\vec{F}_1,\vec{F}_1';\vec{F}_2,\vec{F}_2';...;\vec{F}_n,\vec{F}_n'\right)$ with vector moments

$$\vec{M}\left(\vec{F}_k,\vec{F}_k'\right)=\vec{M}_k\ ,\ k=\overline{1,n}. \qquad (3.26)$$

Determine a simpler couple system that is equivalent to the original one.

By sequentially applying corollary 2 to the original couples, we get

$$\left(\vec{F}_1,\vec{F}_1{}';\vec{F}_2,\vec{F}_2{}';\ldots;\vec{F}_n,\vec{F}_n{}'\right)\sim\left(\vec{Q},\vec{Q}'\right); \tag{3.27}$$

$$\vec{M}\left(\vec{Q},\vec{Q}'\right)=\vec{M}=\sum_{k=1}^{n}\vec{M}_k \; . \tag{3.28}$$

It means that a system of n couples applied to a rigid body is equivalent to a single couple with a vector moment equal to the geometrical sum of vector moments for the original couple system.

The equilibrium of a couple system.

It is evident that a system of ***n*** couples applied to a rigid body is in balance if a single couple to which the original system may be reduced is equivalent to zero. The vector form of the equilibrium condition for a system of n couples follows from relations (3.26), (3.27), and (3.28):

$$\sum_{k=1}^{n}\vec{M}\left(\vec{F}_k,\vec{F}_k{}'\right)=\sum_{k=1}^{n}\vec{M}_k=0 \; . \tag{3.29}$$

4 General force system

4.1 Total vector and total moment

Let us consider a force system $\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right)$.

The vector sum of the system forces is called a **total vector**

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{k=1}^{n} \vec{F}_k \ . \qquad (4.1)$$

The vector sum of the forces moments about a given center is called a **total moment about the center O:**

$$\vec{M}_O = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) = \sum_{k=1}^{n} \vec{r}_k \times \vec{F}_k \ , \qquad (4.2)$$

wherein $\vec{r}_k$ is the position vector of the force $\vec{F}_k$ relative to the center O.

There are two main determination methods of the total force $\vec{F}$ and total moment $\vec{M}_O$.

The **first** method is geometrical.

Since a total vector of a force system is not related to any center, so it can be constructed from an arbitrary point.

To find the total force $\vec{F}$ and the total moment $\vec{M}_O$ in accordance with formulas (4.1), (4.2), we can use the method which was earlier applied to determine a resultant of the concurrent force system (Chapter 2). Let us consider this procedure when used to determine the total moment about the chosen center O.

The vector $\vec{M}_O\left(\vec{F}_1\right)$ is laid off to some convenient scale. We should remember that a force moment about a center is a vector applied to the center.

Then the vector $\vec{M}_O\left(\vec{F}_2\right)$ is drawn from the tip of the vector $\vec{M}_O\left(\vec{F}_1\right)$ in a head-to-tail fashion, the vector $\vec{M}_O\left(\vec{F}_3\right)$ starts from the tip of the previous vector, and so on. The last vector $\vec{M}_O\left(\vec{F}_n\right)$ is drawn on the chosen scale so that its tail coincides with the head of the vector

$\overrightarrow{M}_O\left(\overrightarrow{F}_{n-1}\right)$. The vector drawn from the tail of the first vector $\overrightarrow{M}_O\left(\overrightarrow{F}_1\right)$ to the head of the last vector $\overrightarrow{M}_O\left(\overrightarrow{F}_n\right)$ represents to the scale the total moment of the force system $\left(\overrightarrow{F}_1, \overrightarrow{F}_2, ..., \overrightarrow{F}_n\right)$. The magnitude of the total moment on the scale is determined by the length of the obtained vector.

The **second** determination method of the total force $\overrightarrow{F}$ and total moment $\overrightarrow{M}_O$ of the force system.

If we project left and right sides of the following one expressions (4.1) and (4.2) onto axes of the Cartesian rectangular coordinate system with the origin in the center O, we obtain expressions for the calculating components of the system total vector $\overrightarrow{F}$:

$$\left.\begin{aligned} F_x &= \sum_{k=1}^{n} F_{k_x}, \\ F_y &= \sum_{k=1}^{n} F_{k_y}, \\ F_z &= \sum_{k=1}^{n} F_{k_z}, \end{aligned}\right\} \tag{4.3}$$

and total moment $\overrightarrow{M}_O$:

$$\left.\begin{aligned} M_{Ox} &= \sum_{k=1}^{n} M_{Ox}\left(\overrightarrow{F}_k\right), \\ M_{Oy} &= \sum_{k=1}^{n} M_{Oy}\left(\overrightarrow{F}_k\right), \\ M_{Oz} &= \sum_{k=1}^{n} M_{Oz}\left(\overrightarrow{F}_k\right). \end{aligned}\right\} \tag{4.4}$$

The total vector magnitude is determined from the following:

$$F=\sqrt{F_x^2+F_y^2+F_z^2}=\sqrt{\left(\sum_{k=1}^{n}F_{k_x}\right)^2+\left(\sum_{k=1}^{n}F_{k_y}\right)^2+\left(\sum_{k=1}^{n}F_{k_z}\right)^2}. \quad (4.5)$$

Its direction is determined with the help of direction cosines

$$cos\left(\vec{F},\ x\right)=\frac{F_x}{F},\ cos\left(\vec{F},\ y\right)=\frac{F_y}{F},\ cos\left(\vec{F},\ z\right)=\frac{F_z}{F}, \quad (4.6)$$

where $\left(\vec{F},\ x\right)$, $\left(\vec{F},\ y\right)$, $\left(\vec{F},\ z\right)$ are the angles between the total vector direction and the positive x, y, z direction.

Similarly, we determine the magnitude and direction of the system $\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right)$ total moment $\vec{M}_O$ about the center O:

$$M_O=\sqrt{M_{Ox}^2+M_{Oy}^2+M_{Oz}^2}, \quad (4.7)$$

$$cos\left(\overrightarrow{M_O},\ x\right)=\frac{M_{Ox}}{M},\ cos\left(\overrightarrow{M_O},\ y\right)=\frac{M_{Oy}}{M},\ cos\left(\overrightarrow{M_O},\ z\right)=\frac{M_{Oz}}{M}, \quad (4.8)$$

wherein M_{Ox}, M_{Oy}, M_{Oz} are set by expressions (4.4).

4.2 The simplest equivalent of general force system

To form the simplest force system equivalent to a general force system, it is necessary to place all forces of the system at a chosen center. We can do it easily in accordance with the corollary of Axiom 2, if line of action of the force passes through the center O. But if the line of action of the force does not pass through the center, we have to use the rule (lemma) of the force translation to a parallel position: the force applied at the point O of a rigid body is equivalent to the same force applied at another point of the body and a couple with the moment equal to the moment of the original force with respect to the new center (Fig. 4.1).

Proof. Let us assume that a force $\vec{F}$ applied at the point A of a rigid body has to be moved to the point B. In accordance with Axiom 2, a balanced force system can be applied to the rigid body at the point B, the forces forming the balanced system beings $\vec{F}'=\vec{F}$, $\vec{F}''=-\vec{F}$.

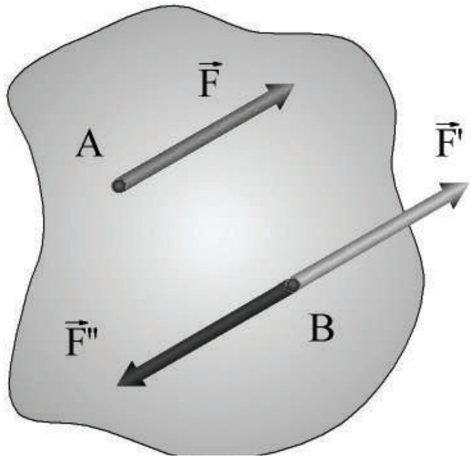


Fig. 4.1

A new force system is equivalent to the original force, i.e. $\left(\vec{F},\ \vec{F}',\ \vec{F}''\right) \sim \vec{F}$. On the other hand, the new force system can be regrouped and represented as a set of two elements: force $\vec{F}'$ applied at the point B and equal to the initial force and a couple $\left(\vec{F}',\ \vec{F}''\right)$. The couple moment is equal to the moment of the initial force about the point B:

$$M\left(\vec{F}',\ \vec{F}''\right) = \overrightarrow{BA} \times \vec{F} = \vec{M}_B\left(\vec{F}\right). \tag{4.9}$$

So, $\vec{F} \sim \left(\vec{F}';\ \vec{F},\vec{F}''\right)$ and the lemma is proven.

Using this result (lemma) we can prove the main theorem of statics.

Theorem. General force system applied to a rigid body can be reduced to an equivalent force-couple system acting at a given point O. The force is equal to the total vector. The couple has a vector moment equal to the total moment of original force system about the point O.

If point O is a chosen center of reduction; and the reduction force system is $\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right)$, then

$$\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right) \sim \left(\vec{F}_O;\ \vec{Q},\vec{Q}'\right), \tag{4.10}$$

$$\vec{F}_O = \sum_{k=1}^{n} \vec{F}_k \ , \qquad (4.11)$$

$$M(\vec{Q},\vec{Q}') = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) = \vec{M}_O \ . \qquad (4.12)$$

It is a mathematical form of the theorem stated above.

Proof. Assume that forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, ..., \vec{F}_n$ are applied to a rigid body (general force system). Apply the lemma of the force translation to a parallel position to the force $\vec{F}_1$. Then (Fig. 4.2) $\vec{F}_1 \sim \left(\vec{F}'_1; \vec{F}_1', \vec{F}_1''\right)$; $\vec{F}_1' = \vec{F}_1$; $\left(\vec{F}_1, \vec{F}_1''\right)$ is a couple with the moment equal to $\vec{M}\left(\vec{F}_1, \vec{F}_1''\right) = \vec{r}_1 \times \vec{F}_1 = \vec{M}_O\left(\vec{F}_1\right)$, wherein $\vec{r}_1$ is a position vector of the force $\vec{F}_1$ point of application with respect to the chosen center O.

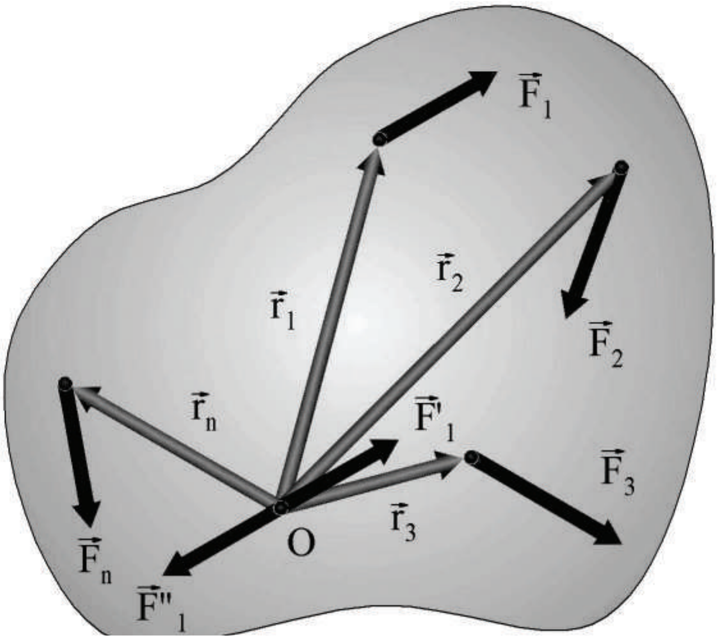


Fig. 4.2

In the same manner the other forces can be translated (Fig. 4.2 illustrates the application of the lemma to the force $\vec{F}_1$ only).

So, for any ***k*** from 1 to n, the following relations are true:

$$\vec{F}_k \sim \left(\vec{F}_k'; \vec{F}_k, \vec{F}_k''\right);$$

$$\vec{F}'_k = \vec{F}_k\,;\ \vec{M}\left(\vec{F}_k,\ \vec{F}''_k\right) = \vec{r}_k \times \vec{F}_k = \vec{M}_O\left(\vec{F}_k\right)\ ,\ k = \overline{1,n}\,. \quad (4.13)$$

As a result, we have

$$\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right) \sim \left(\vec{F}'_1;\ \vec{F}_1, \vec{F}''_1;\ \vec{F}'_2;\ \vec{F}_2, \vec{F}''_2;\ ...,\ \vec{F}'_n;\ \vec{F}_n, \vec{F}''_n\right). \quad (4.14)$$

After regrouping the forces at the right side of the relation, we obtain

$$\left(\vec{F}_1,\ \vec{F}_2, ...,\ \vec{F}_n\right) \sim$$

$$\sim\left(\vec{F}'_1,\ \vec{F}'_2,\ ...,\ \vec{F}'_n;\ \vec{F}_1, \vec{F}''_1;\ \vec{F}_2, \vec{F}''_2;\ ...;\ \vec{F}_n, \vec{F}''_n\right). \quad (4.15)$$

On the other hand, the set of forces $\left(\vec{F}'_1,\ \vec{F}'_2,\ ...,\ \vec{F}'_n\right)$ form a concurrent force system applied at the point O and, in accordance with the theorem about the resultant of such a force system the system can be replaced by a single force equal to the total vector of the system, i.e.

$$\left(\vec{F}'_1,\ \vec{F}'_2,\ ...,\ \vec{F}'_n\right) \sim \vec{F}_O\,, \quad (4.16)$$

wherein

$$\vec{F}_O = \sum_{k=1}^{n} \vec{F}'_k = \sum_{k=1}^{n} \vec{F}_k\ . \quad (4.17)$$

4.3 Total vector and total moment dependence on the center of reduction position

It follows from the definition of the total vector (4.1) that for any chosen reduction center, the total vector is equal to the vector sum of the system forces. So the total vector is independent of the reduction center position, i.e. the total vector is invariant with respect to the center of reduction.

Let us consider the relation between the total moments of force system determined about two different centers.

Assume that a general force system $\left(\vec{F}_1,\ \vec{F}_2, ...,\ \vec{F}_k, ...,\ \vec{F}_n\right)$ is applied to a rigid body. The system's total moment about point O is

$$\vec{M}_O = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) = \sum_{k=1}^{n} \vec{r}_k \times \vec{F}_k \ . \tag{4.20}$$

The system's total moment about point O_1 is

$$\vec{M}_{O_1} = \sum_{k=1}^{n} \vec{M}_{O_1}\left(\vec{F}_k\right) = \sum_{k=1}^{n} \vec{r}_k' \times \vec{F}_k \ . \tag{4.21}$$

Here $\vec{F}_k$ is a force with arbitrary number $\left(k = \overrightarrow{1,\ n}\right)$, $\vec{r}_k$ is position vector of the force $\vec{F}_k$ point of application relative to the center O, $\vec{r}_k'$ is the position vector of force $\vec{F}_k$ application point with respect to the center O_1 (Fig. 4.3):

$$\vec{r}_k' = \overrightarrow{O_1O} + \vec{r}_k \ . \tag{4.22}$$

If we put expression (4.22) in to expression (4.21,) we obtain

$$\vec{M}_{O_1} = \sum_{k=1}^{n}\left(\overrightarrow{O_1O} + \vec{r}_k\right) \times \vec{F}_k =$$

$$= \sum_{k=1}^{n} \overrightarrow{O_1O} \times \vec{F}_k + \sum_{k=1}^{n} \vec{r}_k \times \vec{F}_k = \overrightarrow{O_1O} \times \sum_{k=1}^{n} \vec{F}_k + \vec{M}_O. \tag{4.23}$$

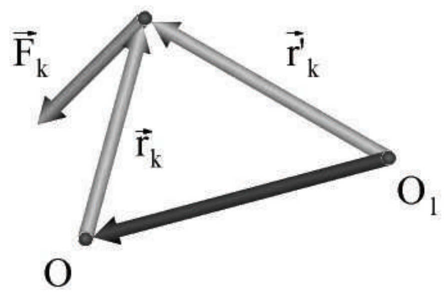


Fig. 4.3

Since $\sum_{k=1}^{n} \vec{F}_k$ is the total vector of the system, the vector product at the right side of the equality (4.23) can be represented as a moment about the center O_1 of the total vector applied at the center O, i.e.

$$\overrightarrow{O_1O} \times \sum_{k=1}^{n} \vec{F}_k = \overrightarrow{O_1O} \times \vec{F}_O = \vec{M}_{O_1}\left(\vec{F}_O\right) . \qquad (4.24)$$

Finally if we take into account expression (4.24), the equality (4.23) will have the following form

$$\vec{M}_{O_1} = \vec{M}_O + \vec{M}_{O_1}\left(\vec{F}_O\right) = \vec{M}_O + \overrightarrow{O_1O} \times \vec{F}_O \ , \qquad (4.25)$$

where $\vec{M}_{O_1}$ is a total moment of the system about the center O_1, $\vec{M}_O$ is a total moment of the system about the center O, $\vec{M}_{O_1}\left(\vec{F}_O\right)$ was described above by (4.24).

So if the reduction center is changed, the total vector of the system will not change but the total moment of the system will be changed by a vector equal to the moment of the total vector applied to the initial center) about the new center O_1.

4.4 Force system invariants

Some value is called invariant with respect to the some parameter (argument) if the value does not vary when the parameter changes.

As was stated above the total vector does not change when the center of reduction changes its position. So the total vector is invariant with respect to the center of reduction position.

The total vector magnitude is called the first invariant of statics:

$$I_1 = \left|\vec{F}_O\right| = \left|\sum_{k=1}^{n} \vec{F}_k\right| = \sqrt{F_{O_x}^2 + F_{O_y}^2 + F_{O_z}^2} \ . \qquad (4.26)$$

The dot product of the total vector and the total moment is called the second invariant of statics

$$I_2 = \vec{F}_O \cdot \vec{M}_O . \qquad (4.27)$$

If we calculate the dot product with respect to the reduction center O_1 using equalities (4.24), (4.25), we obtain

$$\vec{F}_{O_1} \cdot \vec{M}_{O_1} = \vec{F}_O \cdot \left(\vec{M}_O + \overrightarrow{O_1O} \times \vec{F}_O\right) = \vec{F}_O \cdot \vec{M}_O \ . \qquad (4.28)$$

In (4.28) $\vec{F}_O \cdot \left(\overline{O_1 O} \times \vec{F}_O\right) = 0$ because it is a triple scalar product involving two equal factors.

Equality (4.28) shows that for arbitrarily chosen centers O and O_1, the dot product of the total vector and the total moment are identical.

Using expression (4.28) in scalar form and taking into consideration that $F_{O_1} = F_O$, we obtain

$$F_{O_1} \cdot M_{O_1} . \cos\left(\vec{F}_{O_1}, \vec{M}_{O_1}\right) = F_O \cdot M_O \cdot \cos\left(\vec{F}_O, \vec{M}_O\right), \quad (4.29)$$

from which

$$M_{O_1} \cdot \cos\left(\vec{F}_{O_1}, \vec{M}_{O_1}\right) = M_O \cdot \cos\left(\vec{F}_O, \vec{M}_O\right). \quad (4.30)$$

Since for any reduction center the total vector is the same, equality (4.30) means the following: for a general force system the projection of the total moment onto the line of action of the total vector is constant at any reduction center.

4.5 The simplest equivalent of a special force system

The main theorem of statics states that a general force system applied to a rigid body is equivalent to a set of forces and a couple (see (4.10) – (4.12)) equal to the total vector $\vec{F}_O$ and the total moment of the system $\vec{M}_O$ about a chosen reduction center. If one of these parameters ($\vec{F}_O$ or $\vec{M}_O$) is equal to 0, the equivalent set $\left(\vec{F}_O; \vec{Q}, \vec{Q}'\right)$ can be reduced.

Let us consider different combinations of the total vector and total moment values as special cases of the general force system reduction described by expressions (4.10) – (4.12).

1. $\vec{F}_O \neq 0$; $\vec{M}_O = 0$. (4.31)

It follows from (4.12) that $\vec{M}\left(\vec{Q}, \vec{Q}'\right) = 0$. Therefore, the couple $\left(\vec{Q}, \vec{Q}'\right) \sim 0$, and from expression (4.10) the following conclusion may be drawn: the initial force system is equivalent to a single force called a **resultant**

$$\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right) \sim \vec{R}\ , \tag{4.32}$$

$$\vec{R} = \vec{F_O} = \sum_{k=1}^{n} \vec{F_k}\ . \tag{4.33}$$

The line of action of the resultant intersects the reduction center O.

2. $\vec{F_O} = 0$; $\vec{M}_O \neq 0$. (4.34)

In this case, expression (4.10) has the following form:

$$\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right) \sim \left(\vec{Q},\ \vec{Q'}\right)\ . \tag{4.35}$$

The system is equivalent to a couple with the vector moment equal to the total moment of the system about the center O

$$\vec{M}\left(\vec{Q},\ \vec{Q'}\right) = \vec{M}_O\ . \tag{4.36}$$

3. $\vec{F_O} = 0$; $\vec{M}_O = 0$.

In this case, (4.10) becomes:

$$\left(\vec{F}_1,\ \vec{F}_2,...,\ \vec{F}_n\right) \sim 0\ , \tag{4.37}$$

i.e. the force system is **balanced.**

4. $\vec{F_O} \neq 0$; $\vec{M}_O \neq 0$. (4.38)

Let us decompose the vector $\vec{M}_O$ into two components: 1) $\vec{M^*}$ collinear with the vector $\vec{F_O}$ and 2) $\vec{M^{**}}$ perpendicular to the vector $\vec{F_O}$ (Fig. 4.4)

$$\vec{M}_O = \vec{M^*} + \vec{M^{**}}\ . \tag{4.39}$$

We shall consider two special cases: a) $\overrightarrow{M^*} = 0$, and b) $\overrightarrow{M^*} \neq 0$.

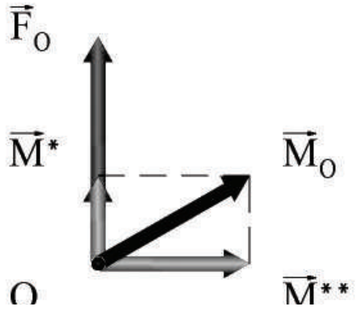


Fig. 4.4

In case a) we have:

4a. $\overrightarrow{F_O} \neq 0$; $\overrightarrow{M_O} \neq 0$; $\overrightarrow{M_O} \perp \overrightarrow{F_O}$. (4.40)

In relation (4.10) the couple $\left(\vec{Q},\ \vec{Q'}\right)$ lies in the plane perpendicular to $\overrightarrow{M_O}$, i.e. in the same plane as the total vector $\overrightarrow{F_O}$. Let us denote the plane by $\prod$ (Fig. 4.5).

In accordance with the properties of a couple, it is possible to transform the couple in any way if its moment remains invariant. We choose a couple of forces $\left(\vec{Q},\ \vec{Q'}\right)$ equal in magnitude to the total vector F_O and apply one of the couple forces $\vec{Q}$, $\vec{Q} = -\overrightarrow{F_O}$ at the given reduction center. Then $\overrightarrow{F_O}, \vec{Q} \sim 0$, and therefore the initial force system is equivalent to the single force $\overrightarrow{Q'}$ equal to the total vector $\overrightarrow{F_O}$. In accordance with the definition of a couple moment, its magnitude is determined by the following expression:

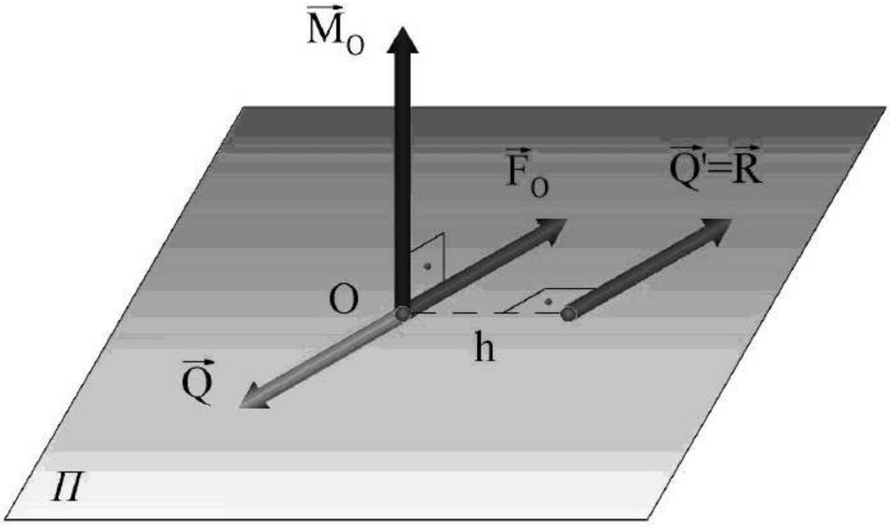
$$M\left(\vec{Q},\ \vec{Q'}\right) = Q' \cdot h \ , \tag{4.41}$$

whence

$$h = \frac{M\left(\vec{Q},\ \vec{Q'}\right)}{Q'} = \frac{M_O}{F_O} \ . \tag{4.42}$$

So the force system resultant is

$$\vec{R} = \vec{Q'} = \vec{F}_O \; ; \; \vec{M}_O\left(\vec{R}\right) = M\left(\vec{Q}, \; \vec{Q'}\right) = \vec{M}_O . \qquad (4.42a)$$



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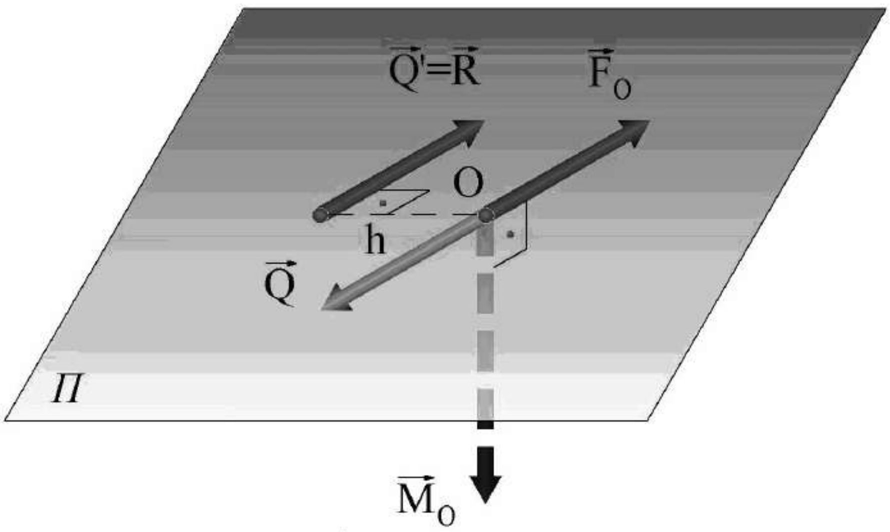


b

Fig. 4.5

If conditions (4.40) are fulfilled the force system is equivalent to the resultant equal to the total vector of the system. The distance between the resultant line of action and the initial reduction center is equal to the quotient of the total moment by the total vector (4.42). We can draw the reduced force system resultant in the following way.

The plane Π perpendicular to the total moment must be drawn at point O.

In the plane Π, the total vector is moved to another line of action placed at a distance ***h*** determined by expression (4.42), so that the vector of the system total moment and the vector moment of a resultant $\vec{R}$ relative to the initial reduction center (Fig. 4.5, a,b) were of the same sense:

$$\left(\vec{F}_1,\ \vec{F}_2,\dots,\ \vec{F}_n\right) \sim \vec{R}\ , \tag{4.43}$$

$$\vec{R} = \vec{F_O} = \sum_{k=1}^{n} \vec{F_k}\ . \tag{4.44}$$

In general case, when the total vector and total moment are non-zero and not mutually perpendicular, the force system can be substituted for a wrench.

Consider this case in detail

4b. $\vec{F_O} \neq 0$; $\vec{M}_O \neq 0$; $\vec{F_O} \cdot \vec{M}_O \neq 0$. (4.45)

Angles between the total vector and the total moment about different reduction centers are random. On the other hand, for a given force system, the projection of the total moment onto the direction of the total vector is an invariant value (4.30). Remember that the total vector of the system is the same for all reduction centers. It means that if condition (4.45) is fulfilled there is such a reduction center O^* that the total vector and the total moment of the system are directed along a common line intersecting the point O^*. At the same time, the magnitude of the total moment is minimal, i.e. in decomposition (4.39) $\overrightarrow{M^*} = 0$ and therefore $\overrightarrow{M_O^*} = \overrightarrow{M^*}$.

So, if condition (4.45) is fulfilled, the following relations hold for the force system

$$\left(\vec{F}_1,\ \vec{F}_2,\dots,\ \vec{F}_n\right) \sim \left(\vec{F}_{O^*};\ \vec{Q},\ \vec{Q'}\right), \tag{4.46}$$

$$\vec{F}_{O^*} = \vec{F_O} = \sum_{k=1}^{n} \vec{F_k}\ , \tag{4.47}$$

$$\overrightarrow{M}\left(\overrightarrow{Q},\ \overrightarrow{Q'}\right) = M^* = \sum_{k=1}^{n} \overrightarrow{M}_{O^*}\left(\overrightarrow{F_k}\right), \tag{4.48}$$

i.e. the initial force system is equivalent to the force equal to the total vector of the system and applied at the center O^*; the couple lying in the plane perpendicular to the total vector; and the couple moment equal to the total moment of the system with respect to the reduction center O^*.

A set of forces equal to the total vector of the system and a couple with the couple plane perpendicular to the total vector is called a **wrench.**

So the point O^* is such that the total moment of the system relative to O^* has a minimal in magnitude and is collinear with the total vector. The location of O^* can be determined proceeding from the condition for the total vector and the total moment to be collinear about O^*.

Let us choose a Cartesian rectangular system with an origin at an arbitrary point O (Fig. 4.6). Let O^* be the required center where the total moment of the force system under consideration is minimal, $\overrightarrow{r^*}$ is the position vector of O^* relative to the origin of the coordinate system. The total vector of the force system and the total moment about the reference origin as well as their projections F_{ox}, F_{oy}, F_{oz}, M_{ox}, M_{oy}, M_{oz} onto the chosen coordinate axes are determined by expressions (4.3), (4.4). Using expression (4.25) the total moment we get

$$\overrightarrow{M^*} = \overrightarrow{M}_{O^*} = \overrightarrow{M}_O + \overrightarrow{O^*O} \times \overrightarrow{F_O} = \overrightarrow{M}_O - \overrightarrow{r^*} \times \overrightarrow{F^*}. \tag{4.49}$$

Remember that $\overrightarrow{F^*} = \overrightarrow{F_O}$, i.e. the total vector of the system is the same for different reduction centers. The collinearity condition for vectors $\overrightarrow{F_O}$ and $\overrightarrow{M^*}$ can be used to determine analytically the location of the reduction center O^* where the system can be reduced to a wrench. This condition can be shown as

$$p \cdot \overrightarrow{F_O} = \overrightarrow{M^*}, \tag{4.50}$$

where p is a scalar coefficient measured in the units of length and called a **wrench parameter**.

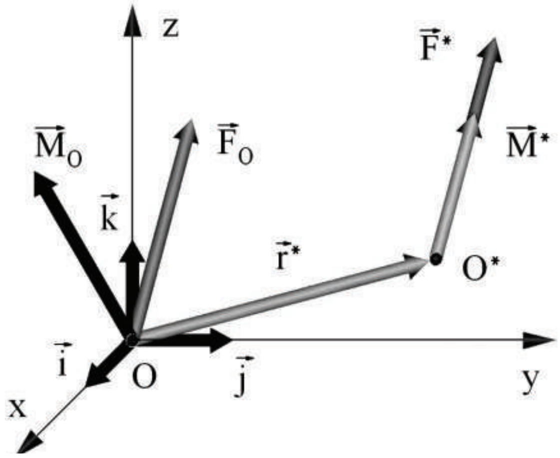


Fig. 4.6

After substituting the value of $\overrightarrow{M^*}$ from (4.49) into (4.50), we get the following expression

$$p \cdot \overrightarrow{F_O} = \overrightarrow{M}_O - \overrightarrow{r}^* \times \overrightarrow{F}^* \ . \qquad (4.51)$$

The position vector $\overrightarrow{r}^*$ specifies the location of the center O^* in a chosen coordinate system. Equation (4.51) where the position vector $\overrightarrow{r}^*$ is unknown has an infinite number of solutions, because for any point on the line intersecting O^* and parallel to the total vector $\overrightarrow{F_O}$, the cross-product $\overrightarrow{r}^* \times \overrightarrow{F}^* = \overrightarrow{M}_O\left(\overrightarrow{F}^*\right)$ is the same. Therefore, the locus of the points $\mathbf{O}^*$ where the system is equivalent to the wrench is a straight line intersecting $\mathbf{O}^*$ and parallel to the total vector $\overrightarrow{\mathbf{F}_O}$. This line is called **central axis of the force system** or **principal screw axis.**

Let us designate the coordinates of the vectors $\overrightarrow{F_O}$, $\overrightarrow{M}_O$, $\overrightarrow{r}^*$ (their projections onto coordinate axes) as

$$\overrightarrow{F_O}\left(\overrightarrow{F_x},\ \overrightarrow{F_y},\ \overrightarrow{F_z}\right),\ \overrightarrow{M_O}\left(\overrightarrow{M_x},\ \overrightarrow{M_y},\ \overrightarrow{M_z}\right),\ \overrightarrow{r}^*(x,\ y,\ z),$$

and then write vector equation (4.51) in a scalar form

$$p\left(\vec{i}F_x+\vec{j}F_y+\vec{k}F_z\right)=\vec{i}M_x+\vec{j}M_y+\vec{k}M_z-\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ F_x & F_y & F_z\end{vmatrix}. \quad (4.52)$$

If we expand the determinant in terms of its first row elements and equate the coefficients of the same unit vectors in both sides of the expressions we shall have the following equations

$$\begin{cases} p\cdot F_x = M_x-\left(y\cdot F_z - z\cdot F_y\right),\\ p\cdot F_y = M_y-\left(z\cdot F_x - x\cdot F_z\right),\\ p\cdot F_z = M_z-\left(x\cdot F_y - y\cdot F_x\right).\end{cases} \quad (4.53)$$

This implies

$$\frac{M_x-\left(y\cdot F_z-z\cdot F_y\right)}{F_x}=\frac{M_y-\left(z\cdot F_x-x\cdot F_z\right)}{F_y}=$$

$$=\frac{M_z-\left(x\cdot F_y-y\cdot F_x\right)}{F_z}=p. \quad (4.53a)$$

It is obvious that the double equality is equivalent to two equations (for example, the equality of the first and the second fractions and the equality of the second and the third fractions). It is the equation of **central axis of force system.**

Though simple transformations of (4.43), we have

$$\left.\begin{aligned} x\cdot F_x\cdot F_z + y\cdot F_y\cdot F_z - z\cdot\left(F_x^2+F_y^2\right) = F_yM_x - F_xM_y,\\ x\cdot F_x\cdot F_y - y\cdot\left(F_x^2+F_z^2\right) + z\cdot F_y\cdot F_z = F_xM_z - F_zM_x.\end{aligned}\right\} \quad (4.54)$$

Either equation (4.54) is linear relative to the current coordinates x, y, z, and specifies a plane. These equations specify a straight line (the intersection line of two planes). This line is the central axis of a force system.

The line position in space (in the coordinate system Oxyz) is given if we know the coordinates of the two points at the line or the coordinates of a single point and the direction of the line. In the first case, the coordinates of two points where the line intersects the coordinate planes are defined. For example, to find the point where the line intersects the plane xOy we suppose that $z = 0$ in (4.54) and then solve the system of equations with respect to x and y. As a result, we have some x_A, y_A which are the coordinates of the point A.

In a similar manner we can determine the point where the line intersects another coordinate plane. For example, to determine the point where the line intersects the plane yOz suppose that $x = 0$ in (4.54) and then solve the obtained system of equations relative to y and z. As a result, we have some y_B, z_B which are the coordinates of the point B. The central axis of the force system passes through the points A $(x_A, y_A, 0)$ and B $(0, y_A, z_A)$.

It is possible to define the central axis of the force system as a line passing through the point where the axis intersects one of the coordinate planes (see the determination procedure described above) and parallel to the total vector $\vec{F}_O$ of the force system.

The wrench is distinguished by the relative position of the total vector $\vec{F}_O$ and the minimal total moment $\vec{M}^*$. If the total vector and the minimal total moment have the same direction, the wrench is said to be right. If the total vector and the minimal total moment have opposite directions, the wrench is said to be left.

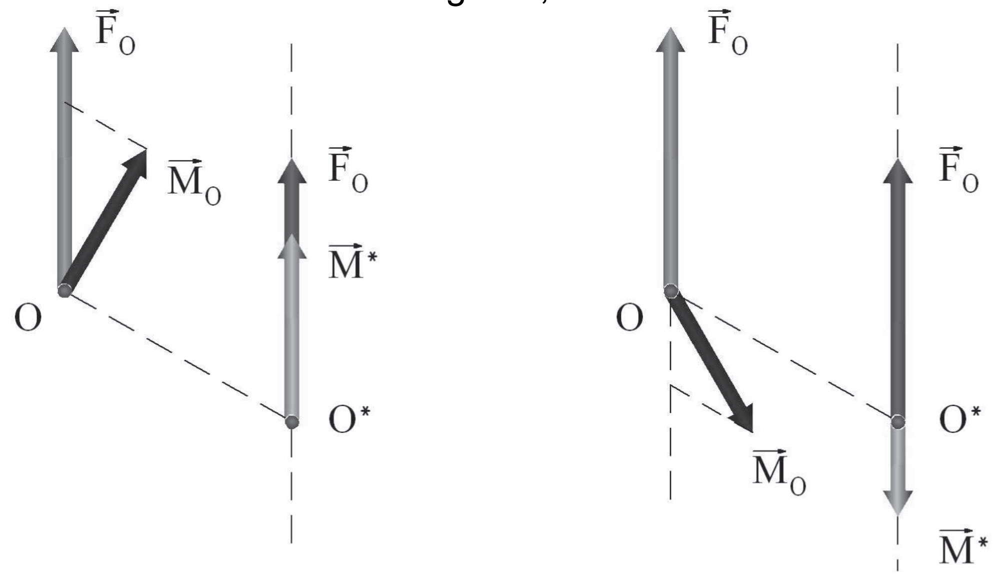
Obviously (see Fig. 4.4),the conditions under which a force system can be reduced to the right wrench can be written as

$$\vec{F}_O \cdot \vec{M}_O > 0 \; . \tag{4.55}$$

Similarly, the conditions under which a force system can be reduced to the left wrench ca be presented as

$$\vec{F}_O \cdot \vec{M}_O < 0 \; . \tag{4.56}$$

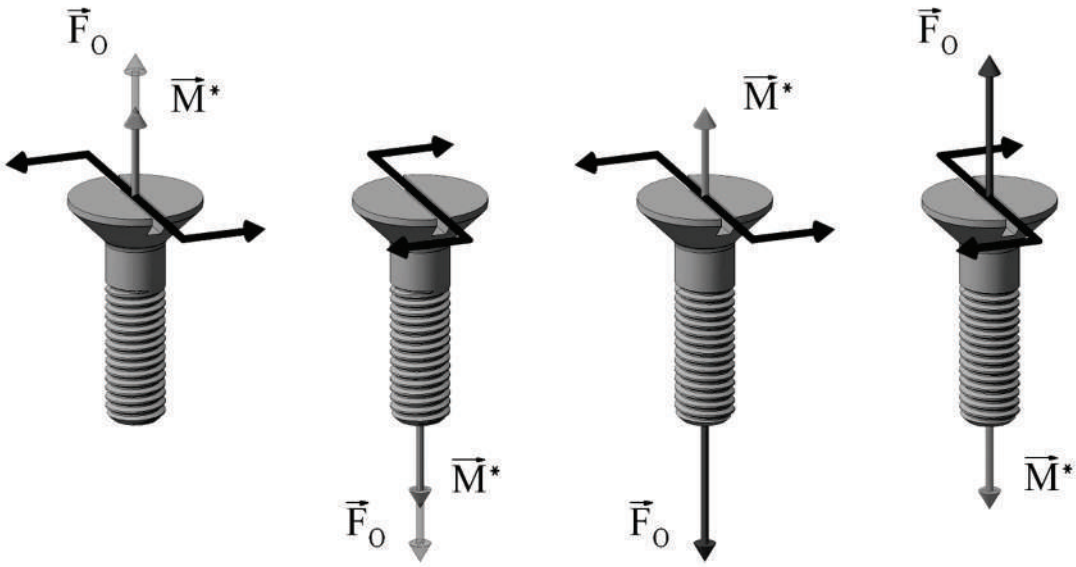
In Fig. 4.7,a the right wrench is shown as a set of the total vector $\vec{F}_O$ of the force system and the minimal total moment $\vec{M}^*$ acting in the same direction. The left wrench consists of the same vectors having the opposite directions as shown in Fig. 4.7, b.



a b

Fig. 4.7

A right wrench can be graphically presented as a set of forces applied to the right-hand screw. If we apply a couple with the moment $\overrightarrow{M^*}$ directed along the axis to the screw, we obtain the motion of the screw in the direction of the total vector $\vec{F}_O$. In this case the total vector and the total moment have the same directions (Fig. 4.8, a, b).



a b c d

Fig. 4.8

The Table 4.1 presents all possible cases of the force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ reduction (all combinations of the total vector and the total moment of the force system under consideration).

Table 4.1

N	$\vec{F_O}$	$\vec{M}_O$	$I_2 = \vec{F_O} \cdot \vec{M}_O$	The simplest equivalent force system
1	$\vec{F_O} \neq 0$	$\vec{M}_O = 0$	$I_2 = 0$	The resultant $\vec{R} = \vec{F_O}$ passing through the chosen reduction center O
2	$\vec{F_O} = 0$	$\vec{M}_O \neq 0$	$I_2 = 0$	The couple $\left(\vec{Q}, \vec{Q}'\right)$, $\vec{M}\left(\vec{Q}, \vec{Q}'\right) = \vec{M_O}$
3	$\vec{F_O} = 0$	$\vec{M}_O = 0$	$I_2 = 0$	A balanced force system
4a	$\vec{F_O} \neq 0$	$\vec{M}_O \neq 0$	$I_2 = 0$	The resultant $\vec{R} = \vec{F_O}$ with the line of action determined by the equation $\vec{M_O}\left(\vec{R}\right) = \vec{M_O}$
4b	$\vec{F_O} \neq 0$	$\vec{M}_O \neq 0$	$I_2 \neq 0$	A wrench

So the first main problem of statics, i. e. the problem of replacing the given force system by a simple equivalent, can be solved with the help of the main theorem of statics both for a general force system and in special cases described above.

Problem 4.1. The system of three forces F_1, F_2, F_3 is applied to the corners of a rectangular parallelepiped as shown in Fig. 4.9. It must be reduced to the simplest possible force system. The sides of the parallelepiped are $a = 0{,}3\,m,\ b = 0{,}5\,m,\ c = 0{,}4\,m$, the force magnitudes are $F_1 = 10\,N,\ F_2 = 20\,N,\ F_3 = 15\,N$.

Let the origin of the coordinate system O be a reduction point. We have to find $\sin\alpha$ and $\cos\alpha$ to determine the total vector $\vec{F}_O$ and total moment $\vec{M}_O$ of the system:

$$sin\alpha = \frac{c}{\sqrt{a^2 + c^2}} = 0,8 \; ; \; cos\alpha = \frac{a}{\sqrt{a^2 + c^2}} = 0,6 \; .$$

The projections of the total vector onto the coordinate axes are

$$F_{Ox} = -F_1 + F_2 \cdot cos\alpha = 2\,N;$$

$$F_{Oy} = F_3 = 15\,N;$$

$$F_{Oz} = -F_2 \cdot sin\alpha = -16\,N.$$

The magnitude of the total vector is

$$F_O = \sqrt{F_{Ox}^2 + F_{Oy}^2 + F_{Oz}^2} = 22\,N \; .$$

We can use the values of the vector projections F_{Ox}, F_{Oy}, F_{Oz} to draw the total vector $\vec{F}_O$ at the scale $K_F = 0,5\,\frac{N}{mm}$.

The projections of the total moment $\vec{M}_O$ onto the coordinate axes are

$$M_{Ox} = -F_2 \cdot sin\alpha \cdot b = -8\;N \cdot m;$$

$$M_{Oy} = -F_1 \cdot c + F_2 \cdot cos\alpha \cdot c = 0,8\;N \cdot m;$$

$$M_{Oz} = -F_2 \cdot cos\alpha \cdot b + F_3 \cdot a = -1,5\;N \cdot m.$$

The magnitude of the total moment of the force system with respect to the origin O is

$$M_O = \sqrt{M_{Ox}^2 + M_{Oy}^2 + M_{Oz}^2} = 8,18\;N \cdot m \; .$$

In Fig. 4.9, we see the total moment of the system about the reduction center O drawn at the scale $K_M = 0,08\,\frac{N \cdot m}{mm}$. Let us make sure that the total vector and total moment are not mutually perpendicular. For this purpose we calculate the dot product of these two vectors

$$\vec{F}_O \cdot \vec{M}_O = F_{Ox} \cdot M_{Ox} + F_{Oy} \cdot M_{Oy} + F_{Oz} \cdot M_{Oz} = 20 \; .$$

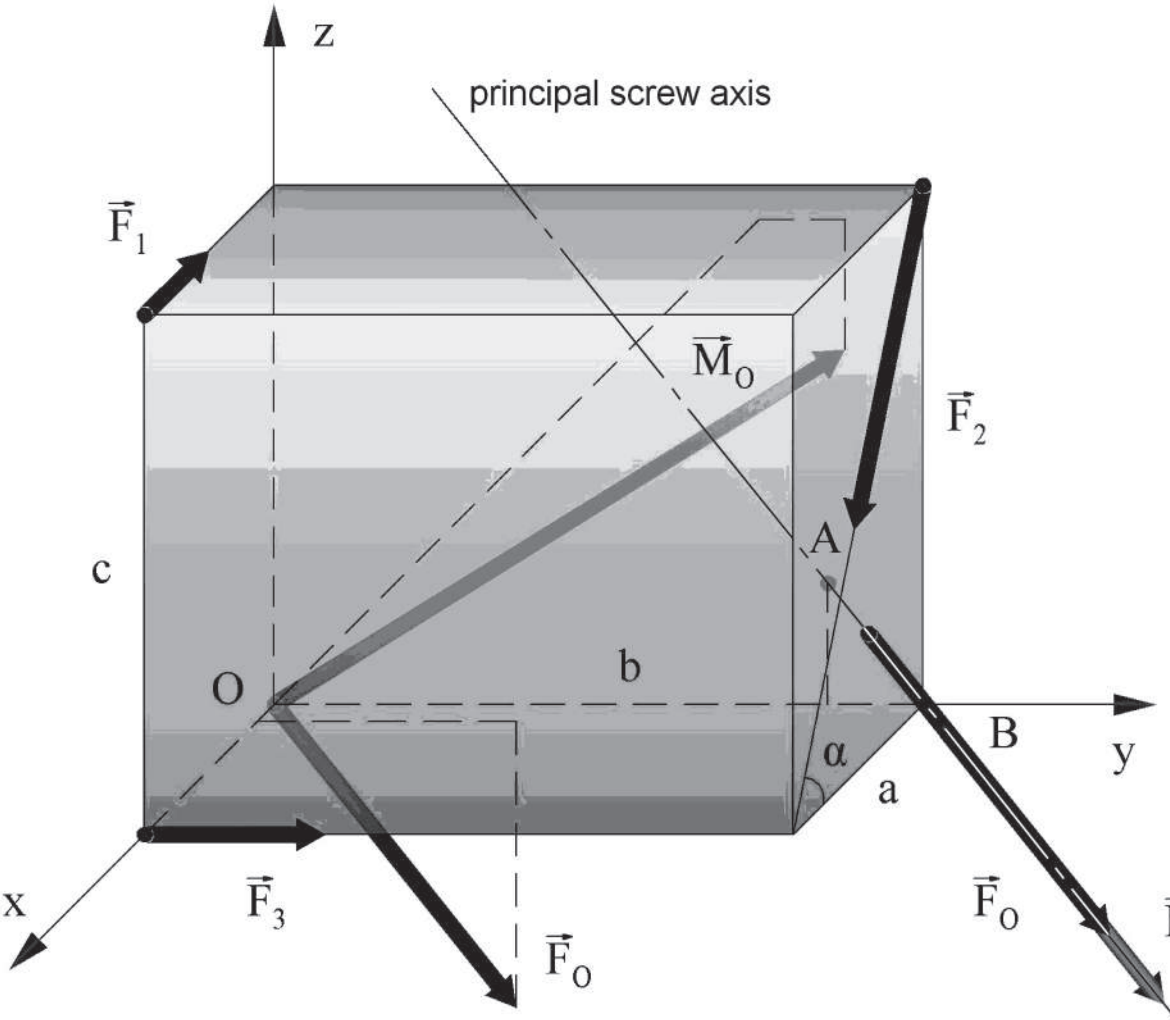


Fig. 4.9

So we have the case 4b: $\vec{F}_O \neq 0$, $\vec{M}_O \neq 0$, $I_2 = \vec{F}_O \cdot \vec{M}_O \neq 0$. Therefore, the force system $\left(\vec{F}_1, \vec{F}_2, \vec{F}_3\right)$ is equivalent to the right wrench because $\vec{F}_O \cdot \vec{M}_O > 0$. Let us obtain an equation for the central axis of the force system under consideration by substituting the values of $F_{Ox}, F_{Oy}, ..., M_{Oz}$ for $F_x, F_y, ..., M_z$ correspondingly into (4.54)

$$\left.\begin{aligned} 32x + 240y + 229z &= 121.6, \\ 30x - 260y - 240z &= -131. \end{aligned}\right\}$$

To find out where the central axis intersects the ***yOx*** plane, we consider ***x=0*** in this system and ***y=42 m***, ***z=0,09 m***. Analogously, we find where the central axis intersects the ***xOy***-plane (***z=0, x=0,01 m, y=0,51 m***). So the central axis intersects the coordinate plane ***yOz*** at the point $A(0;\,0,42;\,0,09)$ and the coordinate ***xOy*** plane at the point $B(0,01;\,0,51;\,0)$.

Conclusion: The system of forces $\left(\vec{F}_1, \vec{F}_2, \vec{F}_3\right)$ is equivalent to the wrench, i.e. to the set of the force $\vec{F}_O$ and the couple with the vector moment $\overrightarrow{M^*}$, with the force and the moment lying at the line called the central axis of the system or the principal screw axis, and passing through points A and B.

The magnitude of the minimal total moment of the system is

$$M^* = \frac{\vec{F}_O \cdot \vec{M}_O}{F_O} = \frac{20}{22} = 0,91\ N \cdot m\ .$$

4.6 The equilibrium conditions for a general force system

Special reduction cases for a general force system were presented in the previous chapter. It was stated in case 3 that a force system is equivalent to zero (balanced) if its total vector $\vec{F}_O$ is zero and the total moment $\vec{M}_O$ about a chosen center O is zero. The requirements are:

$$\vec{F}_O = \sum_{k=1}^{n} \vec{F}_k = 0\ , \tag{4.57}$$

$$\vec{M}_O = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) = 0, \tag{4.58}$$

therefore both sufficient conditions for the equilibrium of a rigid body under the action of a general force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$. As we know, the total vector remains invariant when the reduction centre is displaced while the total moment of the system will be changed by a vector equal to the moment about a new different centre. On this basis, if (4.57) and (4.58) are valid for the chosen reduction center, they are valid for any other reduction center.

At the same time, equations (4.57), (4.58) are necessary equilibrium conditions for a general force system. Assume that a body affected by a force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ is in the state of equilibrium but one of the vectors $\vec{F}_O$ and $\vec{M}_O$ is not equal to zero. Thus, one of the conditions (4.57) – (4.58) is not valid. We arrive at a contradiction because a body can not be in the state of equilibrium under the action of a single force or a single couple with the moment $\vec{M}_O$.

Hence, a necessary and sufficient condition for a general force system to be in equilibrium is that the total vector and the total moment about a chosen center are equal to zero (see (4.57) – (4.58)) as a vector form of these condition. If we bring the chosen center O to the origin of the coordinate system, we obtain the general force system equilibrium conditions in the analytical (coordinate) form:

$$\left.\begin{aligned}
F_{O_x} &= \sum_{k=1}^{n} F_{k_x} = F_{1_x} + F_{2_x} + ... + F_{n_x} = 0,\\
F_{O_y} &= \sum_{k=1}^{n} F_{k_y} = F_{1_y} + F_{2_y} + ... + F_{n_y} = 0,\\
F_{O_z} &= \sum_{k=1}^{n} F_{k_z} = F_{1_z} + F_{2_z} + ... + F_{n_z} = 0,\\
M_{O_x} &= \sum_{k=1}^{n} M_x\left(\vec{F}_k\right) = M_x\left(\vec{F}_1\right) + M_x\left(\vec{F}_2\right) + ... + M_x\left(\vec{F}_n\right) = 0,\\
M_{O_y} &= \sum_{k=1}^{n} M_y\left(\vec{F}_k\right) = M_y\left(\vec{F}_1\right) + M_y\left(\vec{F}_2\right) + ... + M_y\left(\vec{F}_n\right) = 0,\\
M_{O_z} &= \sum_{k=1}^{n} M_z\left(\vec{F}_k\right) = M_z\left(\vec{F}_1\right) + M_z\left(\vec{F}_2\right) + ... + M_z\left(\vec{F}_n\right) = 0.
\end{aligned}\right\} \quad (4.59)$$

A rigid body can be at equilibrium under the action of a general force system in three cases:

- the body is free;
- the body is partially restricted;
- the body is completely fixed.

In the first case, a free body under the action of forces can move arbitrarily in space. Only active forces can form in the set of forces under the action of which a rigid body is in balance in the position of interest. So, in this case, (4.59) can be used to verify if a body can be in the state of equilibrium in the given position:

to determine a force or a couple that balance all other loads;

to define the positions of equilibrium.

In other worlds, a body is in the state of equilibrium if equalities (4.59) are valid in the position of our interest. If at least one equality from (4.59) is not valid it means that the body is not in balance in the position of our interest.

It is possible to secure the body equilibrium in the position of our interest by changing the loads acting up on it.

In the second case (if a body partially restricted), the constraints do not always guarantee its univalent equilibrium in the position of our interest. Remember that to generate correct equilibrium equations, we should take into consideration all the forces (active and reactive) acting on a partially restricted body. So equilibrium equations of a partially restricted body must contain the following unknowns: reactive forces, unknown loads (forces or couples) equilibrating given loads, and geometrical parameters determining the equilibrium position of the body.

In the third case, the constraints applied to the body provide its equilibrium in the given position under the action of any force system. In this case, the equilibrium equations (4.59) are used to determine the unknown reactive forces only.

4.7 Categories of equilibrium

It is possible to obtain equilibrium conditions for a specific force system as a special case of a general situation. An appropriate coordinate system must be used for each case.

1. The three-dimensional system of parallel forces.

Choose a rectangular Cartesian coordinate system so that one of the axes is parallel to the lines of action. We denote this axis as Oz. The projections of the forces onto the axes ***Ox*** and ***Oy*** are zero, and the force moments about ***Oz*** are zero too. It means that equations 1, 2 and 6 from system (4.59) are arithmetic identities $0 \equiv 0$.

So, for a rigid body to be at equilibrium under the action of the three-dimensional system of parallel (to the O_2 axis) forces there are three necessary and sufficient conditions:

$$\left.\begin{aligned}&\sum_{k=1}^{n} F_{k_z}=0,\\&\sum_{k=1}^{n} M_x\left(\vec{F}_k\right)=0,\\&\sum_{k=1}^{n} M_y\left(\vec{F}_k\right)=0.\end{aligned}\right\}\qquad(4.60)$$

2. The two-dimensional system of parallel forces.

Combine a force plane with the coordinate plane ***xOz*** of the coordinate system (see the previous case). For this force system, we have two equilibrium equations

$$\left.\begin{aligned}&\sum_{k=1}^{n} F_{k_z}=0,\\&\sum_{k=1}^{n} M_y\left(\vec{F}_k\right)=0.\end{aligned}\right\}\qquad(4.61)$$

The force moments about the axis Ox are equal to zero (the force and an axis are in the same plane) the second equation in (4.60) is an identity $0 \equiv 0$.

3. The three-dimensional concurrent force system.

Let the coordinate system origin O be at the point of concurrency. Then the moment of each force about any axis is zero and therefore the last three equations in (4.59) are identities $0 \equiv 0$.

So in a three-dimensional space, equilibrium conditions for the forces concurrent at the point are the following:

$$\left.\begin{aligned}&\sum_{k=1}^{n} F_{k_x}=0,\\&\sum_{k=1}^{n} F_{k_y}=0,\\&\sum_{k=1}^{n} F_{k_z}=0.\end{aligned}\right\}\qquad(4.62)$$

The same results were obtained when we studied a concurrent force system (see (2.2)).

4. The two-dimensional concurrent force system.

Suppose that a force plane coincides with the coordinate plane ***xOy*** described in case 3. Then

$$\left.\begin{aligned} \sum_{k=1}^{n} F_{k_x} = 0, \\ \sum_{k=1}^{n} F_{k_y} = 0. \end{aligned}\right\} \tag{4.63}$$

5. The two-dimensional general force system (coplanar force system).

Suppose that all forces of the system are in the plane xOy of a rectangular Cartesian coordinate system. So the projections of the forces onto the axes Oz and the force moments about axes Ox and Oy are zero. It means that equations 3, 4 and 5 in (4.59) are arithmetic identities $0 \equiv 0$. Therefore in the coordinate system described above, the sufficient and necessary equilibrium conditions for a coplanar force system are

$$\left.\begin{aligned} \sum_{k=1}^{n} F_{k_x} = 0, \\ \sum_{k=1}^{n} F_{k_y} = 0, \\ \sum_{k=1}^{n} M_z\left(\vec{F}_k\right) = 0. \end{aligned}\right\} \tag{4.64}$$

Conditions (4.64) can be used to determine unknown reactions. In this case, in order to be statically definable the system (4.64) must include no more than three unknowns.

In turn, if the system includes three unknowns it must be simultaneous and have a single solution. It means that none of the equations of this system is a consequence of the other two. Under the conditions formulated above any three equations are sufficient and necessary equilibrium conditions for a planar force system.

Along with conditions (4.64), we get

$$\left.\begin{aligned}&\sum_{k=1}^{n} F_{k_x} = 0,\\&\sum_{k=1}^{n} M_{A_z}\left(\vec{F}_k\right) = 0,\\&\sum_{k=1}^{n} M_{B_z}\left(\vec{F}_k\right) = 0.\end{aligned}\right\} \tag{4.65}$$

The segment AB is not perpendicular to the axis Ox

$$\left.\begin{aligned}&\sum_{k=1}^{n} M_{A_z}\left(\vec{F}_k\right) = 0,\\&\sum_{k=1}^{n} M_{B_z}\left(\vec{F}_k\right) = 0,\\&\sum_{k=1}^{n} M_{C_z}\left(\vec{F}_k\right) = 0.\end{aligned}\right\} \tag{4.66}$$

Points A, B, C belong to different lines. Let us prove that conditions (4.65) and (4.66), after fulfilling the appropriate requirements for the points A, B, C, are sufficient and necessary equilibrium conditions for a planar force system.

Conditions (4.65). Necessity. Force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right) \sim 0$. We have to prove that all conditions (4.65) hold. For a balanced force system, the resultant force $\vec{F}$ and the resultant moment about any center (e. g. A, M_A) are equal to zero (see (4.57), (4.58)), i.e.

$$\vec{F} = \sum_{k=1}^{n} \vec{F}_k = 0\ , \tag{4.67}$$

$$\vec{M}_A = \sum_{k=1}^{n} \vec{M}_A\left(\vec{F}_k\right) = 0\ . \tag{4.68}$$

After projecting equality (4.67) onto the axis Ox, we have the first equation in (4.65).

The resultant moments of the force system relate to the two centers A and B as

$$\overrightarrow{M}_B = \overrightarrow{M}_A + \overrightarrow{BA} \times \overrightarrow{F} . \quad (4.69)$$

According to equations (4.67), (4.68), $\overrightarrow{M}_B = 0$. Therefore the projections of the resultant moments onto the axis Az are zero, i.e. $\overrightarrow{M}_{A_z} = 0$, $\overrightarrow{M}_{B_z} = 0$. Hence conditions (4.65) are necessary.

Sufficiency. We need to prove that under condition (4.65) the force system is balanced (equal to zero), i.e. the resultant force and the resultant moment about any point are zero.

Projecting equality (4.69) onto the axis z perpendicular to the force plane (plane xAy), we obtain

$$M_{B_z} = M_{A_z} + \left(\overrightarrow{BA} \times \overrightarrow{F}\right)_z . \quad (4.70)$$

Using (4.65) we get

$$\sum_{k=1}^{n} F_{k_x} = F_x = 0 , \quad (4.71)$$

$$\left(\overrightarrow{BA} \times \overrightarrow{F}\right)_z = 0. \quad (4.72)$$

Equalities (4.71) and (4.72) are sufficient equilibrium conditions if the axis ***x*** is not perpendicular to the segment AB. If the segment AB is perpendicular to the axis ***x***, both (4.71) and (4.72) are valid for a non-zero resultant force $\overrightarrow{F}$ perpendicular to the axis x and collinear to the vector $\overrightarrow{BA}$. In this case, the force system is equivalent to the resultant force and conditions (4.60) are necessary but not sufficient for the force system to be in equilibrium.

Conditions (4.66). Their **necessity** immediately follows from equalities (4.68). To prove their sufficiency of the conditions (4.66) let us apply the relations between the projections of the force system resultant moments about the points A, B, C onto the axis z perpendicular to the force plane

$$M_{B_z} = M_{A_z} + \left(\overrightarrow{BA} \times \overrightarrow{F}\right)_z , \quad (4.73)$$

$$M_{C_z} = M_{A_z} + \left(\overrightarrow{CA} \times \overrightarrow{F}\right)_z \ . \tag{4.74}$$

Since A, B, C are chosen arbitrarily in the force plane and the vector moments $\overrightarrow{M}_A$, $\overrightarrow{M}_B$, $\overrightarrow{M}_C$ are perpendicular to the plane, we see that for the force system under consideration, a resultant moment about any point is zero.

After multiplying equalities (4.74) by a nonzero scalar coefficient α in accordance with (4.66) and subtracting the result from equality (4.73), we get

$$\left[\left(\overrightarrow{BA} - \alpha \cdot \overrightarrow{CA}\right) \times \overrightarrow{F}\right]_z = 0 \ . \tag{4.75}$$

If the vectors $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are non collinear (it means that points A, B, C belong to different lines), then equation (4.75) has a unique solution $\overrightarrow{F} = 0$ for any α. Otherwise,

$$\overrightarrow{BA} - \alpha \cdot \overrightarrow{CA} = 0 \ , \tag{4.76}$$

and equality (4.75) is true for a nonzero resultant force $\overrightarrow{F}$.

In this case, equalities (4.66) are necessary but not sufficient conditions for the force system equilibrium, and this force system can be reduced by the resultant force.

The sufficiency of conditions (4.66) is proven.

So, to solve a problem, we can use the basic form of equilibrium equations for a coplanar force system (4.64) as well as their forms (4.65) or (4.66).

4.8 Solving problems on the equilibrium of system of bodies

Equilibrium conditions for a general force system (4.59) or equilibrium conditions for a coplanar force system (4.64) – (4.66) are formulated as necessary and sufficient for a rigid body. For deformable bodies including joined ones, these conditions are necessary but not sufficient.

Indeed, by using the principle of solidification to solve the problem of a deformable body equilibrium (or the equilibrium of a system of deformable bodies) we shall treat this body as rigid and write appropriate equilibrium conditions.

There is no one-side action of force in nature, i.e. there is a reaction to every action. In a deformable body or in connected bodies (system of bodies), the forces of interaction between different parts are internal if we view the body (system) as a whole. These internal forces act in pairs, so in accordance with the principle of action-and-reaction the forces in a pair are equal in magnitude and act along the same line in opposite directions. The sum of these forces is always equal to zero. The sum of vector moments about any center of interaction of two separated parts of a body is zero. Therefore, for a set of internal forces $\left(\vec{F}^i\right)$ in any mechanical system, the resultant force and the resultant moment about any center are equal to zero:

$$\vec{F}^i = \sum_{k=1}^{n} \vec{F}_k^i = 0, \tag{4.77}$$

$$\vec{M}_O^i = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k^i\right) = 0 \;. \tag{4.78}$$

It means that in accordance with the principle of solidification and equalities (4.77), (4.78), only external forces must be included in the equilibrium conditions for a system of bodies.

For a general force system, the equilibrium conditions have the following vector form

$$\vec{F}^e = \sum_{k=1}^{n} \vec{F}_k^e = 0, \tag{4.79}$$

$$\vec{M}_O^e = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k^e\right) = 0 \;. \tag{4.80}$$

The equilibrium conditions stated above for a specific force system are valid for an appropriate body system (general, coplanar etc.). The external forces must be included in equilibrium equations.

When we analyze the rest (or motion) of a mechanical system we must determine the internal forces acting in the system.

To determine all the unknown reactions, it is not enough to know the equilibrium conditions for a mechanical system even if this system is statically definable. Moreover, the unknown internal reactions are not included into these equations because they belong to internal forces. So it is impossible to construct a set of equations sufficient for the determination of all unknown forces (both external and internal) if we consider only equilibrium equations for the mechanical system. So to determine all unknown forces (both external and internal) when solving on the body system equilibrium, we use the **method of section,** which consists of the following – divide the system of bodies into parts by cutting the internal constraints between the bodies of the system.

For each adjacent part at the section, apply the reactive forces corresponding to the type of the destroyed constraint. For the whole system these forces are internal and must have the features of internal forces ((4.77), (4.78)), but for any separate part, these forces are external.

For each part, write equilibrium equations, taking into consideration all the forces acting on the part (including the reactions of destroyed internal constraints).

Add the mathematical expression of the features for internal forces to the conditions of equilibrium. If the original material system is statically definable, then the number of equations obtained is sufficient to find all unknown reactions.

The following example illustrates the method of section. Let us consider a plane hinge-rod structure consisting of two rods AC and CD connected with the help of a pin joint C. The fixed pin support A and rollers B and D maintain the structure in the state of equilibrium.

Section is made in the internal pin C. It is possible to write a set of three equilibrium equations for each part (Fig. 4.10). The FBD on sketch a) illustrates the principle of solidification, FBD on sketch b) and c) illustrates the method of sections, FBD on sketch b) and d) represents the method of sections too.

We do not need to write all the equations because in this case they are dependent or repeated. It is enough to form six equilibrium equations and two additional ones for the relations between the reactions in the pin C such as:

three equations for the FBD a) and three equations for the FBD b);

three equations for the FBD a) and three equations for the FBD c);

three equations for the FBD b) and three equations for the FBD c) and additional equations:

$$x_C' = x_C \ , \ y_C' = y_C \ . \tag{4.81}$$

three equations for the FBD b) and three equations for the FBD d) and additional equations:

$$x'_C + x_C = 0 \ , y'_C + y_C = 0 \ . \qquad (4.82)$$

The last set of equations (including additional conditions (4.82)) is universal because it holds for pin connections with any number of rods, but the third set of equations (including (4.81)) works only for a pin connection of two rods. For example, for a pin connection of three rods (Fig. 4.11) the additional equations are

$$x_{C_1} + x_{C_2} + x_{C_3} = 0 \ , \ y_{C_1} + y_{C_2} + y_{C_3} = 0 \ . \qquad (4.83)$$

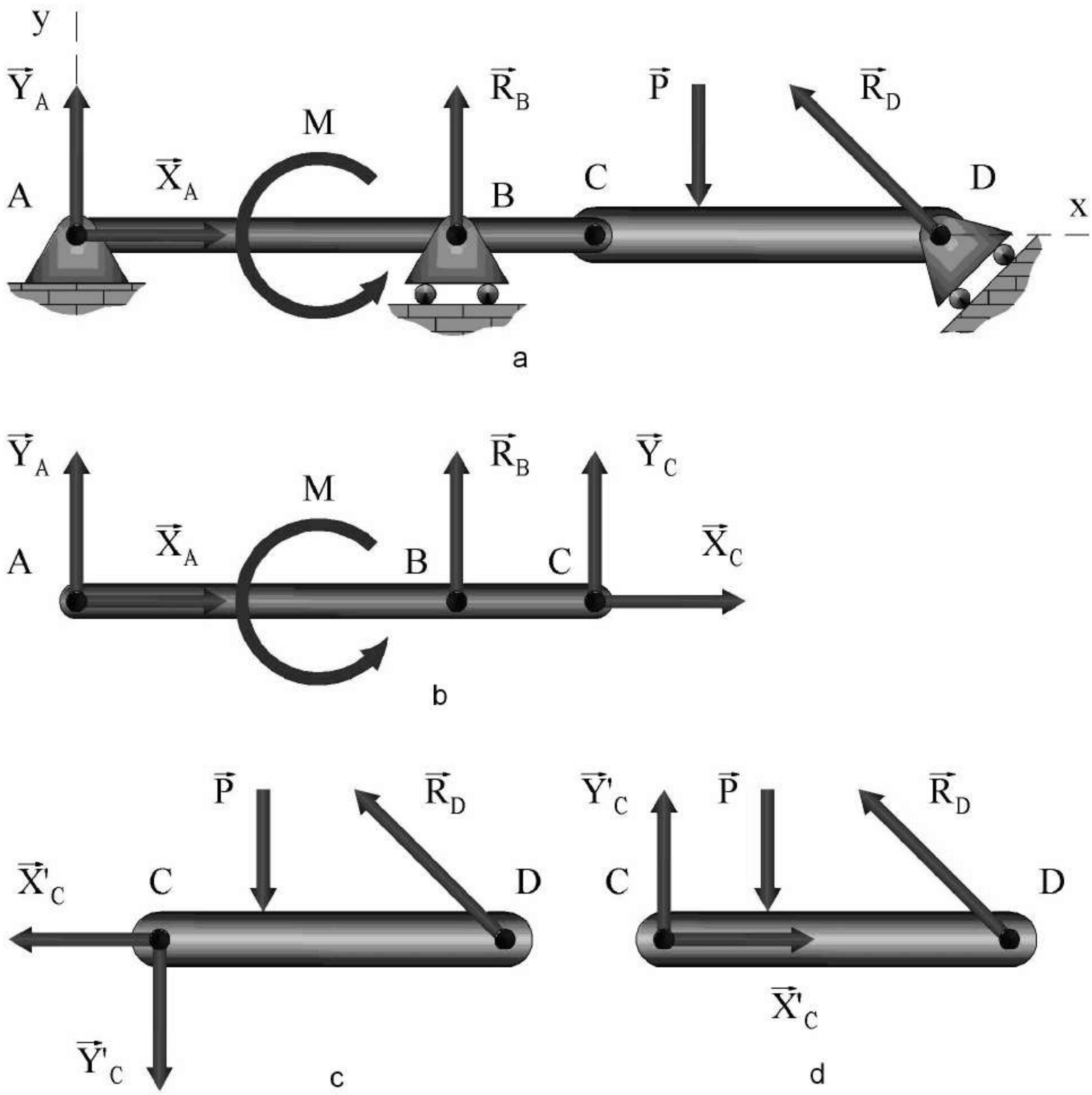


Fig. 4.10

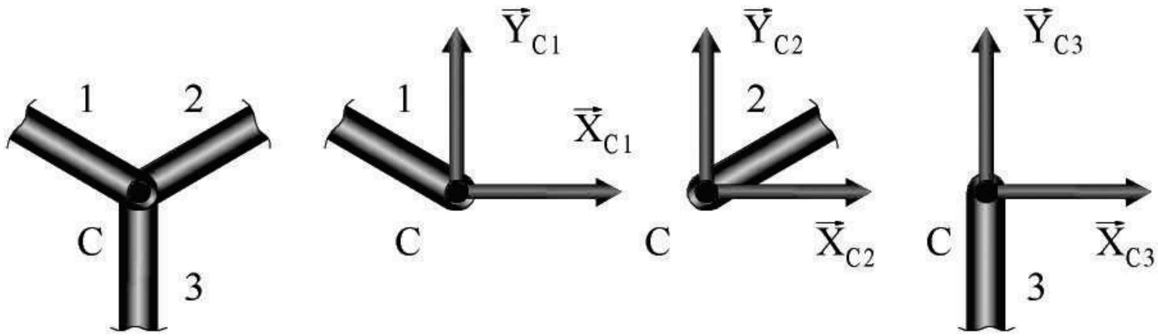


Fig. 4.11

In conclusion, let us note that when constructing a free-body diagram, we direct the reactions arbitrarily along the line of action (for example, in the positive direction of the appropriate coordinate axis). The assigned direction may prove to be wrong after the algebraic signs of components are calculated. If the sign of the component turn out negative, this means that a component actually acts in the direction opposite to the original one.

5 The center of parallel forces. The gravity center of a rigid body

In this chapter, we will consider a parallel force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ acting on a rigid body. It is supposed that the system can be reduced to an equivalent one consisting of a resultant force. For this to be true it is necessary and sufficient that the total vector is not zero:

$$\vec{F} = \sum_{k=1}^{n} \vec{F}_k \neq 0 \ . \qquad (5.1)$$

The necessity follows from condition (5.1) since the resultant force coincides with the total vector in magnitude and direction

$$\vec{R} = \vec{F} = \sum_{k=1}^{n} \vec{F}_k \ . \qquad (5.2)$$

To prove the sufficiency of condition (5.1), consider the special cases of the force system reduced to the simplest form (see Chapter 4.5). In cases 1 and 4a mentioned above, the system is reduced to the resultant force:

1) $\vec{F} \neq 0$, $\vec{M}_O = 0$; (5.3)

4a) $\vec{F} \neq 0$, $\vec{M}_O \neq 0$, $\vec{F} \perp \vec{M}_O$. (5.4)

For a parallel force system, the total vector $\vec{F}$ is collinear with the system of forces. The total moment $\vec{M}_O$ about a chosen center O is equal to the resultant couple moment and perpendicular to the couple plane. Therefore, if the total moment about the center O is not zero, it is perpendicular to the total vector $\vec{F}$.

So the parallel force system under consideration is equivalent to the resultant force and the condition (5.1) holds

$$\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right) \sim \vec{R} \ . \qquad (5.5)$$

Varignon's theorem. If a force system acting upon a rigid body is equivalent to the resultant force, the resultant force moment about any chosen center is equal to the sum of the moments of the system forces about the same center.

There are several ways to prove this theorem. For example, we can prove it by analyzing conditions 4.5 (see Chapter 4.5, the special cases of the force system reduction to the simplest form).

Here the simplest proof of the theorem is presented. Let the general force system be equivalent to the resulting force (5.5). If the vector opposite to the resultant force is added to the original force system, the obtained force system is equivalent to zero:

$$\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n, -\vec{R}\right) \sim 0 \ . \qquad (5.6)$$

So the total moment of a new system about any point O should be zero

$$\vec{M}_O\left(\vec{F}_1\right) + \vec{M}_O\left(\vec{F}_2\right) + ... + \vec{M}_O\left(\vec{F}_n\right) + \vec{M}_O\left(-\vec{R}\right) = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) - \\ -\vec{M}_O\left(\vec{R}\right) = 0,$$

since

$$\vec{M}_O\left(-\vec{R}\right) = \vec{r} \times \left(-\vec{R}\right) = -\vec{r} \times \vec{R} = -\vec{M}_O\left(-\vec{R}\right), \qquad (5.7)$$

where $\vec{r}$ is a position vector of the application point of the resulting force with respect to the center O.

It follows from equation (5.7)

$$\vec{M}_O\left(\vec{R}\right) = \sum_{k=1}^{n} \vec{M}_O\left(\vec{F}_k\right) . \qquad (5.8)$$

This completes the proof of the Varignon theorem.

5.1 The center of parallel force system

Let the set of parallel forces $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ be applied to the body. This force is equivalent to resultant force $\vec{R}$. If we draw parallel lines at the force application points and then rotate each point by the angle α about its axis in the same direction, the new set of forces is parallel and has a resultant force collinear with new forces. If the force system rotates by an arbitrary angle about the parallel axes, the resultant force should rotate the same angle about the axis parallel to the axes of the forces of system. The axis of rotation of the resultant intersects some specific point that is called as **center of the parallel force**.

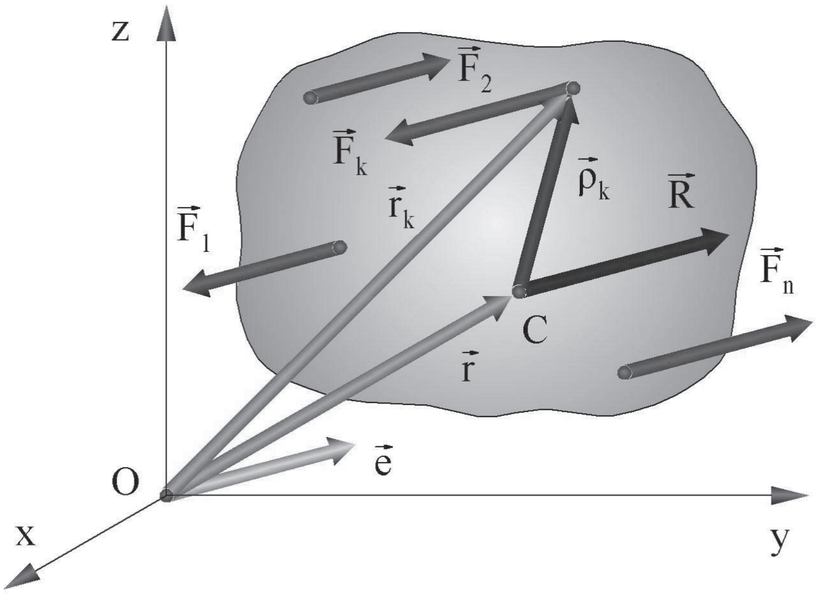


Fig. 5.1

Let us take in consideration the unit vector $\vec{e}$ collinear with the forces of system. Then for any force the next would be true

$$\vec{F}_k = \vec{e} \cdot F_k^* \ , \ k = \overline{1,n} , \tag{5.9}$$

where F_k^* is projection onto the unit vector $\vec{e}$ of force with number k. It is clear that $F_k^* > 0$ if the force has the same direction with unit vector $\vec{e}$ and $F_k^* > 0$ in opposite case.

It is clear from definitions of the center of parallel force system that its position does not depend on the unit vector $\vec{e}$ direction. This feature may be used for the determination of the center's coordinates with respect to the chosen coordinate system $Oxyz$. Let point C (Fig. 5.1) is the center of a parallel force system. Assuming that the force system is equivalent to the resultant force only (it means the Varignon's theorem condition hold) we can say

$$\vec{M}_C\left(\vec{R}\right)=\sum_{k=1}^{n}\vec{M}_C\left(\vec{F}_k\right). \tag{5.10}$$

Obviously,

$$\vec{M}_C\left(\vec{R}\right)=0, \tag{5.11}$$

$$\vec{M}_C\left(\vec{F}_k\right)=\vec{\rho}_k\times\vec{F}_k,\ k=\overline{1,n}, \tag{5.12}$$

$$\vec{\rho}_k=\vec{r}_k-\vec{r},\ k=\overline{1,n}, \tag{5.13}$$

where $\vec{\rho}_k$, $\vec{r}_k$ are the position vectors of the force $\vec{F}_k$ point of application with respect to the points O and C; $\vec{r}$ is a required position vector of the parallel force center C.

Using equation (5.13) to make substitutions in (5.12) and (5.11) in (5.10), we get

$$\sum_{k=1}^{n}\left(\vec{r}_k-\vec{r}\right)\times\vec{F}_k=0. \tag{5.14}$$

In accordance with (5.9), the latter equation takes the form

$$\left(\sum_{k=1}^{n}\vec{r}_k\cdot F_k^*-\vec{r}\cdot\sum_{k=1}^{n}F_k^*\right)\times\vec{e}=0. \tag{5.15}$$

This equation is true for the unit vector $\vec{e}$ of any direction and any angle of rotation of the forces about the parallel axes. So it follows from equation (5.15) that the expression in brackets is equal to zero

$$\sum_{k=1}^{n}\vec{r}_k\cdot F_k-\vec{r}\cdot\sum_{k=1}^{n}F_k^*=0. \tag{5.16}$$

This equation can be used to determine a required vector position of the center of the parallel force system. From (5.16) we have

$$\vec{r} = \frac{\sum_{k=1}^{n} \vec{r}_k \cdot F_k^*}{\sum_{k=1}^{n} F_k^*} . \qquad (5.17)$$

Projecting both sides of equality (5.17) onto the axes of the rectangular coordinate system (see Fig. 5.1), we get formulas for the coordinates of the center:

$$x = \frac{\sum_{k=1}^{n} x_k \cdot F_k^*}{\sum_{k=1}^{n} F_k^*} ; \; y = \frac{\sum_{k=1}^{n} y_k \cdot F_k^*}{\sum_{k=1}^{n} F_k^*} ; \; z = \frac{\sum_{k=1}^{n} z_k \cdot F_k^*}{\sum_{k=1}^{n} F_k^*} . \qquad (5.18)$$

The expression $\sum_{k=1}^{n} \vec{r}_k \cdot F_k^*$ in equation (5.17) is called the first static moment of a parallel force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ with respect to the center O; and the expressions $\sum_{k=1}^{n} x_k \cdot F_k^*$, $\sum_{k=1}^{n} y_k \cdot F_k^*$, $\sum_{k=1}^{n} z_k \cdot F_k^*$ in (5.18) are called the first static moment of a parallel force system $\left(\vec{F}_1, \vec{F}_2, ..., \vec{F}_n\right)$ with respect to the coordinate planes yOz , xOz , xOy respectively.

5.2 The center of gravity of a rigid body

Gravitational force is the force exerted on the body by the earth. The numerical value of this force is equal to the body weight. The vector of the gravitational force is directed along the cord with the help of which the body is suspended to a fixed point. This direction is called vertical; the plane perpendicular to the vertical is called horizontal.

If we mentally decompose a body into elementary particles, the force of gravity acting on each particle should act through the point coinciding with the particle. The set of gravitational forces of elementary particles can with sufficient accuracy be considered a system of parallel forces (for example, if we have two particles located one kilometer apart on the surface of the Earth, the angle between two forces of gravity is 32°).

It is clear that this force system is equivalent to the resultant force only if all forces of the system have the same direction.

The resultant of the gravitational forces of the elementary particles forming a rigid body acts through a point called the parallel force center. The center of parallel gravitational forces of the rigid body particles coincides with the center of gravity. The location of the center of gravity can be determined with the help of equation (5.17)

$$\vec{r}_C = \frac{\sum_{k=1}^{n} \vec{r}_k \Delta P_k}{\sum_{k=1}^{n} \Delta P_k}, \tag{5.19}$$

where r_k is a position vector of the elementary particle k; ΔP_k is its weight, n is the number of the elementary volumes (particles) forming the body.

Equation (5.19) determines the approximate value of the center of gravity position vector for a rigid body. The accuracy of the result will increase if the volume of elementary particles diminishes.

When $n \to \infty$ $(\Delta P_k \to 0)$, the sums of the numerator and the denominator of formula (5.19) are integral and we can write

$$\vec{r}_C = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \vec{r}_k \Delta P_k}{\sum_{k=1}^{n} \Delta P_k} = \frac{\lim_{n\to\infty} \sum_{k=1}^{n} \vec{r}_k \Delta P_k}{\lim_{n\to\infty} \sum_{k=1}^{n} \Delta P_k} =$$

$$= \frac{\int_{(P)} \vec{r} \cdot dP}{\int_{(P)} dP} = \frac{1}{P} \int_{(P)} \vec{r} \cdot dP, \tag{5.20}$$

where ***P*** is a body weight, $\vec{r}$ is a position vector of the point with the coordinates x, y, z; ***dP*** is the weight of an infinitesimal particle.

If the body is not homogeneous, its specific weight is a function of the point coordinates

$$\gamma = \gamma(x, y, z); \; dP = \gamma(x, y, z) \cdot dV, \tag{5.21}$$

where ***dV*** is an infinitesimal value, and the formula (5.20) can be rewritten as

$$\vec{r} = \frac{1}{P}\iiint\limits_{(V)} \vec{r}\gamma\left(x,\ y,\ z\right)\cdot dV\ . \tag{5.22}$$

After projecting both sides of equation (5.20) onto the axes x, y, z of a Cartesian coordinate system, we obtain expressions for the center of gravity coordinates:

$$\left.\begin{aligned} x_C &= \frac{1}{P}\iiint\limits_{(V)} x\cdot\gamma(x,y,z)dV, \\ y_C &= \frac{1}{P}\iiint\limits_{(V)} y\cdot\gamma(x,y,z)dV, \\ z_C &= \frac{1}{P}\iiint\limits_{(V)} z\cdot\gamma(x,y,z)dV. \end{aligned}\right\} \tag{5.23}$$

For a homogeneous body, $\gamma_V(x,y,z) = \gamma_V = const$ and relations (5.23) can be simplified:

$$\left.\begin{aligned} x_C &= \frac{1}{V}\iiint\limits_{(V)} x\cdot dxdydz, \\ y_C &= \frac{1}{V}\iiint\limits_{(V)} y\cdot dxdydz, \\ z_C &= \frac{1}{V}\iiint\limits_{(V)} z\cdot dxdydz. \end{aligned}\right\}. \tag{5.24}$$

Note that $P = \gamma\cdot V$, $dV = dxdydz$. The center of gravity of a homogeneous body is called the **centroid of volume**.

It is known that a body is called a plate, shell, or two-dimensional body if one of its character sizes (for example, thickness) is substantially smaller than the other two.

For a uniform surface, we have $\Delta P = \gamma_S dS$, $P = \gamma_S\cdot S$ (S is the area of the surface under consideration).

So the coordinates of the center of gravity of a uniform shell are

$$x_C = \frac{1}{S}\iint_{(S)} x dS \ ; \ y_C = \frac{1}{S}\iint_{(S)} y dS \ ; \ z_C = \frac{1}{S}\iint_{(S)} z dS \ . \qquad (5.25)$$

The center of gravity of a uniform shell is called the **centroid of a surface**.

For a thin uniform plate, the infinitesimal area is defined as $dS = dxdy$. Here it is supposed that the plate lies in the coordinate plane ***xOy*** so for expressions (5.25) we get:

$$x_C = \frac{1}{S}\iint_{(S)} x dx dy \ ; \ y_C = \frac{1}{S}\iint_{(S)} y dx dy \ . \qquad (5.26)$$

If one of the character sizes of a body is substantially greater than the other two dimensions, the body is called a uniform rod. Let us consider a rod element enclosed between two sections perpendicular to the axial line of the rod. The weight of the element ΔP_k is proportional to the length $\Delta \ell$ of the axial line arc. Hence $\Delta P_k = \gamma_L \Delta \ell_k$, $P = \gamma_L L$ where L is the rod length, γ_L is linear weight (the rod unit length weight). The coordinates of rod center of gravity are

$$x_C = \frac{1}{L}\int_{(L)} x d\ell \ , \ y_C = \frac{1}{L}\int_{(L)} y d\ell \ , \ z_C = \frac{1}{L}\int_{(L)} z d\ell \ . \qquad (5.27)$$

The center of gravity of a uniform rod is called the **centroid of a line**.

The terms centroid of volume, surface, or line are used because only the geometrical features of a body are used to define the centre of gravity, while volume, surface, and line integrals are evaluated.

5.3 The coordinates of the center of gravity

To determine the coordinates of the gravity center for a rigid body, we must evaluate the volume, surface and line integrals. But under some conditions, the location of the gravity center can be determined by very simple methods. Let us consider some of them.

Theorems of symmetry:

Theorem 1. If a body is uniform and symmetric with respect to a plane, the body's centroid lies in this plane.

Theorem 2. If a body is uniform and symmetric with respect to a line, the body's centroid lies in this axis.

Theorem 3. If a body is uniform and symmetric with respect to a center O, the body's centroid coincides with O.

A body is said to be symmetric with respect to a plane, axis, or center if for every particle on one side of the plane, axis, or center there exists a particle of equal weight on the other side of the plane so that the segment of the line connecting these two points is perpendicular to the plane and is divided into two equal parts by this plane, axis, or center).

Method of dividing body into n parts

To determine the position of the center of gravity a body can be divided into parts with a known (or easily found) weight and gravity center. For a body divided into n parts, we have

$$\vec{r_C} = \frac{P_1\vec{r}_1 + P_2\vec{r}_2 + ... + P_n\vec{r}_n}{P}, \qquad (5.28)$$

where $\vec{r}_C$ is a position vector of the whole body center of gravity, $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$ are position vector's of different parts $P_1, P_2, ..., P_n$ are weights of the parts, $\boldsymbol{P}$ is the body weight.

The expressions used to determine the coordinates of the gravity center follow from equation (5.28). It is supposed that the origin of the position vector coincides with the origin of the coordinate system):

$$x_C = \frac{1}{P}\sum_{k=1}^{n} P_k x_k\ ,\ y_C = \frac{1}{P}\sum_{k=1}^{n} P_k y_k\ ,\ z_C = \frac{1}{P}\sum_{k=1}^{n} P_k z_k\ . \qquad (5.29)$$

For a uniform plate that lies in the coordinate plane xOy, from (5.29) we have

$$x_C = \frac{1}{S}\sum_{k=1}^{n} S_k x_k\ ,\ y_C = \frac{1}{S}\sum_{k=1}^{n} S_k y_k\ ,\ z_C = 0\ , \qquad (5.30)$$

where S is the plate area, x_k, y_k $(k = 1, 2, ..., n)$ are the gravity center coordinates of the separated parts; S_k $(k = 1, 2, ..., n)$ are the areas of the parts.

Method of negative weights

This method is used to determine the centre of gravity of a body with holes. The method of dividing the body into n parts can be applied, but the holes are assumed to have negative weight, i.e.

$$\vec{r}_C = \frac{P \cdot \vec{r} - P_1\vec{r}_1 - P_2\vec{r}_2 - ... - P_k\vec{r}_k}{P - P_1 - P_2 - ... - P_k} \qquad (5.31)$$

Here $\vec{r}_C$ is a required position vector of the centre of gravity of a body with holes; ***P*** is the weight of the body without holes; $\vec{r}_1, \vec{r}_2, ..., \vec{r}_k$, $P_1, P_2, ..., P_k$ are position vectors of the gravity centers and the weights of the body parts equivalent to the holes, respectively.

Method based on the Pappus-Guldinus theorems

Theorem 1. The area of a surface of revolution is equal to the product of the arc length and the length of the circle generated by the gravity centre. It is supposed that the surface is generated by rotating a plane curve about an axis external to the plane curve and lying in the same plane:

$$S = 2\pi x_C \cdot L, \qquad (5.32)$$

where ***S*** is the area of the surface of revolution, ***L*** is the arc length of a plane curve, x_C is a coordinate of the curve centre of gravity (it is supposed that the curve lies in the plane ***xOy*** and rotates about the axis Oy).

Theorem 2. The volume of a body of revolution generated by rotating a plane figure about an external axis is equal to the product of the area of a plane figure and the distance travelled by its geometrical centroid:

$$V = 2\pi x_C \cdot S\ . \qquad (5.33)$$

where ***V*** is the volume of a body of revolution, ***S*** is the area of the plane figure, x_C is the abscissa of the gravity centre of the figure laying in the plane ***xOy*** and rotating about the axis ***Oy***.

BIBLIOGRAPHY

1. Beer F. P., Johnston E. R., Eisenberg E. R. and G. H. Staab, Vector Mechanics for Engineers: Statics. New-York: McGraw-Hill, 2004. – 621 p.
2. Meriam J.L., Krage L.G. Engineering Mechanics: Statics. New-York: John Willey and sons, 1987. – 454 p.
3. Бутенин Н.В. Курс теоретической механики: в 2 т. / Н.В. Бутенин, Я.Л. Лунц, Д.Р. Меркин. – М.: Наука, 1976. – Т. 1. – 272 с.
4. Бухгольц Н.Н. Основной курс теоретической механики: в 2 ч. / Н.Н. Бухгольц. – М.: Наука, 1966. – Ч. 1. – 466 с.
5. Лойцянский Л.Г. Курс теоретической механики: в 2 т. / Л.Г. Лойцянский, А.И. Лурье. – М.: Наука, 1982. – Т. 1. – 352 с.
6. Сборник задач по теоретической механике / Н.А. Бражниченко, В.Л. Канн, Б.Л. Минцберг и др. – М.: Высш. шк., 1974. – 520 с.

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Зв. план, 2010
Підписано до друку 27.12.2010
Формат 60×84 1/16. Папір офс. № 2. Офс. друк
Ум. друк. арк. 4,6. Обл.-вид. арк. 5,25. Наклад 100 прим.
Замовлення 437. Ціна вільна

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