MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
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SWEPTBACK WING STRESS STATE

Guidance Manual for Course Project

Kharkiv “KhAI” 2013

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V15

Посібник дає уявлення про розрахунки статично невизначених систем на прикладі стрілоподібного крила із силовим кореневим трикутником. Припускається, що студент володіє знаннями з вищої математики, опору матеріалів і будівельної механіки.

Вихідними даними курсового проекту є базові геометричні параметри конструкції, властивості матеріалів і величина зовнішнього навантаження.

Метою роботи є аналіз розподілення зусиль по балках кореневого трикутника. Отримані результати можуть бути використані для подальшого проектування конструкції крила.

Для студентів старших курсів авіаційних спеціальностей.

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Guidance manual illustrates how statically indeterminate systems are calculated by the example of a sweptback wing with a root triangle. The authors assume that students already have appropriate knowledge of further mathematics, mechanics of materials, and structural mechanics.

Given data for course project are basic geometrical parameters of structure, materials properties, and external loads.

The project is intended to analyze distribution of internal loads among root triangle beams. Obtained results could be further used to design wing structure.

Guidance manual will serve advanced students in aerospace engineering.

Figures 44. Tables 11. Ref.: 4 titles

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LIST OF ACRONYMS AND ABBREVIATIONS

EL	elementary state
FFB	front fuselage beam
FR	fuselage rib
FSC	front spar continuation
F-state	loaded state
RFB	rear fuselage beam
RR	root rib
SB	self-balanced state
WB	wingbox

INTRODUCTION

This guidance manual describes the process following the stage of conceptual design of aircraft. Before the detailed design process could be started, a designer must know loads applied at structural elements, distribution of these loads, and internal force factors acting in structural elements. This data could not be obtained using numerical methods (like finite element analysis) because detailed geometric parameters of the structure are still unknown at this stage. Distribution of forces among structural elements depends on their rigidities, so it can be calculated using analytical methods of structural mechanics based on elementary parameters of the structure such as typical thicknesses, cross section areas, material data, etc.

We shall use a few methods in this project including force method and the Papkovich theorem. All of them are based on the number of assumptions and simplifications that is why obtained results are not absolutely accurate but still more than sufficient for starting detailed design process. Calculations will give us the distribution of normal forces, bending moments, and shear flows acting in the structural elements under consideration.

Designers use these results to design structure. After that, strength engineers make checking calculations of stress-strain state, usually using numerical methods.

This course project is intended to study analytical methods of strength calculation of spatial structures consisting of plane beams and stiffened shell elements by calculating stress state of sweptback wingbox.

The information presented in this guidance manual is also covered in [1]. More information about general methods of structural analysis, such as force method, is available in [2, 3].

1. STRUCTURAL LAYOUT AND DESIGN MODEL OF SWEPTBACK WINGBOX

Fig. 1.1 shows the structural layout of an ordinar sweptback wing with two spars.

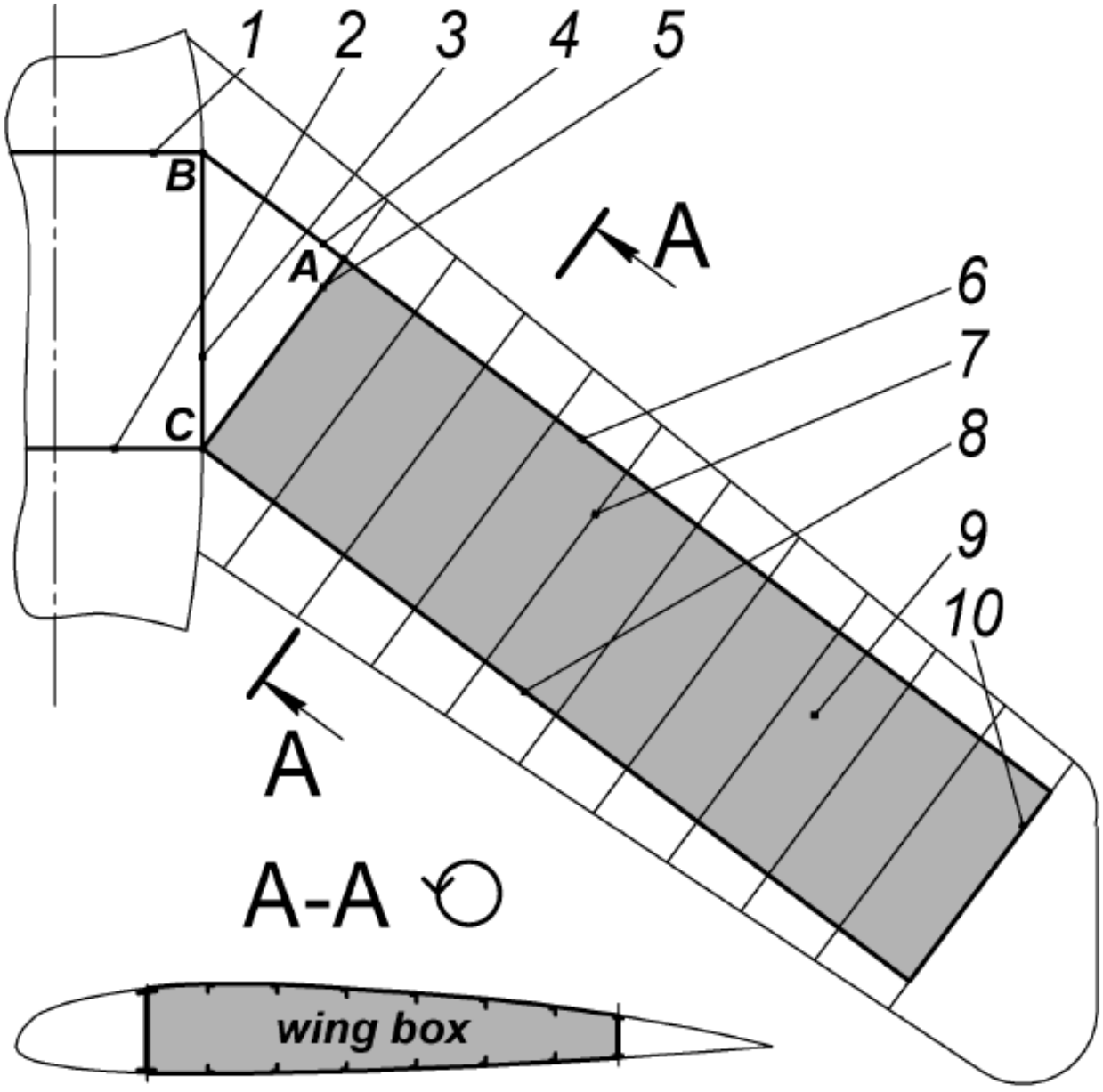


Figure 1.1. Sweptback wing structure:
1 – front fuselage beam (FFB); 2 – rear fuselage beam (RFB); 3 – fuselage rib (FR); 4 – front spar continuation (FSC); 5 – root rib (RR); 6 – front spar (FS); 7 – rib; 8 – rear spar (RS); 9 – wingbox; 10 – end rib

Air pressure that loads the wing is carried mostly by the wingbox because the structure before the front spar and behind the rear spar usually transmits

the load to the wingbox. This means that such structural elements as slats, aileron, or flaps have no influence on the wingbox rigidity and load carrying ability.

Typically, wing structure has longitudinal elements that support the wing skin (stringers). To simplify the model, we assume that the skin carries only shear loads while tensile loads are carried only by spar caps. In such a model, areas of spar caps represent not the actual areas but the sum of cross section areas of skin, stringers, and caps themselves. Thus, it is assumed that the wingbox consist only of spars and upper and lower skins. Cross section of the wingbox model is shown in Fig. 1.2.

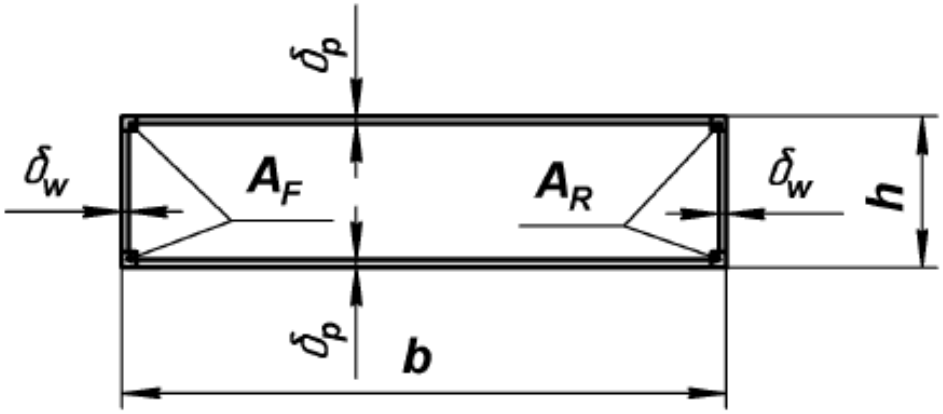


Figure 1.2. Cross section of wingbox:
A_F, A_R – cross section areas of front and rear spars; δ_w – thickness of spar webs; δ_p – thickness of upper and lower skins (panels)

Root triangle is a structure consisting of a root rib, a front spar continuation, and a fuselage rib (see triangle ABC in Fig. 1.1).

We do not consider skins in root triangle in order to simplify the model. In real structure, lower skin could indeed be not load-carrying due to landing gear situated in root triangle. To crutially simplify calculations, we remove upper skin though it becomes a shortcoming of the model. It should be noted that this fact does not have much effect on final results, especially for structures with thin skin.

Design model is an idealized spatial system of structural elements jointed by ideal connections. For this system and for its components, the structural mechanics laws are valid. Some assumptions made in the model are listed below:

1. Wingbox and all its elements deform linearly, so relation between stress and strain is linear, and displacements are small.
2. Two-spar wingbox is a membrane shell with a cross section that is absolutely rigid in its plane and is flexible outside it.

3. Wing ribs are located continuously by the length of the wingbox; they are absolutely rigid in their planes and flexible outside them.
4. Axial loads (normal stresses) are carried only by spar caps.
5. Spar webs and skins carry only shear loads (there are only shear flows in them, no normal stresses).
6. Elements of root triangle and fuselage structure (RR, FR, FSC, FFB, RFB) are planar beams; they are finitely rigid in their planes and absolutely flexible outside them.
7. Upper and lower skins of a root triangle ***ABC*** do not carry loads.
8. Fuselage structure composed of beams FR, FFB, RFB is a spatial statically determinate system.

Outer wing (composed of wingbox and FSC) is attached to the fuselage rib at points ***B*** and ***C*** (Fig. 1.3).

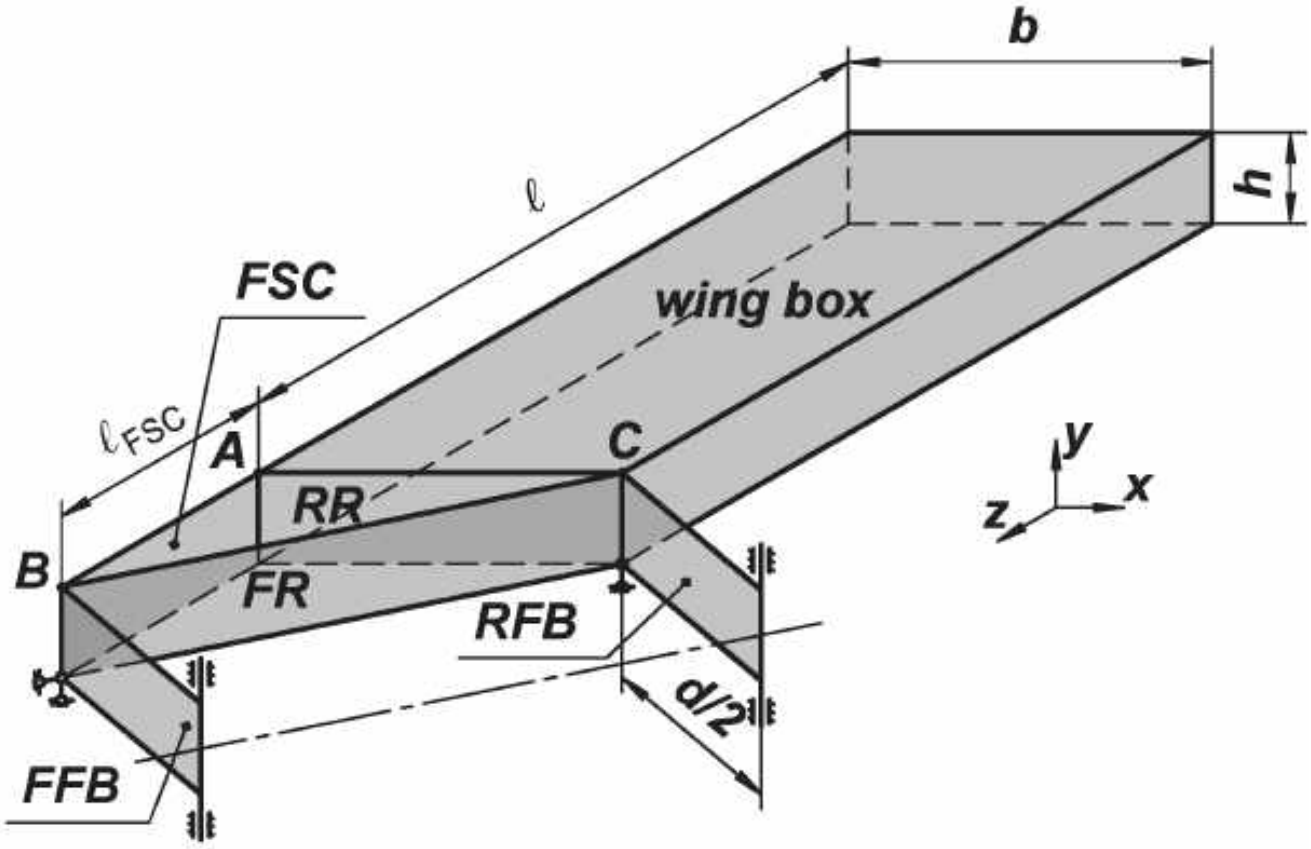


Figure1.3. Design model of wing

System as a whole has the same degree of static indeterminacy as the outer wing because the fuselage structure is statically determinate. There are two ideal cylindrical hinges at point ***B*** and two ideal spherical hinges at point ***C*** (Fig. 1.4).

The wingbox jointed with FSC is a spatial disk, so it has six degrees of freedom. Each cylindrical hinge constrains two degrees of freedom in the system; each spherical hinge takes away three degrees of freedom. So we shall possibly have a four times statically indeterminate structure. It is desirable

to introduce some simplifications to the model in order to make calculations easier. We can exclude constraints 1 and 2 (see Fig. 1.4) because leaving only one vertical constraint is enough for points B and C. It is valid becasue FSC and RR are both planar beams which do not have normal stresses in transverse direction according to the beam theory.

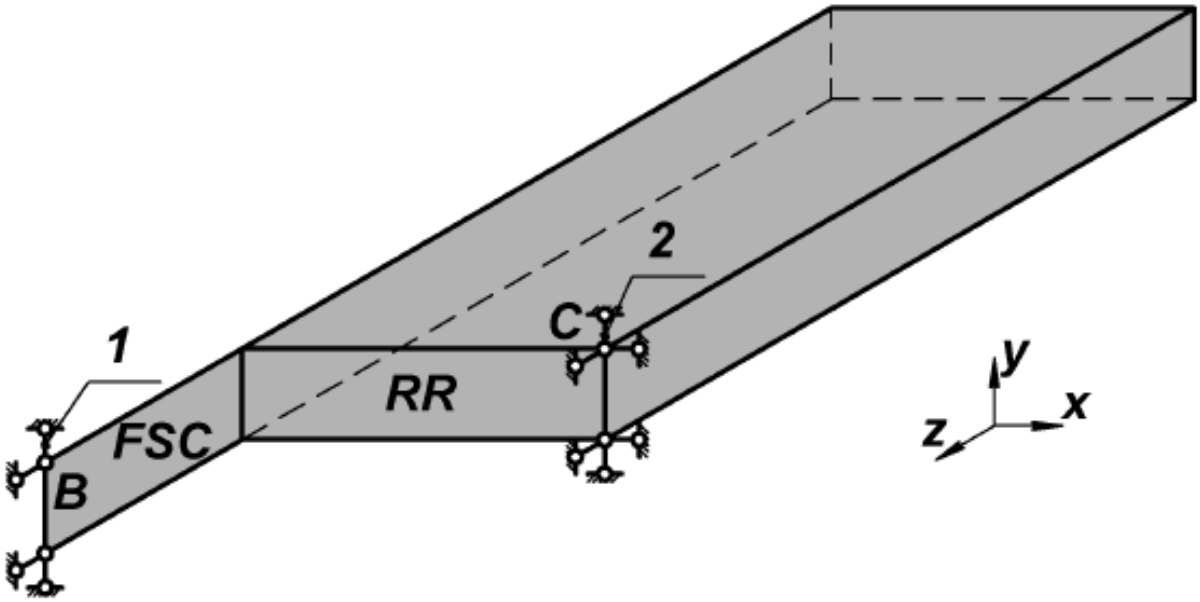


Figure 1.4. Design model of outer wing

Fuselage rib is a statically determinate beam. FR is attached to the fuselage in the way shown in Fig. 1.5.

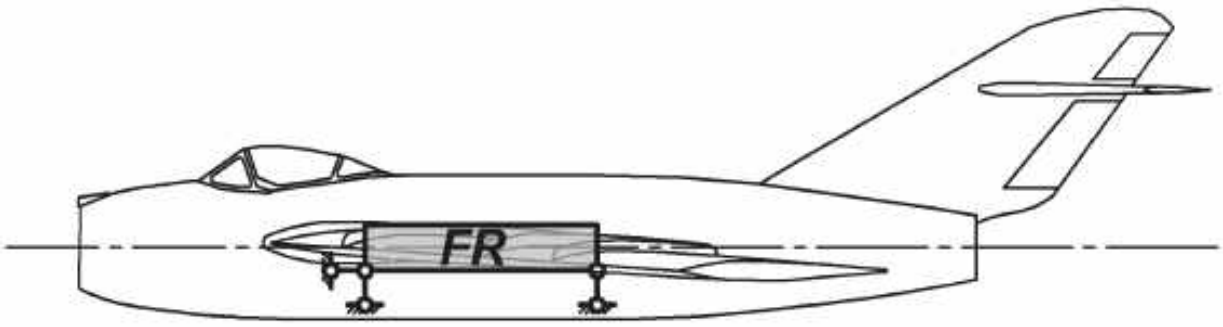


Figure 1.5. Fuselage rib fixation

Fig. 1.6 shows how FFB and RFB are attached to fuselage frames. Due to the symmetry of the aircraft, we can consider one half of the structure. Thus, we should apply constraints to the plane of aircraft symmetry: vertical deflections are allowed, but horizontal deflections (outside the plane of symmetry) are not.

The input data for course project include:

a) areas of front and rear spar caps, A_F and A_R (see Fig. 1.4);

b) moments of inertia of front and rear fuselage beams, fuselage rib, and root rib, I_{FFB}, I_{RFB}, I_{FR}, I_{RR} correspondingly;
c) thickness of front and rear spar webs, δ_w;
d) thickness of upper and lower skins (panels), δ_p;
e) distance from the front spar to the point of load application, x_D (it is assumed that aerodynamic forces are reduced to the resultant force, Fig. 1.7);
f) wing sweep angle, χ;
g) wingbox length, width, and height, ℓ, b, h correspondingly;
h) fuselage diameter, d;
i) resultant aerodynamic force, F;
j) material properties.

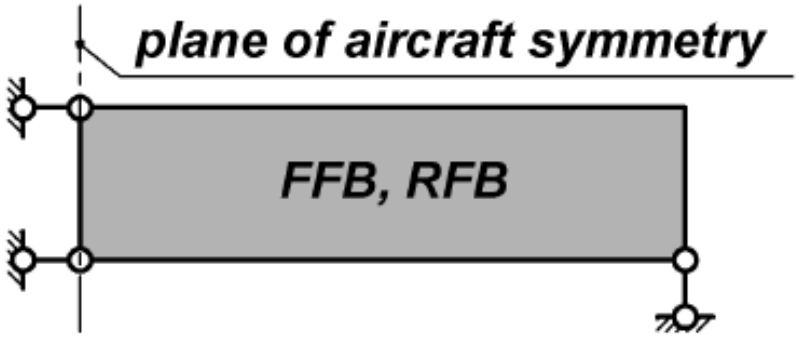


Figure 1.6. Front and rear fuselage beams fixation

Moment of inertia of FSC is the same as moment of inertia of the front spar. The length of FSC can be calculated as $\ell_{FSC} = b \cdot \tan\chi$.

Fig. 1.7 shows the diagram of a sweptback wing with dimensions, and Fig. 1.2 gives a cross section of a wingbox.

Task variants for the course project include the parameters A_F, A_B, δ_w, δ_p, and x_D as well as the type of basic system. Other parameters and material properties are invariant.

All beams (i.e. RR, FR, FSC, FFB, and RFB), spar webs, and caps are made of steel 30ХГСА. Upper and lower skins are made of aluminum alloy Д16Т. Properties of these materials (Young's modulus E, shear modulus G and yeild stress σ_y) are given in Table 1.1.

The results of the calculations include:
a) diagrams of normal forces distribution in the spar caps, diagrams of shear flows distribution in spar webs, upper and lower skins;

b) diagrams of bending moments distribution in the elements of a root triangle (RR, FR, FSC) as well as in FFB and RFB;
c) forces in the constraints.

The course project is mainly intended to find out how loads are distributed among structural members of the root triangle. Thus, the most important results are the diagrams of bending moments in five beams of the root triangle.

Table 1.1

Materials properties

Material	E, MPa	G, MPa	σ_y, MPa
30ХГСА	$2 \cdot 10^5$	$7.7 \cdot 10^4$	1250
Д16Т	$7.2 \cdot 10^4$	$2.8 \cdot 10^4$	280

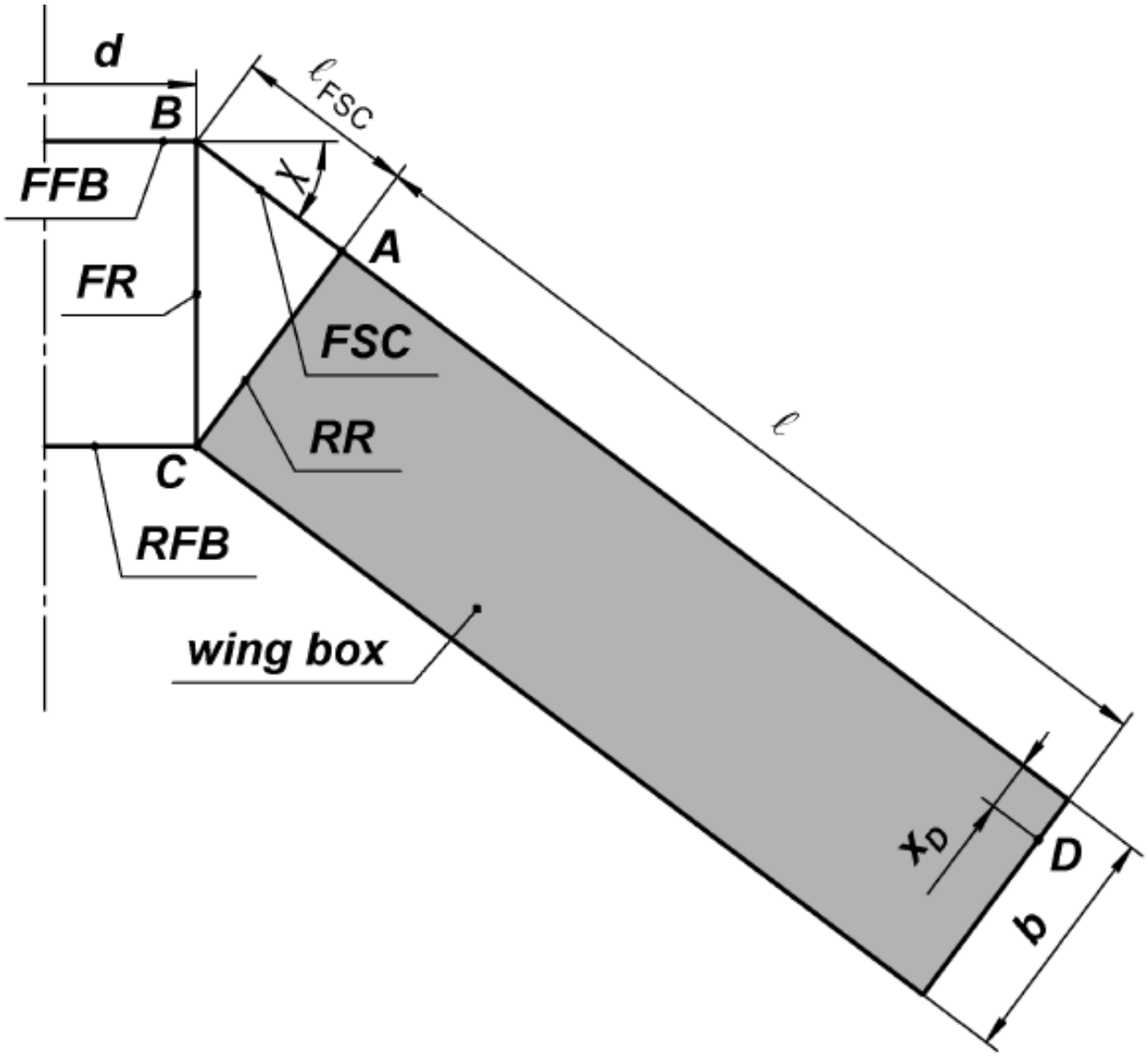


Figure 1.7. Sweptback wing dimensions

2. ANALYSIS OF THE GIVEN SYSTEM

In order to check the stability and find out the degree of static indeterminacy, the given system must be properly analysed. Firstly, we carry out the kinematic analysis, and then check the results with statical analysis.

Outer wing (wingbox and front spar continuation) is attached to the fuselage structure (front fuselage beam, rear fuselage beam, and fuselage rib). This fuselage structure is immovable and statically determinate, so we should conduct the analysis only for the outer wing.

Outer wing is attached to the fuselage rib by eight rods (1...3, 7...11). It consists of two disks, one spatial (wingbox) and one planar (FSC). How the front spar continuation is connected to the wingbox (more exactly to the front spar) could be represented by three rods 4..6 (Fig. 2.1).

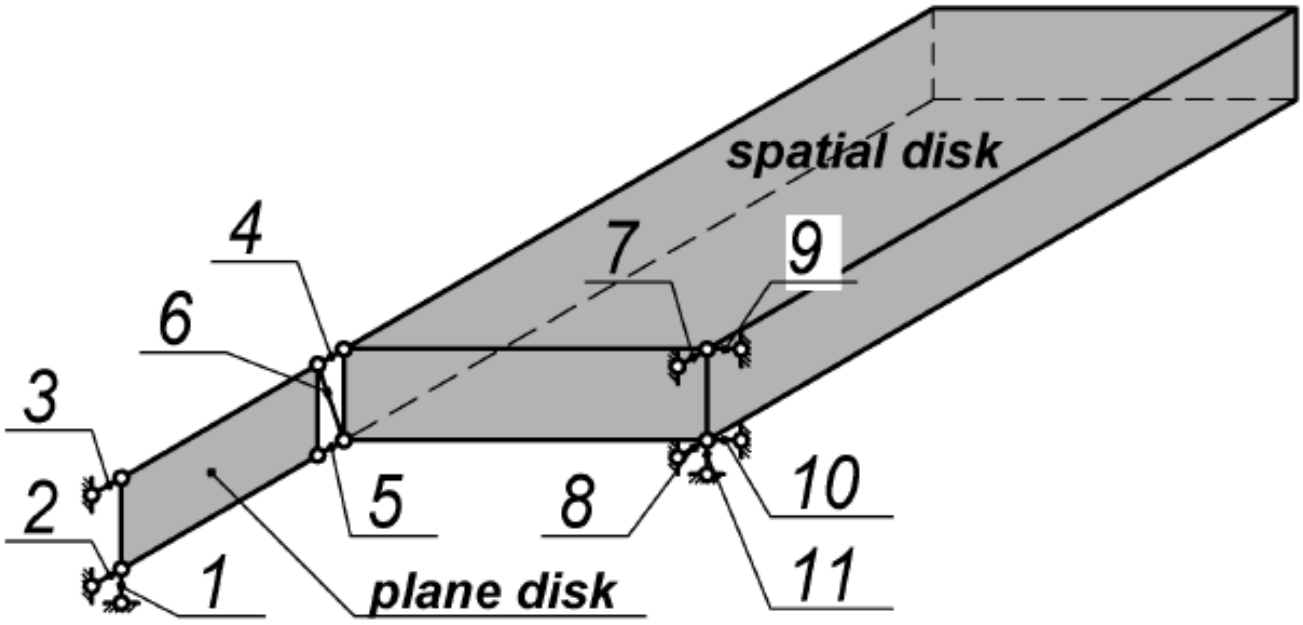


Figure 2.1. Numbering of constraints

The movability of the system (Mov) can be calculated using the formula

$$Mov = 6 \cdot D_{spatial} + 3 \cdot D_{planar} - C - C_0, \quad (2.1)$$

where $D_{spatial}$, D_{planar} is a number of spatial and planar disks in the system, correspondingly;

C is a number of rods (constraints) in the system;

C_0 is a number of support rods (constraints).

Substituting $D_{spatial} = 1$; $D_{planar} = 1$; $C = 3$; $C_0 = 8$ into (2.1), we obtain

$$Mov = 6 \cdot 1 + 3 \cdot 1 - 3 - 8 = -2. \qquad (2.2)$$

This result means that the system is possibly two times statically indeterminate.

Being a necessary but not sufficient condition of geometrical stability, kinematic analysis assesses the system quantitatively. Thus, we have to carry out a statical analysis to qualitatively assess the system.

For this we have to compose the matrix of coefficients for the equations of static equilibrium. Instead of ideal rods, we apply forces S_i (i varies from 1 to 11, Fig. 2.2).

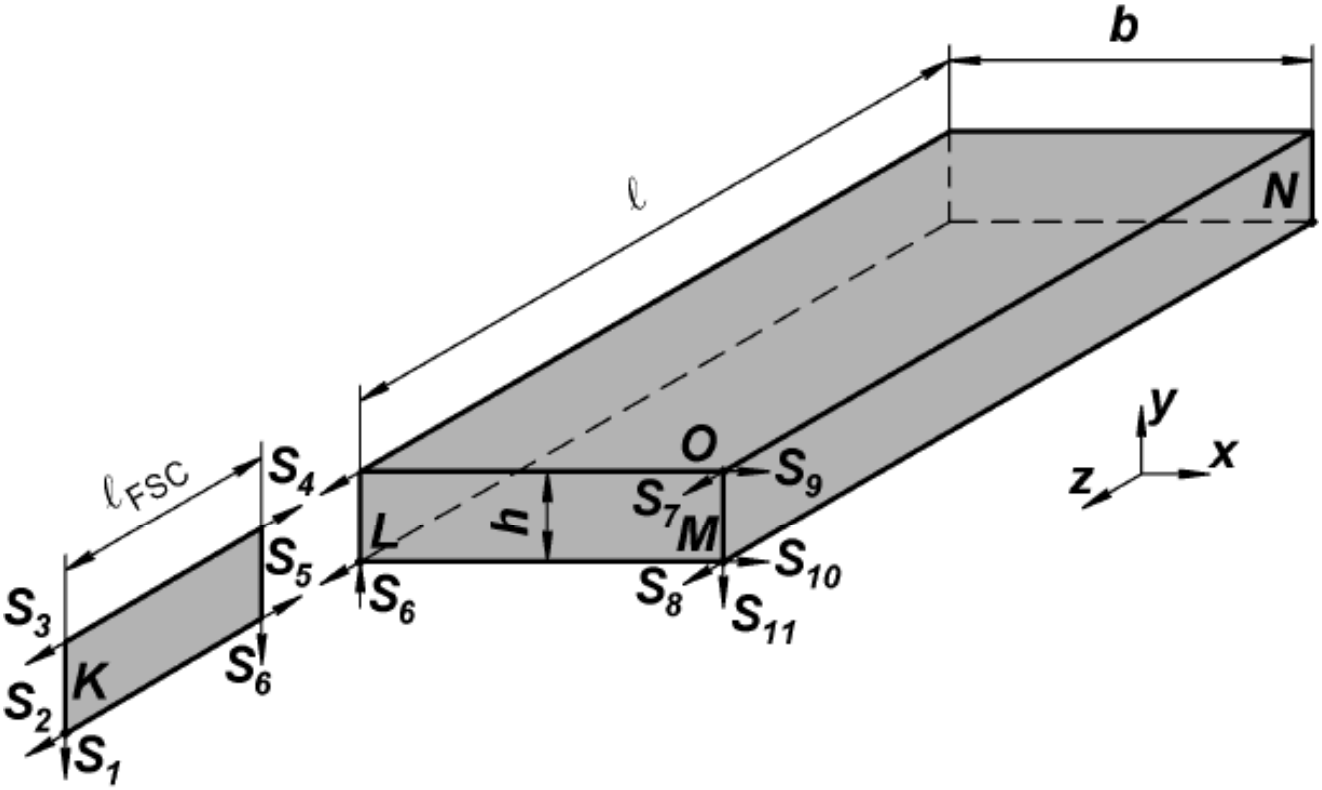


Figure 2.2. Forces in constraints

We have three equilibrium equations for FSC,

$$\begin{aligned} &\sum F_y = 0, \quad S_1 + S_6 = 0; \\ &\sum F_z = 0, \quad S_2 + S_3 - S_4 - S_5 = 0; \\ &\sum M_K = 0, \quad S_3 \cdot h - S_4 \cdot h - S_6 \cdot \ell_{FSC} = 0, \end{aligned} \qquad (2.3)$$

and six equations for the wingbox,

$$
\begin{aligned}
&\sum F_x = 0, \quad && S_9 + S_{10} = 0 ; \\
&\sum F_y = 0, \quad && S_6 - S_{11} = 0 ; \\
&\sum F_z = 0, \quad && S_4 + S_5 + S_7 + S_8 = 0 ; \\
&\sum M_{LM} = 0, \quad && S_4 \cdot h + S_7 \cdot h = 0 ; \\
&\sum M_{MO} = 0, \quad && S_4 \cdot b + S_5 \cdot b = 0 ; \\
&\sum M_{MN} = 0, \quad && S_6 \cdot b + S_9 \cdot h = 0 .
\end{aligned}
\tag{2.4}
$$

Now we have the system of linear equations. The matrix of coefficients for it has $m = 11$ columns and $n = 9$ rows:

$$
\begin{array}{c}
\begin{array}{ccccccccccc}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11}
\end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h & -h & 0 & -\ell_{FSC} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & h & 0 & 0 & h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & h & 0 & 0
\end{pmatrix}
\end{array}
\tag{2.5}
$$

To be geometrically stable, a system must have a matrix which rank is equal to the number of rows or number of columns, whichever is less. Concerning our problem, we must have non-zero determinants for any of 55 possible square matrices which could be obtained by striking out any two columns in (2.5).

In the course project, the statical analysis is used to check the stability of the basic system. In this case, columns which should be crossed out correspond to constraints which are removed to get the basic system, and the only condition required is that the deterimant of the obtained square matrix is non-zero.

3. CHOICE OF BASIC SYSTEM

We have proved that outer wings are attached to the fuselage structure in a statically indeterminate way. The degree of static indeterminacy is two. Thus, we have to choose some method of structural mechanics for statically indeterminate problems.

We can solve the problem by different methods (for example, the one covered in [4]) but we choose the force method because it suits our purpose best. The order of calculation for our problem is typical for any statically indeterminate problem solved by the force method:

1. Choose basic system.
2. Calculate normal forces and shear flows in the outer wing in loaded state (also called F-state). Forces and shear flows in the wingbox are calculated using the elementary and self-balanced state of the wingbox.
3. Calculate bending moments in the root triangle in loaded state.
4. Calculate normal forces, shear flows, and bending moments in the basic system in the first unit state.
5. Calculate normal forces, shear flows, and bending moments in the basic system in the second unit state.
6. Calculate coefficients of equations canonical for the force method; determine forces X_1 and X_2.
7. Draw total diagrams of distribution of normal forces, shear flows, bending moments, and constraint forces.
8. Perform deformation check.

To apply the force method, the chosen basic system must be geometrically stable and statically determinate. To choose it, we have to remove two redundant constraints conforming these conditions. We can remove any 2 of 11 constraints (the total number of possible basic systems is $C_{11}^2 = 55$), but the basic system should be geometrically stable. Some of possible 55 basic systems are not stable.

To prove that the basic system conforms conditions, we have to strike out two columns (those corresponding to removed constraints) in (2.5) and to find its determinant. The determinant must be non-zero.

Fig. 3.1 (a-f) shows examples of basic systems. The points where constraints were removed are written near the figures.

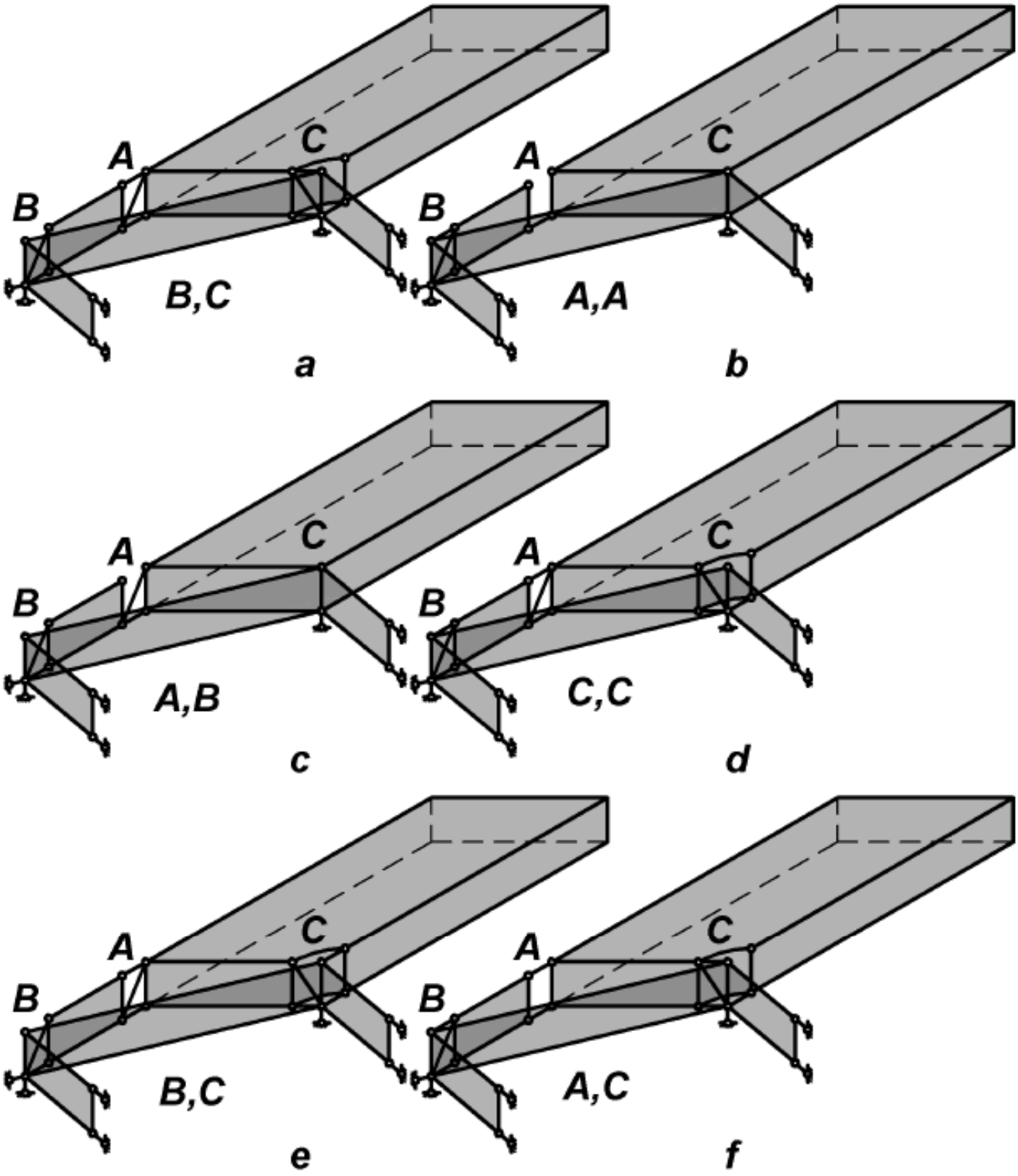


Figure 3.1. Examples of basic system

As an example, we shall consider the basic system with constraints 1 and 3 (at point B) removed (Fig. 3.2).

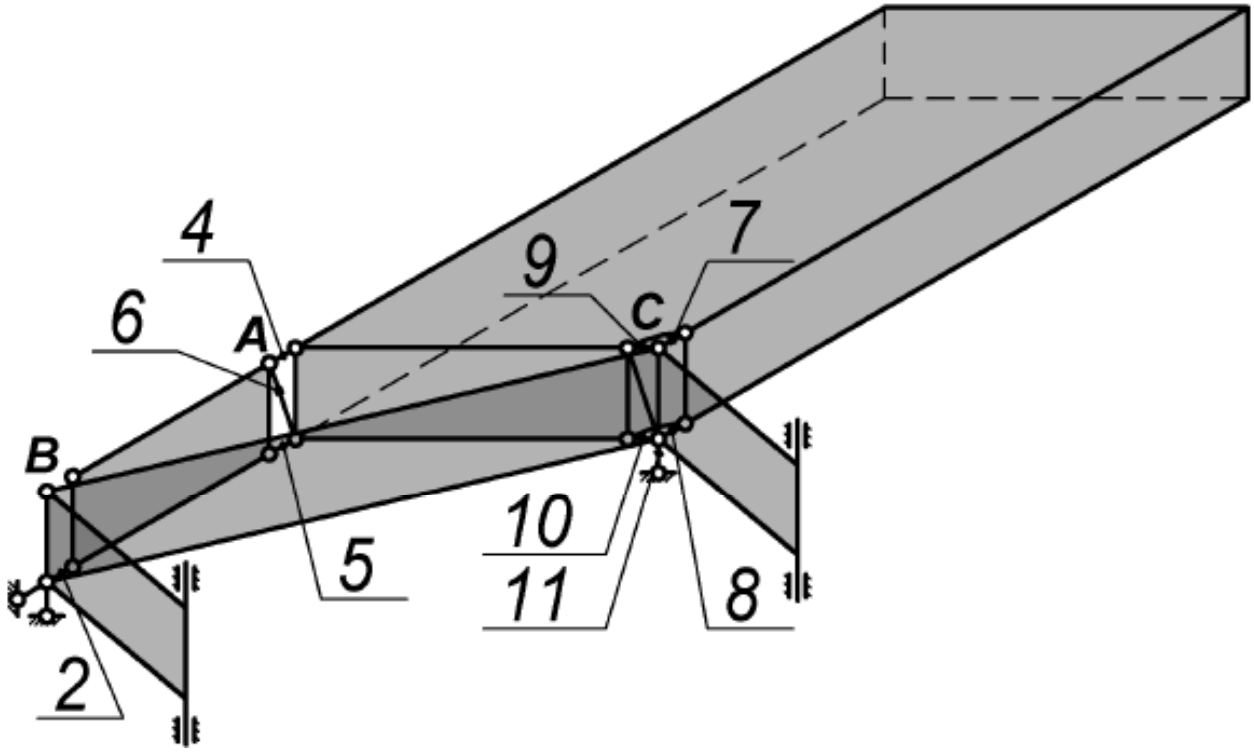


Figure 3.2. Chosen basic system

To prove that the chosen basic system conforms to conditions, we strike out first and third columns in (2.5) and find the determinant of the obtained matrix:

$$\begin{vmatrix} S_2 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 & S_{10} & S_{11} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h & 0 & -\ell_{FSC} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & h & 0 & 0 \end{vmatrix} = b \cdot h^3 \qquad (3.1)$$

The determinant is non-zero, so the chosen basic system is geometrically stable. The system of equilibrium equations has a unique solution and is sufficient to find the forces in constraints for the basic system.

To obtain the equivalent system from the basic system, we apply external load (F) and forces X_1 and X_2 instead of removed constraints. These forces are unknown and have to be calculated. The equivalent system for this basic system is shown in Fig. 3.3.

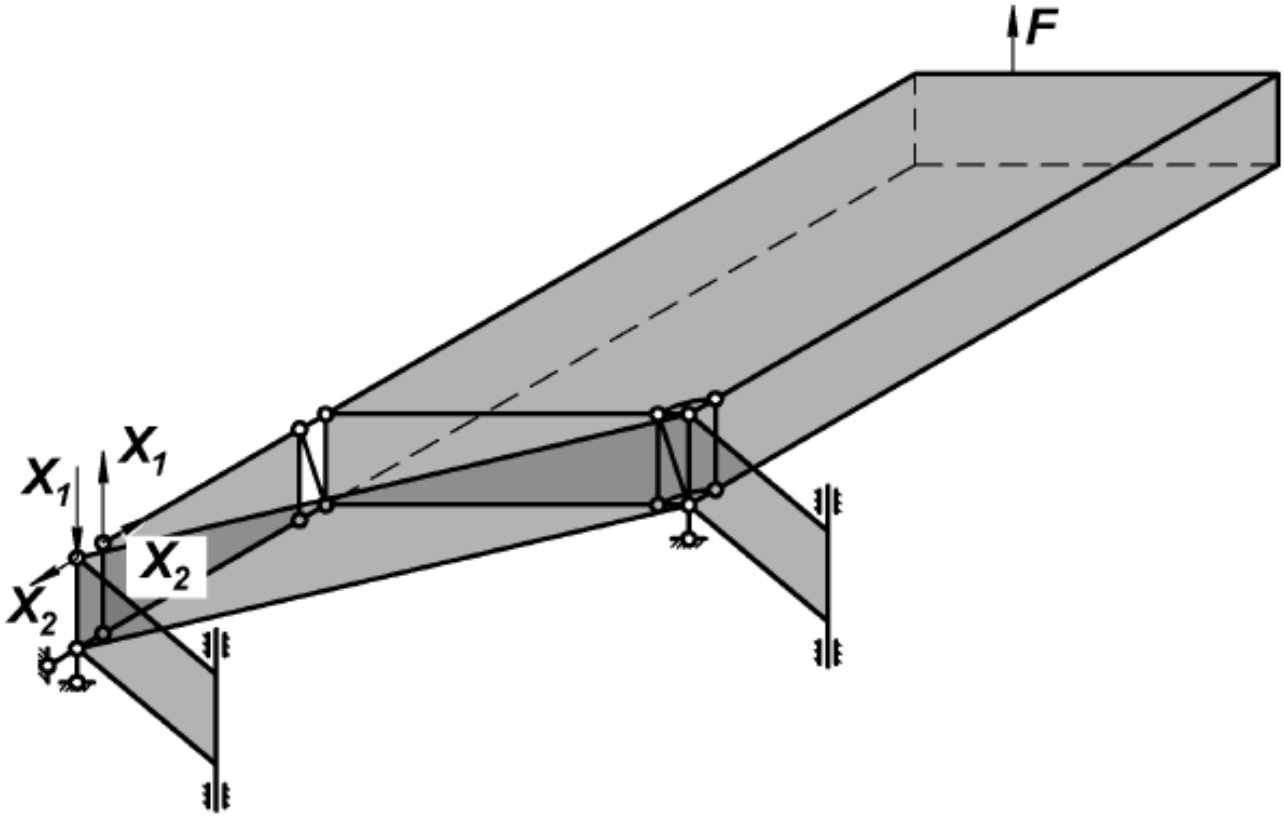


Figure 3.3. Equivalent system

4. BASIC SYSTEM IN LOADED STATE

4.1. *Constraint Forces*

The basic system in loaded state (F-state) is shown in Fig. 4.1,a.

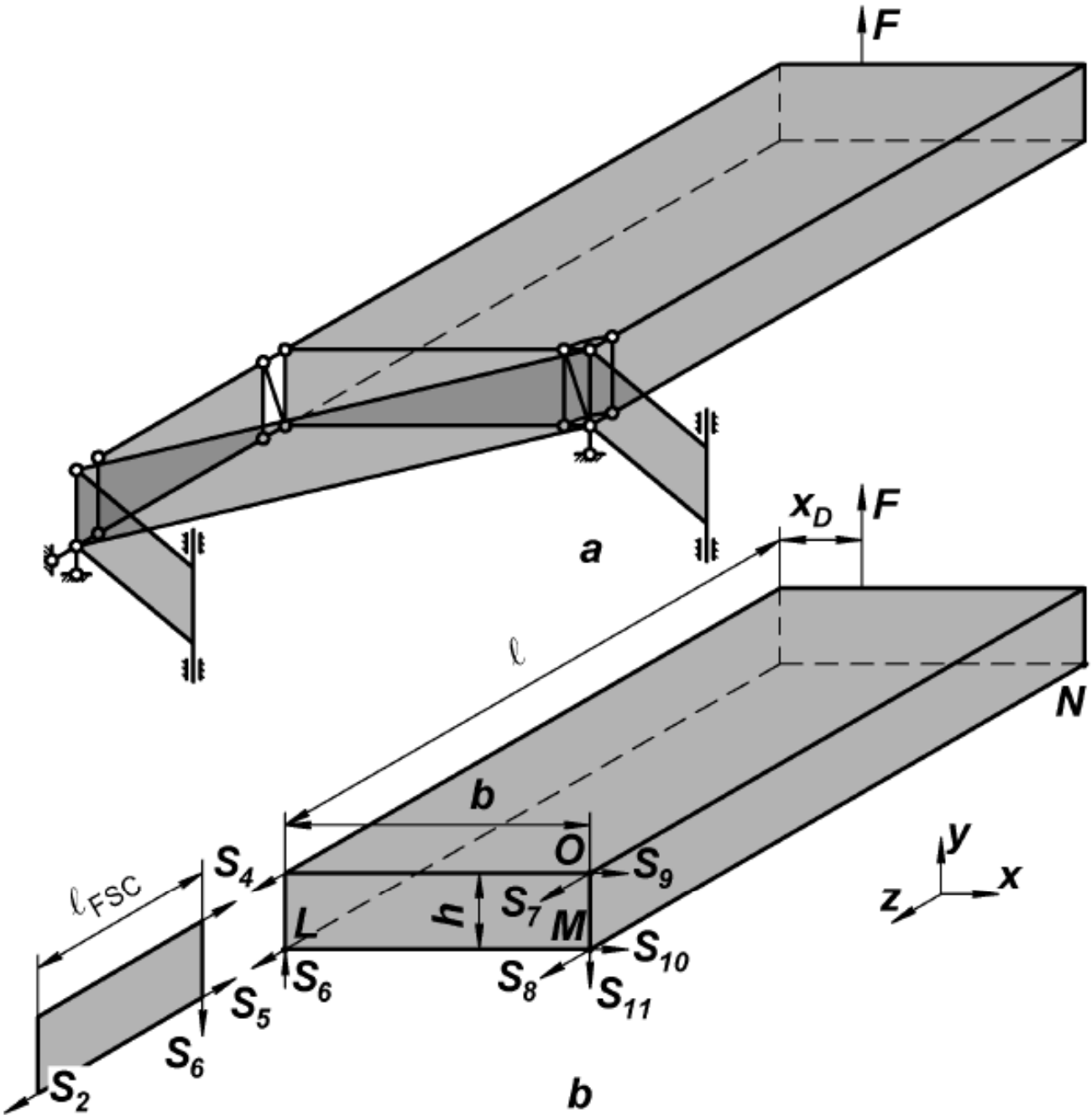


Figure 4.1. Basic system in F-state and constraint forces:
a – basic system and external load; b – constraint forces in outer wing

Let us determine values and directions of forces S_i in constraints. For the purpose, we use equilibrium equations for FSC and wingbox (Fig. 4.1,b). We write them so that to instantly find an unknown value with each following equation. For FSC, first two equations are:

$$\begin{aligned} &\sum F_y = 0, \quad S_6 = 0; \\ &\sum M_L = 0, \quad S_4 \cdot h = 0 \;\Rightarrow\; S_4 = 0. \end{aligned} \tag{4.1}$$

The following four ones are for a wingbox:

$$\begin{aligned} &\sum M_{LM} = 0, \quad S_7 \cdot h + F \cdot \ell = 0 \;\Rightarrow\; S_7 = -\frac{F \cdot \ell}{h}; \\ &\sum F_y = 0, \qquad S_{11} - F = 0 \;\Rightarrow\; S_{11} = F; \\ &\sum M_{MO} = 0, \quad S_5 \cdot b = 0 \;\Rightarrow\; S_5 = 0; \\ &\sum F_z = 0, \qquad S_7 + S_8 = 0 \;\Rightarrow\; S_8 = \frac{F \cdot \ell}{h}. \end{aligned} \tag{4.2}$$

The sum of forces for front spar continuation is

$$\sum F_z = 0, \quad S_2 = 0; \tag{4.3}$$

Finally, we use two remaining equations for the wingbox to get

$$\begin{aligned} &\sum M_{MN} = 0, \quad S_9 \cdot h + F \cdot (b - x_D) = 0 \Rightarrow \\ &\Rightarrow S_9 = -\frac{F \cdot (b - x_D)}{h}; \\ &\sum F_x = 0, \qquad S_9 + S_{10} = 0 \;\Rightarrow S_{10} = \frac{F \cdot (b - x_D)}{h}. \end{aligned} \tag{4.4}$$

Real direction of forces is shown in Fig. 4.2.

Next we have to find normal forces in spar caps and shear flows in skins and spar webs. Actually, we have a two-spar wingbox with a rectangular cross section loaded by given forces. Let us check whether this structure is statically determinate. For this purpose, we cut the wingbox by the plane $z = const$ (Fig. 4.3) and consider equilibrium of the cut portion. Internal force factors acting in the section are forces N_i in spar caps and shear flows q_i in skins and webs. Totally there are eight unknown values, but we have only six independent equilibrium equations. So, the two-spar wingbox with a rectangular

cross section is twice statically indeterminate. This problem can be solved by different methods, but we shall use the Papkovich theorem which was derived specifically for a wingbox. This theorem claims that real stress state of a wingbox can be considered as sum of two states, elementary (EL state, the one that conforms equilibrium equations) and self-balanced (SB state, the one calculated from the condition of minimum complementary potential energy).

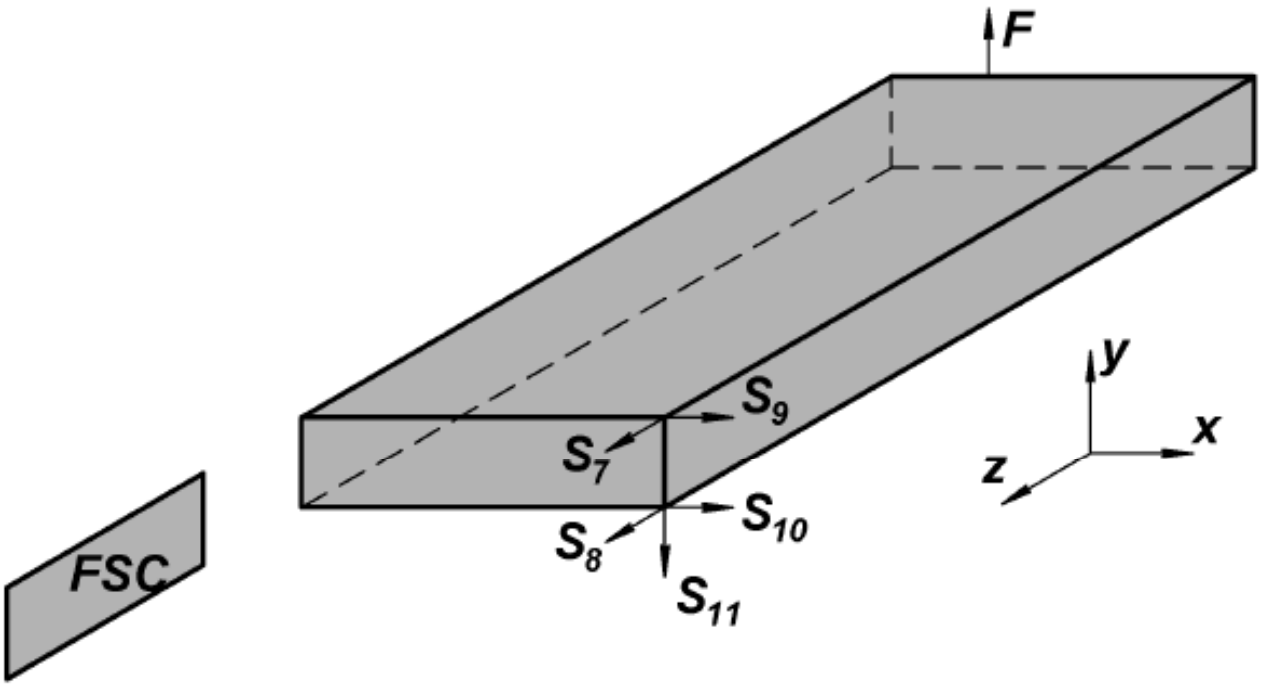


Figure 4.2. Constraint forces in basic system

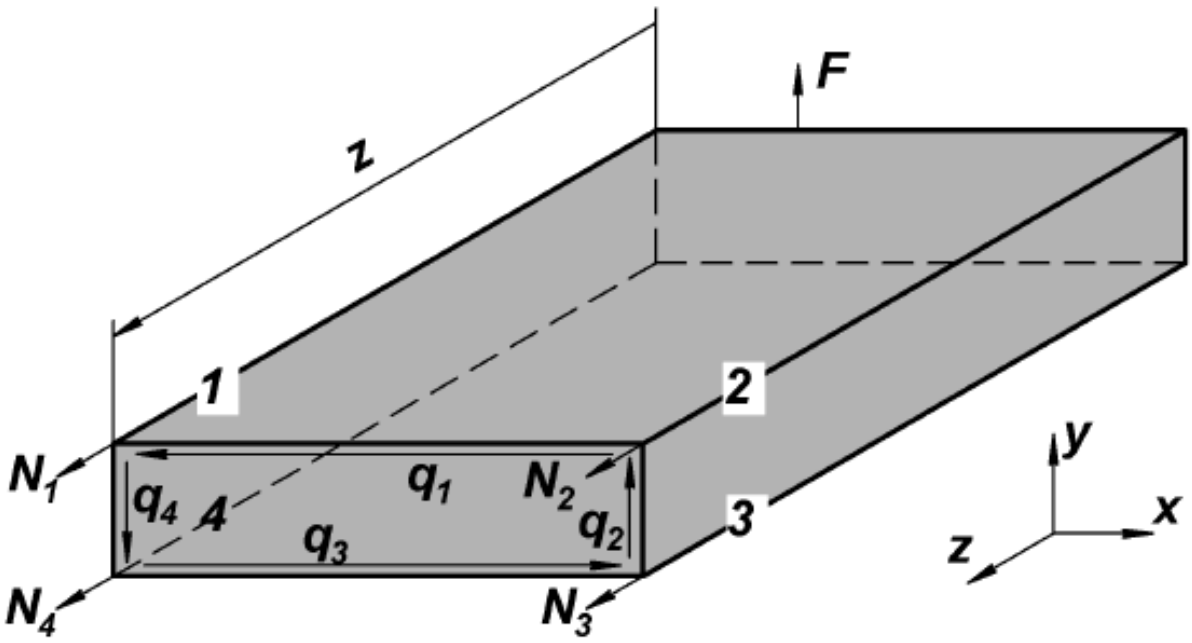


Figure 4.3. Cut portion of wingbox

Let us set the notation of wingbox parts. Spar caps 1 and 4 have their cross section areas, A_1 and A_4, so that $A_1 = A_4 = A_F$. So are spar caps 2 and 3, $A_2 = A_3 = A_R$.

The similar principle works with thicknesses of wingbox skins: $\delta_1 = \delta_3 = \delta_w$ and $\delta_2 = \delta_4 = \delta_p$, respectively.

4.2. *Elementary State*

Elementary state of a wingbox is calculated using the elementary beam theory of bending and free torsion of the thin-walled once-closed bar. Two-spar wingbox is a thin-walled bar with once-closed contour of a rectangular cross section. Skins and spar webs of the wingbox do not carry normal forces, and spar caps carry normal forces only.

Normal Forces in the Caps

Normal stresses in the i-th spar cap $\sigma_{z\,i}^{EL}$ are calculated by the formula

$$\sigma_{z\,i}^{EL} = -\frac{M_x \cdot y_i}{I_x} = -\frac{F \cdot z \cdot y_i}{I_x}, \qquad (4.5)$$

where $M_x = F \cdot z$ is a bending moment in the considered cross section $(0 \le z \le \ell)$;

$y_i = \pm\frac{h}{2}$ are coordinates of the i-th spar cap $\left(i = \overline{1,4}\right)$;

$I_x = \sum_{i=1}^{4} A_i \cdot y_i^2 = \sum_{i=1}^{4} A_i \cdot \left(\frac{h}{2}\right)^2$ is a moment of inertia of the cross section in centroidal principal axes;

A_i is a cross section area of the i-th spar cap $\left(i = \overline{1,4}\right)$.

Let us determine normal forces using (4.5):

$$N_{z\,i}^{EL} = \sigma_{z\,i}^{EL} \cdot A_i = \mp\frac{F \cdot z \cdot h}{2 \cdot I_x} \cdot A_i =$$

$$= \mp \frac{F \cdot z \cdot h \cdot A_i}{2 \cdot \sum_{i=1}^{4} A_i \left(h/2\right)^2} = \mp C \cdot A_i \cdot z \,, \qquad (4.6)$$

where $C = \dfrac{F \cdot h}{2 \cdot \sum_{i=1}^{4} A_i \left(h/2\right)^2}$ is a constant coefficient.

Values of $N_{z\,i}^{EL}$ depend on the z-coordinate. That is why we have to calculate them for several z-values: $z = 0;\ 1;\ 2;\ 3\ m$. It is convenient to write the values of $N_{z\,i}^{EL}$ in the table (Table 4.1).

Table 4.1

Calculation of normal forces in elementary state

z, m	0	1	2	3
N_{z1}^{EL}	0			
N_{z2}^{EL}	0			
N_{z3}^{EL}	0			
N_{z4}^{EL}	0			

Fig. 4.4 shows distribution diagrams of normal forces.

Shear Flows in the Cross Section

Sign convention: We can assume that the positive direction of shear flow is clockwise.

Shear flows in the elementary state can be determined by the formula:

$$q_i^{EL} = q_{p\,i}^{EL} + q_0^{EL}, \quad i = \overline{1,4}\ ; \qquad (4.7)$$

$$q_{p\,i}^{EL} = \frac{F \cdot S_x(s)}{I_x} = \frac{F \cdot S_x(s)}{\sum_{i=1}^{4} A_i \cdot \left(\dfrac{h}{2}\right)^2}, \qquad (4.8)$$

where $q_{p\,i}^{EL}$ are shear flows in the opened contour;

F is a projection of the resultant aerodynamic force on the y -axis (in our case, the projection of the resultant aerodynamic force is equal to the resultant aerodynamic force);

$S_x(s)$ is a current value of the cross section static moment in centroidal principal axes (Fig. 4.5);

q_0^{EL} is a shear flow at the origin of angular coordinate S origin (point 0 in Fig. 4.5) (it is invariable over cross section);

I_x is a cross section moment of inertia in centroidal principal axes.

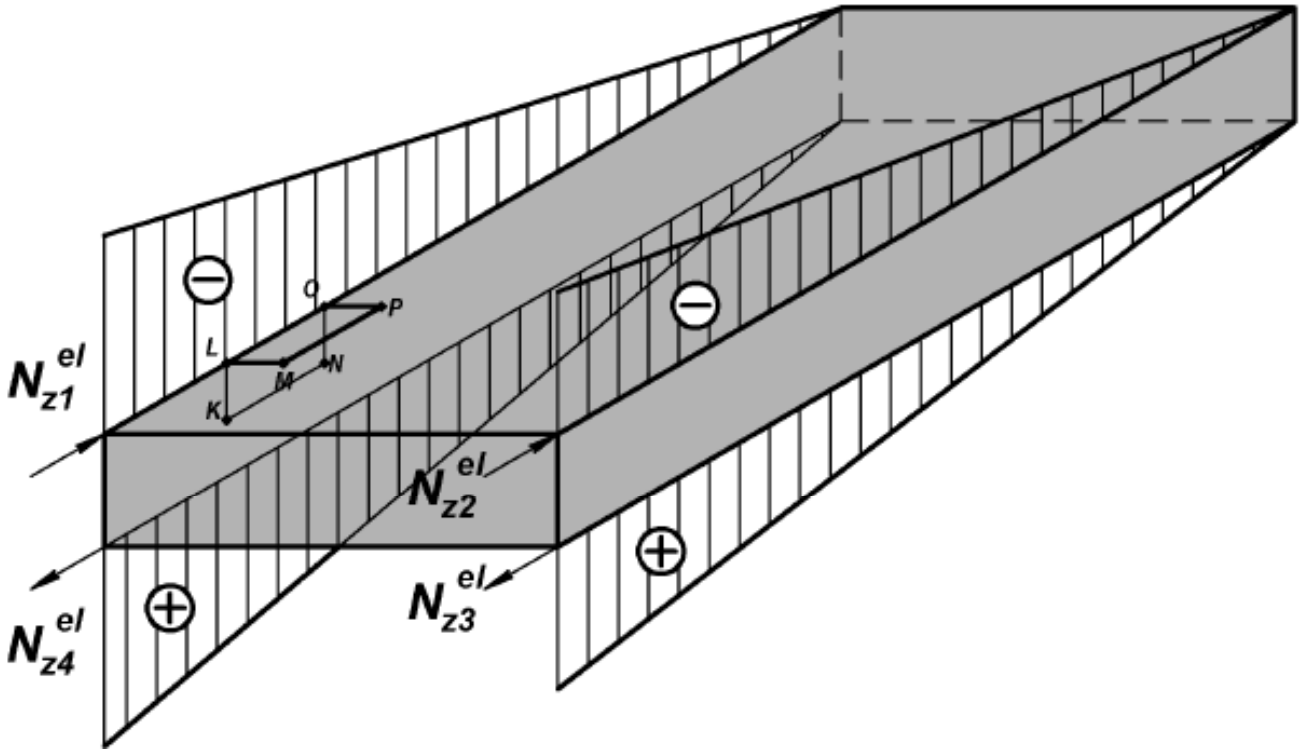


Figure 4.4. Diagrams of normal forces distribution

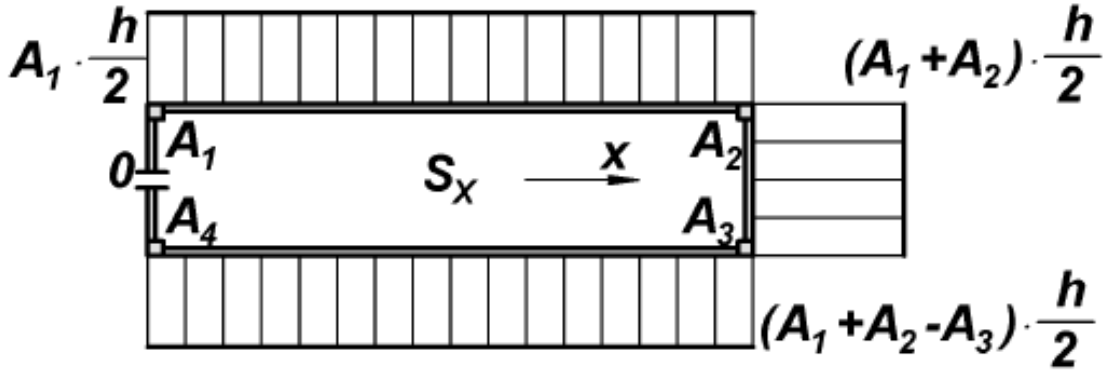


Figure 4.5. Static moment

To calculate elementary shear flows, we have to:

1. Build the diagrams of current values of a cross section static moment $S_x(s)$. To do this, we have first to determine the origin of the angular coordinate s (see Fig. 4.5).
2. Calculate the value of a shear flow $q_{p\,i}^{EL}$ by (4.8) and determine its direction. For this purpose, we have to consider equilibrium of the element ***KLMNOP*** (Fig. 4.6; this element is also shown in Fig. 4.4). This element is a portion of a spar cap with adjacent skin and spar web. The length of this element is dz. The distribution diagram of shear flows $q_{p\,i}^{EL}$ is shown in Fig. 4.7,a.
3. Calculate values of shear flows q_0^{EL}.

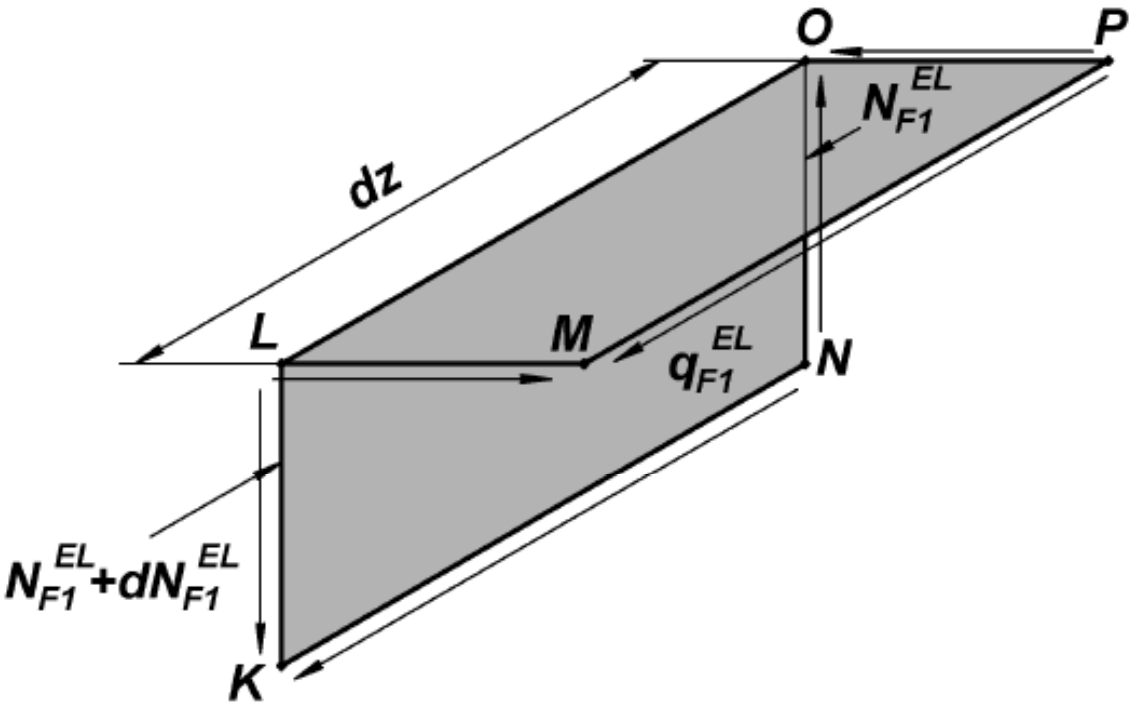


Figure 4.6. Portion of spar cap with adjacent skin and web

We can choose the direction of q_0^{EL} arbitrary, for example, anticlockwise (Fig. 4.7,b). From the equilibrium condition, we get

$$\sum M_3 = 0,\ \int_p q_F^{EL}\cdot\rho\cdot ds - \int_p q_0^{EL}\cdot\rho\cdot ds + F\left(b - x_d\right) = 0\ . \quad (4.9)$$

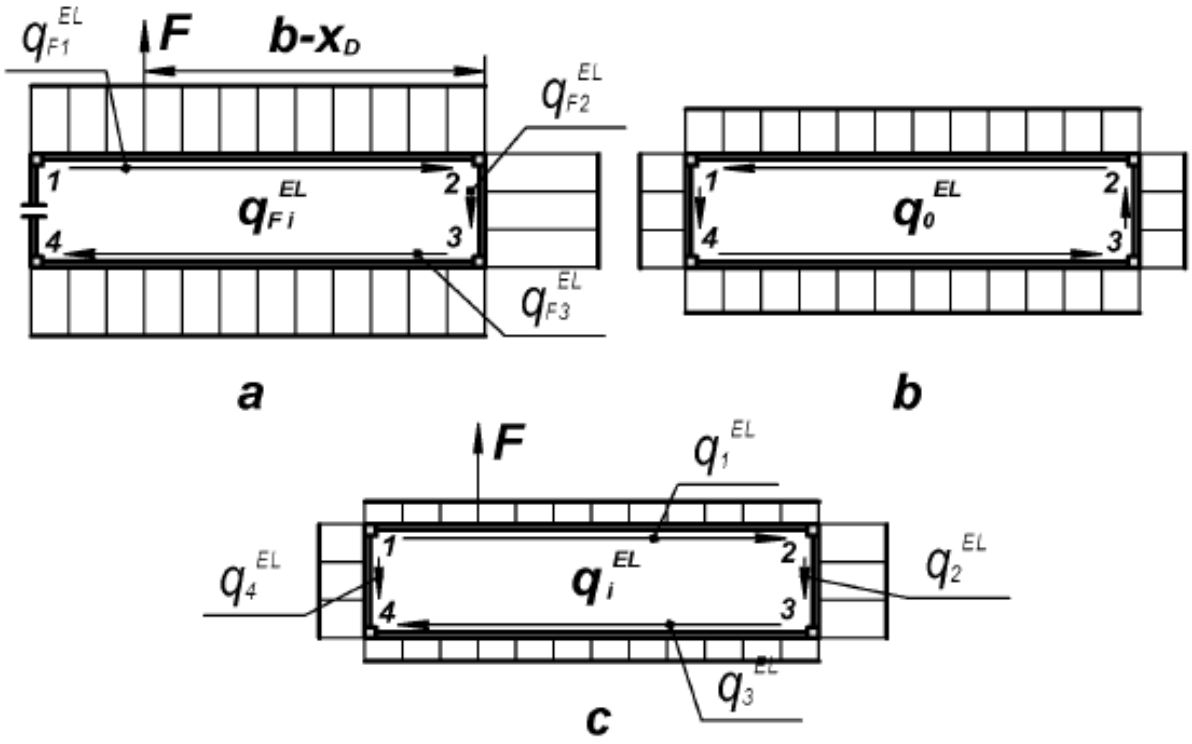


Figure 4.7. Shear flows in elementary state:
a – opened contour; b – closed contour; c – total shear flows

Hence,

$$q_0^{EL} = \frac{q_{F1}^{EL} \cdot b \cdot h + F \cdot (b - x_d)}{2 \cdot b \cdot h}, \qquad (4.10)$$

where p is the perimeter of the contour;

ρ is the perpendicular to the contour element ds.

If $q_0^{EL} < 0$, we must reverse the direction of shear flows q_0^{EL}.

Then we calculate the total shear flows in the elementary state q_i^{EL} using (4.7) and build its distribution diagram (see Fig. 4.7,c).

To check calculations, we can write these equilibrium equations:

$$\begin{aligned} &\sum F_y = 0, \quad F - q_4^{EL} \cdot h - q_2^{EL} \cdot h = 0; \\ &\sum M_1 = 0, \quad F \cdot x_d - q_2^{EL} \cdot h \cdot b - q_3^{EL} \cdot h \cdot b = 0. \end{aligned} \qquad (4.11)$$

The elementary shear flows q_i^{EL} are invariable of coordinate z, because the resultant force F is a constant value along the wingbox.

4.3. *Self-balanced State of Wingbox*

Let us show that for two-spar wingboxs, a self-balanced state exists, for which the resultant vectors of internal forces and moments are zero. Considering the cut portion of wingbox, we write six equilibrium equations (directions of N_i and q_i are chosen arbitrary as shown in Fig. 4.3):

$$\begin{aligned}
&\sum P_z = 0, && N_1 + N_2 + N_3 + N_4 = 0 ; \\
&\sum M_{S-R} = 0, && N_1 \cdot h + N_2 \cdot h = 0 ; \\
&\sum M_{Q-R} = 0, && N_1 \cdot b + N_4 \cdot b = 0 ; \\
&\sum P_x = 0, && -q_1 \cdot b + q_3 \cdot b = 0 ; \\
&\sum M_{S-T} = 0, && q_1 \cdot b \cdot h + q_2 \cdot h \cdot b = 0 ; \\
&\sum P_y = 0, && q_2 \cdot h - q_4 \cdot h = 0 .
\end{aligned} \tag{4.12}$$

Hence, $N_1 = N_3 = -N_2 = -N_4$, or in other words, normal forces in spar caps in self-balanced state are equal by number but opposite by direction:

$$N_{z\,i}^{SB}(z) = (-1)^{i+1} \cdot N_z^{SB}(z) . \tag{4.13}$$

The N_z^{SB} (normal force in spar cap) is a function of z-coordinate.

From (4.12), we obtain that $q_1 = q_3 = -q_2 = -q_4 = q^{SB}(z)$, or, in other words,

$$q_i^{SB}(z) = (-1)^{i+1} \cdot q^{SB}(z) . \tag{4.14}$$

To find the relation between $q^{SB}(z)$ and $N_z^{SB}(z)$, we cut the portion of spar cap with adjacent skin and spar web (Fig. 4.8, this portion is also shown in Fig. 4.4). For this portion, the sum of projections of the forces onto the z-axis must be zero:

$$\sum F_z = 0, \quad dN_z^{SB}(z) - 2 \cdot q^{SB}(z) \cdot dz = 0 . \tag{4.15}$$

Hence,

$$q^{SB}(z) = \frac{1}{2} \cdot \frac{dN_z^{SB}(z)}{dz} . \tag{4.16}$$

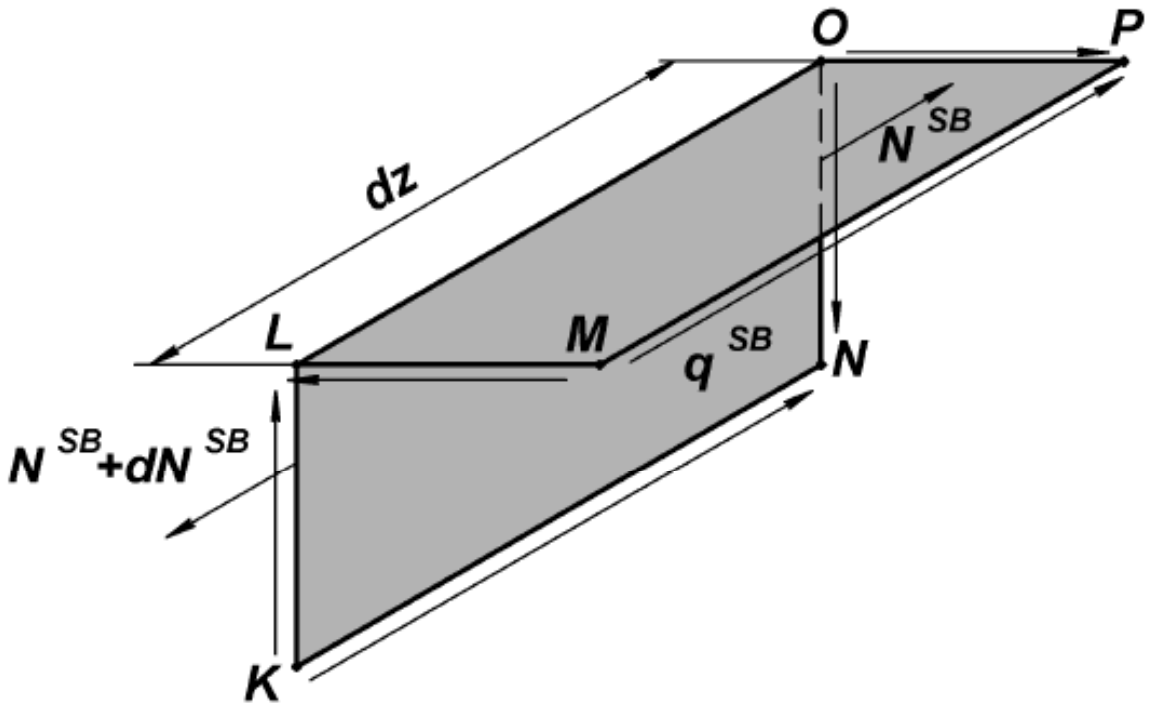


Figure 4.8. Portion of spar cap with adjacent skin and spar web

In the considered structure portion, shear flows $q^{SB}(z)$ that give the projection onto the z-axis are codirectional because the resultant vector of moments must be zero.

Taking into account the signs of $q_i^{SB}(z)$ and $N_{z\,i}^{SB}(z)$, we obtain

$$q_i^{SB}(z) = -\frac{1}{2} \cdot \frac{dN_{z\,i}^{SB}(z)}{dz}, \quad i = \overline{1,4}\,. \tag{4.17}$$

For the first (i=1) spar cap is subjected to tension (force $N_{z\,1}^{SB}$), the self-balanced state of the wingbox is shown in Fig. 4.9.

Forces acting in the wingbox elements in F-state are the sum of forces in elementary and self-balanced state:

$$N_{z\,i}^{(F)}(z) = N_{z\,i}^{EL}(z) + (-1)^{i+1} \cdot N_z^{SB}(z); \tag{4.18}$$

$$q_i^{(F)}(z,s) = q_{p\,i}^{EL} + q_0^{EL} + \frac{(-1)^{i+1}}{2} \cdot \frac{dN_z^{SB}(z)}{dz}\,. \tag{4.19}$$

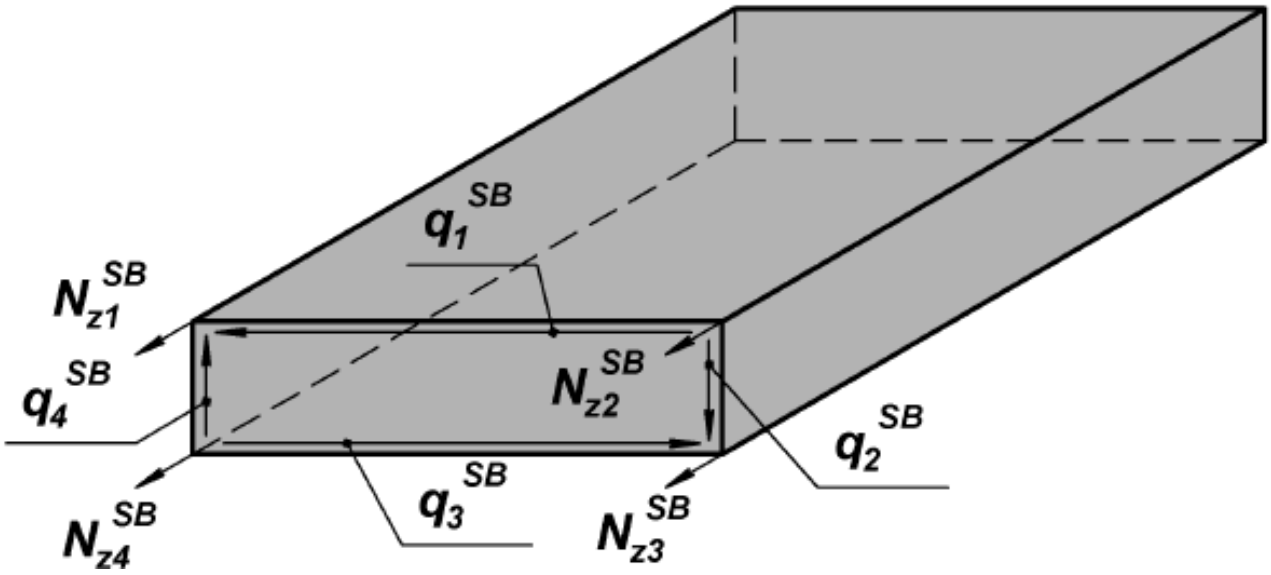


Figure 4.9. Self-balanced state of wingbox

Here, N_z^{SB} is an unknown function. To determine this function, we use the Castigliano variational principle (principle of minimum complementary potential energy). The principle claims that for linearly deformed systems, only real stress state gives the minimum complemenrary potential energy. Let us write the expression for potential energy of the wingbox. Because $\sigma_x = \sigma_y = 0$ and only one shear stress τ_{zs} (shear along the s-axis) is non-zero, we get

$$U = \frac{1}{2} \cdot \int_V \left(\sigma_z \cdot \varepsilon_z + \tau_{zs} \cdot \gamma_{zs} \right) \cdot dV, \tag{4.20}$$

where

$$\begin{gathered} \sigma_{z\,i} = \frac{N_{z\,i}^{(F)}(z)}{A_i}, \qquad \varepsilon_{z\,i} = \frac{\sigma_{z\,i}}{E_i} = \frac{N_{z\,i}^{(F)}(z)}{E_i \cdot A_i}, \\ dV_i = A_i \cdot dz, \\ \tau_{zs\,i} = \frac{q_i^{(F)}(z,s)}{\delta_i}, \qquad \gamma_{zs\,i} = \frac{\tau_{zs\,i}}{G_i} = \frac{q_i^{(F)}(z,s)}{G_i \cdot \delta_i}, \\ dV_i = \delta \cdot S_i \cdot dz. \end{gathered} \tag{4.21}$$

Substituting this into (4.20), we obtain

$$U = \int_0^{\ell} \left(\sum_{i=1}^{4} \frac{\left(N_{z\,i}^{(F)}(z) \right)^2}{2 \cdot E_i \cdot A_i} + \sum_{i=1}^{4} \frac{\left(q_i^{(F)}(z,s) \right)^2 \cdot S_i}{2 \cdot G_i \cdot \delta_i} \right) dz \,, \qquad (4.22)$$

where S_i are equal to the width of the skins and the height of the spar webs ($S_1 = S_3 = b$, $S_2 = S_4 = h$), and ℓ is a length of the wingbox.

The integrand in (4.22) can be written as:

$$\Phi = \sum_{i=1}^{4} \frac{\left(N_{z\,i}^{(F)}(z) \right)^2}{2 \cdot E_i \cdot A_i} + \sum_{i=1}^{4} \frac{\left(q_i^{(F)}(z,s) \right)^2 \cdot S_i}{2 \cdot G_i \cdot \delta_i} =$$

$$= \sum_{i=1}^{4} \frac{\left(N_{z\,i}^{EL}(z) + (-1)^{i+1} \cdot N_z^{SB} \right)^2}{2 \cdot E_i \cdot A_i} +$$

$$+ \sum_{i=1}^{4} \frac{\left(q_i^{EL} + \frac{(-1)^{i+1}}{2} \cdot \frac{dN_z^{SB}}{dz} \right)^2 \cdot S_i}{2 \cdot G_i \cdot \delta_i} \,. \qquad (4.23)$$

The integrand in (4.22) is a function of z-coordinate and of functions N_z^{SB} and $N_z^{SB\prime} = \frac{dN_z^{SB}}{dz}$, so we have a functional

$$U = \int_0^{\ell} \Phi\left(z, N_z^{SB}, N_z^{SB\prime} \right) dz \,. \qquad (4.24)$$

Thus, to find the function N_z^{SB}, we have to minimize the functional Φ. We can write the Euler condition for the minimum of the functional $\Phi\left(z, N_z^{SB}, N_z^{SB\prime} \right)$:

$$\frac{\partial \Phi}{\partial N_z^{SB}} - \frac{d}{dz}\left(\frac{\partial \Phi}{\partial N_z^{SB\prime}} \right) = 0 \,. \qquad (4.25)$$

Let us determine the summands in (4.25) using (4.23):

$$\frac{\partial \Phi}{\partial N_z^{SB}} = \sum_{i=1}^{4} \frac{\left(N_{z\,i}^{EL}(z) + (-1)^{i+1} \cdot N_z^{SB}(z)\right) \cdot (-1)^{i+1}}{E_i \cdot A_i} =$$

$$= (-1)^{2\cdot(i+1)} \cdot N_z^{SB}(z) \cdot \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot N_{z\,i}^{EL}(z)}{E_i \cdot A_i} =$$

$$= N_z^{SB}(z) \cdot A_N + \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot N_{z\,i}^{EL}(z)}{E_i \cdot A_i}, \qquad (4.26)$$

where the numerical coefficient A_N is

$$A_N = \sum_{i=1}^{4} \frac{1}{E_i \cdot A_i}. \qquad (4.27)$$

Next,

$$\frac{\partial \Phi}{\partial N_z^{SB'}} = \sum_{i=1}^{4} \frac{\left(q_i^{EL} + \frac{(-1)^{i+1}}{2} \cdot \frac{dN_z^{SB}}{dz}\right) \cdot S_i \cdot \frac{(-1)^{i+1}}{2}}{G_i \cdot \delta_i} =$$

$$= N_z^{SB'}(z) \cdot \sum_{i=1}^{4} \frac{S_i}{4 \cdot G_i \cdot \delta_i} + \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot q_i^{EL} \cdot S_i}{2 \cdot G_i \cdot \delta_i} =$$

$$= N_z^{SB'}(z) \cdot A_q + \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot q_i^{EL} \cdot S_i}{2 \cdot G_i \cdot \delta_i}, \qquad (4.28)$$

where numerical coefficient A_q is

$$A_q = \sum_{i=1}^{4} \frac{S_i}{4 \cdot G_i \cdot \delta_i}. \qquad (4.29)$$

Finally,

$$\frac{d}{dz}\left(\frac{\partial \Phi}{\partial N_z^{SB'}}\right) = \frac{dN_z^{SB'}(z)}{dz} \cdot A_q + \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot \frac{dq_i^{EL}}{dz} \cdot S_i}{2 \cdot G_i \cdot \delta_i} =$$

$$= N_z^{SB''}(z) \cdot A_q + \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot S_i}{2 \cdot G_i \cdot \delta_i} \cdot \frac{dq_i^{EL}}{dz}. \quad (4.30)$$

Substituting the derivatives (4.26) and (4.30) into the Euler equation (4.25), we obtain

$$N_z^{SB} \cdot A_N - N_z^{SB''} \cdot A_q +$$

$$+ \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot N_{z\,i}^{EL}}{E_i \cdot A_i} - \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot S_i}{2 \cdot G_i \cdot \delta_i} \cdot \frac{dq_i^{EL}}{dz} = 0. \quad (4.31)$$

Thus,

$$N_z^{SB''} - \frac{A_N}{A_q} \cdot N_z^{SB} =$$

$$= \frac{1}{A_q} \cdot \left(\sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot N_{z\,i}^{EL}}{E_i \cdot A_i} - \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot S_i}{2 \cdot G_i \cdot \delta_i} \cdot \frac{dq_i^{EL}}{dz} \right). \quad (4.32)$$

Let us denote

$$\alpha = \sqrt{\frac{A_N}{A_q}} \quad (4.33)$$

and

$$f(z) = \frac{1}{A_q} \cdot \left(\sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot N_{z\,i}^{EL}}{E_i \cdot F_i} - \sum_{i=1}^{4} \frac{(-1)^{i+1} \cdot S_i}{2 \cdot G_i \cdot \delta_i} \cdot \frac{dq_i^{EL}}{dz} \right). \quad (4.34)$$

Substituting this into (4.32), we get

$$N_z^{SB''}(z) - \alpha^2 \cdot N_z^{SB}(z) = f(z). \quad (4.35)$$

The solution of this equation can be written as

$$N_z^{SB}(z) = C_1 \cdot \sinh \alpha z + C_2 \cdot \cosh \alpha z + N_{z\ partial}, \quad (4.36)$$

where C_1, and C_2 are coefficients obtained from the boundary conditions, α is a coefficient calculated from wingbox geometry and material properties data,

$N_{z\ partial}$ is a partial solution dependent on the external load type. The partial solution for the single resultant force loading is zero.

To calculate normal forces $N_{z\,i}^{(F)}$ ($i=\overline{1,4}$) in spar caps, we can use three equilibrium equations: $\sum M_x = 0$, $\sum M_y = 0$, $\sum F_z = 0$. Therefore, the problem is once statically indeterminate. So, there will be only one self-balanced state of the wingbox.

Let us deterimne the coefficients C_1 and C_2 in (4.36) from the boundary conditions for a basic system in F-state (see Fig. 4.1). First, we calculate the coefficients A_N by (4.27), A_q by (4.29), and α by (4.33). In the chosen basic system (see Fig. 4.2), the wingbox is loaded by forces S_7 and S_8 obtained from equilibrium equations. The forces acting in spar caps (4.18) are the sums of forces in elementary and self-balanced states. For the first spar cap, we have

$$N_{z1}^{(F)}(z) = N_{z1}^{EL}(z) + N_{z1}^{SB}(z). \tag{4.37}$$

Let us specify boundary conditions for (4.36) which are applicable for the first spar cap:

$$z=0,\quad N_{z1}^{SB} = C_1\cdot 0 + C_2\cdot 1 = 0 \Rightarrow C_2 = 0;$$

$$z=\ell,\quad N_{z1}^{(F)} = N_{z1}^{EL}\Big|_{z=\ell} + C_1\cdot \sinh\alpha\ell = 0 \Rightarrow$$

$$\Rightarrow C_1 = \frac{N_{z1}^{(F)} - N_{z1}^{EL}}{\sinh\alpha\ell}. \tag{4.38}$$

Then,

$$N_{z\,i}^{SB}(z) = (-1)^{i+1}\cdot C_1\cdot \sinh\alpha z, \tag{4.39}$$

and shear flows will be equal to

$$q_i^{SB}(z) = -\frac{dN_{z\,i}^{SB}(z)}{2\cdot dz} =$$

$$= -\frac{(-1)^{i+1}\cdot\alpha\cdot\left(N_{z1}^{(P)} - N_{z1}^{EL}\right)}{2\cdot\sinh\alpha\ell}\cdot\cosh\alpha z = \tag{4.40}$$

$$= (-1)^{i+1}\cdot B\cdot\cosh\alpha z,$$

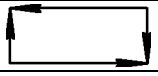
where the numerical coefficient

$$B = -\frac{\alpha \cdot \left(N_{z1}^{(F)} - N_{z1}^{EL} \right)}{2 \cdot \sinh \alpha \ell} . \qquad (4.41)$$

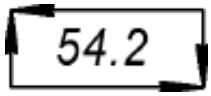
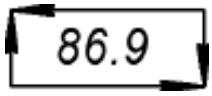
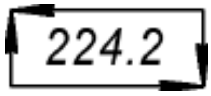
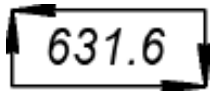
To find the direction of shear flows $q^{SB}(z)$ in skins and spar webs, we cut the portion and consider its equilibrium. To build the distribution diagrams of normal forces and shear flows, we calculate the values of $N_{z\,i}^{SB}$ and $q_{z\,i}^{SB}$ in different sections (for example, at four z-values from 0 to the length of the wingbox). It is convenient to present these calculations as a table (Table 4.2). We also have to put the signs of normal forces and shear flows into this table.

Table 4.2

Calculation of normal forces and shear flows in self-balanced state

z, m	0	1	2	3
αz	0			
$\sinh \alpha z$	0			
$\cosh \alpha z$	1			
N_z^{SB}	0	±		
q_z^{SB}				

It is not convenient to determine directions of shear flows by some signs, because in this case, we have to define the sign rule which could be difficult to imagine and understand. It is more natural to draw the directions of shear flows directly into the table. Students should always include units of variables listed in the table. For example, the filled row for q_z^{SB} could look like:

q_z^{SB}, kN/m				

4.4. *Diagrams in Loaded State*

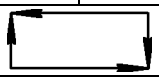
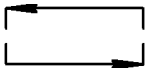
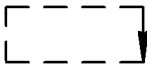
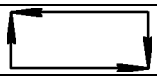
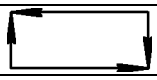
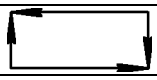
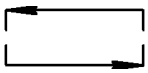
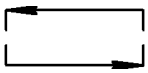
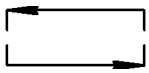
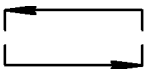
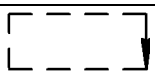
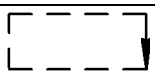
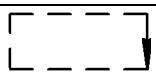
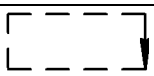
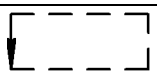
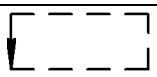
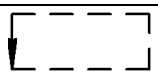
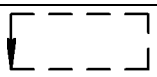
To build distribution diagrams of normal forces and shear flows in the loaded state, we have simply to sum two diagrams for elementary and self-balanced state:

$$N_{zi}^{(F)} = N_{zi}^{EL} + N_{zi}^{SB};$$
$$q_i^{(F)} = q_i^{EL} + q_i^{SB} = q_{pi}^{EL} + q_0^{EL} + q_i^{SB}, \quad i = \overline{1,4}. \tag{4.42}$$

Using values from Tables 4.1 and 4.2, we can determine $N_{z\,i}^{(F)}(z)$ and $q_{z\,i}^{(F)}(z)$ by (4.42) and fill Table 4.3. Total distribution diagrams of normal forces in spar caps $N_{z\,i}^{(F)}(z)$ in F-state are illustrated by Fig. 4.10, and shear flows in webs and skins – by Fig. 4.11.

Table 4.3

Calculation of normal forces and shear flows in F-state

z, m	0	1	2	3
$N_{z1,4}^{EL}$	$\mp$	$\mp$	$\mp$	$\mp$
$N_{z2,3}^{EL}$	$\mp$	$\mp$	$\mp$	$\mp$
N_z^{SB}	0	$\pm$	$\pm$	$\pm$
$N_{z1}^{(F)}; N_{z4}^{(F)}$	0	$\mp$	$\mp$	$\mp$
$N_{z2}^{(F)}; N_{z3}^{(F)}$	0	$\mp$	$\mp$	$\mp$
q_0^{EL}				
$q_{F\,1,3}^{EL}$				
$q_{F\,2}^{EL}$				
$q_{F\,4}^{EL}$				
q^{SB}				
$q_{1,3}^{(F)}$				
$q_2^{(F)}$				
$q_4^{(F)}$				

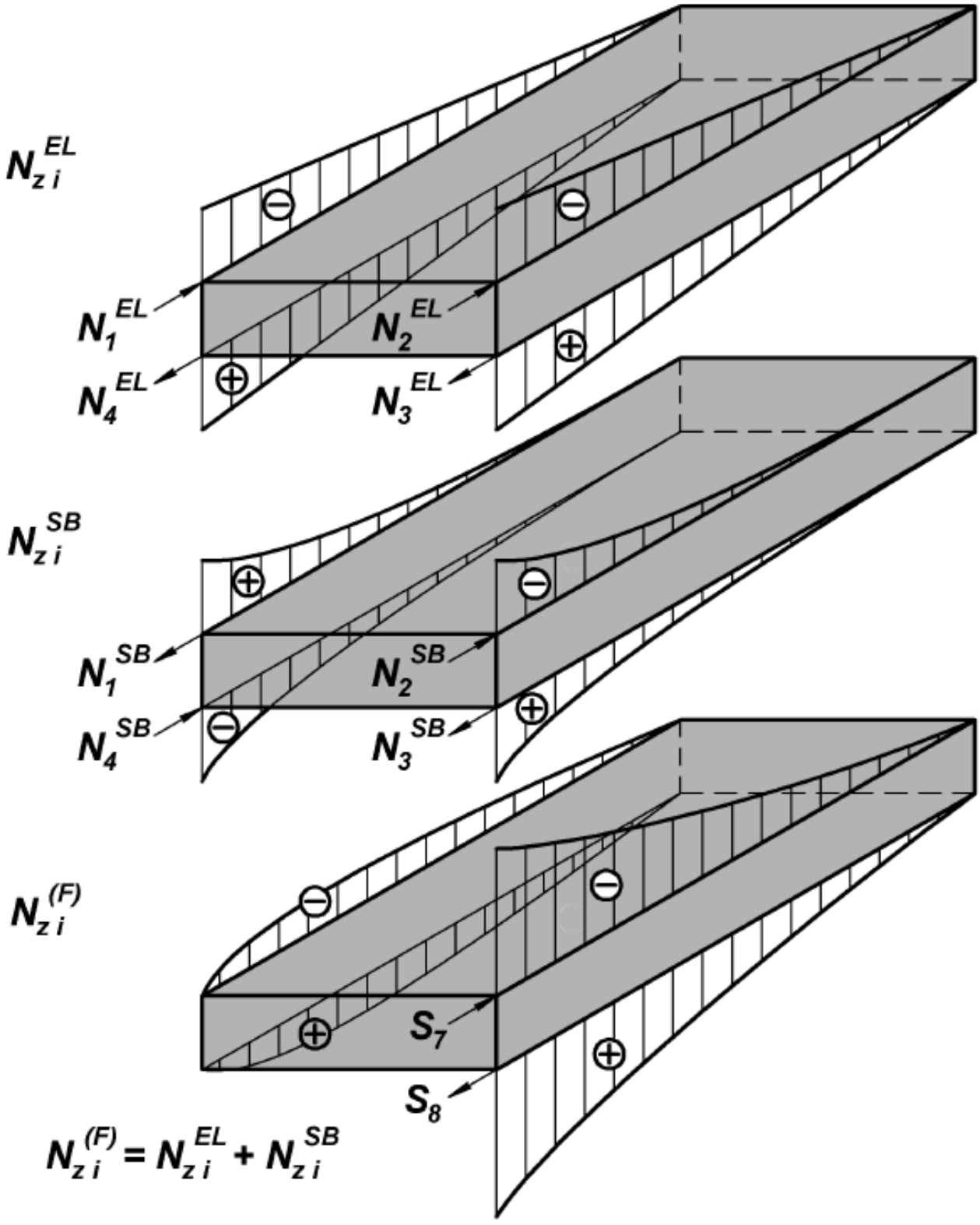


Figure 4.10. Normal forces in spar caps in F-state, in elementary state and in self-balanced state

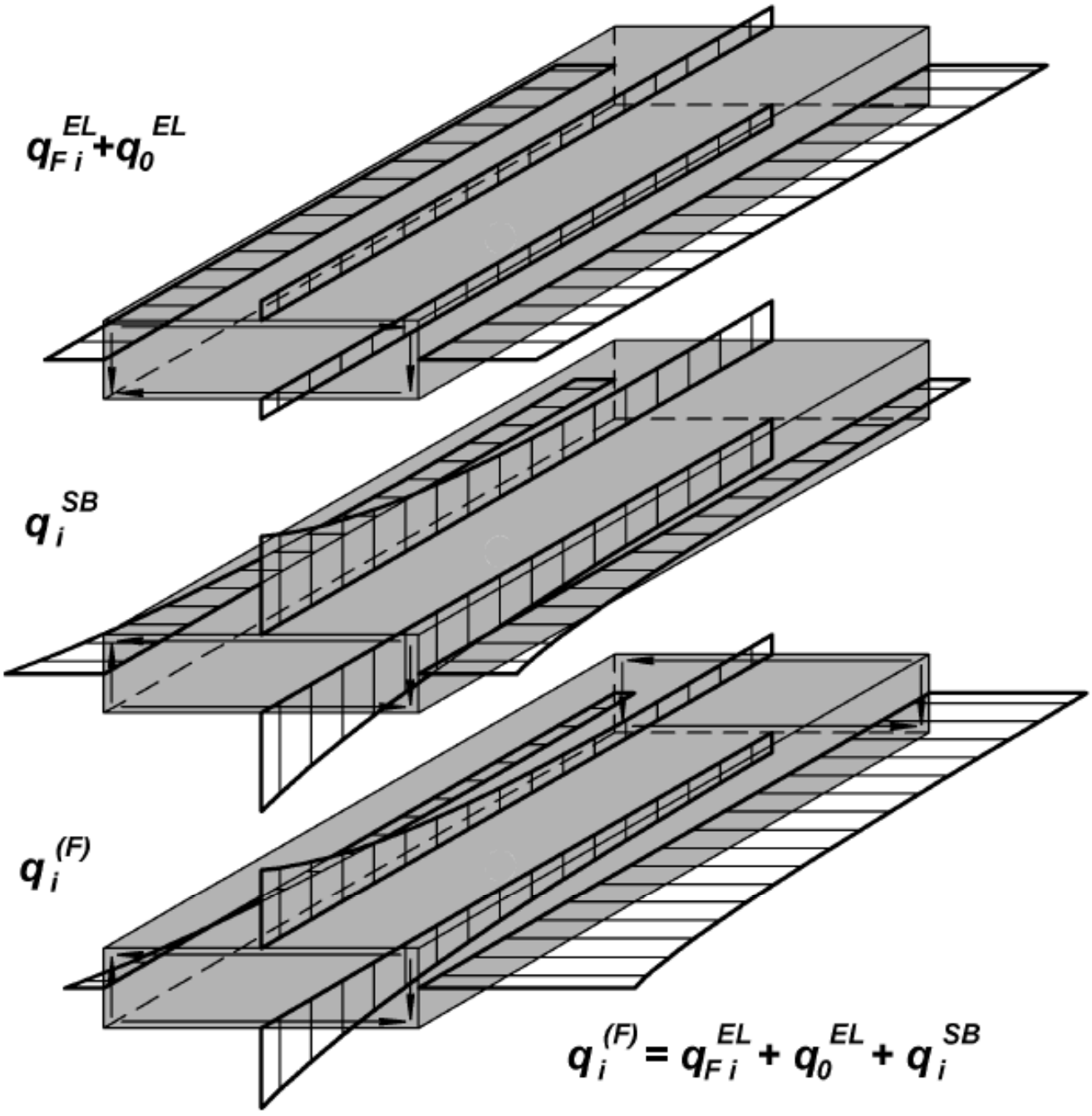


Figure 4.11. Shear flows in skins and spar webs in elementary state, in self-balanced state and in F-state

4.5. *Loaded State of Root Triangle*

Finding the distribution of bending moments along root triangle beams, we have to consider moments that are transferred from the outer wing structure to the root triangle structure. Conventionally, moments are shown in figures as vectors perpendicular to the plane in which couple of forces acts. The law of signs for moments can be formulated as *when we look from the end of the vector, we see the anticlockwise direction of the force couple*.

For the chosen basic system (see Fig. 4.1,a), we have forces S_7, S_8, S_9, S_{10} and S_{11} at point C (see Fig. 4.2). The root triangle structure is loaded by the same forces but with opposite direction. Fig. 4.12 shows the top view of the wingbox, the front spar continuation, and the root triangle structure. Forces S_7 and S_8 give a moment $M_1 = S_7 \cdot h$, and forces S_9 and S_{10} give a moment $M_2 = S_9 \cdot h$. These moments load the rear fuselage beam, the fuselage rib, and the root rib at point C.

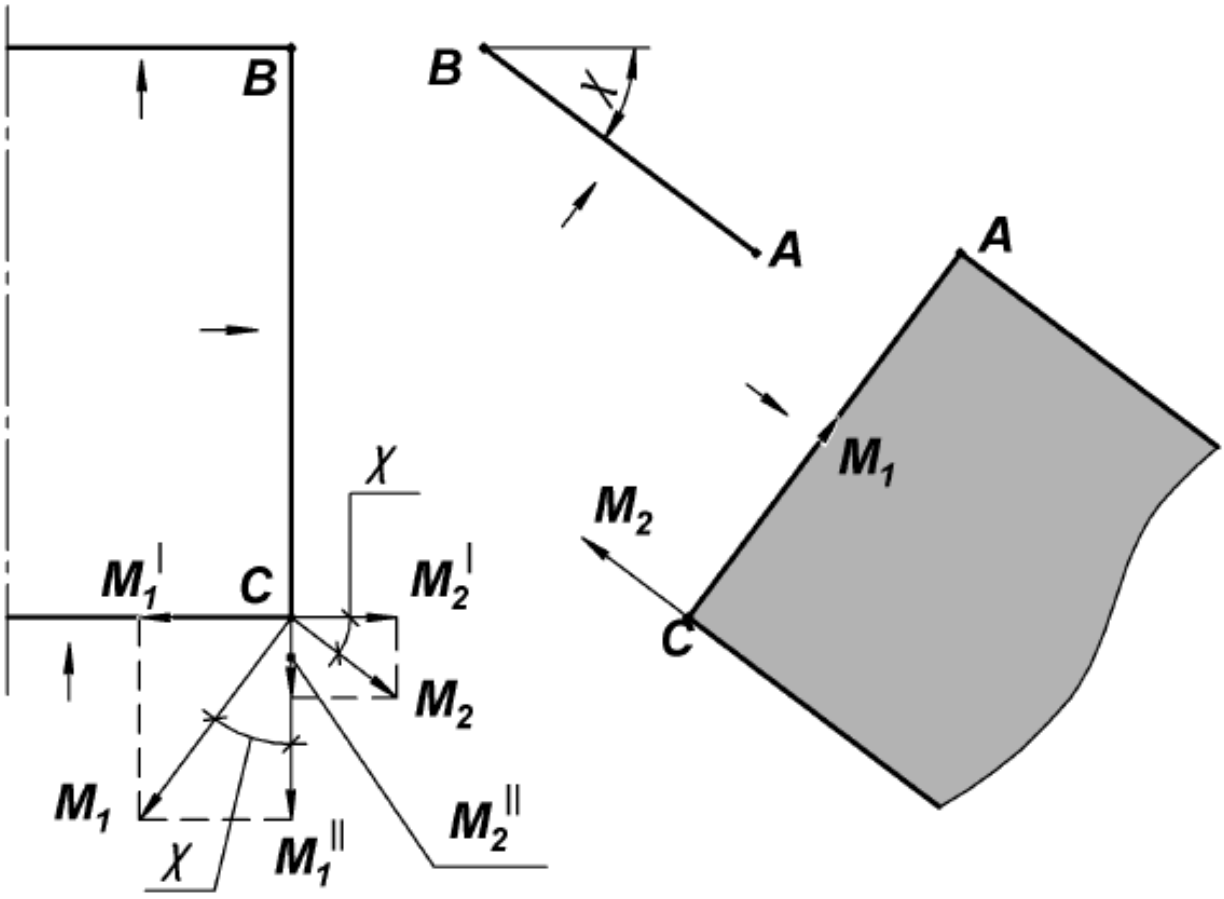


Figure 4.12. Loading of root triangle structure

In the chosen basic system, the front spar continuation and the front fuselage beam are not loaded (because corresponding constraint forces are zero).

Moments M_1 and M_2 at point C can be resolved into projections, $M_1' = M_1 \cdot \sin\chi$, $M_1'' = M_1 \cdot \cos\chi$ and $M_2' = M_2 \cdot \cos\chi$, $M_2'' = M_2 \cdot \sin\chi$ correspondingly. The beam does not carry forces that act outside its plane, or in other words, the beam carries only moments whose vectors are perpendicular to the beam plane. Thus, the moments M_1' and M_2' load only fuselage rib, and the moments M_1'' and M_2'' load only rear fuselage beam.

Let us build distribution diagrams of bending moments in root triangle beams in F-state. Directions from which we look at beams when we draw diagrams are shown in Fig. 4.12 (small arrows).

The root rib is loaded by shear flows $q_i^{(F)}$ (see Fig. 4.11) and known forces S_9, S_{10}, S_{11} (see Fig. 4.2). The equation of the root rib moment ($\sum M_a$, see point a in Fig. 4.13) is:

$$\begin{aligned} &M_{RR}(x) = -q_4^{(F)} \cdot h \cdot x + q_1^{(F)} \cdot x \cdot h\ ; \\ &M_{RR}(0) = 0\ ; \\ &M_{RR}(b) = \left(q_1^{(F)} - q_4^{(F)}\right) \cdot h \cdot b = M_2\ . \end{aligned} \qquad (4.43)$$

The bending moment diagram is shown in Fig. 4.13.

Front spar continuation and front fuselage beam are not loaded, so the bending moments in them are zero (see Fig. 4.13).

Fuselage rib is loaded at point C by the difference of moments $M_1' = M_1 \cdot \sin\chi$ and $M_2' = M_2 \cdot \cos\chi$. It is also loaded by the reaction force at point B, but to build the diagram we do not have to calculate it: we just have to know that the diagram of bending moment produced by the concentrated force is linear. The fuselage rib is fixed at point B with a single hinge, so the moment at this point is zero. So, the fuselage rib moment equation is linear: $M_{FR} = \left(M_1' - M_2'\right) \cdot \frac{x}{b}$. The diagram is shown in Fig. 4.13.

Rear fuselage beam is loaded at point C by the sum of moments $M_1'' = M_1 \cdot \cos\chi$ and $M_2'' = M_2 \cdot \sin\chi$. The RFB is fixed at the plane of

symmetry in such a way that there are no vertical constraints and no vertical forces, consequently. Thus, the bending moment in it is constant. The diagram is shown in Fig. 4.13.

So, we have found all normal forces and shear flows in the wingbox as well as bending moments in the root triangle structure in the F-state. The following step is to find these internal force factors in two unit states.

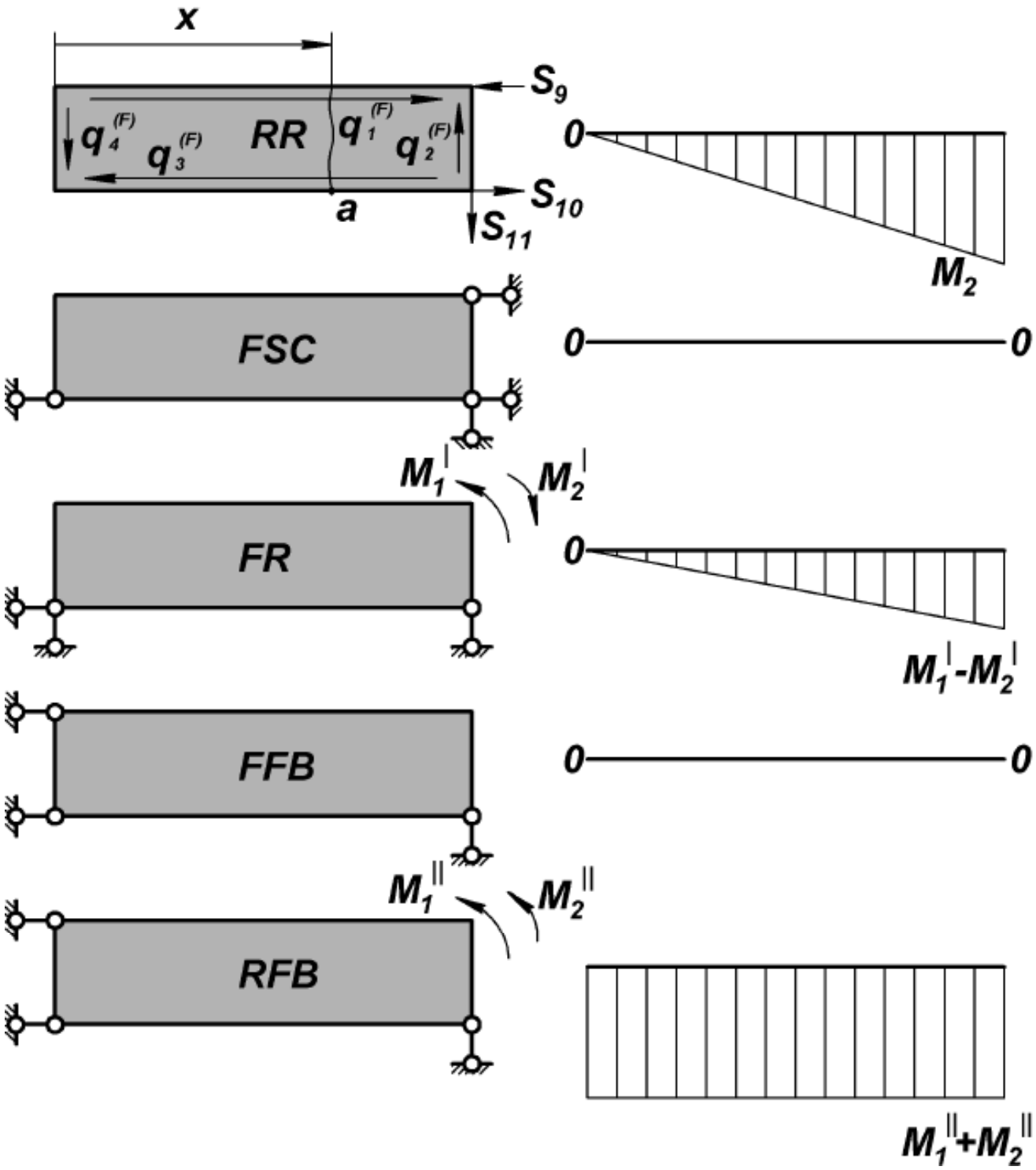


Figure 4.13. Bending moment diagrams in root triangle beams in F-state

5. FIRST UNIT STATE OF BASIC SYSTEM

In the first unit state, we apply the unit force $\overline{X}_1 = 1$ instead of the vertical constraint $\#1$ (see Fig. 2.1). Fig. 5.1 shows the basic system in the first unit state.

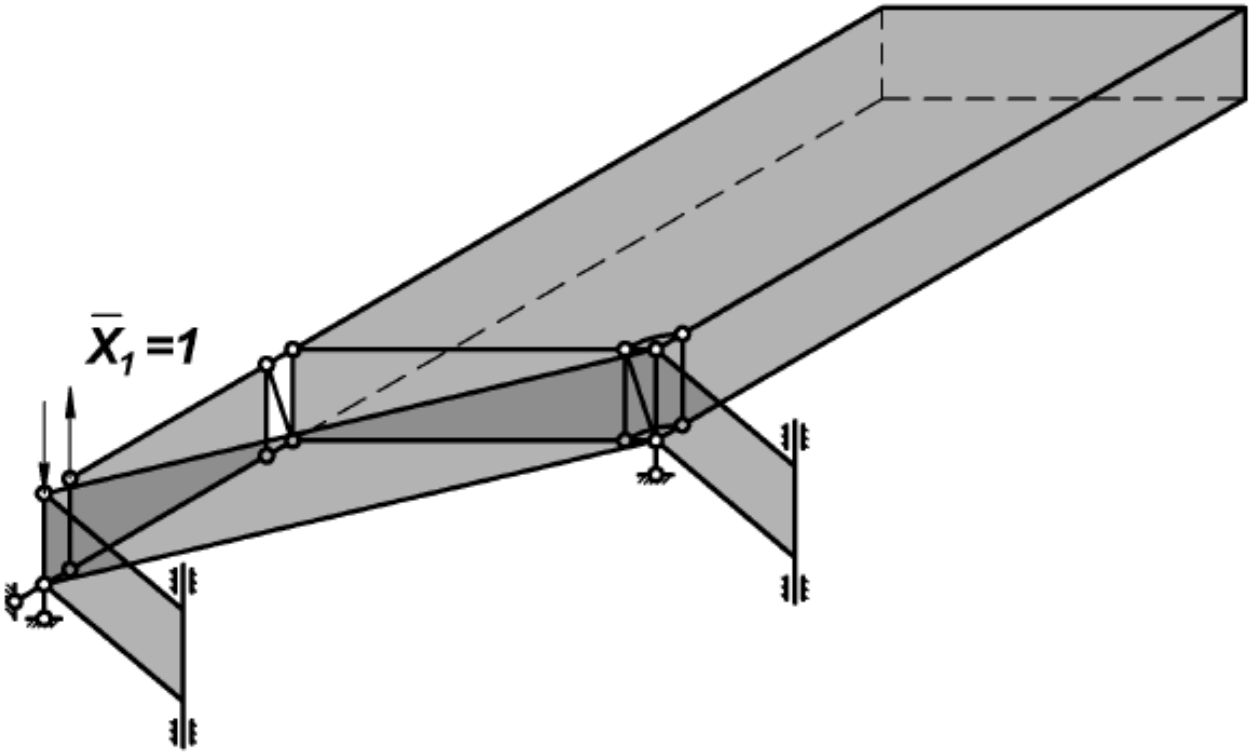


Figure 5.1. Basic system in first unit state

To find the values of constraint forces $\overline{S}_i^{(1)} = 1$, we have to write equilibrium equations for the front spar continuation and the wingbox. Solving these equations, we get

$$\begin{gathered}
\overline{S}_6^{(1)} = \overline{S}_{11}^{(1)} = 1; \\
\overline{S}_4^{(1)} = \overline{S}_5^{(1)} = \overline{S}_7^{(1)} = \overline{S}_8^{(1)} = \frac{\ell_{FSC}}{h}; \\
\overline{S}_2^{(1)} = 0; \\
\overline{S}_9^{(1)} = \overline{S}_{10}^{(1)} = \frac{b}{h}.
\end{gathered} \tag{5.1}$$

Directions of constraint forces applied to the wingbox and FSC are shown in Fig. 5.2.

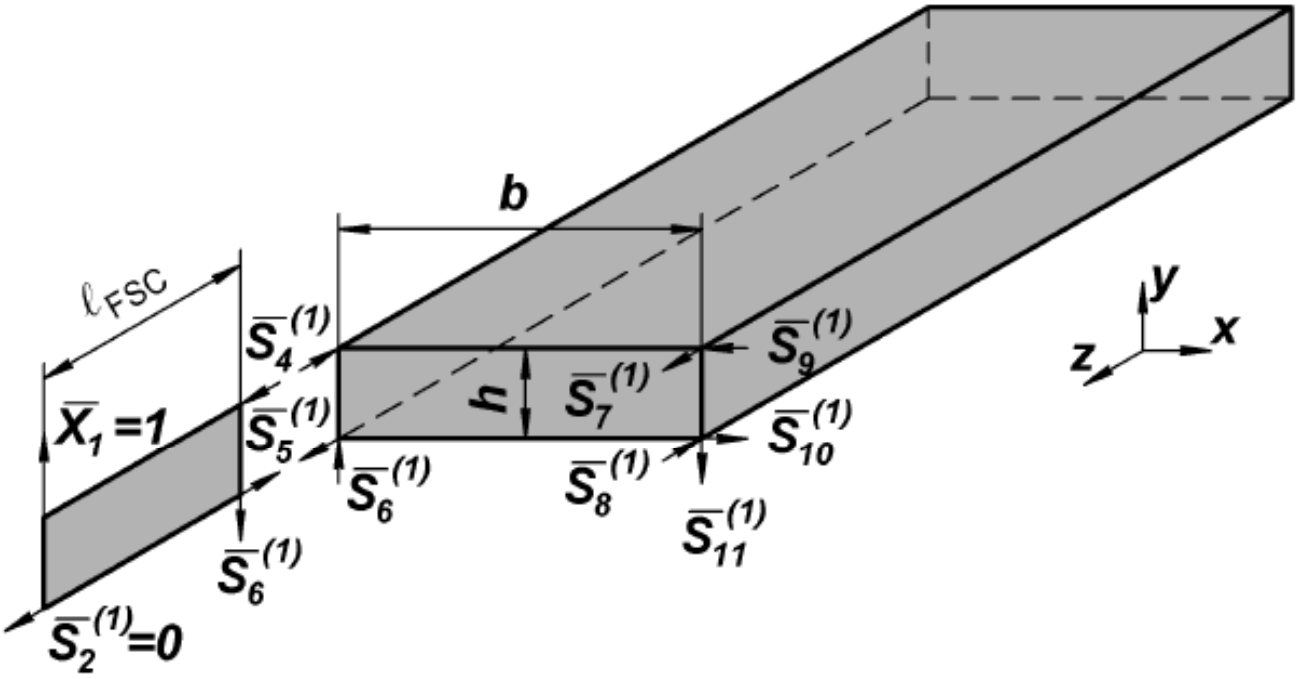


Figure 5.2. Constraint and external forces in first unit state

For some basic systems, it can appear that the wingbox is not loaded at all (root rib and other beams are, however, loaded). In this case, normal forces and shear flows are zero, and we shall consider bending moments in the root triangle only.

In case of a loaded wingbox (see Fig. 5.2), loading forces are self-balanced (sum of forces is equal to zero). Thus, the stress state of the wingbox can be considered self-balanced, so no elementary forces and shear flows exist. Hence, forces and shear flows in the wingbox are equal to self-balanced ones. Normal forces can be found as (4.36):

$$\bar{N}_z^{(1)}(z) = \bar{C}_1^{(1)} \cdot \sinh \alpha z + \bar{C}_2^{(1)} \cdot \cosh \alpha z \,. \tag{5.2}$$

Constants $\bar{C}_1^{(1)}$ and $\bar{C}_2^{(1)}$ are found from boundary conditions:

$$z = 0, \quad \bar{N}_{zi}^{(1)} = \bar{C}_2^{(1)} = 0 \,;$$

$$z = \ell, \quad \bar{N}_{z1}^{(1)} = \bar{C}_1^{(1)} \cdot \sinh \alpha \ell = -\bar{S}_4^{(1)} \;\Rightarrow$$

$$\Rightarrow \quad \bar{C}_1^{(1)} = -\frac{\bar{S}_4^{(1)}}{\sinh \alpha \ell} \,. \tag{5.3}$$

Substituting this into (5.2), we get

$$\overline{N}_{zi}^{(1)}(z) = (-1)^{i+1} \cdot \overline{C}_1^{(1)} \cdot \sinh \alpha z = \frac{(-1)^i \cdot \overline{S}_4^{(1)}}{\sinh \alpha \ell} \cdot \sinh \alpha z \,. \quad (5.4)$$

Shear flows are:

$$\overline{q}_i^{(1)}(z) = \frac{1}{2} \cdot \frac{d\overline{N}_{zi}^{(1)}(z)}{dz} =$$

$$= -\frac{(-1)^{i+1} \cdot \alpha \cdot \overline{S}_4^{(1)}}{2 \cdot \sinh \alpha \ell} \cdot \cosh \alpha z = (-1)^{i+1} \cdot \overline{B}^{(1)} \cdot \cosh \alpha z \,. \quad (5.5)$$

where $\overline{B}^{(1)} = -\frac{\alpha \cdot \overline{S}_4^{(1)}}{2 \cdot \sinh \alpha \ell}$.

Direction of shear flows $\overline{q}_i^{(1)}(z)$ can be found by way of considering equilibrium of a small portion of the spar cap with adjacent skin and web, as it was done in the F-state (see Fig. 4.6).

To build distribution diagrams of normal forces and shear flows, we have to find values of $\overline{N}_{zi}^{(1)}(z)$ and $\overline{q}_i^{(1)}(z)$ in different sections (at four z-values from 0 to the length of the wingbox). It is convenient to present these calculations as a table (see Table 5.1).

Table 5.1

Calculation of normal forces and shear flows in first unit state

z, m	0	1	2	3
αz	0			
$\sinh \alpha z$	0			
$\cosh \alpha z$	1			
$\overline{N}_z^{(1)}$	0			
$\overline{q}^{(1)}$	[redacted]	[redacted]	[redacted]	[redacted]

Fig. 5.3 shows distribution diagrams of normal forces and shear flows in the wingbox in the first unit state.

Forces from the wingbox are transferred to the root triangle structure. Forces $\overline{S}_7^{(1)}$ and $\overline{S}_8^{(1)}$ give the moment $\overline{M}_1^{(1)} = \overline{S}_7^{(1)} \cdot h$, and forces $\overline{S}_4^{(1)}$ and

$\overline{S}_5^{(1)}$ give the moment $\overline{M}_3^{(1)} = \overline{S}_4^{(1)} \cdot h$ ($\overline{M}_3^{(1)}$ is equal to $\overline{M}_1^{(1)}$ but has the opposite direction). Forces $\overline{S}_9^{(1)}$ and $\overline{S}_{10}^{(1)}$ give the moment $\overline{M}_2^{(1)} = \overline{S}_9^{(1)} \cdot h$ (Fig. 5.4).

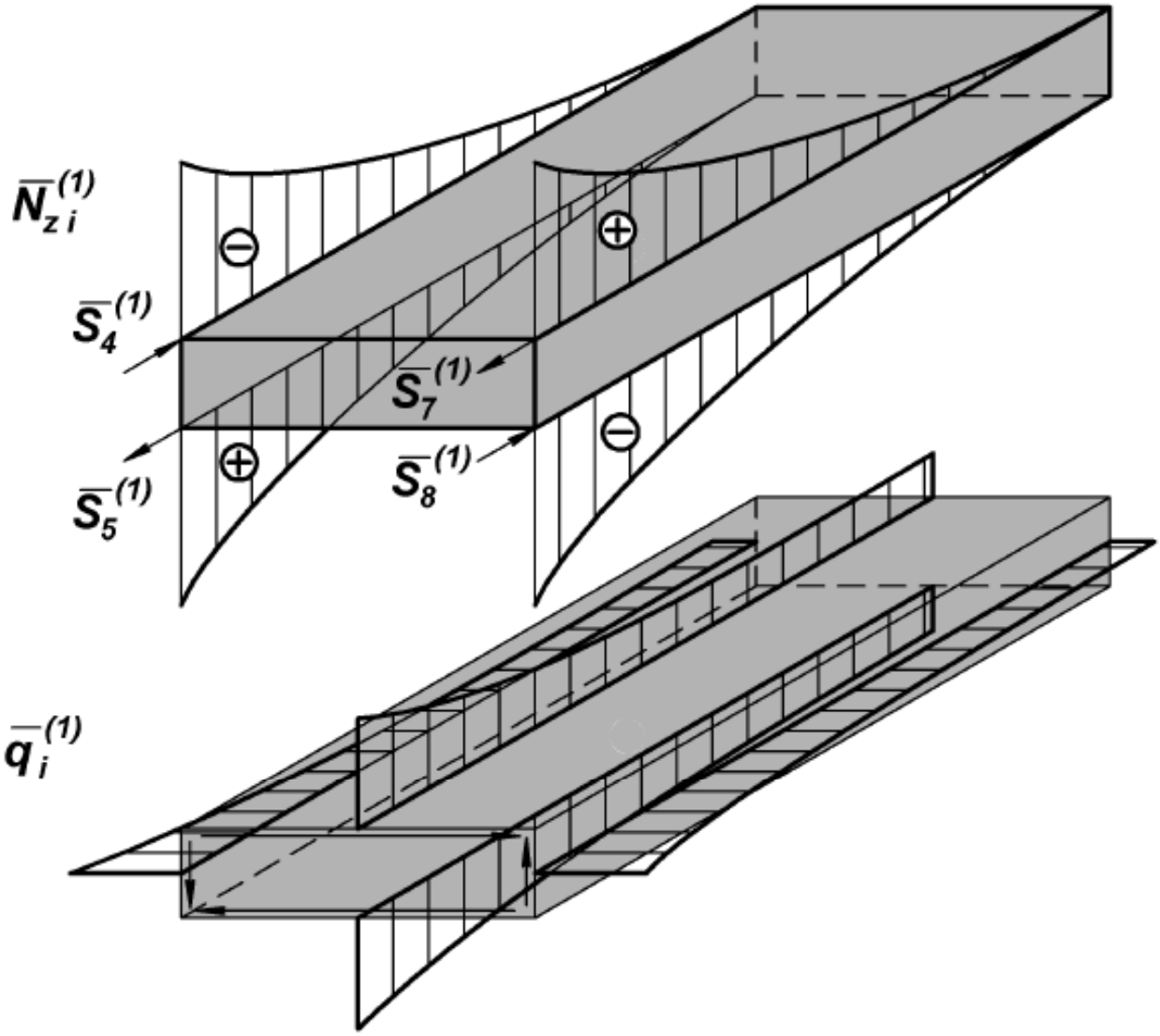


Figure 5.3. Diagrams of distribution of normal forces and shear flows in first unit state

In Fig. 5.4, moments $\overline{M}_1^{(1)}$ and $\overline{M}_2^{(1)}$ can be resolved into projections as it was done in the F-state. After that we can build distribution diagrams of bending moments in root triangle beams and in fuselage beams.

The root rib is loaded by shear flows $\overline{q}_i^{(1)}$ and forces $\overline{S}_6^{(1)}$, $\overline{S}_9^{(1)}$, $\overline{S}_{10}^{(1)}$ and $\overline{S}_{11}^{(1)}$ (Fig. 5.5). The root rib moment equation ($\sum M_a$, see point a in Fig. 5.5) is:

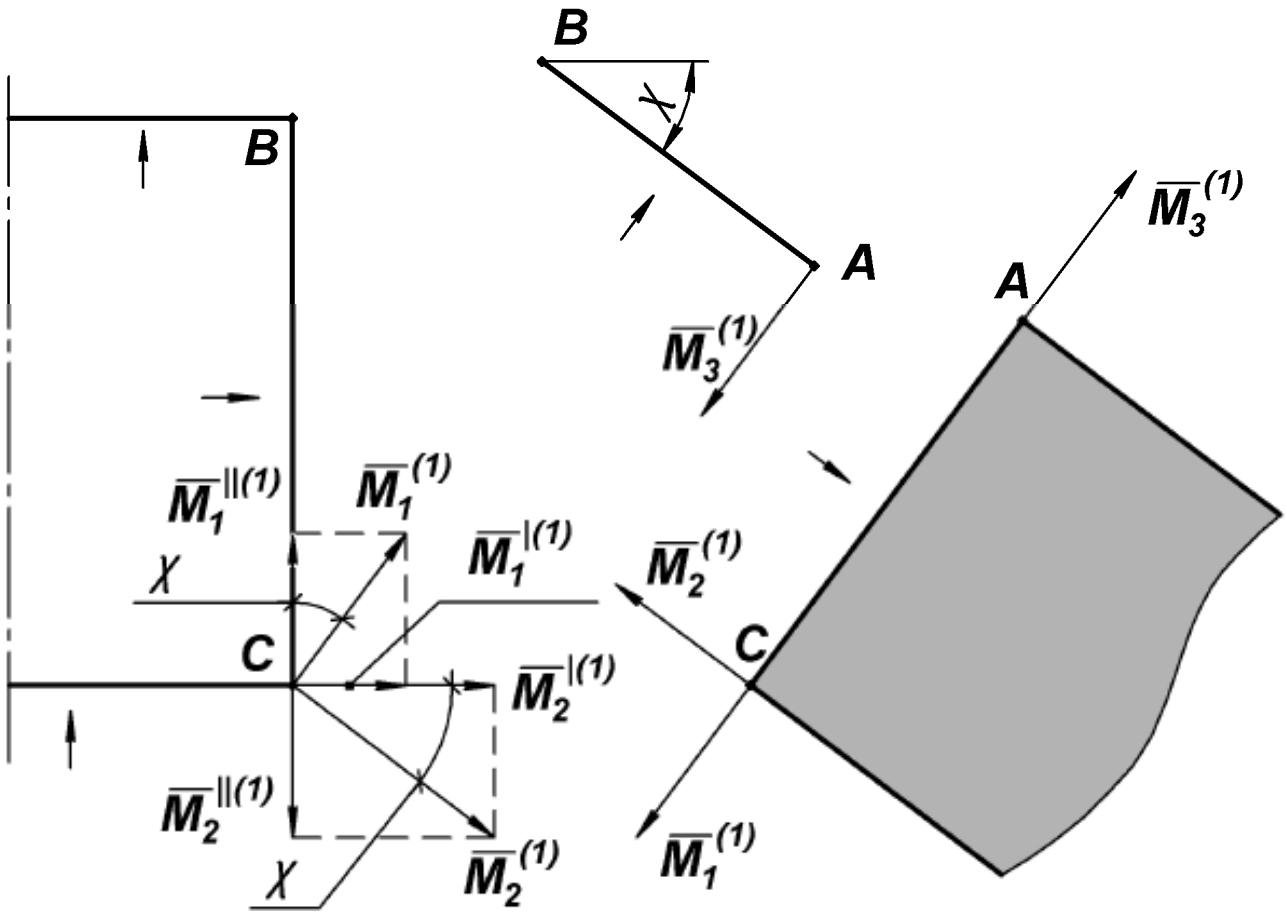


Figure 5.4. Loading of root triangle structure in first unit state

$$\overline{M}_{RR}^{(1)}(x) = \overline{q}_4^{(1)} \cdot h \cdot x - \overline{q}_1^{(1)} \cdot x \cdot h + \overline{S}_6^{(1)} \cdot x \ ;$$

$$\overline{M}_{RR}^{(1)}(0) = 0 \ ; \qquad (5.6)$$

$$\overline{M}_{RR}^{(1)}(b) = \overline{S}_6^{(1)} \cdot b = \overline{M}_1^{(1)}.$$

The bending moment diagram is linear (see Fig. 5.5).

Front spar continuation is loaded by the force $\overline{X}_1 = 1$ (see Fig. 5.2) at the left end, so the FSC moment equation ($\sum M_a$, see point *a* in Fig. 5.5) is:

$$\overline{M}_{FSC}^{(1)}(x) = \overline{X}_1 \cdot x \ ;$$

$$\overline{M}_{FSC}^{(1)}(0) = 0 \ ; \qquad (5.7)$$

$$\overline{M}_{FSC}^{(1)}(\ell_{FSC}) = \overline{M}_3^{(1)}.$$

The diagram is shown in Fig. 5.5.

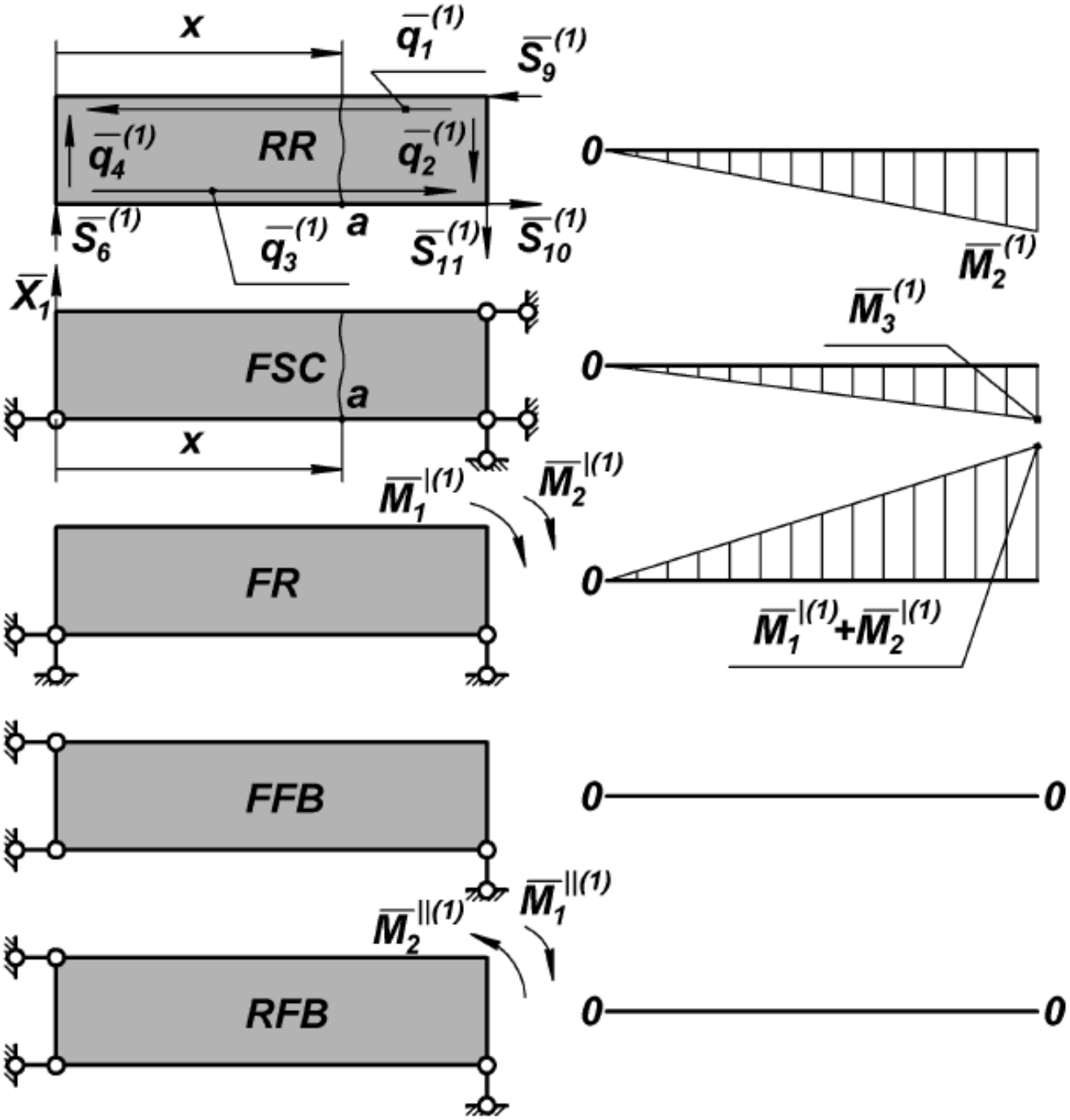


Figure 5.5. Diagrams of distribution of bending moments in first unit state

Fuselage rib is loaded by the sum of moments, $\bar{M}_1^{\prime(1)} = \bar{M}_1^{(1)} \cdot \sin\chi$ and $\bar{M}_2^{\prime(1)} = \bar{M}_2^{(1)} \cdot \cos\chi$, at the right end. The FR fixing at the left end does not carry moments, so the value of the bending moment at the left end is zero. The diagram is linear (see Fig. 5.5).

Front fuselage beam is not loaded, so the bending moment diagram is zero (see Fig. 5.5).

Rear fuselage beam is loaded by the sum of two moments, $\overline{\overline{M}}_1^{\prime\prime(1)}$ and $\overline{\overline{M}}_2^{\prime\prime(1)}$, at the left end (these moments have opposite directions). Let us show that they are equal by value:

$$\begin{aligned}\overline{\overline{M}}_1^{\prime\prime(1)} &= \overline{\overline{M}}_1^{(1)} \cdot \cos\chi = \overline{S}_7^{(1)} \cdot h \cdot \cos\chi = \\ &= \frac{\ell_{FSC}}{h} \cdot h \cdot \frac{b}{\ell_{FR}} = \frac{b \cdot \ell_{FSC}}{\ell_{FR}}; \\ \overline{\overline{M}}_2^{\prime\prime(1)} &= \overline{\overline{M}}_2^{(1)} \cdot \sin\chi = \overline{S}_9^{(1)} \cdot h \cdot \sin\chi = \\ &= \frac{b}{h} \cdot h \cdot \frac{\ell_{FSC}}{\ell_{FR}} = \frac{b \cdot \ell_{FSC}}{\ell_{FR}}.\end{aligned} \tag{5.8}$$

Here ℓ_{FR} is the length of the fuselage rib. Hence, the sum of the moments $\overline{\overline{M}}_1^{\prime\prime(1)}$ and $\overline{\overline{M}}_2^{\prime\prime(1)}$ is zero, so RFB is not loaded and the bending moment diagram is zero (see Fig. 5.5).

6. SECOND UNIT STATE OF BASIC SYSTEM

In the second unit state, we apply the unit force $\overline{X}_2 = 1$ (Fig. 6.1) instead of the constraint $\#3$ (see Fig. 2.1).

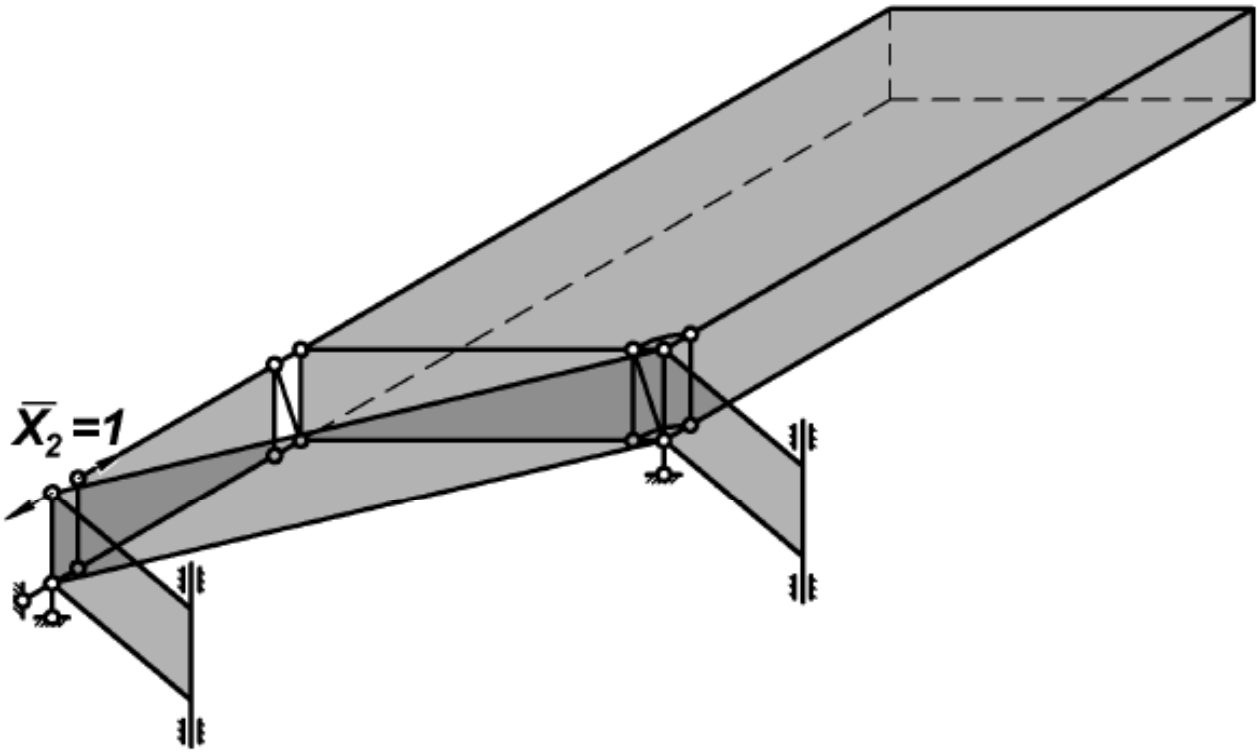


Figure 6.1. Basic system in second unit state

The order of calculations for the second unit state is the same as for the first one. Constraint forces are found from equilibrium equations:

$$\begin{aligned} &\overline{S}_6^{(2)} = \overline{S}_9^{(2)} = \overline{S}_{10}^{(2)} = \overline{S}_{11}^{(2)} = 0\ ; \\ &\overline{S}_2^{(2)} = \overline{S}_4^{(2)} = \overline{S}_5^{(2)} = \overline{S}_7^{(2)} = \overline{S}_8^{(2)} = 1\ . \end{aligned} \tag{6.1}$$

Fig. 6.2 shows directions of these forces.

Wingbox is loaded by the self-balanced system of forces ($\overline{S}_4^{(2)}$, $\overline{S}_5^{(2)}$, $\overline{S}_7^{(2)}$, and $\overline{S}_8^{(2)}$), so, by analogy with the first unit state, only self-balanced normal forces and shear flows exist in the wingbox. Normal forces can be found as:

$$\overline{N}_z^{(2)}(z) = \overline{C}_1^{(2)} \cdot \sinh \alpha z + \overline{C}_2^{(2)} \cdot \cosh \alpha z\ . \tag{6.2}$$

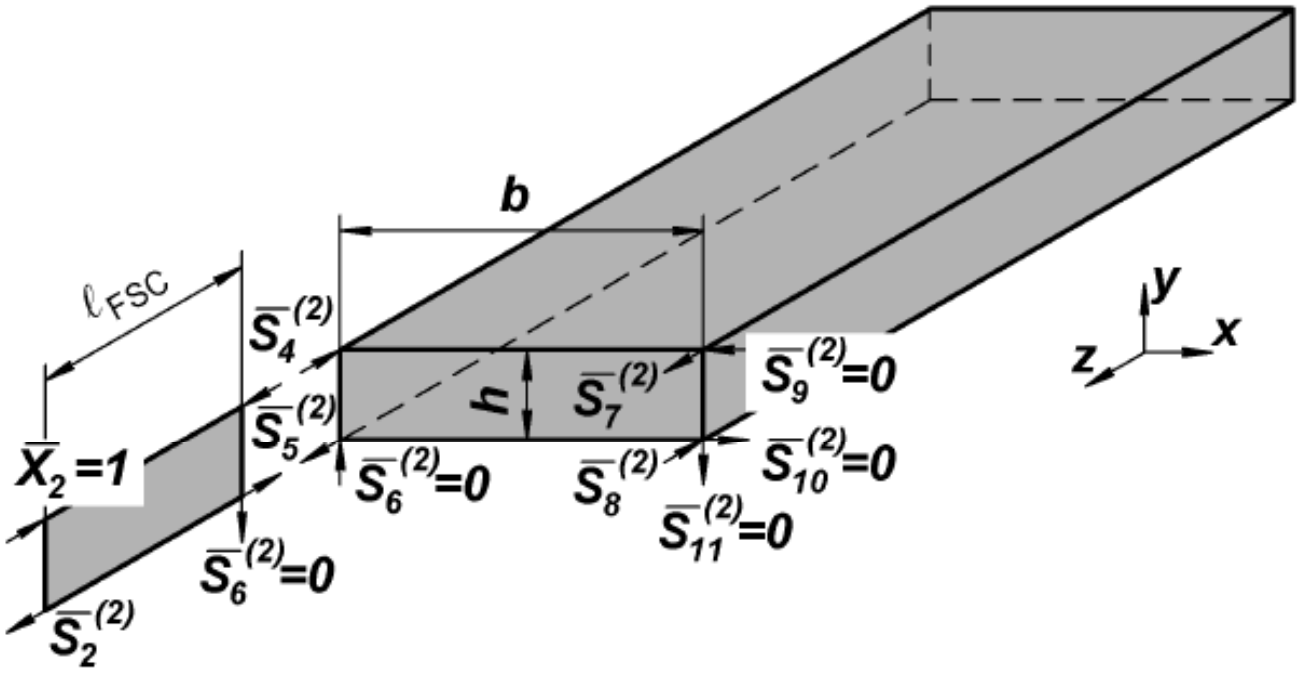


Figure 6.2. Constraint and external forces in second unit state

Constants $\bar{C}_1^{(2)}$ and $\bar{C}_2^{(2)}$ are found from the boundary conditions:

$$z=0,\quad \bar{N}_{zi}^{(2)}=\bar{C}_2^{(2)}=0\ ;$$

$$z=\ell,\quad \bar{N}_{z1}^{(2)}=\bar{C}_1^{(2)}\cdot\sinh\alpha\ell=-\bar{S}_4^{(2)}\ \Rightarrow$$

$$\Rightarrow\quad \bar{C}_1^{(2)}=-\frac{\bar{S}_4^{(2)}}{\sinh\alpha\ell}. \tag{6.3}$$

Substituting this into (6.2), we get

$$\bar{N}_{zi}^{(2)}(z)=(-1)^{i+1}\cdot\bar{C}_1^{(2)}\cdot\sinh\alpha z=\frac{(-1)^{i}\cdot\bar{S}_4^{(2)}}{\sinh\alpha\ell}\cdot\sinh\alpha z\ . \tag{6.4}$$

Shear flows are equal to

$$\bar{q}_i^{(2)}(z)=\frac{1}{2}\cdot\frac{d\bar{N}_{zi}^{(2)}(z)}{dz}=-\frac{(-1)^{i+1}\cdot\alpha\cdot\bar{S}_4^{(2)}}{2\cdot\sinh\alpha\ell}\cdot\cosh\alpha z=$$

$$=(-1)^{i+1}\cdot\bar{B}^{(2)}\cdot\cosh\alpha z\ , \tag{6.5}$$

where $\overline{B}^{(2)} = -\dfrac{\alpha \cdot \overline{S}_4^{(2)}}{2 \cdot \sinh \alpha \ell}$.

Direction of shear flows $\overline{q}_i^{(2)}(z)$ can be found by way of considering equilibrium of a small portion of spar cap with adjacent skin and web, as it was done in the F-state (see Fig. 4.6).

Distribution diagrams of normal forces and shear flows are built after the values $\overline{N}_{zi}^{(2)}(z)$ and $\overline{q}_i^{(2)}(z)$ were calculated in different sections (at four z-values from 0 to the length of the wingbox). It is convenient to present these calculations as a table (Table 6.1). The diagrams are shown in Fig. 6.3.

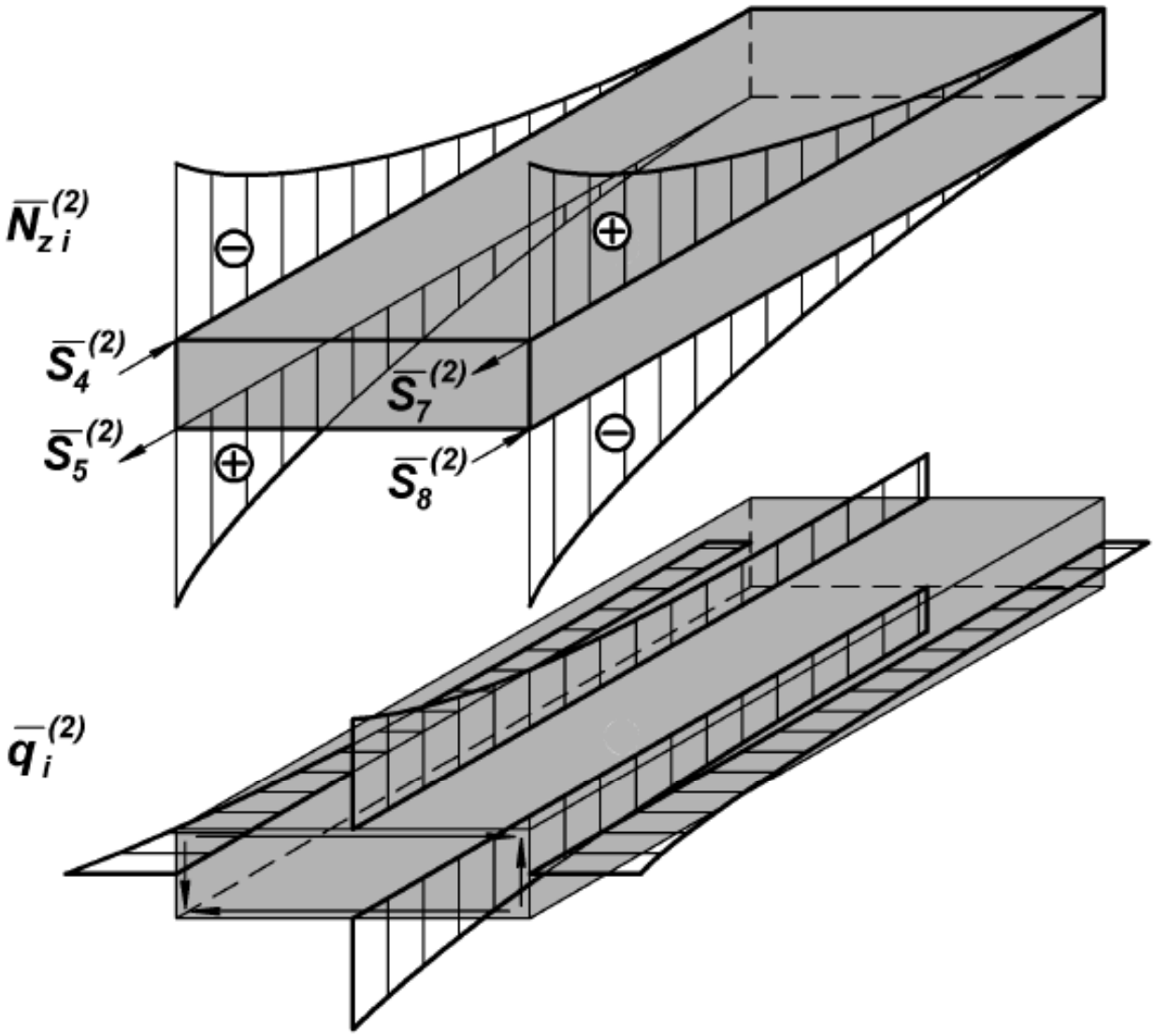


Figure 6.3. Diagrams of distribution of normal forces and shear flows in second unit state

Table 6.1

Calculation of normal forces and shear flows in the 2nd unit state

z, m	0	1	2	3
αz	0			
$\sinh \alpha z$	0			
$\cosh \alpha z$	1			
$\overline{N}_z^{(2)}$	0			
$\overline{q}^{(2)}$				

Forces $\overline{S}_4^{(2)}$, $\overline{S}_5^{(2)}$, $\overline{S}_7^{(2)}$, and $\overline{S}_8^{(2)}$ transferred from the wingbox to the root triangle structure give moments that are equal in value but opposite in direction. Forces $\overline{S}_7^{(2)}$ and $\overline{S}_8^{(2)}$ give the moment $\overline{M}_1^{(2)} = \overline{S}_7^{(1)} \cdot h$; and forces $\overline{S}_4^{(2)}$ and $\overline{S}_5^{(2)}$ give the moment $\overline{M}_3^{(3)} = \overline{S}_4^{(1)} \cdot h$ (Fig. 6.4).

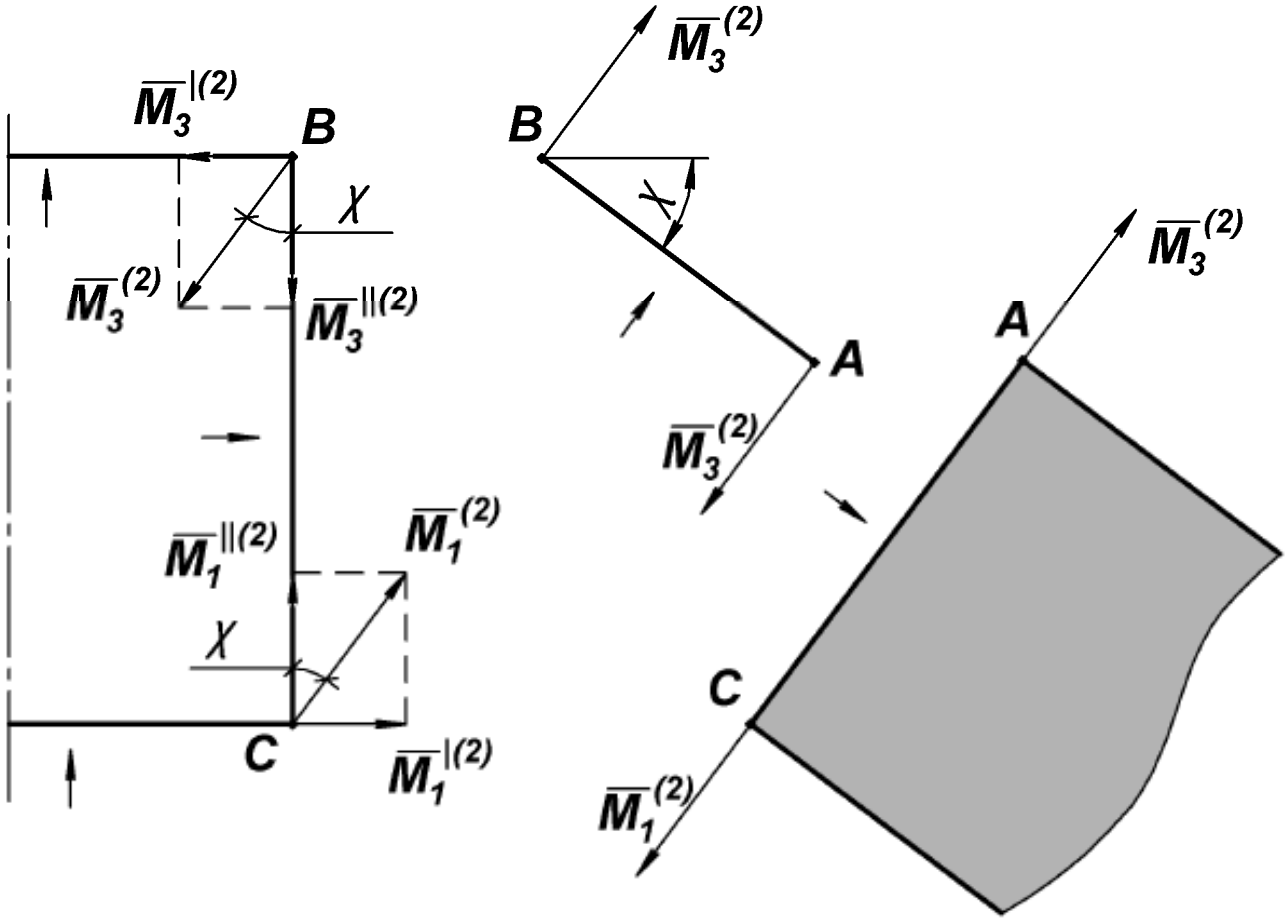


Figure 6.4. Loading of root triangle structure in second unit state

As we did it for the first unit state, we resolve moments $\overline{M}_1^{(2)}$ and $\overline{M}_3^{(2)}$ into projections to find the moments that load beams of the root triangle and the fuselage structure.

Root rib is loaded only by shear flows $\overline{q}_i^{(2)}$. Thus, the bending moment is zero (Fig. 6.5).

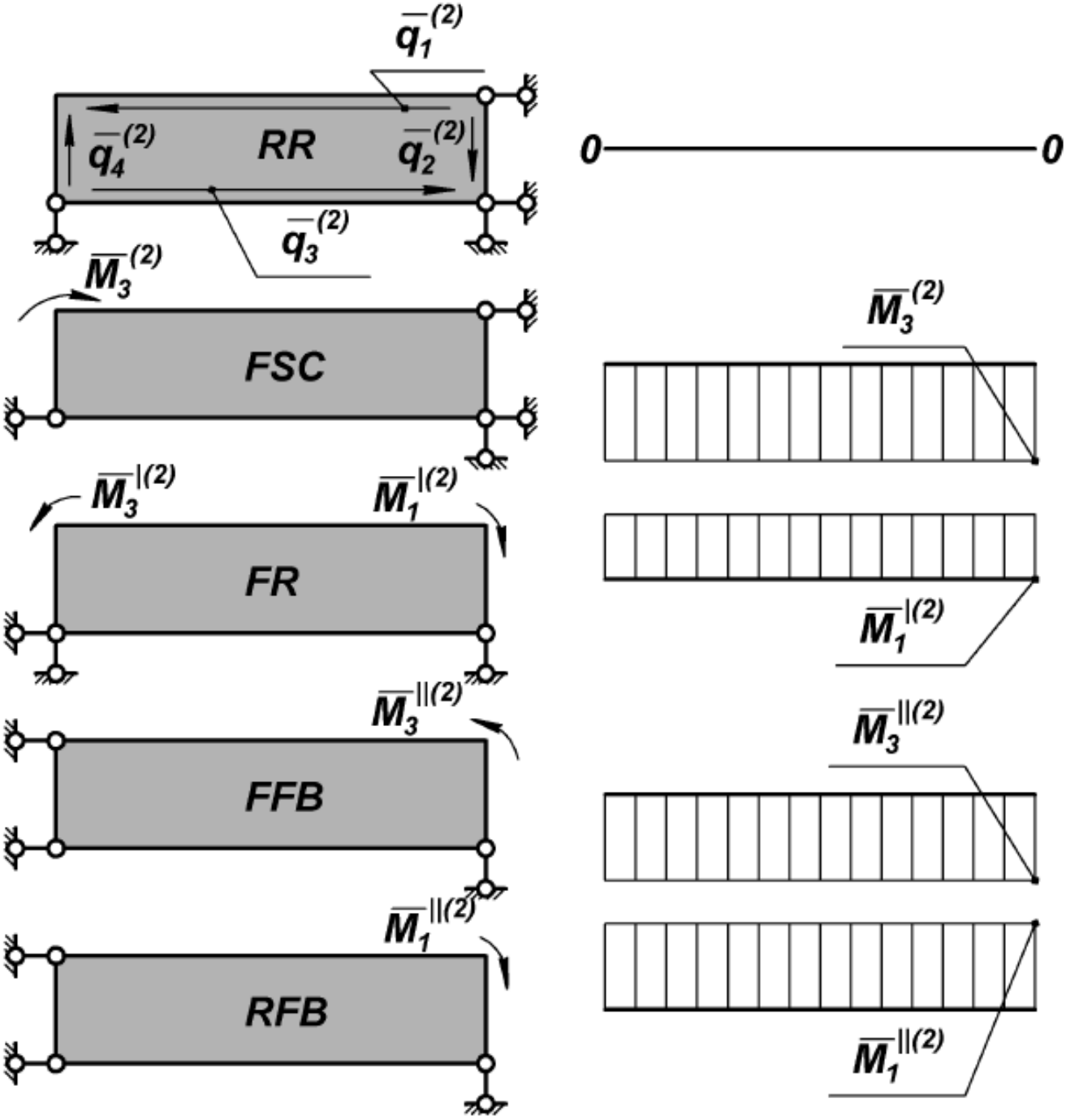


Figure 6.5. Diagrams of distribution of bending moments in second unit state

Front spar continuation is loaded by two moments $\overline{M}_3^{(2)}$ applied at both ends, so the bending moment is constant (see Fig. 6.5).

Fuselage rib is loaded by the moment $\overline{M}_3^{\prime(2)}$ at the left end and by the moment $\overline{M}_1^{\prime(2)}$ at the right end. These moments are equal, so the bending moment is constant (see Fig. 6.5).

Front fuselage beam is loaded by the moment $\overline{M}_3^{\prime\prime(2)}$ at the right end. At the left end, FFB is fixed so that there are no vertical constrains. Thus, the bending moment diagram is constant (see Fig. 6.5).

Rear fuselage beam is loaded and fixed analogously to the FFB (it is loaded by the moment $\overline{M}_1^{\prime\prime(2)}$). The diagram is also constant (see Fig. 6.5).

7. DETERMINATION OF CANONICAL EQUATIONS COEFFICIENTS. TOTAL DIAGRAMS

7.1. *System of Canonical Equations*

The system of canonical equations for the force method for two-times statically indeterminate problem is

$$\begin{cases} \delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \Delta_{1F} = 0 ; \\ \delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \Delta_{2F} = 0 . \end{cases} \quad (7.1)$$

Here δ_{jk} and Δ_{jF} are generalized displacements. *Generalized displacement* means the displacement between two points in the direction of the corresponding generalized force. *Generalized force* here is a couple of unit forces. We have two generalized forces, the first one being a couple of forces $\overline{X}_1$ (one applied to FSC, and the other to FR and FFB), and the second one being a couple of forces $\overline{X}_2$.

(7.1) means that generalized displacements (at the directions of unit forces) are zero at points where constraints were removed.

We understand the indices of δ in (7.1) as follows. δ_{jk} is a generalized displacement at the point (where the unit force $\overline{X}_j$ was applied instead of a constraint) in the direction of the unit force $\overline{X}_j = 1$ in the k-th unit state (when the basic system is loaded by the unit force $\overline{X}_k = 1$), $j = \overline{1,2}\ ;\ k = \overline{1,2}$.

Δ_{jF} is a generalized displacement at the point (where the unit force $\overline{X}_j$ was applied instead of a constraint) in the direction of $\overline{X}_j = 1$ in the F-state (when the basic system is loaded by the external load F), $j = \overline{1,2}$.

δ_{jk} and Δ_{jF} are the sums of values for the wingbox and for the root triangle structure:

$$\delta_{jk} = \delta_{jk}^{WB} + \delta_{jk}^{RT} \ ; \quad \Delta_{jF} = \Delta_{jF}^{WB} + \Delta_{jF}^{RT} \ . \quad (7.2)$$

These values are determined by the Mohr's formula:

$$\delta_{jk}^{WB} = \sum_{i=1}^{4} \int_0^{\ell} \frac{\overline{N}_{zi}^{(j)} \cdot \overline{N}_{zi}^{(k)}}{E_i \cdot A_i} dz + \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{\overline{q}_i^{(j)} \cdot \overline{q}_i^{(k)}}{G_i \cdot \delta_i} ds dz ;$$

$$\Delta_{jF}^{WB} = \sum_{i=1}^{4} \int_0^{\ell} \frac{\overline{N}_{zi}^{(j)} \cdot N_{zi}^{(F)}}{E_i \cdot A_i} dz + \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{\overline{q}_i^{(j)} \cdot q_i^{(F)}}{G_i \cdot \delta_i} ds dz ;$$

$$\delta_{jk}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(j)} \cdot \overline{M}_n^{(k)}}{E_n \cdot I_n} dx ; \qquad (7.3)$$

$$\Delta_{jF}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(j)} \cdot M_n^{(F)}}{E_n \cdot I_n} dx ,$$

where s is a length of the skin (b) or height of the spar webs (h);

ℓ_n is a length of the root triangle beam (FR, RR, FFB, RFB, FSC), $n = \overline{1,5}$;

$E_i \cdot A_i$ are rigidities of spar caps which carry normal forces $\left(i = \overline{1,4}\right)$;

$G_i \cdot \delta_i$ are rigidities of skins and spar webs which carry shear loads $\left(i = \overline{1,4}\right)$;

$E_n \cdot I_n$ are rigidities of root triangle beams, which carry bending moments $\left(n = \overline{1,5}\right)$;

$\overline{N}_{zi}^{(1)}, \overline{N}_{zi}^{(2)}, N_{zi}^{F}$ are normal forces acting in spar caps in the first unit state, in the second unit state, and in the F-state;

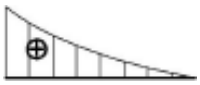
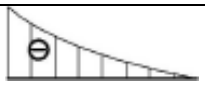
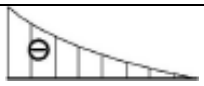
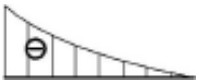
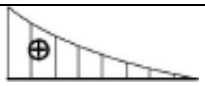
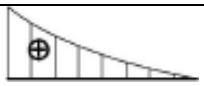
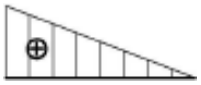
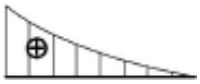
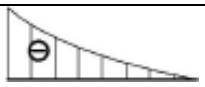
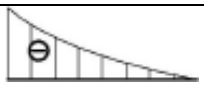
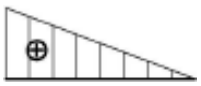
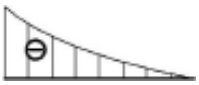
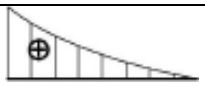
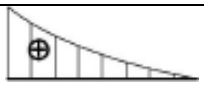
$\overline{q}_i^{(1)}, \overline{q}_i^{(2)}, q_i^{F}$ are shear flows acting in skins and spar webs in the first unit state, in the second unit state, and in the F-state;

$\overline{M}_n^{(1)}, \overline{M}_n^{(2)}, M_n^{(F)}$ are bending moments acting in the root triangle beams in the first unit state, in the second unit state, and in the F-state.

It is convenient to combine all diagrams (N_{zi}^{EL}, N_{zi}^{SB}, $\overline{N}_{zi}^{(j)}$, q_i^{EL}, q_i^{SB}, $\overline{q}_i^{(j)}$, $M_n^{(F)}$, $\overline{M}_n^{(j)}$) into tables (Tables 7.1–7.3). We do so to avoid errors then performing calculations and to make them more obvious.

Table 7.1

Distribution diagrams of normal forces in spar caps

	F-state		1st unit state	2nd unit state
Cap #	$N_{z\,i}^{EL}$	$N_{z\,i}^{SB}$	$\overline{N}_{z\,i}^{(1)}$	$\overline{N}_{z\,i}^{(2)}$
1	$-C \cdot A_F \cdot z$	 $C_1 \cdot \sinh \alpha z$	 $-\overline{C}_1^{(1)} \cdot \sinh \alpha z$	 $-\overline{C}_1^{(2)} \cdot \sinh \alpha z$
2	$-C \cdot A_R \cdot z$	 $-C_1 \cdot \sinh \alpha z$	 $\overline{C}_1^{(1)} \cdot \sinh \alpha z$	 $\overline{C}_1^{(2)} \cdot \sinh \alpha z$
3	 $C \cdot A_R \cdot z$	 $C_1 \cdot \sinh \alpha z$	 $-\overline{C}_1^{(1)} \cdot \sinh \alpha z$	 $-\overline{C}_1^{(2)} \cdot \sinh \alpha z$
4	 $C \cdot A_F \cdot z$	 $-C_1 \cdot \sinh \alpha z$	 $\overline{C}_1^{(1)} \cdot \sinh \alpha z$	 $\overline{C}_1^{(2)} \cdot \sinh \alpha z$

Comment. C is the coefficient from (4.6).

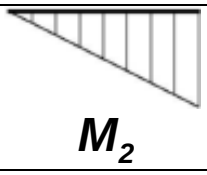
Table 7.2

Shear flows in wingbox

	F-state		1st unit state	2nd unit state
	$q_F^{EL} + q_0^{EL}$	q_i^{SB}	$\overline{q}_i^{(1)}$	$\overline{q}_i^{(2)}$
1. Upper skin	$q_{F1}^{EL} + q_0^{EL}$	$B \cdot \cosh \alpha z$	$\overline{B}^{(1)} \cdot \cosh \alpha z$	$\overline{B}^{(2)} \cdot \cosh \alpha z$
2. Rear spar web	$q_{F2}^{EL} + q_0^{EL}$	$B \cdot \cosh \alpha z$	$\overline{B}^{(1)} \cdot \cosh \alpha z$	$\overline{B}^{(2)} \cdot \cosh \alpha z$
3. Lower skin	$q_{F1}^{EL} + q_0^{EL}$	$B \cdot \cosh \alpha z$	$\overline{B}^{(1)} \cdot \cosh \alpha z$	$\overline{B}^{(2)} \cdot \cosh \alpha z$
4. Front spar web	q_0^{EL}	$B \cdot \cosh \alpha z$	$\overline{B}^{(1)} \cdot \cosh \alpha z$	$\overline{B}^{(2)} \cdot \cosh \alpha z$

Table 7.3

Distribution diagrams of bending moments in root triangle and fuselage beams

	F-state	1st unit state	2nd unit state
1. RR	 M_2	$\overline{M}_2^{(1)}$	
2. FSC		$\overline{M}_3^{(1)}$	$\overline{M}_3^{(2)}$
3. FR	$M_1' - M_2'$	$\overline{M}_1'^{(1)} + \overline{M}_2'^{(1)}$	$\overline{M}_1'^{(2)}$
4. FFB			$\overline{M}_3''^{(2)}$
5. RFB	$M_1'' + M_2''$		$\overline{M}_1''^{(2)}$

After filling the tables, we have to calculate six coefficients of canonical equations: Δ_{1F}, Δ_{2F}, δ_{11}, δ_{12}, δ_{21}, and δ_{22} (note that $\delta_{12} = \delta_{21}$).

7.2. *Example of Calculating Coefficients in Canonical Equations*

Let us discuss how to find δ_{11} and Δ_{1F}. They are calculated by formulae:

$$\delta_{11} = \delta_{11}^{WB} + \delta_{11}^{RT}; \quad \delta_{11}^{WB} = \delta_{11}^{WB,N} + \delta_{11}^{WB,q};$$

$$\delta_{11}^{WB,N} = \sum_{i=1}^{4} \int_0^{\ell} \frac{\overline{N}_{zi}^{(1)} \cdot \overline{N}_{zi}^{(1)}}{E_i \cdot A_i} dz;$$

$$\delta_{11}^{WB,q} = \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{\overline{q}_i^{(1)} \cdot \overline{q}_i^{(1)}}{G_i \cdot \delta_i} ds\, dz; \qquad (7.4)$$

$$\delta_{11}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(1)} \cdot \overline{M}_n^{(1)}}{E_n \cdot I_n} dx.$$

To calculate them, we need some integration formulae:

$$\int \sinh \alpha x \, dx = \frac{1}{\alpha} \cdot \cosh \alpha x \, ;$$

$$\int \cosh \alpha x \, dx = \frac{1}{\alpha} \cdot \sinh \alpha x \, ;$$

$$\int \sinh^2 \alpha x \, dx = \frac{1}{2\alpha} \cdot \sinh \alpha x \cdot \cosh \alpha x - \frac{1}{2} \cdot x \, ; \qquad (7.5)$$

$$\int \cosh^2 \alpha x \, dx = \frac{1}{2\alpha} \cdot \sinh \alpha x \cdot \cosh \alpha x + \frac{1}{2} \cdot x \, ;$$

$$\int x \cdot \sinh \alpha x \, dx = \frac{x \cosh \alpha x}{\alpha} - \frac{\sinh \alpha x}{\alpha^2} \, .$$

First, we calculate $\delta_{11}^{WB,N}$:

$$\delta_{11}^{WB,N} = \sum_{i=1}^{4} \int_0^{\ell} \frac{\overline{N}_{zi}^{(1)} \cdot \overline{N}_{zi}^{(1)}}{E_i \cdot A_i} dz = \int_0^{\ell} \frac{\left(\overline{N}_{z1}^{(1)}\right)^2}{E \cdot A_1} + \int_0^{\ell} \frac{\left(\overline{N}_{z2}^{(1)}\right)^2}{E \cdot A_2} +$$

$$+ \int_0^{\ell} \frac{\left(\overline{N}_{z3}^{(1)}\right)^2}{E \cdot A_3} + \int_0^{\ell} \frac{\left(\overline{N}_{z4}^{(1)}\right)^2}{E \cdot A_4} \, . \qquad (7.6)$$

According to the given data, we have $A_1 = A_4 = A_F$, $A_2 = A_3 = A_R$. Absolute values $\overline{N}_{zi}^{(1)}$ of normal forces in the self-balanced state are also equal: $\left|\overline{N}_{z1}^{(1)}\right| = \left|\overline{N}_{z2}^{(1)}\right| = \left|\overline{N}_{z3}^{(1)}\right| = \left|\overline{N}_{z4}^{(1)}\right|$. Therefore,

$$\delta_{11}^{WB,N} = \frac{2}{E \cdot A_F} \int_0^{\ell} \left(\overline{N}_{z1}^{(1)}\right)^2 + \frac{2}{E \cdot A_R} \int_0^{\ell} \left(\overline{N}_{z1}^{(1)}\right)^2 =$$

$$= \frac{2}{E} \cdot \left(\frac{1}{A_F} + \frac{1}{A_R} \right) \cdot \int_0^{\ell} \left(\overline{C}_1^{(1)} \cdot \sinh \alpha z \right)^2 dz =$$

$$= \frac{2 \cdot \left(\bar{C}_1^{(1)}\right)^2 \cdot \left(A_F + A_R\right)}{E \cdot A_F \cdot A_R} \int_0^{\ell} \sinh^2 \alpha z \, dz =$$

$$= \frac{2 \cdot \left(\bar{C}_1^{(1)}\right)^2 \cdot \left(A_F + A_R\right)}{E \cdot A_F \cdot A_R} \times$$

$$\times \left[\frac{1}{2\alpha} \cdot \sinh \alpha z \cdot \sinh \alpha z - \frac{z}{2} \right]_{z=\ell} = \ldots \qquad (7.7)$$

Then calculating $\delta_{11}^{WB,q}$ by (7.4), we have to take into account that shear flows are not variable by the s-coordinate. Thus,

$$\int_0^{\ell} \int_0^{s_i} \frac{\bar{q}_i^{(1)} \cdot \bar{q}_i^{(1)}}{G_i \cdot \delta_i} ds_i \, dz = \begin{cases} b \cdot \int_0^{\ell} \frac{\bar{q}_i^{(1)} \cdot \bar{q}_i^{(1)}}{G_i \cdot \delta_i} dz, \quad i = 1,3 \ (skins), \\ h \cdot \int_0^{\ell} \frac{\bar{q}_i^{(1)} \cdot \bar{q}_i^{(1)}}{G_i \cdot \delta_i} dz, \quad i = 2,4 \ (webs). \end{cases} \qquad (7.8)$$

We know that spar webs are made of steel 30ХГСА (shear modulus is G_S), and upper and lower skin is made of aluminum alloy Д16Т (shear modulus is G_A). Absolute values $\bar{q}_i^{(1)}$ of shear flows in self-balanced state are equal: $\left|\bar{q}_1^{(1)}\right| = \left|\bar{q}_2^{(1)}\right| = \left|\bar{q}_3^{(1)}\right| = \left|\bar{q}_4^{(1)}\right|$. Substituting (7.8) into (7.4), we get

$$\delta_{11}^{WB,q} = b \cdot \int_0^{\ell} \frac{\left(\bar{q}_1^{(1)}\right)^2}{G_A \cdot \delta_2} dz + h \cdot \int_0^{\ell} \frac{\left(\bar{q}_2^{(1)}\right)^2}{G_S \cdot \delta_1} dz +$$

$$+ b \cdot \int_0^{\ell} \frac{\left(\bar{q}_3^{(1)}\right)^2}{G_A \cdot \delta_2} dz + h \cdot \int_0^{\ell} \frac{\left(\bar{q}_4^{(1)}\right)^2}{G_S \cdot \delta_1} dz =$$

$$= 2 \cdot \left(\overline{B}^{(1)}\right)^2 \cdot \left(\frac{b}{G_A \cdot \delta_2} + \frac{h}{G_S \cdot \delta_1}\right) \times$$

$$\times \int_0^{\ell} \left(\overline{B}^{(1)} \cdot \cosh \alpha z\right)^2 dz =$$

$$= 2 \cdot \left(\overline{B}^{(1)}\right)^2 \cdot \left(\frac{b}{G_A \cdot \delta_2} + \frac{h}{G_S \cdot \delta_1}\right) \times$$

$$\times \left(\frac{1}{2\alpha} \cdot \sinh \alpha z \cdot \cosh \alpha z + \frac{z}{2}\right)\Bigg|_{z=\ell} = \ldots \qquad (7.9)$$

Then, we calculate δ_{11}^{RT} from (7.4) as

$$\delta_{11}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(1)} \cdot \overline{M}_n^{(1)}}{E_n \cdot I_n} dx = \int_0^{\ell_{RR}} \frac{\left(\overline{M}_{RR}^{(1)}\right)^2}{E_S \cdot I_{RR}} dx +$$

$$+ \int_0^{\ell_{FSC}} \frac{\left(\overline{M}_{FSC}^{(1)}\right)^2}{E_S \cdot I_{FSC}} dx + \int_0^{\ell_{FR}} \frac{\left(\overline{M}_{FR}^{(1)}\right)^2}{E_S \cdot I_{FR}} dx +$$

$$+ \int_0^{\ell_{FFB}} \frac{\left(\overline{M}_{FFB}^{(1)}\right)^2}{E_S \cdot I_{FFB}} dx + \int_0^{\ell_{RFB}} \frac{\left(\overline{M}_{RFB}^{(1)}\right)^2}{E_S \cdot I_{RFB}} dx. \qquad (7.10)$$

To find these integrals, we can use analytical methods or the Vereschagin's rule (it will be described later).

Δ_{1F} is calculated as follows:

$$\Delta_{1F} = \Delta_{1F}^{WB} + \Delta_{1F}^{RT}; \; \Delta_{1F}^{WB} = \Delta_{1F}^{WB,N} + \Delta_{1F}^{WB,q};$$

$$\Delta_{11}^{WB,N} = \sum_{i=1}^{4} \int_0^{\ell} \frac{\overline{N}_{zi}^{(1)} \cdot N_{zi}^{(F)}}{E_i \cdot A_i} dz;$$

$$\Delta_{11}^{WB,q} = \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{\overline{q}_i^{(1)} \cdot q_i^{(F)}}{G_i \cdot \delta_i} ds\, dz \;;$$

$$\Delta_{11}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(1)} \cdot M_n^{(F)}}{E_n \cdot I_n} dx \;. \qquad (7.11)$$

All calculations are similar to those we perfomed to find δ_{11}, but here we have to multiply diagrams in the unit state by diagrams in the F-state:

$$\Delta_{1F}^{WB,N} = 2 \cdot \int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot N_{z1}^{(F)}}{E \cdot A_1} dz + 2 \cdot \int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot N_{z2}^{(F)}}{E \cdot A_2} dz =$$

$$= \frac{2}{E} \cdot \left(\int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot N_{z1}^{(F)}}{A_1} dz + \int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot N_{z2}^{(F)}}{A_2} dz \right) =$$

$$= \frac{2}{E} \cdot \left(\int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot \left(N_{z1}^{EL} + N_{z1}^{SB} \right)}{A_1} dz + \int_0^{\ell} \frac{\overline{N}_{z1}^{(1)} \cdot \left(N_{z2}^{EL} + N_{z2}^{SB} \right)}{A_2} dz \right) =$$

$$= \int_0^{\ell} \frac{2 \cdot \overline{C}_1^{(1)} \cdot \sinh \alpha z \cdot \left(A \cdot A_1 \cdot z + C_1 \cdot \sinh \alpha z \right)}{E \cdot A_1} dz +$$

$$+ \int_0^{\ell} \frac{2 \cdot \overline{C}_1^{(1)} \cdot \sinh \alpha z \cdot \left(A \cdot A_2 \cdot z - C_1 \cdot \sinh \alpha z \right)}{E \cdot A_2} dz =$$

$$= \frac{2 \cdot \overline{C}_1^{(1)}}{E} \cdot \left(2A \cdot \int_0^{\ell} z \cdot \sinh \alpha z\, dz + \right.$$

$$\left. + \frac{C_1}{A_1} \cdot \int_0^{\ell} \sinh^2 \alpha z\, dz - \frac{C_1}{A_2} \cdot \int_0^{\ell} \sinh^2 \alpha z\, dz \right) =$$

$$= \frac{2 \cdot \overline{C}_1^{(1)}}{E} \cdot \left(2A \cdot \int_0^{\ell} z \cdot \sinh \alpha z \, dz + \right.$$

$$\left. + \frac{C_1 \cdot (A_2 - A_1)}{A_1 \cdot A_2} \cdot \int_0^{\ell} \sinh^2 \alpha z \, dz \right) =$$

$$= \frac{2 \cdot \overline{C}_1^{(1)}}{E} \cdot \left(2A \cdot \left(\frac{z}{\alpha} \cdot \cosh \alpha z - \frac{1}{\alpha^2} \cdot \sinh \alpha z \right) \Bigg|_{z=\ell} + \right.$$

$$\left. + \frac{C_1 \cdot (A_2 - A_1)}{A_1 \cdot A_2} \cdot \left(\frac{1}{2\alpha} \cdot \sinh \alpha z \cdot \cosh \alpha z - \frac{z}{2} \right) \Bigg|_{z=\ell} \right) = \ldots \qquad (7.12)$$

Now we calculate $\Delta_{1F}^{WB,q}$:

$$\Delta_{1F}^{WB,q} = \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{\overline{q}_i^{(1)} \cdot q_i^{(F)}}{G_i \cdot \delta_i} ds \, dz = b \cdot \int_0^{\ell} \frac{\overline{q}_1^{(1)} \cdot q_1^{(F)}}{G_A \cdot \delta_2} dz +$$

$$+ h \cdot \int_0^{\ell} \frac{\overline{q}_2^{(1)} \cdot q_2^{(F)}}{G_S \cdot \delta_1} dz + b \cdot \int_0^{\ell} \frac{\overline{q}_3^{(1)} \cdot q_3^{(F)}}{G_A \cdot \delta_2} dz + h \cdot \int_0^{\ell} \frac{\overline{q}_4^{(1)} \cdot q_4^{(F)}}{G_S \cdot \delta_1} dz =$$

$$= \frac{2b}{G_A \cdot \delta_2} \cdot \int_0^{\ell} \overline{q}_1^{(1)} \cdot \left(q_1^{EL} + q_1^{SB} \right) dz +$$

$$+ \frac{h}{G_S \cdot \delta_1} \cdot \left(\int_0^{\ell} \overline{q}_2^{(1)} \cdot \left(q_2^{EL} + q_2^{SB} \right) dz + h \cdot \int_0^{\ell} \overline{q}_4^{(1)} \cdot \left(q_4^{EL} + q_4^{SB} \right) dz \right) =$$

$$= \frac{2b}{G_A \cdot \delta_2} \cdot \int_0^{\ell} \overline{B}^{(1)} \cdot \cosh \alpha z \cdot \left(q_1^{EL} + B \cdot \cos h \alpha z \right) dz +$$

$$+ \frac{h}{G_S \cdot \delta_1} \cdot \left(\int_0^{\ell} \overline{B}^{(1)} \cdot \cosh \alpha z \cdot \left(q_2^{EL} - B \cdot \cosh \alpha z \right) dz + \right.$$

$$\left. + \int_0^{\ell} \overline{B}^{(1)} \cdot \cosh \alpha z \cdot \left(q_4^{EL} - B \cdot \cosh \alpha z \right) dz \right) =$$

$$= \frac{2 \cdot b \cdot \overline{B}^{(1)} \cdot q_1^{EL}}{G_A \cdot \delta_2} \int_0^{\ell} \cosh \alpha z \, dz + \frac{2 \cdot b \cdot B \cdot \overline{B}^{(1)}}{G_A \cdot \delta_2} \int_0^{\ell} \cosh^2 \alpha z \, dz +$$

$$+ \frac{h \cdot B^{(1)} \cdot q_2^{EL}}{G_S \cdot \delta_2} \cdot \int_0^{\ell} \cosh \alpha z \, dz - \frac{h \cdot B \cdot \overline{B}^{(1)}}{G_S \cdot \delta_2} \int_0^{\ell} \cosh^2 \alpha z \, dz +$$

$$+ \frac{h \cdot B^{(1)} \cdot q_4^{EL}}{G_S \cdot \delta_2} \int_0^{\ell} \cosh \alpha z \, dz - \frac{h \cdot B \cdot \overline{B}^{(1)}}{G_S \cdot \delta_2} \int_0^{\ell} \cosh^2 \alpha z \, dz =$$

$$= \left(\frac{2 \cdot b \cdot \overline{B}^{(1)} \cdot q_1^{EL}}{G_A \cdot \delta_2} + \frac{h \cdot B^{(1)} \cdot \left(q_2^{EL} + q_4^{EL} \right)}{G_S \cdot \delta_2} \right) \cdot \int_0^{\ell} \cosh \alpha z \, dz +$$

$$+ \left(\frac{2 \cdot b \cdot B \cdot \overline{B}^{(1)}}{G_A \cdot \delta_2} - \frac{2 \cdot h \cdot B \cdot \overline{B}^{(1)}}{G_S \cdot \delta_2} \right) \cdot \int_0^{\ell} \cosh^2 \alpha z \, dz =$$

$$= \left(\frac{2 \cdot b \cdot \overline{B}^{(1)} \cdot q_1^{EL}}{G_A \cdot \delta_2} + \frac{h \cdot B^{(1)} \cdot \left(q_2^{EL} + q_4^{EL} \right)}{G_S \cdot \delta_2} \right) \cdot \frac{1}{\alpha} \cdot \cosh \alpha z \Big|_{z=0}^{\ell} +$$

$$+ \left(\frac{2 \cdot b \cdot B \cdot \overline{B}^{(1)}}{G_A \cdot \delta_2} - \frac{2 \cdot h \cdot B \cdot \overline{B}^{(1)}}{G_S \cdot \delta_2} \right) \times$$

$$\times \left(\frac{1}{2\alpha} \cdot \sinh \alpha z \cdot \cosh \alpha z + \frac{z}{2} \right) \Big|_{z=\ell} = \ldots \qquad (7.13)$$

At last, we find Δ_{1P}^{RT} :

$$\Delta_{1F}^{RT} = \sum_{n=1}^{5} \int_0^{\ell_n} \frac{\overline{M}_n^{(1)} \cdot M_n^{(F)}}{E_n \cdot I_n} dx = \int_0^{\ell_{RR}} \frac{\overline{M}_{RR}^{(1)} \cdot M_{RR}^{(F)}}{E_S \cdot I_{RR}} dx +$$

$$+ \int_0^{\ell_{FSC}} \frac{\overline{M}_{FSC}^{(1)} \cdot M_{FSC}^{(F)}}{E_S \cdot I_{FSC}} dx + \int_0^{\ell_{FR}} \frac{\overline{M}_{FR}^{(1)} \cdot M_{FR}^{(F)}}{E_S \cdot I_{FR}} dx +$$

$$+\int_0^{\ell_{FFB}} \frac{\overline{M}_{FFB}^{(1)} \cdot M_{FFB}^{(F)}}{E_S \cdot I_{FFB}}\,dx + \int_0^{\ell_{RFB}} \frac{\overline{M}_{RFB}^{(1)} \cdot M_{RFB}^{(F)}}{E_S \cdot I_{RFB}}\,dx\,. \qquad (7.14)$$

These integrals can be calculated using analytical methods or the Vereschagin's rule. The latter is a more obvious way but it can be used only when at least one of two moments is a linear function or a constant value. In our case, this condition is always fulfilled, so we can use Vereschagin's rule for all beams.

To calculate the integral $\int_0^{\ell} M_1(x) \cdot M_2(x)\,dx$, we have only to find the centre of gravity coordinate $x_{c.g.1}$ for the first diagram $M_1(x)$ and multiply the area of the first diagram A_1 by the value of M_2 at the coordinate $x_{c.g.1}$:

$$\int_0^{\ell} M_1(x) \cdot M_2(x)\,dx = \pm A_1 \cdot M_2\left(x_{c.g.1}\right). \qquad (7.15)$$

In these formulae, we use "+" when M_1 and M_2 diagrams are co-directed, and "-" when they have opposite directions.

For example, for diagrams shown in Fig. 7.1, we get (for $\ell = 2\ m$)

$$\int_0^{\ell} M_1(x) \cdot M_2(x)\,dx = -\frac{\ell \cdot M_1(\ell)}{2} \cdot M_2\left(\frac{2}{3}\ell\right) =$$

$$= -\frac{2 \cdot 30}{2} \cdot \frac{10}{3} = -100\ \left(kN^2 \cdot m^3\right). \qquad (7.16)$$

Figure 7.1. Example diagrams of bending moments M_1 and M_2

If any diagram (or both) is not triangular (Fig. 7.2), we have to consider it as the sum of two diagrams and perform calculations by the following formulae:

$$\int_0^{\ell} M_1 \cdot M_2 \, dx = \int_0^{\ell} \left(M_{11} + M_{12}\right) \cdot \left(M_{21} + M_{22}\right) dx =$$

$$= \int_0^{\ell} M_{11} \cdot M_{21} \, dx + \int_0^{\ell} M_{11} \cdot M_{22} \, dx +$$

$$+ \int_0^{\ell} M_{12} \cdot M_{21} \, dx + \int_0^{\ell} M_{12} \cdot M_{22} \, dx \, . \qquad (7.17)$$

All integrands in this final formula are moments with triangle or constant diagrams. Thus, we can calculate them by the Vereschagin's rule.

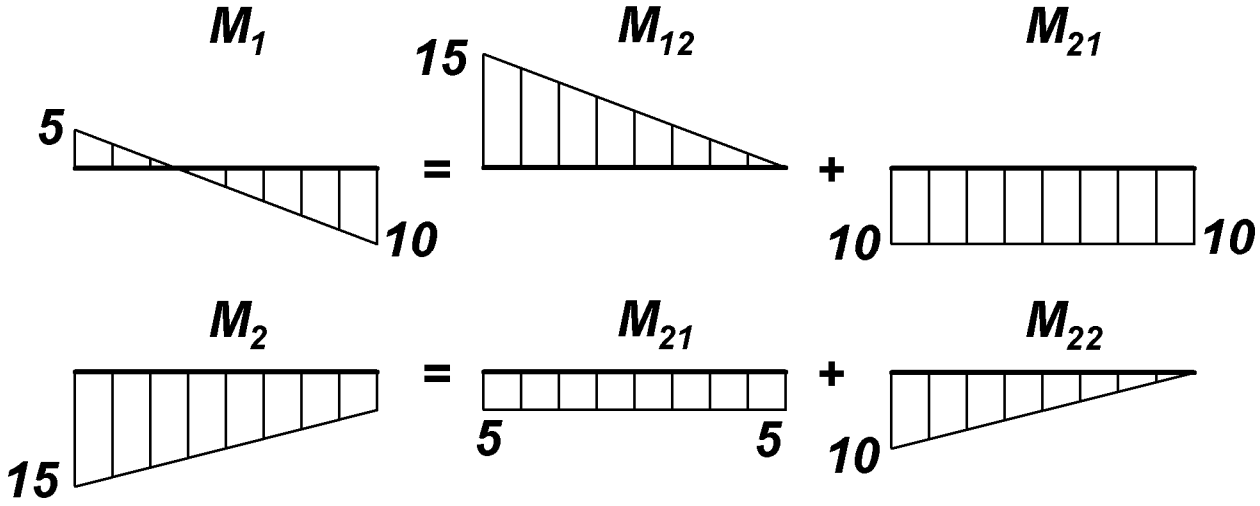


Figure 7.2. Example diagrams of bending moments M_1 and M_2

7.3. *Total diagrams*

After we calculated coefficients in canonical equations δ_{jk}, Δ_{jF}, we can find the unknown values X_1 and X_2 from (7.1).

We start with calculating total values of normal forces, shear flows, bending moments, and constraint forces. To do this, we simply sum the corresponding values in F-state, first unit state (multiplied by X_1), and second unit state (multiplied by X_2):

$$
\begin{aligned}
N_{z\,i}^{\Sigma} &= N_{z\,i}^{(F)} + \overline{N}_{z\,i}^{(1)} \cdot X_1 + \overline{N}_{z\,i}^{(2)} \cdot X_2, \qquad i = \overline{1..4};\\
q_i^{\Sigma} &= q_i^{(F)} + \overline{q}_i^{(1)} \cdot X_1 + \overline{q}_i^{(2)} \cdot X_2, \qquad i = \overline{1..4};\\
M_n^{\Sigma} &= M_n^{(F)} + \overline{M}_n^{(1)} \cdot X_1 + \overline{M}_n^{(2)} \cdot X_2, \qquad n = \overline{1..5};\\
S_i^{\Sigma} &= S_i + \overline{S}_i^{(1)} \cdot X_1 + \overline{S}_i^{(2)} \cdot X_2, \qquad i = \overline{1..11}.
\end{aligned}
\tag{7.18}
$$

It is convenient to present these calculations as a table (Tables 7.4 and 7.5).

Table 7.4

Calculation of total normal forces and total shear flows

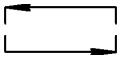
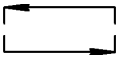
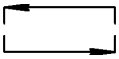
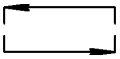
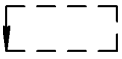
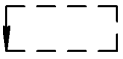
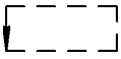
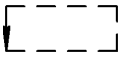
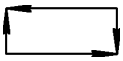
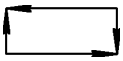
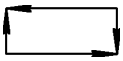
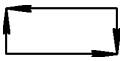
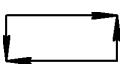
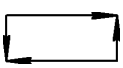
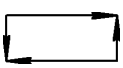
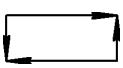
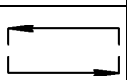
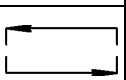
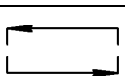
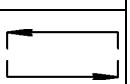
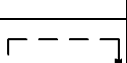
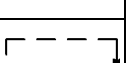
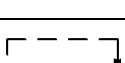
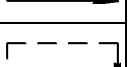
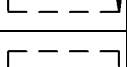
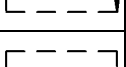
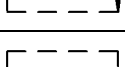
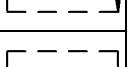
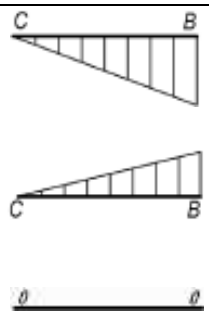
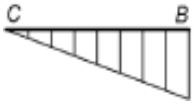
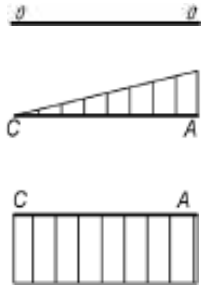
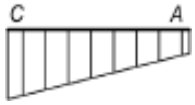
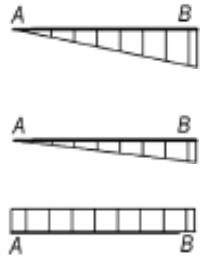
z, m	0	1	2	3
$N_{z\,1,4}^{(F)}$	$\mp$	$\mp$	$\mp$	$\mp$
$N_{z\,2,3}^{(F)}$	$\mp$	$\mp$	$\mp$	$\mp$
$\overline{N}_z^{(1)} \cdot X_1$	0	$\pm$	$\pm$	$\pm$
$\overline{N}_z^{(2)} \cdot X_2$	0	$\mp$	$\mp$	$\mp$
$N_{z\,1,4}^{\Sigma} = N_{z\,1,4}^{(F)} + \overline{N}_{z\,1,4}^{(1)}\, X_1 + \overline{N}_{z\,1,4}^{(2)}\, X_2$	0	$\mp$	$\mp$	$\mp$
$N_{z\,2,3}^{\Sigma} = N_{z\,2,3}^{(F)} + \overline{N}_{z\,2,3}^{(1)}\, X_1 + \overline{N}_{z\,2,3}^{(2)}\, X_2$	0	$\mp$	$\mp$	$\mp$
$q_{1,3}^{(F)}$				
$q_2^{(F)}$				
$q_4^{(F)}$				
$\overline{q}^{(1)} \cdot X_1$				
$\overline{q}^{(2)} \cdot X_2$				
$q_{1,3}^{\Sigma} = q_{1,3}^{(F)} + \overline{q}_{1,3}^{(1)} \cdot X_1 + \overline{q}_{1,3}^{(2)} \cdot X_2$				
$q_2^{\Sigma} = q_2^{(F)} + \overline{q}_2^{(1)} \cdot X_1 + \overline{q}_2^{(2)} \cdot X_2$				
$q_4^{\Sigma} = q_4^{(F)} + \overline{q}_4^{(1)} \cdot X_1 + \overline{q}_4^{(2)} \cdot X_2$				

Table 7.5

Calculation of total distribution diagrams of bending moments in beams

Root triangle beam	F-state, 1st unit state, 2nd unit state,	Total diagrams	Formulae for the moments for each of states
1. RR			$M^{F}_{(B)} = M_2$; $\overline{M}^{(1)}_{(B)} = \overline{M}^{(1)}_{2} \cdot X_1$; $\overline{M}^{(2)} = 0$; $M^{\Sigma}_{(B)} = M_2 - \overline{M}^{(1)}_{2} \cdot X_1$.
2. FSC			$M^{F}_{(B)} = 0$; $\overline{M}^{(1)}_{(A)} = \overline{M}^{(1)}_{3} \cdot X_1$; $\overline{M}^{(2)}_{(C)} = \overline{M}^{(2)}_{(A)} = \overline{M}^{(2)}_{3} \cdot X_2$; $M^{\Sigma}_{(A)} = \overline{M}^{(2)}_{3} \cdot X_2 - \overline{M}^{(1)}_{3} \cdot X_1$; $M^{\Sigma}_{(C)} = \overline{M}^{(2)}_{3} \cdot X_2$.
3. FR			$M^{F}_{(B)} = M'_1 - M'_2$; $\overline{M}^{(1)}_{(B)} = (\overline{M}'^{(1)}_{1} + \overline{M}'^{(1)}_{2}) \cdot X_1$; $\overline{M}^{(2)}_{(B)} = \overline{M}^{(2)}_{(A)} = \overline{M}'^{(2)}_{1} \cdot X_2$; $M^{\Sigma}_{(A)} = \overline{M}'^{(2)}_{1}$; $M^{\Sigma}_{(B)} = M'_1 - M'_2 - \overline{M}'^{(2)}_{1} \cdot X_2 + (\overline{M}'^{(1)}_{1} + \overline{M}'^{(1)}_{2}) \cdot X_1$.

End of Table 7.5

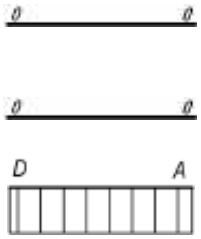
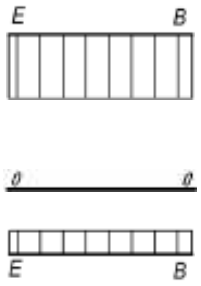
Root triangle beam	F-state, 1st unit state, 2nd unit state,	Total diagrams	Formulae for the moments for each of states
4. FFB			$M^F = 0$; $\overline{M}^{(1)} = 0$; $\overline{M}^{(2)}_{(D)} = \overline{M}^{(2)}_{(A)} = \overline{M}''^{(2)}_{3} \cdot X_2$; $M^{\Sigma}_{(D)} = M^{\Sigma}_{(A)} = \overline{M}''^{(2)}_{3} \cdot X_2$;
5. RFB			$M^F_{(E)} = M^F_{(B)} = M''_1 + M''_2$; $\overline{M}^{(1)} = 0$; $\overline{M}^{(2)}_{(B)} = \overline{M}^{(2)}_{(E)} = \overline{M}''^{(2)}_{1} \cdot X_2$; $M^{\Sigma}_{(E)} = M^{\Sigma}_{(B)} = M''_1 + M''_2 - \overline{M}''^{(2)}_{1} \cdot X_2$.

Fig. 7.3 shows total distribution diagrams of normal forces N^{Σ}_{zi} and shear flows q^{Σ}_{i}. Total distribution diagrams of bending moments in the root triangle and fuselage beams are shown in Table 7.4 (refer to Fig. 7.4 to see total constraint forces according their actual directions).

After we calculated total normal forces, shear flows, and bending moments, we have to perform the deformation check of the stress state of the wingbox. The check shows whether calculations are correct and accurate or not. For this perpose, we have to calculate the displacement which we already know. For example, we can consider any mutual displacement at points A, B, or C as they should be equal to zero.

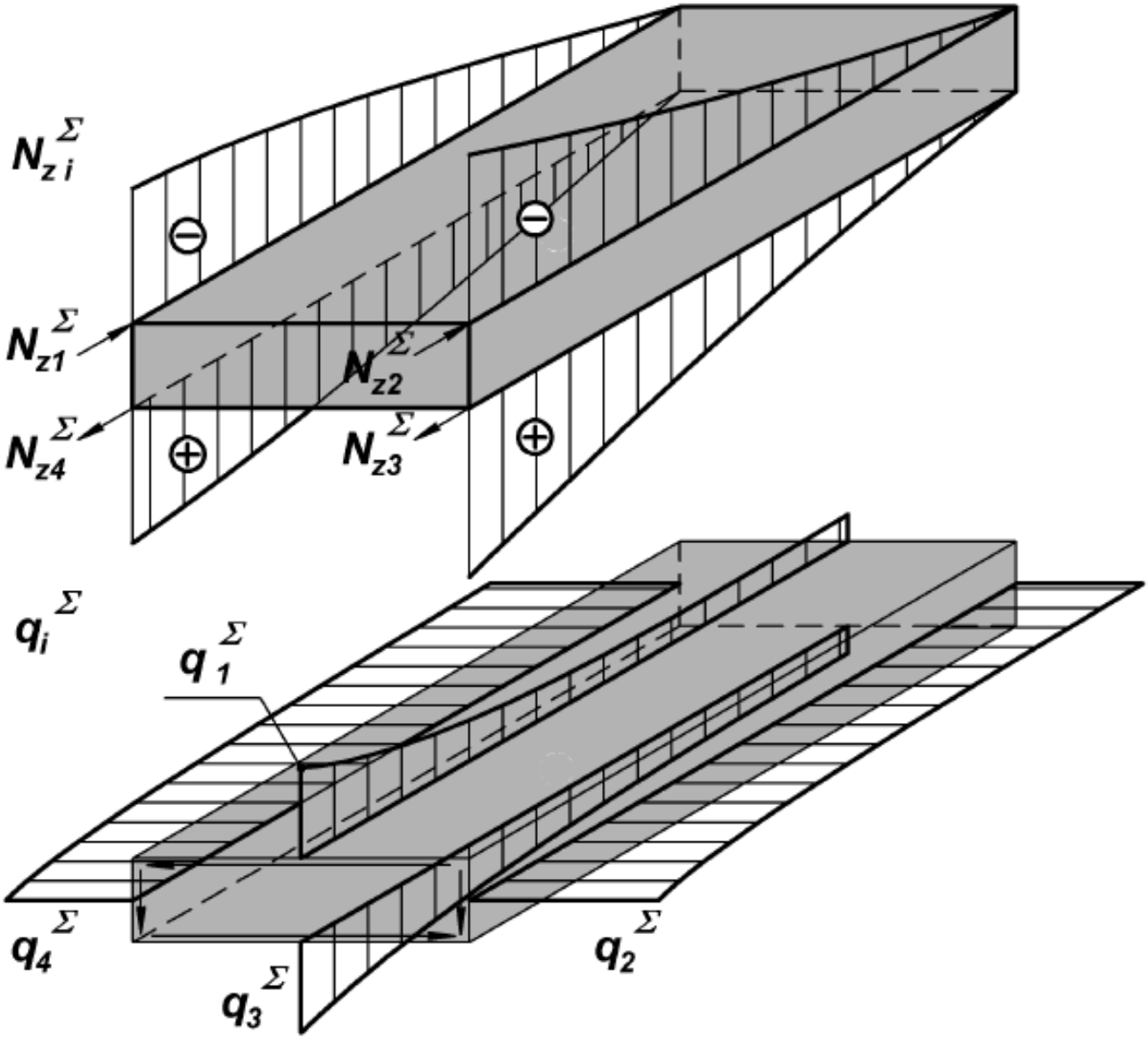


Figure 7.3. Total distribution diagrams of normal forces and shear flows

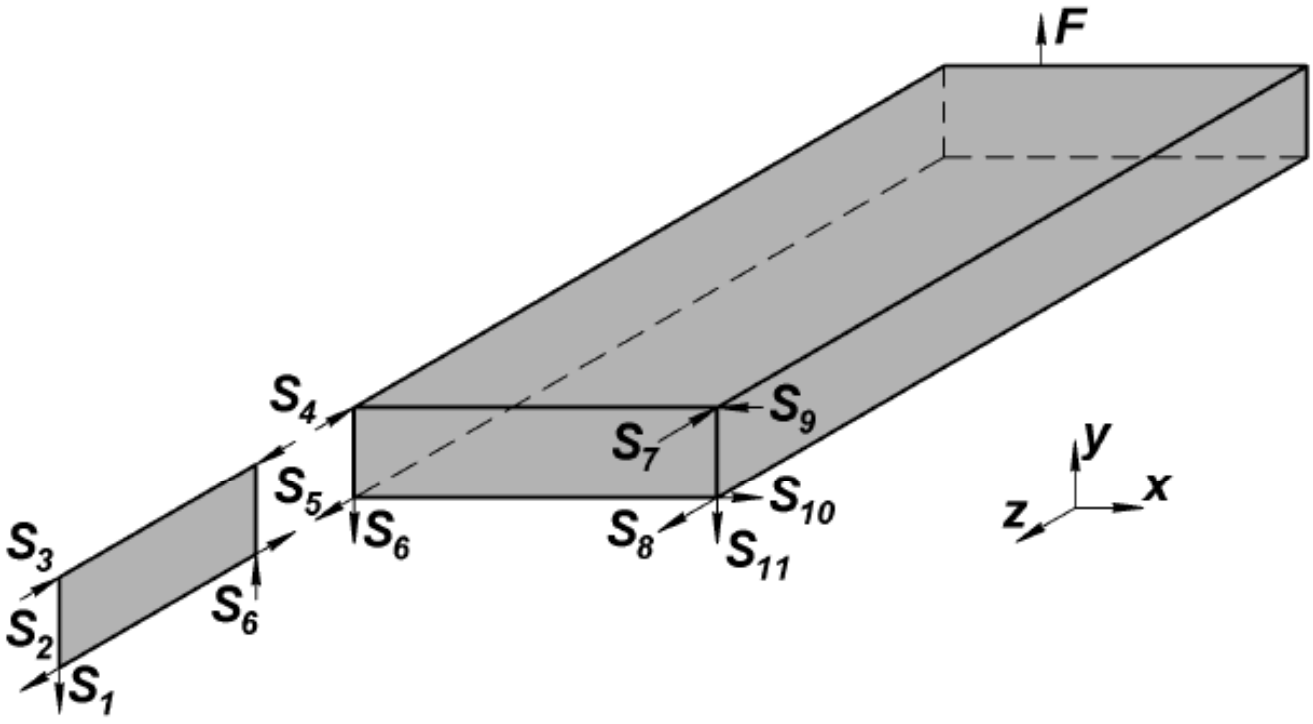


Figure 7.4. Total forces in constraints

8. DEFORMATION CHECK

To perform the deformation check, we should choose a basic system. It has to differ from the basic system in the F-state.

For example, we choose a basic system in which constraints $\#3$ and $\#11$ (Fig. 2.1) are removed (Fig. 8.1). Then we have to apply the unit force $\overline{X}_d = 1$ at points B or C instead of one thrown out constraint in the same direction. We apply the force $\overline{X}_d = 1$ at point B. In this case, we have to find the generalized displacement at point B at the direction of force $\overline{X}_d = 1$. Point B belongs to FSC, FR, and FFB.

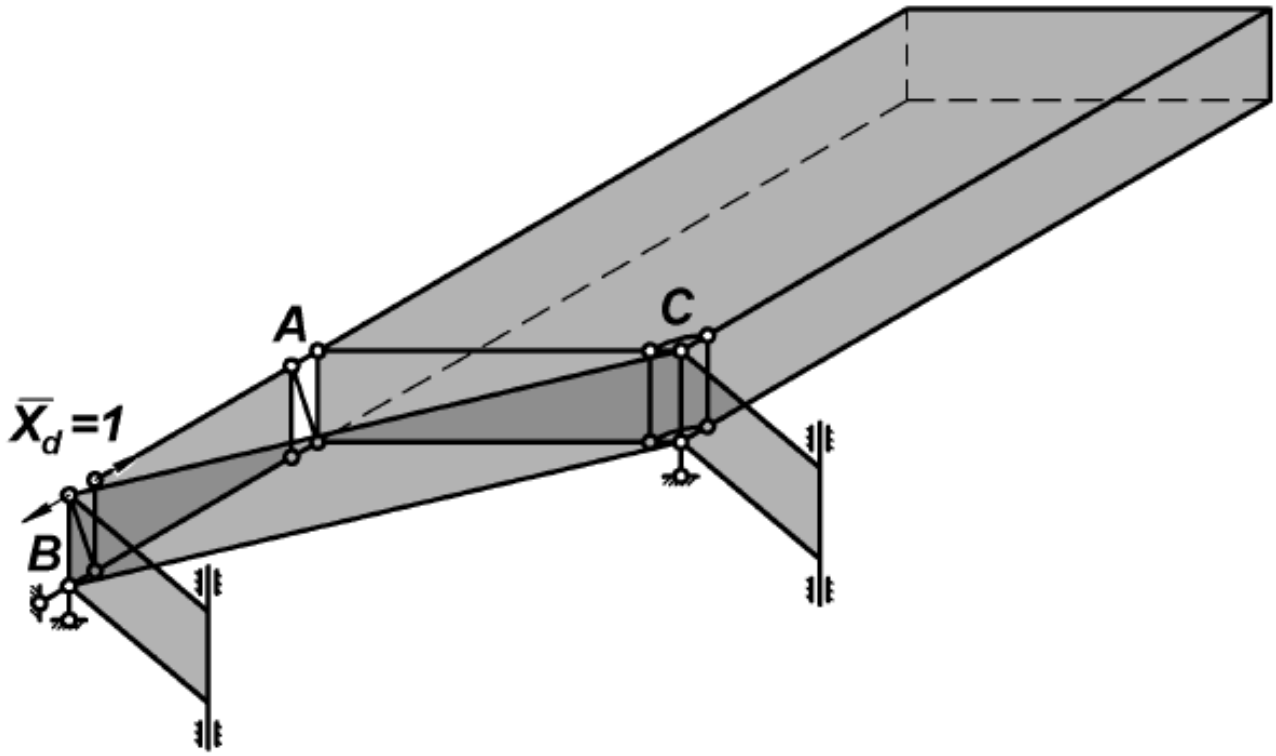


Figure 8.1. Basic system for deformation check

Let us determine values and directions of forces $\overline{S}_i^{(d)}$ in the constraints. We find them from the equilibrium equations for the FSC and the wingbox (Fig. 8.2):

$$\begin{gathered}\overline{S}_1^{(d)} = \overline{S}_6^{(d)} = \overline{S}_9^{(d)} = \overline{S}_{10}^{(d)} = 0\ ; \\ \overline{S}_2^{(d)} = \overline{S}_4^{(d)} = \overline{S}_5^{(d)} = \overline{S}_7^{(d)} = \overline{S}_8^{(d)} = 1\ .\end{gathered} \tag{8.1}$$

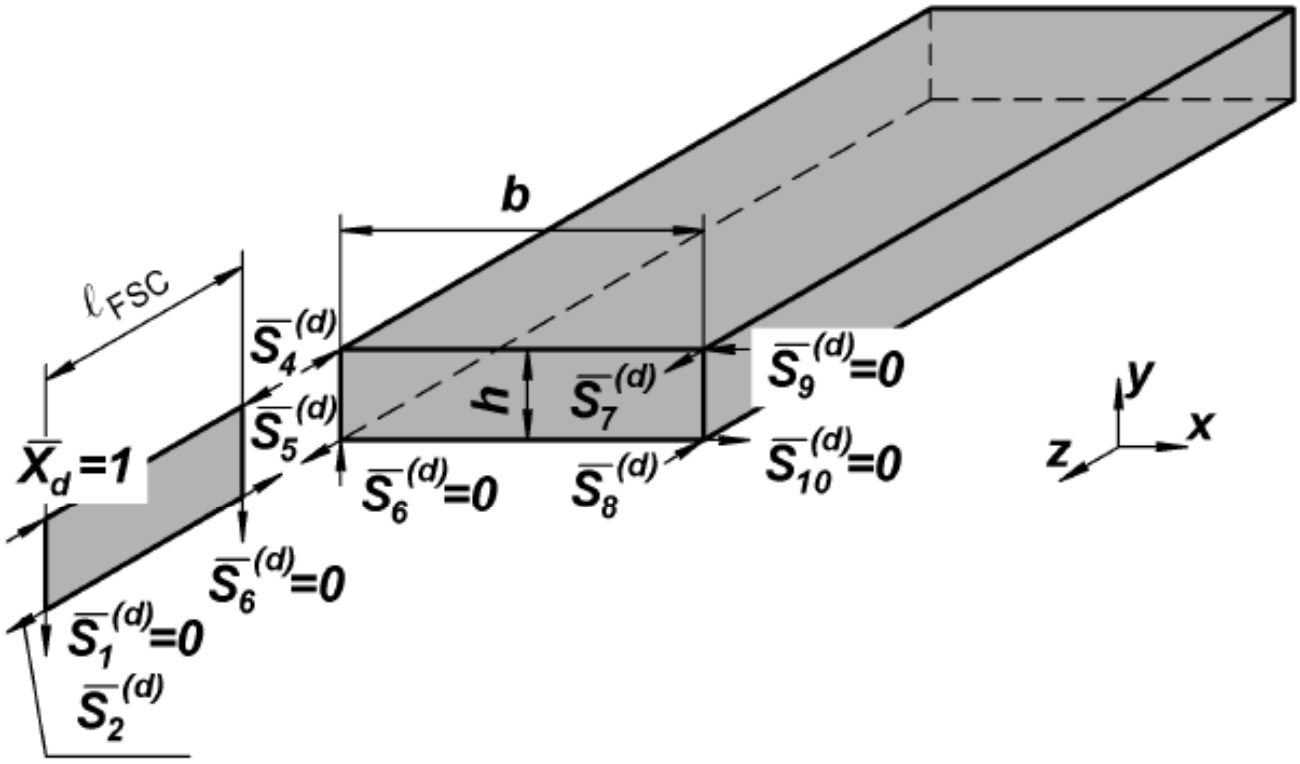


Figure 8.2. Constraint forces in basic system for deformation check

In the state $\bar{X}_d = 1$, the wingbox is loaded by the self-balanced system of forces, and normal forces and shear flows are

$$\bar{N}_{zi}^{(d)}(z) = (-1)^{i+1}\bar{C}_1^{(d)} \cdot \sinh \alpha z \; ; \qquad \bar{q}_i^{(d)} = (-1)^{i+1}\bar{B} \cdot \cosh \alpha z \; , \tag{8.2}$$

where

$$\bar{C}_1^{(d)} = -\frac{\bar{S}_4^{(d)}}{\sinh \alpha \ell} = -\frac{1}{\sinh \alpha \ell} \; ; \qquad \bar{B}^{(d)} = -\frac{\alpha \cdot \bar{S}_4^{(d)}}{2 \cdot \sinh \alpha \ell} = -\frac{\alpha}{2 \cdot \sinh \alpha \ell} \; . \tag{8.3}$$

Fig. 8.3 shows distribution diagrams of normal forces $\bar{N}_{zi}^{(d)}$ and shear flows $\bar{q}_i^{(d)}$.

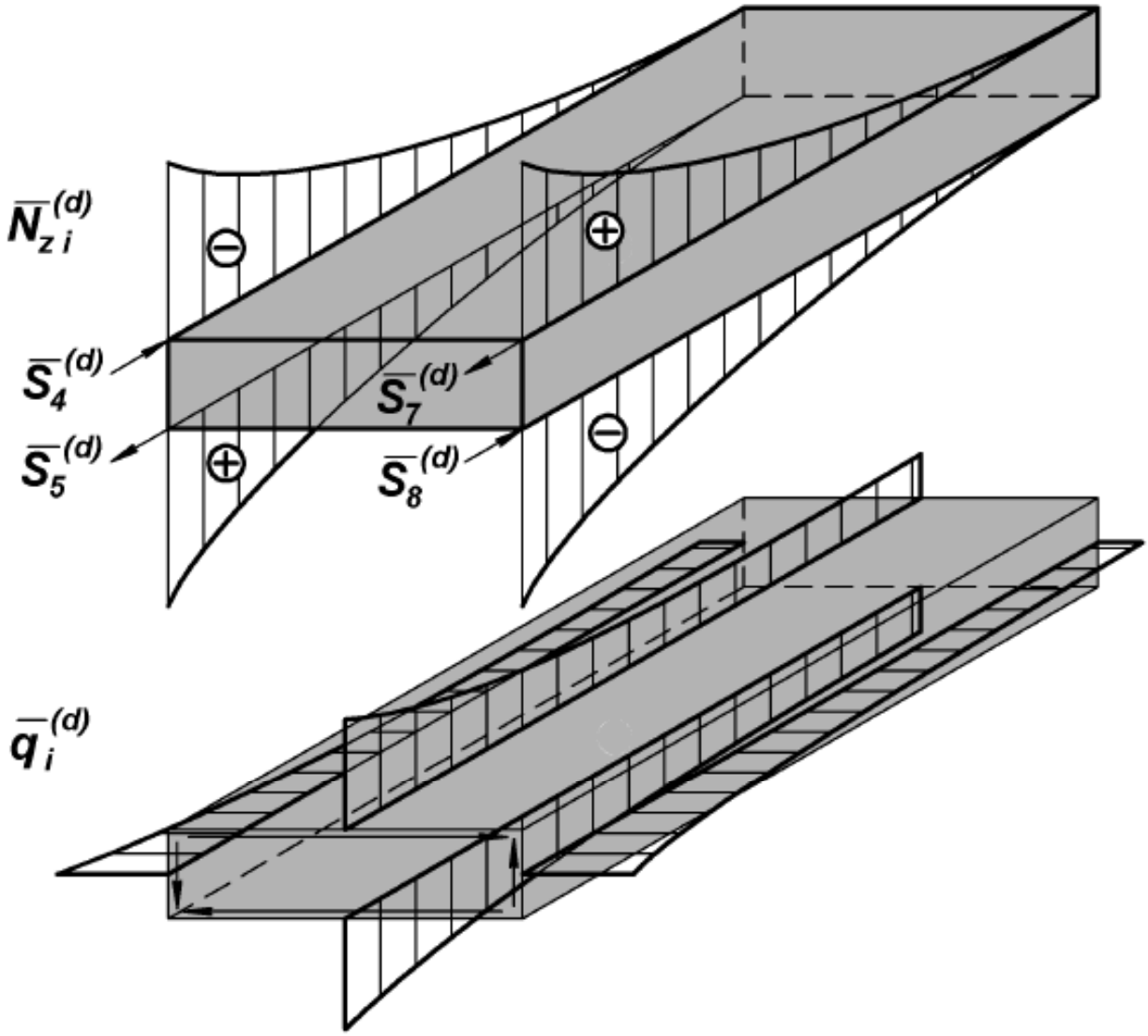


Figure 8.3. Diagrams of distribution of normal forces and shear flows in state $\overline{X}_d = 1$

Force couples at points A, B, and C (see Fig. 8.2) give moments

$$\overline{M}_{1,3}^{(d)} = \overline{X}^{(d)} \cdot h = \overline{S}_4^{(d)} \cdot h = \overline{S}_7^{(d)} \cdot h = h\,. \qquad (8.4)$$

Bending moments $\overline{M}^{(d)}$ are transferred from the wingbox to the root triangle structure at points B and C, so we have to resolve them into projections perpendicular to the beams FFB, FR, and RFB (Fig. 8.4). Fig. 8.5 shows distribution diagrams of bending moments in the beams.

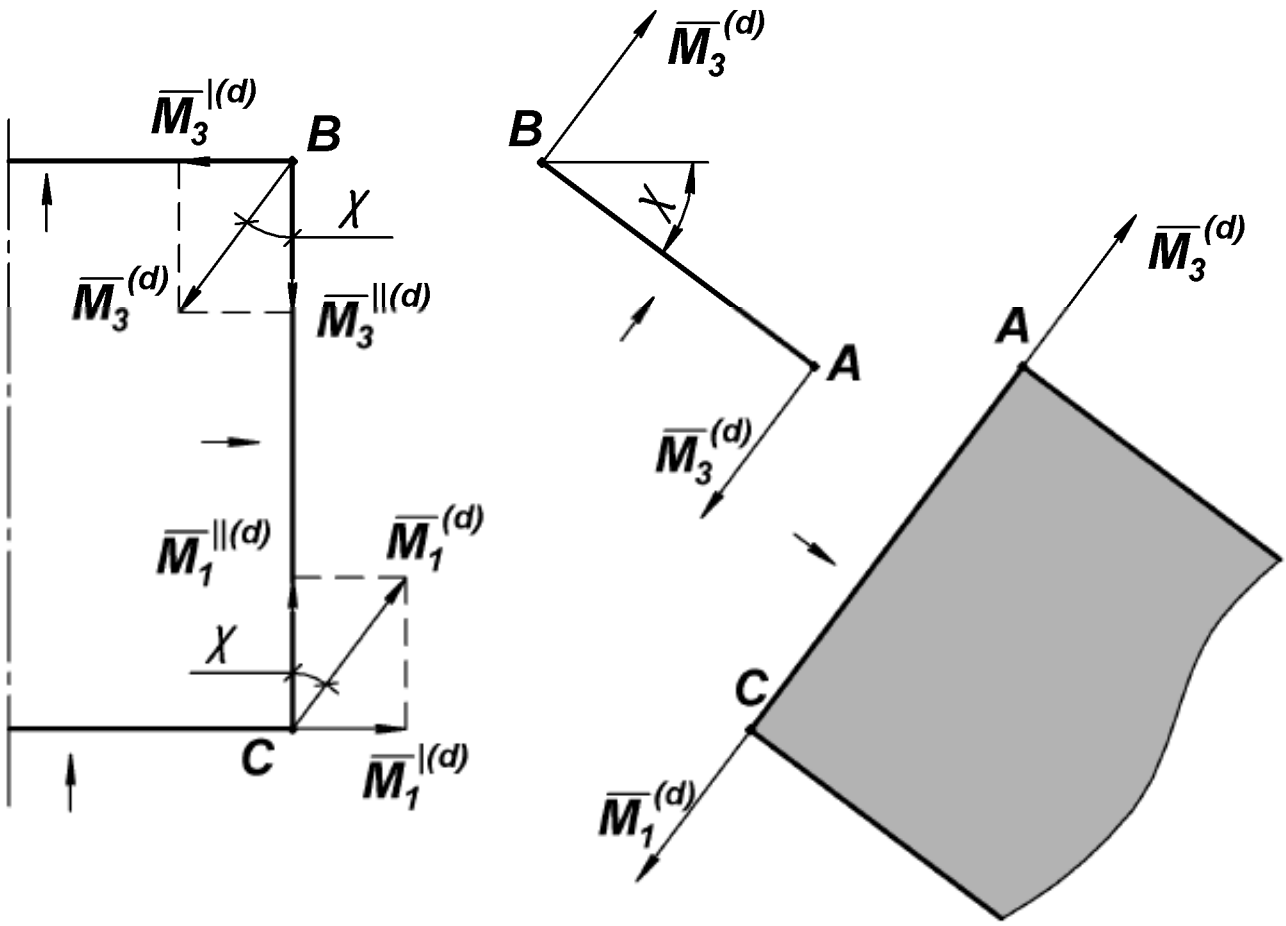


Figure 8.4. Loading of root triangle structure in state $\overline{X}_d = 1$

Let us find generalized horizontal displacement at point B. It is calculated by the Mohr's formula

$$\Delta_B = \sum_{i=1}^{4} \int_0^{\ell} \frac{N_{zi}^{\Sigma}(z) \cdot \overline{N}_{zi}^{(d)}(z)}{E_i \cdot A_i} dz +$$

$$+ \sum_{i=1}^{4} \int_0^{\ell} \int_0^{s} \frac{q_i^{\Sigma}(z) \cdot \overline{q}_i^{(d)}(z)}{G_i \cdot \delta_i} ds dz + \sum_{n=1}^{5} \int_0^{\ell_n} \frac{M_n^{\Sigma} \cdot \overline{M}_n^{(d)}}{E_n \cdot I_n} dx \ . \qquad (8.5)$$

This displacement should be equal to zero.

The terms of (8.5) are calculated in the similar way as the coefficients in canonical equations. However, in this case we have to consider two states, F-state of the basic system and the unit state of basic system chosen for deformation check.

We have to find the percentage of calculation errors by the formula

$$\gamma = \frac{2 \cdot |D_1 - D_2|}{|D_1| + |D_2|} \cdot 100\% . \qquad (8.6)$$

where D_1 is the sum of all positive terms in (8.5), and D_2 is the sum of all negative terms in (8.5).

The percentage of errors must not exceed 3%.

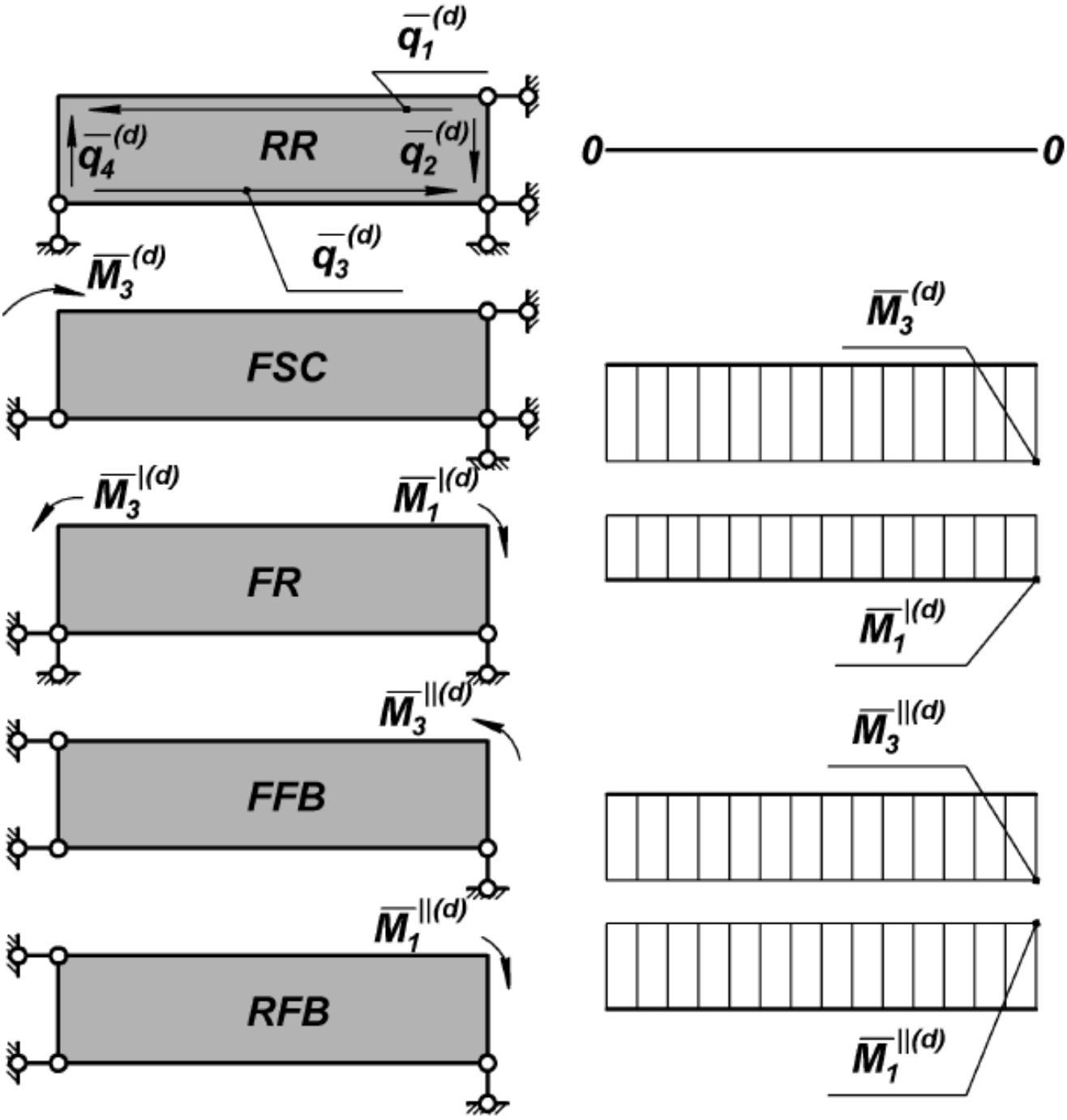


Figure 8.5. Diagrams of distribution of bending moments in state $\bar{X}_d = 1$

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