MINISTRY OF EDUCATION AND SCIENCE,
National aerospace university named after N. Ye. Zhukovskiy
“Kharkov Aviation Institute”

THEORETICAL MECHANICS.
DYNAMICS

Tutorial

Kharkov “KhAI” 2013

UDK [531+531.1] (075.8)
T33

Розглянуто динаміку точки, загальні теореми динаміки: про змінення кількості руху, про змінення моменту кількості руху і про змінення кінетичної енергії, а також рівняння плоско-паралельного руху. У конспективній формі наведено відомості з теоретичного курсу, основні формули і пояснення до них. Подано розв’язання задач різної складності.

Для студентів механічних та інших спеціальностей (з повною та скороченою програмою з теоретичної механіки).

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T33 Theoretical mechanics. Kinematics: Tutorial for self-education / V. N. Pavlenko, I. V. Bunyaeva, S. S. Ternovskaya et al. – Kharkov: National Aerospace University named after N. Ye. Zhukovskiy «Kharkov Aviation Institute», 2013. – 184 p.

The workbook considers dynamics of a particle, general principles of dynamics: Force-Linear momentum, Moment-Angular Momentum, Work-Energy, plane motion kinetics is presented. The information from the theoretical course, the basic formulas and their explanations are given in the workbook. The problems of different complexity are presented.

For college students studying theoretical mechanics (for full-time and for reduced courses of study).

Figs. 154. Table 1. Bibliography: 3 names

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CONTENT

LECTURE 1
1. INTRODUCTION

Mechanics is the branch of the physical sciences that deals with the mechanical motion of bodies, i.e. changing of relative position of bodies in space in the course of time.

Our course is subdivided into four parts: statics, kinematics, dynamics, and analytical mechanics.

Statics is the branch of mechanics, which treats of bodies that are at rest or in the state of uniform motion. Statics studies the laws of composition of forces and the conditions of equilibrium of engineering structures under the action of forces.

Kinematics is study of the geometry of motion without regard to the forces the cause that motion.

Dynamics deals with the action of forces in producing or modifying the motion of bodies.

1.1. Basic conceptions: space, time, frame of reference

As you know, motion is changing with time of the particle (or body) position with respect to the position of some other particle (or body). Therefore, to observe the motion we use two base notions: space and time.

In Newtonian mechanics ideal model of space is used. The model can be visualized by non-limiting rigid body. It is supposed that the space affects the other physical phenomena, but the space itself is not affected by those phenomena.

Such space is called **absolute space.** The absolute space is Euclid space, it means that Euclid's geometry is valid in the space**.**

To determine particle (body) position in space we choose at least one frame of reference consisting of two components: a datum or origin and a system of three linear independent directions (coordinate axes). In the Newtonian mechanics it is postulated that there is at least one fixed frame of reference in which all Newton's laws are valid. Such frame of reference is called **absolute or inertial**. For our purposes we can choose as the inertial frame of reference the heliocentric reference (or geocentric reference). Frames of reference where objects violate Newton's first law are called **noninertial**.

The second ideal model of Newtonian mechanics is the **absolute time.** It is supposed that time runs at the same rate for all the observers in the absolute space.

1.2. Axioms of dynamics

The first three axioms known as Newton's laws of motion:

1. A particle isolated from other bodies remains at rest or continues to move in straight line with a constant velocity if the resultant force acting on the particle is zero.

In other words, a particle initially at rest is predicted to remain at rest if the resultant force acted on is zero, and an particle in motion remains in motion with the same velocity in the same direction:

$$if\ \sum_{i=1}^{n}\vec{F}_i = 0\ then\ \vec{v} = 0,\ or\ \vec{v} = const.$$

The converse of Newton's first law is also true: if we observe an object moving with constant velocity along a straight line, then the total force on it must be zero**:**

$$if\ \vec{v} = 0,\ or\ \vec{v} = const,\ then \sum_{i=1}^{n}\vec{F}_i = 0.$$

2. **A free particle acted on by a single force is accelerated; the acceleration is in the direction of the force and is directly proportional to the force and inversely proportional to the mass of the particle**

$$\boxed{m\vec{W} = \vec{F}.} \qquad (1.1)$$

The property, by virtue of which a particle tends to remain at rest or in uniform rectilinear motion, and to resist being accelerated, is called **inertia**. **Inertial mass** is a measure of inertia of the particle which is its resistance to rate of change of velocity when a force is applied. An object with small inertial mass changes its motion more readily, and an object with large inertial mass does so less readily.

In the solution of problems Eq. (1.1) is usually expressed in scalar component form using one of the coordinate systems developed in kinematics (Cartesian, natural, polar, cylindrical). Eq. (1.1), or any one of its component forms is usually referred to as the **equation of free particle motion in inertial frame of reference**. The equation of motion gives the instantaneous value of the acceleration corresponding to the instantaneous values of the forces which are acting.

3. **To every action there is always an equal and contrary reaction**.

The axiom states that forces there are always in pairs.

It means that if body 1 (Fig.1.1) acts on body 2 with the force $\vec{\boldsymbol{F}}_{12}$ and body 2 acts on body 1 with the force $\vec{\boldsymbol{F}}_{21}$, these forces satisfy the equation

$$\vec{F}_{21} = -\vec{F}_{12}$$

and act along the same line.

It is important to remember that the forces of action and reaction (e.g. $\vec{F}_{21}$ and $\vec{F}_{12}$) do not form a balanced system of forces because they are applied to different bodies.

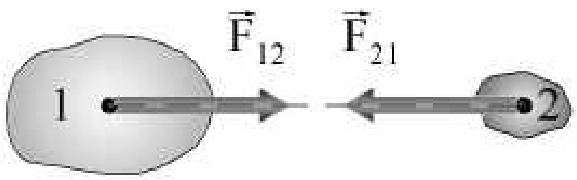


Fig. 1.1

4***. Principle of superposition.***

The resulting acceleration caused by two or more forces is the geometrical sum of the accelerations which would have been caused by each force individually.

Assume that a force system $\left(\vec{F}_1, \vec{F}_2, .., \vec{F}_n\right)$ acts on a particle (concurrent force system). Each force produces acceleration $\vec{W}_i$:

$$\vec{W}_i = \frac{\vec{F}_i}{m}.$$

Resultant force of the concurrent force system is

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i.$$

Resulting acceleration is

$$\vec{W} = \frac{\vec{F}}{m} = \sum_{i=1}^{n} \frac{\vec{F}_i}{m} = \sum_{i=1}^{n} \vec{W}_i. \tag{1.2}$$

4. ***The parallelogram law.*** **Two forces applied at one point of a body (Fig. 1.2) have as their resultant a force applied at the same point and represented by the diagonal of a parallelogram constructed with the two given forces as its sides, i.e. a force system** $\left(\vec{F}_1, \vec{F}_2\right)$ **is equivalent to its resultant** $\vec{F}$.

Magnitude of the resultant can be determined in accordance with cos-theorem

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}\,. \tag{1.3}$$

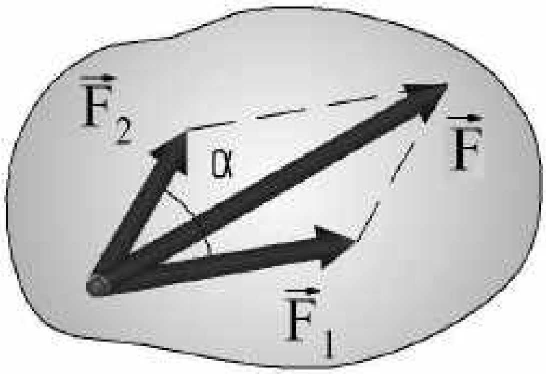


Fig. 1.2

2. A SINGLE PARTICLE DYNAMICS

2.1. Different forms of free particle equation of motion in inertial frame of reference

Consider the equation of free particle motion in inertial frame of reference (1.1). There are different forms of the equation.

Remembering that $\vec{W} = \dfrac{d^2\vec{r}}{dt^2} = \dfrac{d\vec{v}}{dt}$ rewrite eq. (1.1) in vector forms:

$$\boxed{m\frac{d^2\vec{r}}{dt^2} = \vec{F}}; \tag{1.4}$$

$$\boxed{m\frac{d\vec{v}}{dt} = \vec{F}}. \tag{1.5}$$

Coordinate forms (scalar forms) are obtained by projecting of vector eq. (1.1) onto coordinate axes:

a) in a Cartesian coordinate system:

$$\vec{W} = W_x\vec{i} + W_y\vec{j} + W_z\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k},$$

$$\boxed{\begin{cases} m\ddot{x} = F_x, \\ m\ddot{y} = F_y, \\ m\ddot{z} = F_z. \end{cases}} \qquad (1.6)$$

b) in a natural coordinate system (useful for curvilinear path):

$$\vec{W} = \vec{W}^{\tau} + \vec{W}^{n} + \vec{W}^{b},$$

$$\vec{W}^{\tau} = \ddot{\sigma}\vec{\tau},\ \vec{W}^{n} = \frac{(\dot{\sigma})^2}{\rho}\vec{n},\ \vec{W}^{b} = 0,$$

$$\begin{cases} m\ddot{\sigma} = F^{\tau}, \\ m\dfrac{(\dot{\sigma})^2}{\rho} = F^{n}, \\ 0 = F^{b}. \end{cases} \qquad (1.7)$$

Nota. Component $m\dfrac{(\dot{\sigma})^2}{\rho}$ is always positive so F^n (Fig.1.3) is in direction of unit vector of principal normal and resultant force acting on the particle is in direction of trajectory concavity.

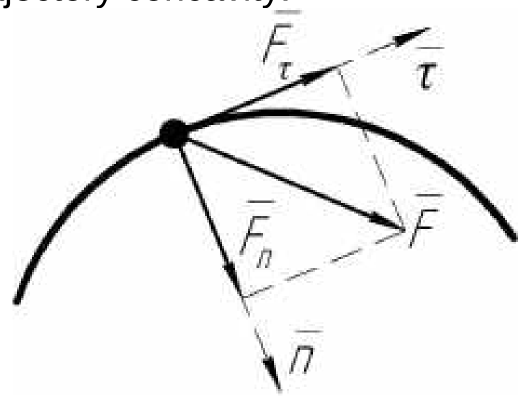


Fig. 1.3

c) in a polar coordinate system (for plane motion of a particle)

$$\vec{W} = \vec{W}^r + \vec{W}^\varphi,$$
$$\vec{W}^r = \left(\ddot{r} - r\dot{\varphi}^2\right)\vec{r}^{\,0},\ \vec{W}^\varphi = \left(r\ddot{\varphi} + 2\dot{\varphi}\dot{r}\right)\vec{p}^{\,0},$$

where $\vec{r}^{\,0}$ is unit vector along the position vector in direction of the position vector increasing, $\vec{p}^{\,0}$ is unit vector, it makes angle 90^0 with $\vec{r}^{\,0}$ in direction of the angle φ increasing (Fig. 1.4). So equation (1.1) in projections on $\vec{r}^{\,0}$ and $\vec{p}^{\,0}$ is:

$$\begin{cases} m\left(\ddot{r} - r\dot{\varphi}^2\right) = F^r, \\ m\left(r\ddot{\varphi} + 2\dot{\varphi}\dot{r}\right) = F^\varphi. \end{cases} \tag{1.8}$$

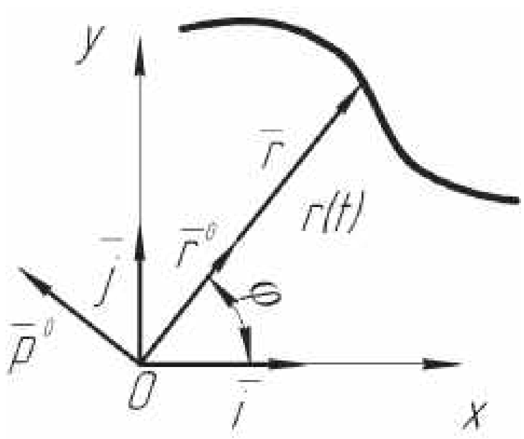


Fig. 1.4

The choice of the appropriate coordinate system is dictated by the type of motion involved and is a vital step in the formulation of any problem.

2.2. Two problems of dynamics

We encounter two types of problems when applying Eq. (1.1). **In the first type** the acceleration is either specified or can be determined directly from known kinematic conditions. The corresponding forces which act on the particle whose motion is specified are then determined by direct substitution into Eq. (1.1). This problem is generally quite **straightforward**.

If motion is given in coordinate form

$$\begin{cases} x = f_1(t), \\ y = f_2(t), \\ z = f_3(t); \end{cases} \tag{1.9}$$

then force produced this motion has components

$$\begin{cases} F_x = m\dfrac{d^2\left(f_1(t)\right)}{dt^2}, \\ F_y = m\dfrac{d^2\left(f_2(t)\right)}{dt^2}, \\ F_z = m\dfrac{d^2\left(f_3(t)\right)}{dt^2}. \end{cases} \tag{1.10}$$

In the second type of problem the forces are specified and the resulting motion is to be determined. If the forces are constant, the acceleration is constant and is easily found from Eq. (1.1). When the forces are functions of time, position, velocity, or acceleration, Eq. (1.1) becomes a differential equation which must be solved to determine the velocity and displacement. The Eq. (1.1) have to supplement the proper quantity of initial conditions to obtain single-valued solution (6 initial conditions for 3D case). This problem is called **inverse**.

Case 1. Force is constant (force of gravity) or function of time (force of interaction between core and magnetizing coil driven by alternating current):

$$\begin{cases} F_x = f_1(t), \\ F_y = f_2(t), \\ F_z = f_3(t). \end{cases} \tag{1.11}$$

Case 2. Force is function of particle coordinates
Force of elasticity

$$\vec{F}_{el} = -c\vec{r}\ , \tag{1.12}$$

$$F_{el\_x} = -cx,$$
$$F_{el\_y} = -cy, \tag{1.13}$$
$$F_{el\_z} = -cz.$$

Gravitational force

$$\vec{F} = -f\frac{m_1 m_2}{r^2}\vec{r} \ , \tag{1.14}$$

$$F_x = -f\frac{m_1 m_2}{r^2}x,$$
$$F_y = -f\frac{m_1 m_2}{r^2}y, \tag{1.15}$$
$$F_z = -f\frac{m_1 m_2}{r^2}z.$$

Case 3. Force is function of particle velocity.

Aerodynamics drag force acting on particle in rectilinear motion is

$$\vec{F}_{ad} = -f(v)\frac{\vec{v}}{v},\ F_{ad\,x} = -f(v)\frac{\dot{x}}{v}. \tag{1.16}$$

For a particle under the action of aerodynamics drag force directly proportional to the speed

$$F_{ad\,x} = -f(v)\frac{\dot{x}}{v} = \left| f(v) = kv \right| = -k\dot{x}. \tag{1.17}$$

For a particle under the action of aerodynamics drag force directly proportional to the speed squired

$$F_{ad\,x} = -f(v)\frac{\dot{x}}{v} = \left| f(v) = kv^2 \right| = -k\dot{x}^2. \tag{1.18}$$

The example of problem about particle motion under the action of aerodynamics drag force will be considered at the end of the lecture.

Lorentz force (Axis Ox is parallel to magnetic field $\vec{H}$) is

$$\vec{F}_L = e\vec{v} \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e\dot{x} & e\dot{y} & e\dot{z} \\ H & 0 & 0 \end{vmatrix}. \tag{1.19}$$

Summarizing. In general case force is function of time, particle position and velocity:

$$\vec{F} = f\left(t, \vec{r}, \dot{\vec{r}}\right), \tag{1.20}$$

$$\begin{cases} F_x = f_1\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right), \\ F_y = f_2\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right), \\ F_z = f_3\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right). \end{cases} \tag{1.21}$$

Equations (1.6) can be rewriting with help of equations (1.21) in the following form

$$\begin{cases} m\ddot{x} = f_1\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right), \\ m\ddot{y} = f_2\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right), \\ m\ddot{z} = f_3\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}\right). \end{cases} \tag{1.22}$$

The system (1.22) is system of second order differential equations with respect to unknown functions $x(t), y(t), z(t)$. This system is termed **main differential equations of particle motion in inertial frame of reference**.

The first integral of the system (1.22) is

$$\begin{cases} \Phi_1\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, C_1, C_2, C_3\right) = 0, \\ \Phi_2\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, C_1, C_2, C_3\right) = 0, \\ \Phi_3\left(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, C_1, C_2, C_3\right) = 0. \end{cases} \tag{1.23}$$

The second integral of the system (1.22) is

$$\begin{cases} \Psi_1\left(t, x, y, z, C_1, C_2, C_3, C_4, C_5, C_6\right) = 0, \\ \Psi_2\left(t, x, y, z, C_1, C_2, C_3, C_4, C_5, C_6\right) = 0, \\ \Psi_3\left(t, x, y, z, C_1, C_2, C_3, C_4, C_5, C_6\right) = 0. \end{cases} \tag{1.24}$$

The equations (1.23) and (1.24) can be satisfied by the substituting the six initial conditions into equations and solving for the constants $C_1, C_2, ..., C_6$:

$$\text{when } t = 0 \begin{cases} x = x_0, y = y_0, z = z_0, \\ \dot{x} = \dot{x}_0, \dot{y} = \dot{y}_0, \dot{z} = \dot{z}_0. \end{cases}$$

Final equations of the particle motion are

$$\begin{cases} x(t) = f_1\left(t, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\right), \\ y(t) = f_2\left(t, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\right), \\ y(t) = f_3\left(t, x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\right). \end{cases} \quad (1.25)$$

So the second problem of dynamics is Cauchy problem or initial value problem.

2.3. Constrained and unconstrained motion

There are two physically distinct types of motion. The first type is *unconstrained* motion where the particle is free of mechanical guides and follows a path determined by its initial motion and by the forces which are applied to it from external sources. An airplane or rocket in flight and an electron moving in a charged field are examples of unconstrained motion.

The second type is *constrained* motion where the path of the particle is partially or totally determined by restraining guides. An ice hockey puck moves with the partial constraint of the ice. A train moving along its track and a collar sliding along a fixed shaft are examples of more fully constrained motion. The forces acting on a particle during constrained motion can be broken into two groups:

– applied from outside sources (applied forces)
– forces on the particle from the constraining guides (reactions)

All forces, both applied and reactions must be accounted for in equation of motion:

$$\boxed{m\vec{W} = \vec{F} + \vec{N},} \quad (1.26)$$

where $\vec{F}$ is resultant of applied forces,

$\vec{N}$ is resultant of reactions.

The choice of a coordinate system is frequently indicated by the number and geometry of the constraints. Thus, if a particle is free to move in space, as is the center of mass of the airplane or rocket in free flight, the particle is said to have *three degrees of freedom* since three independent coordinates are required to specify the position of the particle at any instant. All three of the scalar components of the equation of motion would have to be applied and integrated to obtain the space coordinates as a function of time. If a particle is constrained to move along a surface, as is the hockey puck or a marble sliding on the curved surface of a bowl, only two coordinates are needed to specify its position, and in this case it is said to have *two degrees of freedom.* If a particle is constrained to move along a fixed linear path, as is the collar sliding along a fixed shaft, its position may be specified by the coordinate measured along the shaft. In this case the particle would have only *one degree of freedom.*

2.4. Examples

2.4.1. Free particle motion

Example 1. A particle of a mass **m=2 kg** moves along a horizontal x-axis under action of the force $F_x = 5\cos(\pi t)$**.** Determine the velocity of the particle at the moment **t=4 s** if at **t_0=0** the velocity is **V_0=0**.

Solution

The particle moves only along x-axis that's why we write the projection of the general equation of free particle motion on x-axis:

$$mW_x = F_x,$$

$$m\ddot{x} = 5\cos\left(0.5t\right),$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{5}{m}\cos\left(0.5t\right),$$

$$d\dot{x} = \frac{5}{m}\cos\left(0.5t\right)dt,$$

$$\int d\dot{x} = \frac{5}{m}\int\cos\left(0.5t\right)dt,$$

$$\dot{x} = \frac{5}{0.5m}\sin\left(0.5t\right) + C_1. \tag{1.27}$$

Initial condition is: at $t=0$ $V(0) = \dot{x}(0) = 0$.

Substituting $t=0$ into equation (1.27) we get:

$$\dot{x}(0) = 0 = \frac{5}{0.5m}\sin\left(0.5 \cdot 0\right) + C_1 = C_1,$$

$$C_1 = 0.$$

So velocity is $V = \dfrac{5}{0.5m}\sin\left(0.5t\right)$. At time $t = 4, s$ velocity of the particle is:

$$V\left(4\right) = \frac{5}{0.5 \cdot 2}\sin\left(0.5 \cdot 4\right) = 4.55\left(\frac{m}{s}\right).$$

Answer: $V(4\,\text{s}) = 4.55\left(\dfrac{\text{m}}{\text{s}}\right)$.

Example 2. A **18-N** body moves in the air and traces out the paths, represented in fig. 10. Its initial velocity is $\mathbf{V_0}$ **= 700 m/s**. The missile launching velocity makes an angle of **75°** with the horizontal. Determine the increase in the altitude reached (in kilometers) and increase in the range of flight, if there is no air resistance.

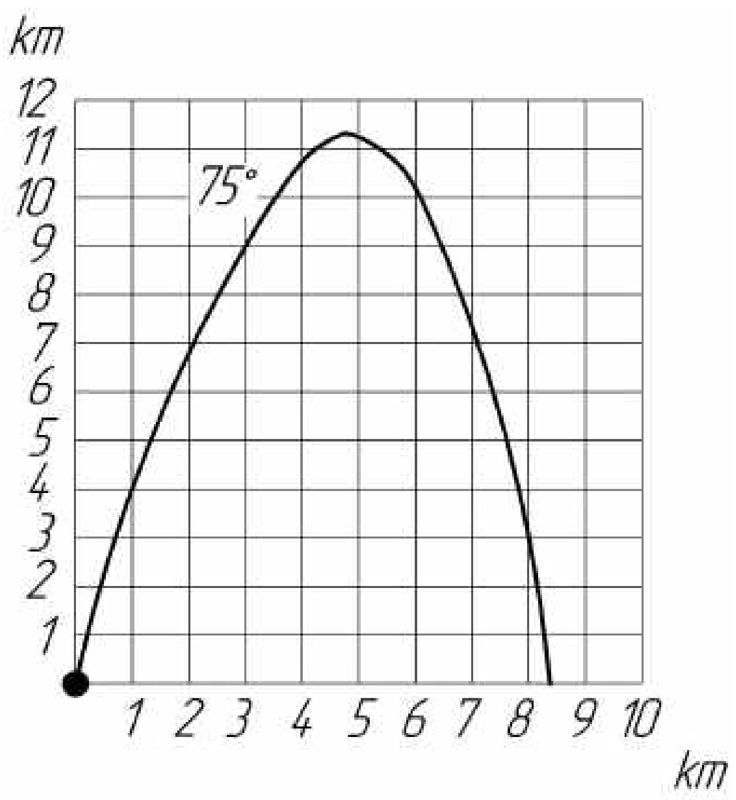


Fig.1.5

Solution

In Fig. 1.5 a real path of the missile is shown (air resistance is took into account). Let us analyze the missile motion if air resistance is neglected. Idealize missile by a particle.

General equation of free particle motion in vector form is (1.1)

$$m\vec{W} = \vec{F}. \tag{1.28}$$

Only gravity force acts on the particle (fig. 1.6) so equation (1.28) is

$$m\vec{W} = \vec{P}. \tag{1.29}$$

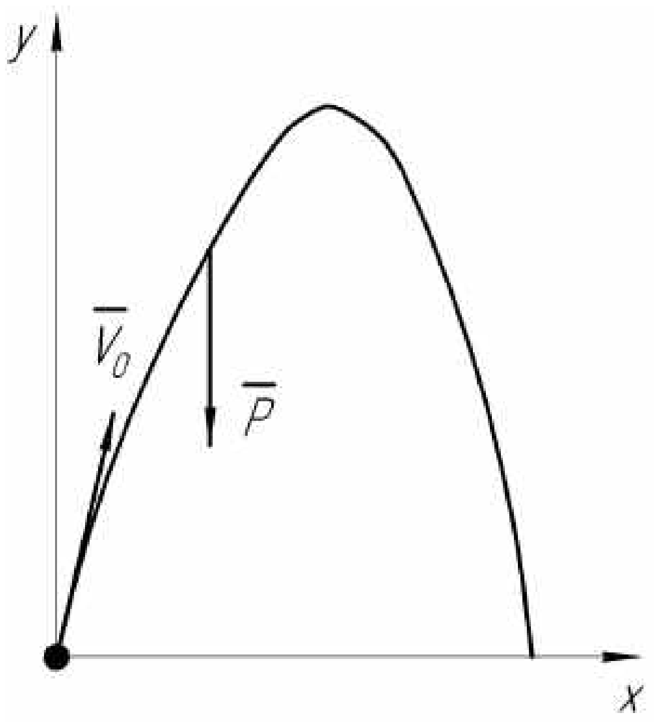


Fig.1.6

Projecting equation (1.29) along x and y we get:

$$x)\ m\ddot{x} = 0, \tag{1.30}$$

$$y)\ \ m\ddot{y} = -P. \tag{1.31}$$

To find the particle velocity components we have to integrate these equations. For equation (1.30):

$$\ddot{x} = 0, \quad \ddot{x} = \frac{d\dot{x}}{dt} = 0,$$

$$d\dot{x} = 0dt \text{ therefore } \dot{x} = C_1,$$

$$\dot{x} = \frac{dx}{dt} = C_1, \tag{1.32}$$

$$x = C_1 t + C_2. \tag{1.33}$$

In the same manner integrate equation (1.31):

$$m\ddot{y} = -P,$$

$$m = \frac{P}{g};$$

$$\ddot{y} = -g,$$

$$\ddot{y} = \frac{d\dot{y}}{dt} = -g;$$

$$\int d\dot{y} = -\int g dt \;\Rightarrow\; \dot{y} = -gt + C_3,$$

$$\dot{y} = -gt + C_3; \tag{1.34}$$

$$\int dy = \int (-gt + C_3) dt,$$

$$y = -g\frac{t^2}{2} + C_3 t + C_4. \tag{1.35}$$

Use initial conditions (particle coordinates and velocity components at time t=0 s) to find the constants of integration:

$$\begin{cases} x(0) = 0, \\ y(0) = 0, \end{cases}$$

$$\begin{cases} \dot{x}(0) = V_0 \cos 75°, \\ \dot{y}(0) = V_0 \sin 75°. \end{cases}$$

Putting to the equations (1.32) – (1.35) we get:

$$\dot{x}(0) = C_1 = V_0 \cos 75°,$$

$$x(0) = C_2 = 0,$$

$$\dot{y}(0) = C_3 = V_0 \sin 75°,$$

$$y(0) = C_4 = 0.$$

Finally equations of particle motion are:

$$x = V_0 \cos 75° t; \tag{1.36}$$

$$y = -g\frac{t^2}{2} + V_0 \sin 75°t. \quad (1.37)$$

Next we find the maximum altitude reached by the missile and the range of flight if there is no air resistance. The particle has maximum altitude (y_{max}) at time t_1 when the velocity y-component $\dot{y}$ is zero:

$$\dot{y} = 0 = -gt_1 + V_0 \sin 75°,$$

$$t_1 = \frac{V_0 \sin 75°}{g}.$$

Substituting t_1 into equation (1.37) we get the maximum altitude:

$$y_{\max} = y(t_1) = -\frac{g}{2}\frac{V_0^2 \sin^2 75°}{g^2} + V_0 \sin 75° \frac{V_0 \sin 75°}{g} =$$

$$= -\frac{V_0^2 \sin^2 75°}{2g} + \frac{V_0^2 \sin^2 75°}{g} = \frac{V_0^2 \sin^2 75°}{2g} = 23.3\,(km).$$

Maximum range of flight (x_{max}) corresponds to the particle position when $t > 0$ and $y = 0$:

$$y = 0 = -g\frac{t_2^2}{2} + V_0 \sin 75°t_2,$$

$$t_2\left(-\frac{g}{2}t_2 + V_0 \sin 75°\right) = 0.$$

There are two roots of the equation: $t_2 = 0.$ and $t_2 = \dfrac{2V_0 \sin 75°}{g}$. The first corresponds to initial position $x = 0$, the second root is flight duration so substituting t_2 into equation (9) we get the maximum flight range:

$$x_{\max} = x(t_2) = V_0 \cos 75° \frac{2V_0 \sin 75°}{g} = \frac{2V_0^2 \cos 75° \sin 75°}{g} = 25\,(km).$$

Now we should compare 2 cases: without air resistance and with it.

The maximum altitude and the flight range in the case with air resistance we can determine using the figure 10: $y_{\max\_AR} = 11.5\,km$, $x_{\max\_AR} = 8.5\,km.$

The increase in altitude is

$$\Delta y = y_{\max} - y_{\max\_AR} = 23.3 - 11.5 = 11.8(km).$$

The increase in range of flight is

$$\Delta x = x_{\max} - x_{\max\_AR} = 25 - 8.5 = 16.5(km).$$

Answer: $\Delta y = 11.8(km), \Delta x = 16.5(km).$

Example 3. A flexible thread, fixed at the point A, passes through a smooth fixed ring O (Fig.1.7). A small ball of mass **m (kg)** is attached to a free end of the thread. The natural length of the thread is **l=AO**. A force equal to **k^{2}m (N)** must be applied to elongate the thread **1 m**. When the thread is stretched along the straight line AB until its length is doubled, then the ball is given a velocity **V$_0$**, perpendicular to AB. Find the path of the ball. Neglect the effect of gravity and assume that the tension in the thread is proportional to its extension.

Solution

Idealize the ball by a particle. Under condition of gravity neglecting there is only one force acting on the particle, it is tension force that is directed along the thread.

General equation of free particle motion in vector form is (1.1)

$$m\vec{W} = \vec{F} = \vec{T}. \qquad (1.38)$$

Choose rectangular coordinate system with origin at O.

Projecting the equation (1.38) on the axes x and y we get:

$$\begin{cases} x)\ m\ddot{x} = T_x, \\ y)\ m\ddot{y} = T_y. \end{cases} \qquad (1.39)$$

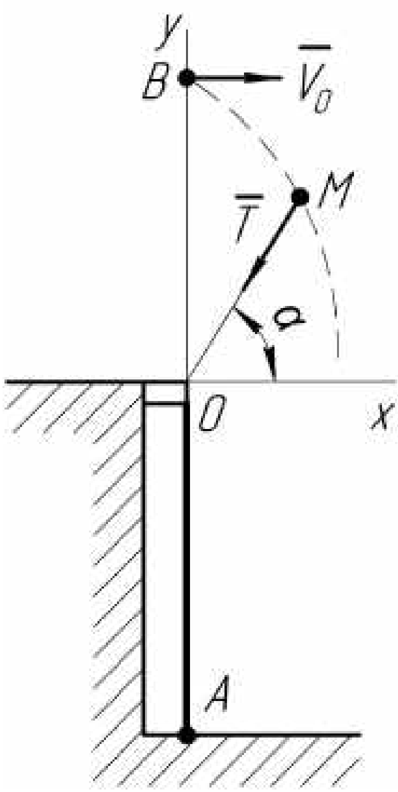


Fig. 1.7

The force $T = k^2 m$ must be applied to elongate the thread 1m. To elongate the thread on $\Delta l = OM$ (m) the force $T = k^2 m \Delta l$ must be applied. The vector $\overline{T}$ is opposite with vector $\overrightarrow{OM}$. Then

$$\overline{T} = -k^2 m \overrightarrow{OM};$$

$$\begin{cases} T_x = -k^2 mOM \cos\alpha = \left|OM \cos\alpha = x\right| = -k^2 mx; \\ T_y = -k^2 mOM \sin\alpha = \left|OM \sin\alpha = y\right| = -k^2 my. \end{cases}$$

System (1.39) can be rewritten as:

$$\begin{cases} m\ddot{x} = -k^2 mx, \\ m\ddot{y} = -k^2 my, \end{cases}$$

or after simplification

$$\begin{cases} \ddot{x} + k^2 x = 0; \\ \ddot{y} + k^2 y = 0. \end{cases} \qquad (1.40)$$

The system obtained is system of the second order homogeneous linear differential equations. They are independent and can be solved separately.

Characteristic equation for the first equation of system (1.40) is:

$$\lambda^2 + k^2 = 0,$$
$$\lambda^2 = -k^2,$$
$$\lambda_{1,2} = \pm ki.$$

Roots are complex values, that's why x-coordinate equation is

$$x = c_1 \cos(kt) + c_2 \sin(kt), \qquad (1.41)$$

where c_1 and c_2 are constants of integration.

To find x-component of velocity we have to differentiate the equation (1.41)

$$v_x = \frac{dx}{dt} = -c_1 k \sin(kt) + c_2 k \cos(kt). \qquad (1.42)$$

Solve the second equation of the system (1.40) in the same manner:

$$\ddot{y} + k^2 y = 0,$$
$$\lambda^2 + k^2 = 0,$$
$$\lambda^2 = -k^2,$$
$$\lambda_{1,2} = \pm ki,$$

$$y = c_3 \cos(kt) + c_4 \sin(kt), \tag{1.43}$$

$$\frac{dy}{dt} = \dot{y} = -c_3 k \sin(kt) + c_4 k \cos(kt). \tag{1.44}$$

To find constants of integration c_1, c_2, c_3, c_4 we have to use initial conditions. According to the statement of the problem at initial moment of time the particle was at point B and the velocity is perpendicular to y-axis, so

$$\text{at t=0: } \begin{cases} x(0) = 0, \\ y(0) = l, \\ \dot{x}(0) = v_0, \\ \dot{y}(0) = 0. \end{cases}$$

Putting t=0 into equations (1.41), (1.42) we get:

$$x(0) = c_1 = 0,$$
$$\dot{x}(0) = c_2 k = V_0 \;\Rightarrow\; c_2 = \frac{V_0}{k}.$$

Putting t=0 into equations (1.43), (1.44) we get:

$$y(0) = c_3 = l,$$
$$\dot{y}(0) = c_4 k = 0 \;\Rightarrow\; c_4 = 0.$$

Finally equations of the particle motion in parametrical form are

$$\begin{aligned} x &= \frac{V_0}{k}\sin(kt); \\ y &= l\cos(kt). \end{aligned} \tag{1.45}$$

To find the path of the ball we need to exclude the parameter t. Rewriting (1.45)

$$\sin(kt) = \frac{xk}{V_0};$$
$$\cos(kt) = \frac{y}{l}. \qquad (1.46)$$

squaring and summing equations (1.46) we obtain:

$$\left(\sin(kt)\right)^2 + \left(\cos(kt)\right)^2 = \left(\frac{xk}{V_0}\right)^2 + \left(\frac{y}{l}\right)^2,$$

but $\left(\sin(kt)\right)^2 + \left(\cos(kt)\right)^2 = 1$ so

$$\frac{x^2k^2}{V_0^2} + \frac{y^2}{l^2} = 1.$$

This is equation of ellipse. The general form of ellipse equation is

$$\frac{\left(x - x_0\right)^2}{a^2} + \frac{\left(y - y_0\right)^2}{b^2} = 1,$$

where x_0, y_0 are coordinates of the center and a, b are semi axes. In our case

$$x_0 = y_0 = 0, a = \frac{V_0}{k}, b = l.$$

Answer: $\frac{x^2k^2}{V_0^2} + \frac{y^2}{l^2} = 1.$

Example 4. Missile of mass m moves vertically. At initial moment of time its velocity was v_0 . If the air drag force is given by $F_{conp} = kv^2$ where k is constant, derive the maximum height and duration of missile lift.

Solution

Equation of free particle motion in vector form is

$$m\vec{W} = \vec{F}_{ad} + \vec{G}. \qquad (1.47)$$

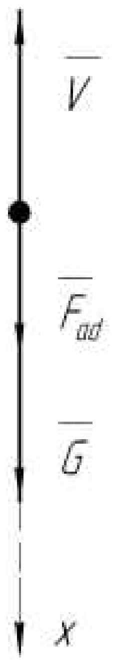


Fig. 1.8

Choose the x direction collinear with direction of missile motion, so that $x = 0$ if $h = 0$. Draw free-body diagram (Fig. 1.8): show air drag force F_{ad}, gravity G. The two forces are directed down therefore choose axis x positive direction down too.

Apply equation of motion in the x-direction to get

$$m\ddot{x} = gm + k\dot{x}^2. \tag{1.48}$$

For determination of duration of missile lift we can rewrite the equation in the following form

$$\ddot{x} = \frac{d\dot{x}}{dt},$$

$$\frac{d\dot{x}}{dt} = g + \frac{k}{m}\dot{x}^2.$$

By separating of the variables we

$$\frac{d\dot{x}}{g\frac{m}{k} + \dot{x}^2} = \frac{k}{m}dt.$$

We can integrate using notation $c^2 = g\frac{m}{k}$

$$\int_{-v_0}^{v} \frac{d\dot{x}}{c^2 + \dot{x}^2} = \int_0^T \frac{k}{m}dt,$$

$$\frac{1}{c}arctg\frac{v}{c} - (\frac{1}{c}arctg\frac{-v_0}{c}) = \frac{k}{m}t$$

or

$$v = c \cdot tg(c\frac{k}{m}t - arctg\frac{v_0}{c}). \tag{1.49}$$

For $t = t_{\max}$ ($t_{\max}$ is duration of missile lift) $v = 0$:

$$0 = c \cdot tg\left(c\frac{k}{m}t_{\max} - arctg\frac{v_0}{c}\right),$$

$$c\frac{k}{m}t_{\max} - arctg\frac{v_0}{c} = 0,$$

$$t_{\max} = \frac{m}{ck}arctg\frac{v_0}{c}. \qquad (1.50)$$

The maximum height of missile lift is obtained by integrating eq. (1.49) with known limits $\left(0, t_{\max}\right)$ for time and $\left(0, x_{\max}\right)$ for distance:

$$v = \frac{dx}{dt} = c \cdot tg(c\frac{k}{m}t - arctg\frac{v_0}{c}),$$

$$dx = c \cdot tg(c\frac{k}{m}t - arctg\frac{v_0}{c})dt,$$

$$\int_0^{x_{\max}} dx = c\int_0^{t_{\max}} (tg(c\frac{k}{m}t - arctg\frac{v_0}{c}))dt;$$

$$x_{\max} = -\left(\frac{m}{k}\ln\left|\cos\left(c\frac{k}{m}t_{\max} - arctg\frac{v_0}{c}\right)\right| - \left(\frac{m}{k}\ln\left|\cos\left(-arctg\frac{v_0}{c}\right)\right|\right)\right),$$

$$x_{\max} = -\frac{m}{k}\ln\left|\cos\left(c\frac{k}{m}t_{\max} - arctg\frac{v_0}{c}\right)\right| + \frac{m}{k}\ln\left|\cos\left(arctg\frac{v_0}{c}\right)\right| =$$

$$= -\frac{m}{k}\ln\left|\frac{\cos\left(c\frac{k}{m}t_{\max} - arctg\frac{v_0}{c}\right)}{\cos\left(arctg\frac{v_0}{c}\right)}\right|. \qquad (1.51)$$

Now we put equation (1.50) into the equation (1.51)

$$x(t_{\max}) = -\frac{m}{k}\ln\left|\frac{\cos\left(c\frac{k}{m}\left(\frac{m}{ck}arctg\frac{v_0}{c}\right) - arctg\frac{v_0}{c}\right)}{\cos(arctg\frac{v_0}{c})}\right| =$$

$$= -\frac{m}{k}\ln\left|\frac{1}{\cos(arctg\frac{v_0}{c})}\right| = -\frac{m}{k}\left(0 - \ln\left|\cos(arctg\frac{v_0}{c})\right|\right) =$$

$$= \frac{m}{k}\ln\left(\cos(arctg\frac{v_0}{c})\right) == \left|\begin{array}{l} \alpha = arctg\frac{v_0}{c},\ c^2 = \frac{k}{gm}, \\ \cos^2\alpha = \frac{1}{1+tg^2\alpha} = \frac{1}{1+\frac{kv_0{}^2}{gm}} = \frac{gm}{gm+kv_0{}^2} \end{array}\right| =$$

$$= \frac{m}{k}\ln\left(\sqrt{\frac{gm}{gm+kv_0{}^2}}\right) = \frac{m}{2k}\ln\left(\frac{gm}{gm+kv_0{}^2}\right).$$

Answer: the maximum height of missile lift is $h = |x(t_{\max})| = \frac{m}{2k}\ln\left(\frac{gm}{gm+kv_0{}^2}\right)$, and duration of missile lift is $t_{\max} = \frac{m}{ck}arctg\frac{v_0}{c}$.

2.4.2. Constrained particle motion

Example 5. A $3 \cdot 10^5$ kg airliner has four engines each of which produces a nearly constant thrust of 180 kN during the takeoff roll. Determine the length s of runway required if the takeoff speed is 220 km/h. Neglect air and rolling resistance.

Solution

Consider airplane as a particle. The airplane moves along straight horizontal line, so the airplane motion is constrained with one degree of freedom.

General equation of constrained motion in vector form is (1.26)

$$m\vec{W} = \vec{F} + \vec{N}.$$

For solution choose one coordinate axis x collinear with airplane trajectory and with origin at airplane initial position (Fig. 1.9).

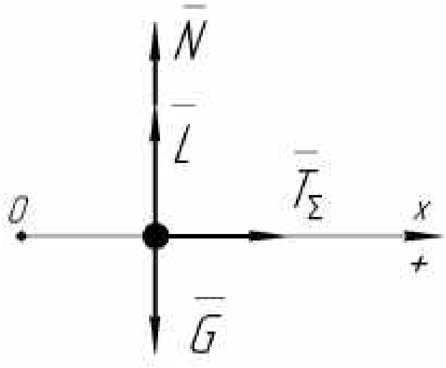


Fig. 1.9

Draw free-body diagram of the airplane treated as a particle:

– applied forces are total thrust force $\vec{T}_\Sigma = 4\vec{T}$, gravity $\vec{G}$, lifting force $\vec{L}$;

– reaction is normal force $\vec{N}$.

Rewrite the equation for the given problem

$$m\vec{W} = \vec{T}_\Sigma + \vec{G} + \vec{L} + \vec{N}. \tag{1.52}$$

In x-direction we get

$$m\ddot{x} = T_\Sigma . \tag{1.53}$$

In the equation there is one unknown only. It is coordinate x as function of time.

Determine the first integral of Eq. (1.53):

$$\ddot{x} = \frac{T_\Sigma}{m}, \tag{1.54}$$

rewrite $\ddot{x}$ as $\dfrac{d\dot{x}}{dt}$: $\dfrac{d\dot{x}}{dt} = \dfrac{T_\Sigma}{m}$, separate variables and integrate

$$d\dot{x} = \frac{T_\Sigma}{m} dt,$$

$$\int d\dot{x} = \int \frac{T_\Sigma}{m} dt,$$

$$\dot{x} = \frac{T_\Sigma}{m} t + C_1, \quad (1.55)$$

where C_1 is the first constant of integration, it has dimension of velocity (m/s).

Determine the second integral of Eq. (1.55):

$$\dot{x} = \frac{dx}{dt} = \frac{T_\Sigma}{m} t + C_1,$$

$$dx = \left(\frac{T_\Sigma}{m} t + C_1\right) dt,$$

$$\int dx = \int \left(\frac{T_\Sigma}{m} t + C_1\right) dt,$$

$$x = \frac{T_\Sigma}{m} \frac{t^2}{2} + C_1 t + C_2, \quad (1.56)$$

where C_2 is the second constant of integration, it has dimension of displacement (m).

Ascertain initial conditions for determination of constants C_1 and C_2 : when $t = 0 s$ then airplane coordinate $x_0 = 0 m$ and velocity $\dot{x}_0 = 0 m/s$ Put the initial conditions into Eq. (1.55) and (1.56)

$$\dot{x}_0 = 0 = \frac{T_\Sigma}{m} 0 + C_1, \quad (1.57)$$

$$x_0 = 0 = \frac{T_\Sigma}{m} \frac{0^2}{2} + C_1 0 + C_2, \quad (1.58)$$

then

$$C_1 = \dot{x}_0 = 0 \ m/s, \quad (1.59)$$

$$C_2 = x_0 = 0 \ m. \quad (1.60)$$

Substituting constants C_1 and C_2 in equations (1.55) and (1.56) we get

$$\dot{x} = \frac{dx}{dt} = \frac{T_\Sigma}{m} t, \tag{1.61}$$

and equation of motion is

$$x = \frac{T_\Sigma}{m} \frac{t^2}{2} = \frac{4 \cdot 180 \cdot 10^3}{3 \cdot 10^5 \cdot 2} t^2 = 1.2t^2. \tag{1.62}$$

Using Eq. (1.61) for airplane velocity determine time when velocity is $220\ km/h = 61.1\ m/s$:

$$t_{takeoff} = \frac{\dot{x}}{T_\Sigma / m} = \frac{61.1,}{4 \cdot 180 \cdot 10^3 / 3 \cdot 10^5} = 25.46 s. \tag{1.63}$$

Put the time into Eq. (1.62) and determine the length of runway

$$s = x\left(t_{takeoff}\right) = 1.2 \cdot 25.46^2 = 778.035 m\ . \tag{1.64}$$

Answer: the length of runway is $778.035 m$.

Example 6. A *54* kg particle initially at rest moves under the action of driving force $F = 27\ N$ inside smooth circular tube ($r = 3m$) in horizontal plane. Determine the horizontal component of normal reaction at $t = 6\ s$ if force direction is always collinear with particle velocity (Fig. 1.10).

Nota. Velocity is always tangent to the particle trajectory.

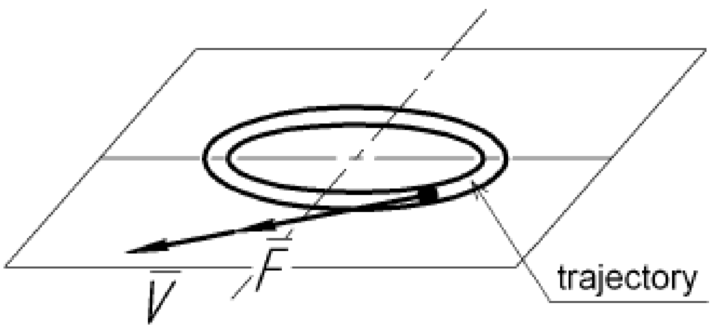


Fig. 1.10

Solution

Motion of the particle is constrained because the circular tube (constraint) specifies the particle trajectory.

General equation of constrained motion in vector form is (1.26)

$$m\vec{W} = \vec{F} + \vec{N}.$$

The trajectory is curvilinear so it is useful to analyze the particle motion in natural coordinate system ($\vec{\tau}, \vec{n}, \vec{b}$) (Fig. 1.11).

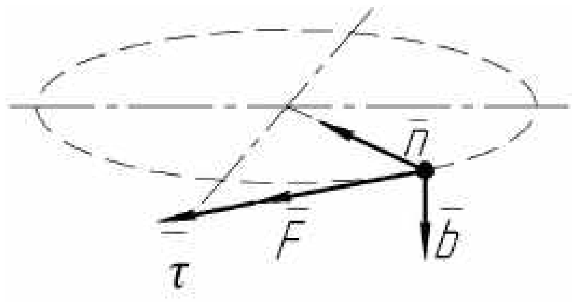


Fig. 1.11

Form free-body diagram (Fig. 1.12):

– applied forces are driving force $\vec{F}$ and gravity $\vec{G}$,

– the tube normal reaction N has two components in the vertical $\vec{N}_{vert}$ and horizontal $\vec{N}_{horiz}$ direction.

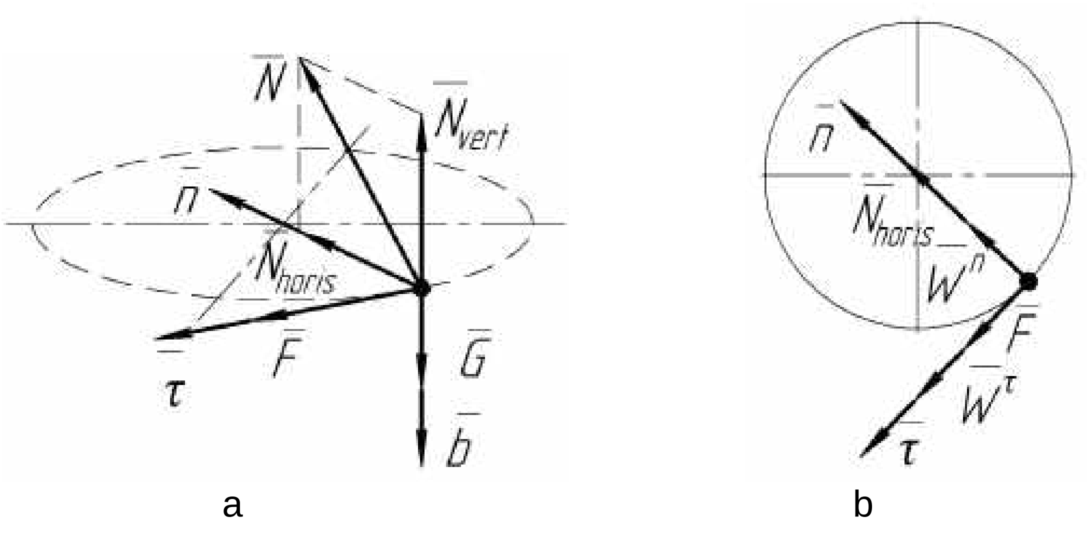


Fig. 1.12

Rewrite the equation (1.26) for the given problem

$$m\vec{W} = \vec{F} + \vec{G} + \vec{N}. \tag{1.65}$$

In scalar form we have

$$\tau)\ \ mW^{\tau} = F,$$

$$n)\ \ mW^{n} = N_{horiz},$$

$$b)\ \ 0 = -G + N_{vert}.$$

Remembering that $W^{\tau} = \ddot{\sigma}$, $W^{n} = \dfrac{(\dot{\sigma})^2}{r}$ we get

$$\tau)\ \ m\ddot{\sigma} = F, \tag{1.66 a}$$

$$n)\ \ m\frac{(\dot{\sigma})^2}{r} = N_{horiz}, \tag{1.66 b}$$

$$b)\ \ 0 = -G + N_{vert}. \tag{1.66 c}$$

In system (1.66) there are three unknowns: σ, N_{horiz}, N_{vert} . It is clear from the second equation that N_{horiz} is function of $\dot{\sigma}$ so first we need to determine function $\dot{\sigma}(t)$ using the τ -equation (1.66 a):

$$\ddot{\sigma} = \frac{F}{m}. \tag{1.67}$$

Present $\ddot{\sigma}$ as

$$\ddot{\sigma} = \frac{d\sigma}{dt},$$

put into (1.67)

$$m\frac{d\dot{\sigma}}{dt} = F,$$

separate variables

$$d\dot{\sigma} = \frac{F}{m}dt,$$

Integrate

$$\int d\dot{\sigma} = \int \frac{F}{m}dt.$$

For constant driving force magnitude we get

$$\dot{\sigma} = \frac{F}{m}t + C, \tag{1.68}$$

consider initial condition: when $t = 0$ then $v_0 = \dot{\sigma}_0 = 0$ so

$$\dot{\sigma}_0 = \frac{F}{m}t_0 + C,$$

or

$$0 = \frac{F}{m}0 + C,$$

from which $C = 0$ and

$$\dot{\sigma} = \frac{F}{m}t.$$

Now determine N_{horiz} using *n*-equation of system (1.66b) at $t = 6\ s$:

$$N_{horiz} = m\frac{(\dot{\sigma})^2}{r} = \frac{m}{r}\left(\frac{F}{m}t\right)^2 = \frac{54}{3}\left(\frac{27}{54}6\right)^2 = 162\,N.$$

Answer: horizontal component of normal force at $t = 6\ s$ is $162\,N$.

Example 7. A 5 kg particle moves from a state of rest along a smooth guide with a radius R situated in a horizontal plane under the action of driving force $F = 0.5t, N$. Determine the velocity of the particle at time 30 s if the force makes a constant angle 50° with the velocity vector.

Solution

Motion of the particle is constrained because the smooth circular guide (constraint) specifies the particle trajectory. Draw sketch illustrating the problem statement (Fig. 1.13, a). Remember that force must point in direction of the trajectory concavity, Fig. 1.13, b presents wrong direction of the force.

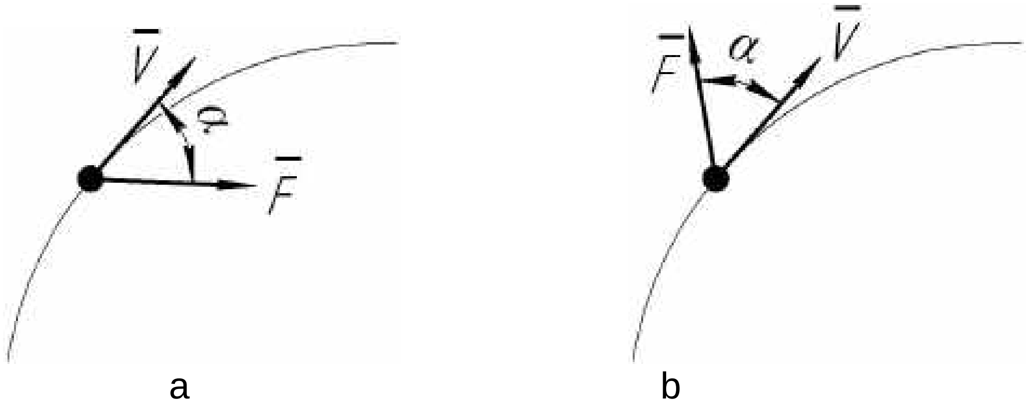


Fig. 1.13

Use natural coordinate system because motion is curvilinear. Tangent axis is directed along the tangent to trajectory, normal one is directed to the center of curvature.

General equation of constrained motion in vector form is (1.26)

$$m\vec{W} = \vec{F} + \vec{N}.$$

Form free-body diagram (Fig. 1.14, a):

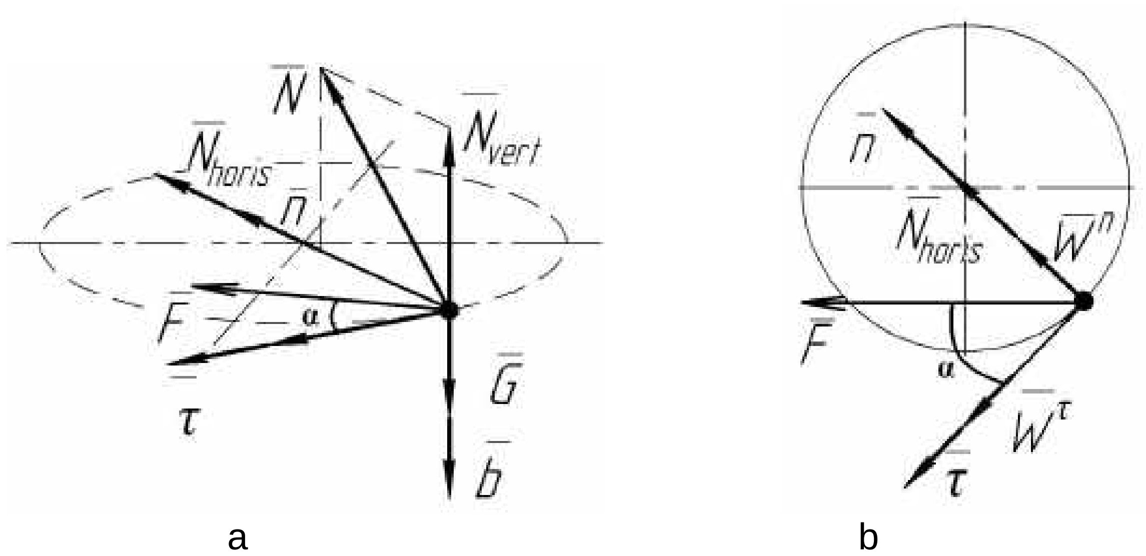


Fig. 1.14

– applied forces are driving force $\vec{F}$ and gravity $\vec{G}$;

– the tube normal reaction N has two components in the vertical $\vec{N}_{vert}$ and horizontal $\vec{N}_{horiz}$ direction.

Projection of the equation onto tangential axis is:

$$\tau)\ mW_\tau = F\cos\alpha .$$

Presenting W_τ as $\dfrac{dV}{dt}$ we get:

$$m\frac{dV}{dt} = 0.5t\cos\alpha,$$
$$\frac{dV}{dt} = \frac{0.5\cos\alpha}{m}t,$$
$$dV = \frac{0.5\cos\alpha}{m}tdt,$$
$$\int dV = \frac{0.5\cos\alpha}{m}\int tdt,$$
$$V = \frac{0.5\cos\alpha}{m}\frac{t^2}{2} + C_1.$$

To find constant of integration C_1 consider initial condition: when $t = 0$ then $v_0 = 0$ so

$$V(0) = 0 = \frac{0.5\cos\alpha}{m}\frac{0}{2} + C_1 = C_1.$$

So $C_1 = 0$ and

$$V = \frac{0.5\cos\alpha}{m}\frac{t^2}{2}.$$

At time t=30 s we have:

$$V(30\text{s}) = \frac{0.5\cos 50^\circ}{5}\frac{30^2}{2} = 28.9\left(\frac{m}{\text{s}}\right).$$

Answer: $V(30\text{s}) = 28.9\left(\dfrac{m}{\text{s}}\right)$.

Example 8. Determine a velocity of a particle M of a conical pendulum which at cord length OM=1 m circumscribe a cone with an angle at vertex $\alpha = 45°$ (Fig. 1.15)

Solution

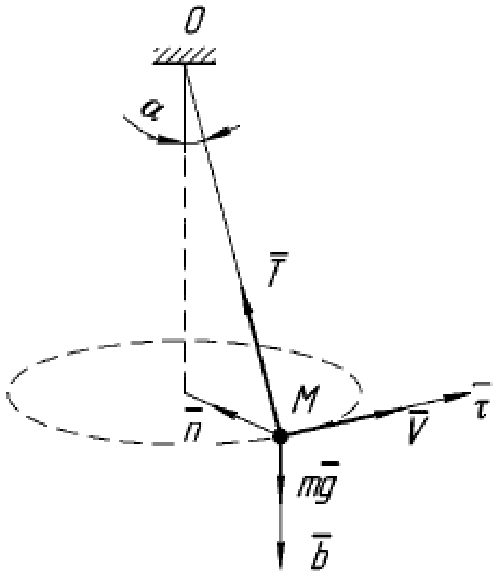


Fig. 1.15

Particle moves curvilinear so natural coordinates (τ, n, b) are used. Motion is constrained by cord.

General equation of constrained motion in vector form is (1.26)

$$m\vec{W} = \vec{F} + \vec{N}.$$

For this problem gravity force represents applied force and tension force is reactive one. So

$$m\vec{W} = \vec{G} + \vec{T}.$$

Projecting on the axes we get:

$$\tau)\ \ mW_\tau = 0, \tag{1.69}$$

$$n)\ \ mW_n = T\sin\alpha, \tag{1.70}$$

$$b)\ \ mW_b = mg - T\cos\alpha. \tag{1.71}$$

Binormal acceleration is always zero: $W_b = 0$. So from equation (1.71) we get:

$$T = \frac{mg}{\cos\alpha}.$$

Substituting this expression to equation (1.70) we obtain:

$$mW_n = \frac{mg}{\cos\alpha}\sin\alpha = mg\,\tan\alpha;$$

$$W_n = g\,\tan\alpha.$$

From the other hand $W_n = \dfrac{V^2}{\rho}$, where $\rho = OM\sin\alpha$ is a radius of the trajectory curvature. So

$$\frac{V^2}{\rho} = g\,\tan\alpha;$$

$$\frac{V^2}{OM\sin\alpha} = g\,\tan\alpha;$$

$$V = \sqrt{g\tan\alpha \cdot OM\sin\alpha}.$$

Substituting values we get $V = \sqrt{9.8\cdot\tan 45°\cdot 1\cdot\sin 45°} = 2.6\left(\dfrac{m}{s}\right)$.

Example 9. A small rocked propelled vehicle of mass m travels down the circular path of effective radius r under the action of its weight and constant trust T from its rocket motor (Fig. 1.16). If the vehicle starts from rest at A determine its speed v when it reaches B and the magnitude N of the force exerted by the guide on the wheels just prior to reaching B. neglect any friction and any loss of mass of the rocket.

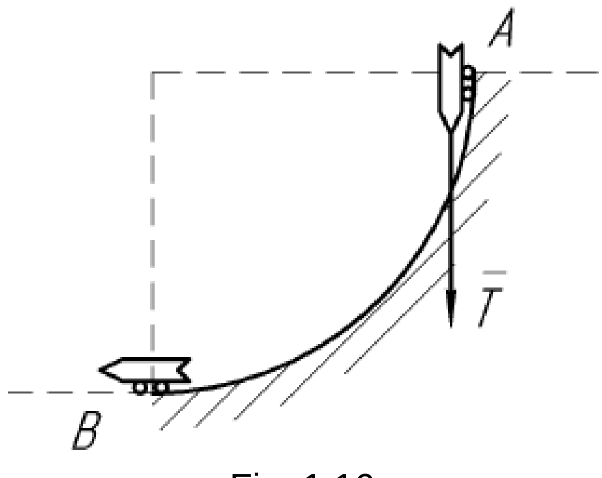


Fig. 1.16

Solution

The vehicle motion is constrained because the vehicle travels circular surface (trajectory is given beforehand). So it is useful to specify the vehicle motion in natural coordinate system ($\vec{\tau}, \vec{n}, \vec{b}$).

Draw free-body diagram of the vehicle treated as a particle: thrust force T , gravity G and reactive force N that is normal reaction of the guide (Fig. 1.17), the reaction is in the vertical plane.

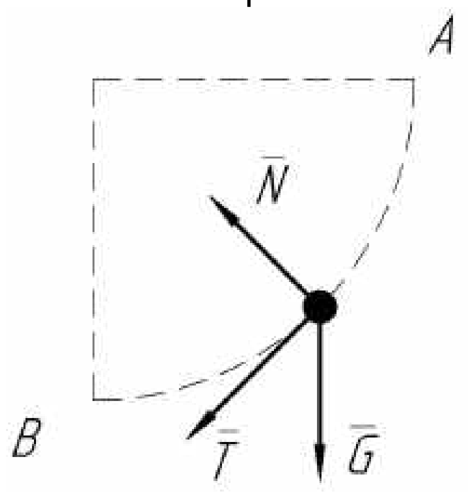


Fig. 1.17

Equation of constrained motion in vector form is

$$m\vec{W} = \vec{T} + \vec{G} + \vec{N}. \tag{1.72}$$

For curvilinear motion acceleration has two components: tangent and normal (Fig. 1.18)

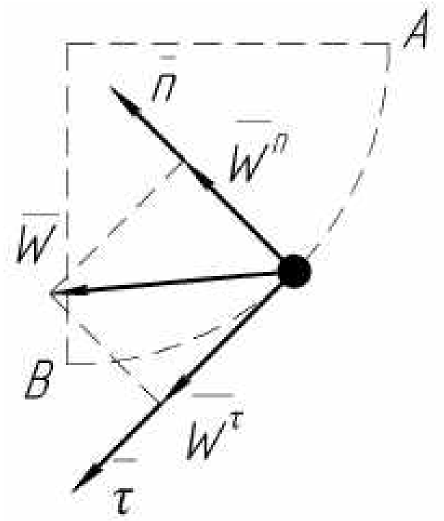


Fig. 1.18

$$\vec{W} = \vec{W}^{\tau} + \vec{W}^{n}.$$

So equation (1.72) in scalar form is the following system:

$$\begin{aligned} &\tau)\;\; m\ddot{\sigma} = T + G \cdot cos\varphi, \\ &n)\;\; m\frac{(\dot{\sigma})^2}{r} = -G \cdot sin\varphi + N, \end{aligned} \qquad (1.73)$$

where σ is path of the vehicle along the circle of radius r, σ is circular arc; φ is angle subtending the arc σ.

There are three unknowns in the system: σ, φ, N , . So the system must be supplement by equation of constraint (circular guide is constraint):

$$\sigma = r \cdot \varphi,\; \dot{\sigma} = r \cdot \dot{\varphi},\; \ddot{\sigma} = r \cdot \ddot{\varphi} \;\; , \qquad (1.74)$$

then

$$\begin{cases} mr\ddot{\varphi} = T + G \cdot Cos\varphi, \\ m\dfrac{(r\dot{\varphi})^2}{r} = -G \cdot Sin\varphi + N. \end{cases} \qquad (1.75)$$

We are asked about velocity at point B, $v_B = \dot{\sigma} = r \cdot \dot{\varphi}_B$. Position of point B can be specified by angle $\varphi_B = \dfrac{\pi}{2}$ as: $\sigma_B = r \cdot \varphi_B$. So to obtain the answer we need to determine function $\dot{\varphi}(\varphi)$ from the system.

Rewrite $\ddot{\varphi}$ as

$$\ddot{\varphi} = \frac{d\dot{\varphi}}{dt} = \frac{d\dot{\varphi}}{dt}\frac{d\varphi}{d\varphi} = \frac{d\varphi}{dt}d\dot{\varphi}\frac{1}{d\varphi} = \dot{\varphi}d\dot{\varphi}\frac{1}{d\varphi}\,. \qquad (1.76)$$

Put Eq.(1.76) into the first equation of the system (1.75):

$$mr\dot{\varphi}d\dot{\varphi}\frac{1}{d\varphi} = T + G \cdot cos\varphi,$$

$$mr\dot{\varphi}d\dot{\varphi} = \left(T + G \cdot cos\varphi\right)d\varphi,$$

$$\int mr\dot{\varphi}d\dot{\varphi} = \int \left(T + G \cdot cos\varphi\right) d\varphi,$$

$$\frac{\dot{\varphi}^2}{2} = \frac{1}{mr}\left(T\varphi + G \cdot sin\varphi\right) + C, \tag{1.77}$$

where C is the constant of integration, it has dimension of velocity squared $\left(\frac{m}{s}\right)^2$.

Ascertain initial condition for determination of constants C: when $\varphi_0 = 0$ then vehicle velocity $\dot{\sigma}_0 = r\dot{\varphi}_0 = 0\, m/s$ and so $\dot{\varphi}_0 = 0\ rad/s$. From Eq. (1.77) we get

$$\frac{\dot{\varphi}_0{}^2}{2} = 0 = \frac{1}{mr}\left(T \cdot 0 + G \cdot sin0\right) + C, \tag{1.78}$$

$$C = 0, \tag{1.79}$$

and

$$\dot{\varphi} = \sqrt{\frac{2}{mr}\left(T\varphi + G \cdot sin\varphi\right)}. \tag{1.80}$$

Vehicle velocity

$$\dot{\sigma} = r\dot{\varphi} = \sqrt{\frac{2r}{m}\left(T\varphi + G \cdot sin\varphi\right)}. \tag{1.81}$$

At point B $\varphi_B = \frac{\pi}{2}$ so

$$v_B = \dot{\sigma}_B = r\dot{\varphi}_B = \sqrt{\frac{2r}{m}\left(T\frac{\pi}{2} + G \cdot sin\frac{\pi}{2}\right)} = \sqrt{r\left(\frac{T\pi}{m} + 2g\right)}. \tag{1.82}$$

Using the second equation from the system (1.75) determine N:

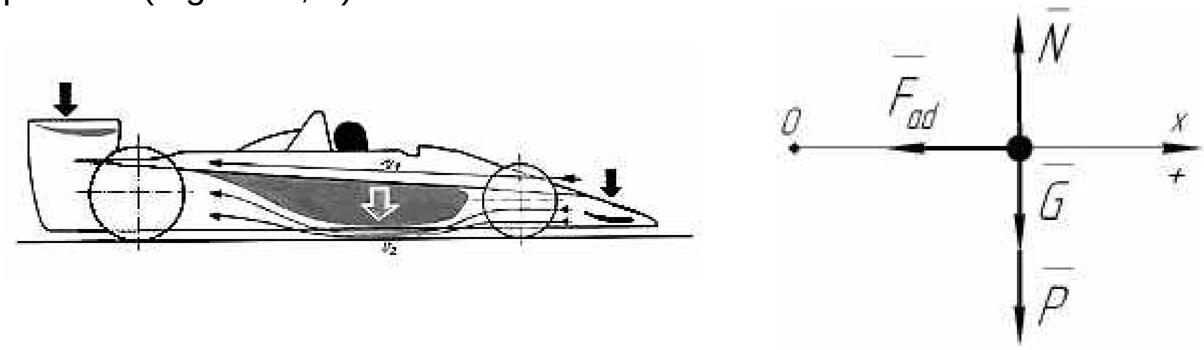
$$N = m\frac{\left(r\dot{\varphi}_B\right)^2}{r} + G \cdot sin\varphi_B = m\frac{r\left(\frac{T\pi}{m} + 2g\right)}{r} + G = T\pi + 3mg. \tag{1.83}$$

Answer: at point B vehicle velocity $v_B = \dot{\sigma}_B = \sqrt{r\left(\frac{T\pi}{m} + 2g\right)}$ and normal force is $N = T\pi + 3mg$.

Example 10. High-speed land racer of mass m moves horizontal. At initial moment of time its velocity was V_0. If the air drag force is $F_{ad} = kv^2$ where k is constant, determine the time *t* required for it to reduce its speed twice and the distance traveled. Neglect dry friction.

Solution

The racer (Fig. 1.19, a) moves along straight horizontal line, so the racer motion is constrained with one degree of freedom. For solution choose one coordinate axis x collinear with racer trajectory and with origin at racer initial position (Fig. 1.19, b).



a b

Fig. 1.19

Draw free-body diagram of the racer treated as a particle: air drag force F_{ad} , gravity G , pressing force P and normal force N (see Fig. 1.19, b).

Equation of motion in vector form is

$$m\vec{W} = \vec{F}_{ad} + \vec{G} + \vec{P} + \vec{N}. \tag{1.84}$$

In x-direction we get

$$m\ddot{x} = -F_{ad} = -k\dot{x}^2. \tag{1.85}$$

In the equation there is one unknown only. It is coordinate x as function of time.

Rewrite (1.56) using equality $\ddot{x} = \frac{d\dot{x}}{dt}$, we get

$$\frac{d\dot{x}}{dt} = -\frac{k}{m}\dot{x}^2 . \tag{1.86}$$

Separate variables and integrate with given limits: initial velocity v_0 , final velocity $v_0/2$:

$$\int_{v_0}^{\frac{v_0}{2}} \frac{d\dot{x}}{\dot{x}^2} = -\int_0^t \frac{k}{m} dt , \tag{1.87}$$

$$-\left(\frac{2}{v_0} - \frac{1}{v_0}\right) = -\frac{k}{m} t,$$

$$t = \frac{1}{v_0}\frac{m}{k} \quad . \tag{1.88}$$

So time of deceleration is function of the initial speed.
Determine the distance:
integrate Eq. (1.87)

$$\int \frac{d\dot{x}}{\dot{x}^2} = -\int \frac{k}{m} dt ,$$

$$-\frac{1}{\dot{x}} = -\frac{k}{m} t + C_1,$$

initial condition for C_1 : when $t = 0$ then $\dot{x} = v_0$, so $C_1 = -\frac{1}{v_0}$ and

$$\frac{1}{v} = \frac{k}{m} t + \frac{1}{v_0},$$

now again separate variables

$$\frac{1}{\frac{dx}{dt}} = \frac{k}{m} t + \frac{1}{v_0},$$

$$\frac{dt}{\frac{k}{m}t+\frac{1}{v_0}}=dx, \tag{1.89}$$

use substitution of variables in Eq. (1.89)

$$z=\frac{k}{m}t+\frac{1}{v_0},\ dz=\frac{k}{m}dt,\ dt=\frac{m}{k}dz$$

and integrate

$$\int\frac{m}{k}\frac{dz}{z}=\int dx,$$

$$\frac{m}{k}\ln|z|=x+C_2.$$

Now replacing z, we get

$$\frac{m}{k}\ln\left|\frac{k}{m}t+\frac{1}{v_0}\right|=x+C_2.$$

Initial condition: when $t=0$ then $\dot{x}=v_0$, so $C_2=\frac{m}{k}\ln\left|\frac{1}{v_0}\right|$ and

$$x=\frac{m}{k}\ln\left|\frac{k}{m}t+\frac{1}{v_0}\right|-\frac{m}{k}\ln\left|\frac{1}{v_0}\right|,$$

on combining the logarithmic terms, we obtain

$$x=\frac{m}{k}\ln\left|\frac{k}{m}t+\frac{1}{v_0}\right|-\frac{m}{k}\ln\left|\frac{1}{v_0}\right|=\frac{m}{k}\ln\left|\frac{k}{m}v_0t+1\right|. \tag{1.90}$$

There is another way of the Eq. (1.57) solution for determination the distance traveled:

$$\frac{d\dot{x}}{dt}\frac{dx}{dx}=\frac{\dot{x}d\dot{x}}{dx}=\frac{d(\dot{x}^2)}{2dx}=-\frac{k}{m}\dot{x}^2,$$

$$\frac{d(\dot{x}^2)}{\dot{x}^2} = -\frac{k}{m}2dx.$$

Integrating we get

$$\ln \dot{x}^2 = -\frac{k}{m}2x + C_3,$$

when $\dot{x} = v_0$ we take x=0 and so $C_3 = \ln v_0^{\ 2}$, and

$$\ln \dot{x}^2 = -\frac{k}{m}2x + \ln v_0^{\ 2},$$

$$\ln \frac{v_0^{\ 2}}{\dot{x}^2} = \frac{k}{m}2x,$$

for $\dot{x} = \frac{v_0}{2}$ we have

$$x = \frac{m}{2k}\ln\frac{v_0^{\ 2}}{\left(\frac{v_0}{2}\right)^2} = \frac{m}{2k}\ln 4\,. \qquad (1.91)$$

Substitute $k = 0,2$ $m = 4000kg$ we get $x = 13680\ m$.

Answer: time of deceleration is function of the initial speed $t = \frac{1}{v_0}\frac{m}{k}$.

Distance traveled during the velocity decreasing is not function of the initial speed value $x = \frac{m}{2k}\ln\frac{v_0^{\ 2}}{\left(\frac{v_0}{2}\right)^2} = \frac{m}{2k}\ln 4$.

2.5. Problem for self solution

Problem 1. A bucket which weighs **280 kgf** descends into a mine with uniform acceleration. During the first **10 sec** it drops **35 m**. Find tension in the cable holding the bucket. (1 kgf=10 N).

Problem 2. An aircraft weighing **2000 kgf** flies horizontally with an acceleration of **5 m/sec^2** and an instantaneous speed of **200 m/sec**. The air drag at this speed is proportional to the square of the speed; when the speed of **1 m/sec** is attained the air drag equals **0.05 kgf**. Assuming that the force of resistance is directed opposite to the velocity, determine the tractive force of the propeller, if the angle between the flight direction and the tractive force is **10°**.

Problem 3. In a test of resistance to motion in an oil bath a small steel ball of mass m (Fig. 1.20) is released from rest at the surface (y = 0). If the resistance to motion is given by **R = kv** where **k** is a constant, derive an expression for the depth **h** required for the ball to reach a velocity **v**.

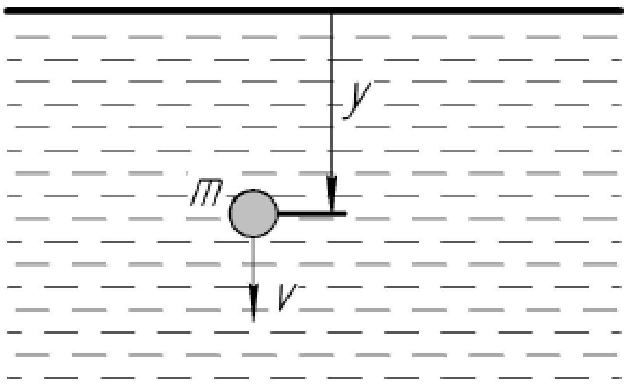


Fig. 1.20

Problem 4. Fig. 1.21 shows the velocity graph of the upward motion of a lift weighing 480 kgf. Find the tensions T_1, T_2, T_3 in the cable holding the lift during the three periods of time: (1) from t = 0 to t = 2 sec; (2) from t = 2 sec to t = 8 sec; (3) from t = 8 sec to t =10 sec.

Answer: T_1 =602.4 kgf; T_2 = 480 kgf; T_3 = 357.6 kgf.

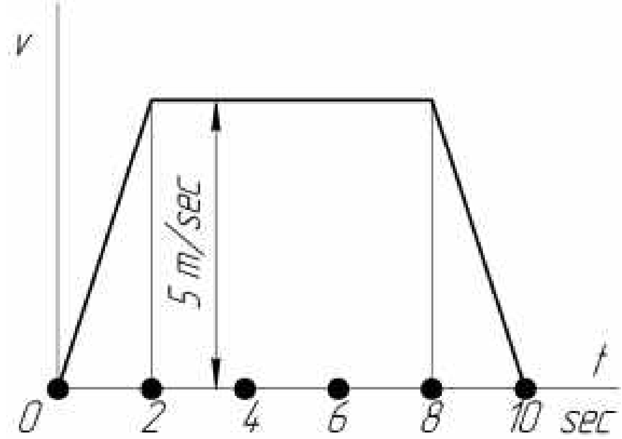


Fig. 1.21

Problem 5. An aircraft dives vertically and attains a speed of 1000 km/h. Reaching this speed the pilot pulls the aircraft out tracing an arc of radius R = 600 m in vertical plane. The weight of the pilot is 80 kgf. He is subjected to pressure from the seat during this flight. Find the maximum pressure exerted on the pilot.

Answer: 1130 kgf.

Problem 6. A body of weight ***P*** is given a push and moves along a rough horizontal plane. It travels the distance ***s = 24.5 m*** during 5 sec and then comes to rest. Determine the coefficient of friction f.

Answer: f = 0.2.

Problem 7. An aircraft flies horizontally. The air resistance is proportional to the square of the speed. At a speed of 1 m/sec the air resistance equals 0.05 kgf. The tractive force is constant and equals 3080 kgf and it makes an angle of 10° with the direction of the movement of the aircraft. Determine the maximum speed of the aircraft.

Answer: V_{max}=246 m/sec.

Problem 8. A body falls from a height without any initial velocity. The air resistance is R = k^2pv^2 where v is the velocity of the body, and p is its weight. Determine the velocity of the body attained after time t. Also find the limiting velocity.

Answer: $v = \frac{1}{k}\frac{e^{kgt} - e^{-kgt}}{e^{kgt} + e^{-kgt}}; v_\infty = \frac{1}{k}.$

Problem 9. Determine the tension P in the cable which will give the 100-lb block a steady acceleration of 5 ft/sec^2 up the incline (Fig. 1.22).

Answer: *P* = 43.8 lb.

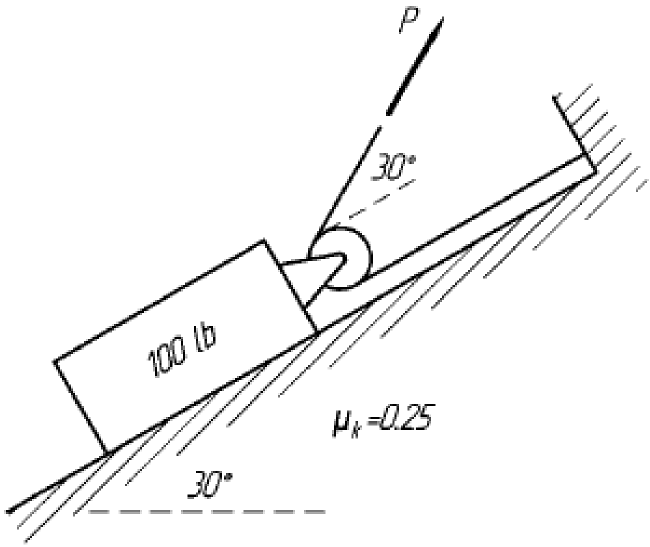


Fig. 1.22

Problem 10. The system is released from rest with the cable taut (Fig. 1.23). Neglect the small mass and friction of the pulley and calculate the acceleration of each body and the cable tension T upon release if (a) $\mu_s = 0.25$, $\mu_k = 0.20$ and (b) $\mu_s = 0.15$, $\mu_k = 0.10$.

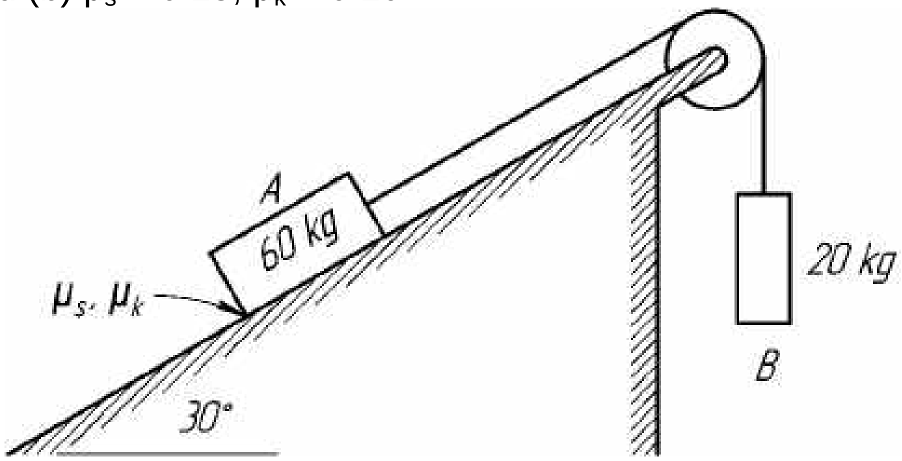


Fig. 1.23

Problem 11. The pulley arrangement of Prob. 16 is modified as shown in the Fig. 1.24. For the friction coefficients $\mu_s = 0.25$ and $\mu_k = 0.20$, calculate the acceleration of each body and the tension T in the cable.

Answer: $a_A = 1.450$ m/s^2 down, $a_B = 0.725$ m/s^2 up, T= 105.4 N.

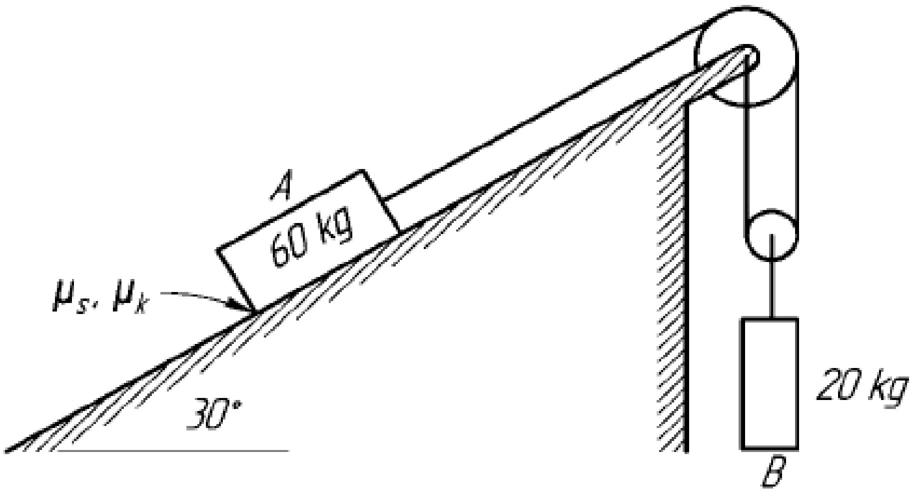


Fig. 1.24

Problem 12. In a test to determine the crushing characteristics of styrofoam packing material, a steel cone of mass m is dropped so that it falls a distance h and then penetrates the material (Fig. 1.25). The resistance **R** of styrofoam to penetration depends upon the cross-sectional area of the penetrating object and thus is proportional to the square of the cone penetration distance **x**, or $R = -kx^2$. If the cone comes to rest at a distance $\boldsymbol{x = d}$, determine the constant **k**in terms of the test conditions and results.

Answer: $k = \dfrac{3mg}{d^3}(h + d).$

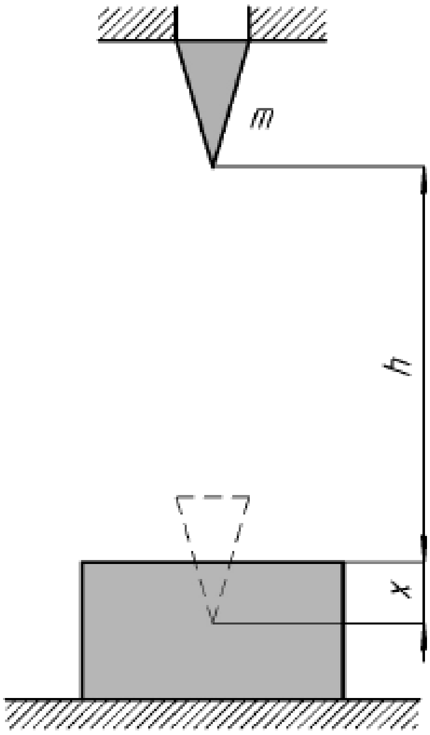


Fig. 1.25

Problem 13. Determine the height ***h*** and tension ***T*** in the cord for the conical pendulum of mass m and length ***l*** which rotates about the vertical axis at the angular rate $\dot{\theta} = \omega$ (Fig. 1.26)

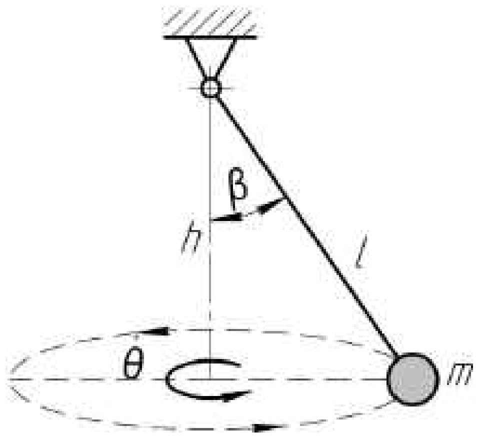


Fig. 1.26

2.6. Short problems

Problem 1. A particle of a mass m=10 kg moves along a curvilinear trajectory under action of a force F=20 N. Determine a velocity of the particle at the moment when a radius of curvature of the trajectory is ρ=12m and an angle between the force and the velocity vector is 35°.

Problem 2. A particle moves along a curvilinear trajectory under action of a force $\overline{F} = 5\overline{\tau} + 0.3\overline{n}$. determine a mass of the particle if at a moment t=20 sec its acceleration is W=0.6 m/sec^2.

Problem 3. A particle of a mass m=2 kg moves in a plane Oxy under action of the force, the projections of which are $F_x = 2\sin(0.5\pi t), F_y = 5\cos(\pi t)$. Determine a magnitude of the particle acceleration at a moment t=1 sec.

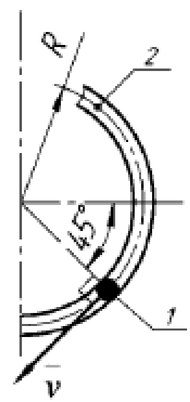


Fig. 1.27

Problem 4. A particle 1 of a mass m=30 kg moves in vertical plane in the tube 2, bent by an arc of a circle of a radius R=12 m (Fig. 1.27). Determine a tangential acceleration of a particle in the given position.

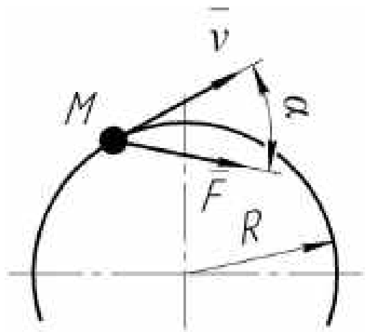


Fig. 1.28

Problem 5. A particle M of a mass m=8 kg moves in horizontal plane along an arc of a radius R=18 m (Fig. 1.28). Determine the angle α in degrees between the force $\overline{F}$ and the velocity $\overline{v}$ at a moment when the velocity of the particle is $v = 3m/\sec$ and tangential acceleration is $W^{\tau} = 0.5m/\sec^2$.

Problem 6. A particle moves along a curvilinear trajectory under action of the force, a tangential component of which is $F_\tau = 0{,}2t^2$ and a normal component is F_n = 8 N. Determine a mass of the particle if at a moment t = 10 s its acceleration is W = 0,7m/sec^2.

Problem 7. A particle with a mass m = 5 kg moves along a curvilinear trajectory under action of a force, the projection of which on a tangent is $F_\tau = 7\,N$, on a normal is $F_n = 0{,}1t^2$. Determine a magnitude of the particle acceleration at a moment t = 12 s.

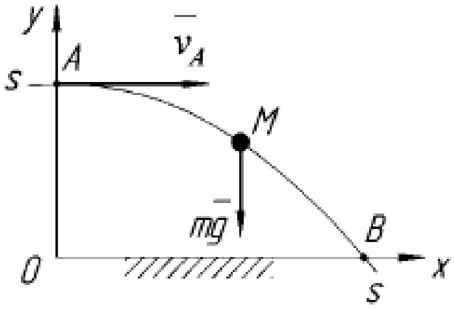


Fig. 1.29

Problem 8. A particle *M* moves along the parabola **s – s** in a vertical plane under action of gravitational force (Fig. 1.29). Determine the particle velocity at the position *B* if at the position *A* its velocity is v_A = 30 m/s and *OA*= 600 m.

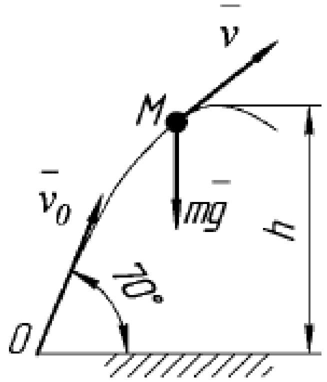


Fig. 1.30

Problem 9. A particle M moves in a vertical plane under action of gravitational force (Fig. 1.30). Determine a maximal height of ascent *h* in km if at initial moment the particle velocity is v_0 = 600 m/s.

Problem 10. A particle of a mass *m* = 15 kg moves from a state of rest along a smooth guide with the radius R situated in a horizontal plane under action of the force *F* = 0,5 *t.* Determine the particle velocity at the moment of time *t* = 30 s if the force makes a constant angle 50° with the velocity vector.

LECTURE 2

2.7. Differential equation of particle motion in noninertial frame of reference

It is know from particle kinematics that often in technique motion of a particle is analyzed with respect to fixed and moving references at the same time. It is compound motion of the particle. However, Newton's second law of motion is valid in *inertial (fixed)* frame of reference only:

$$m\overrightarrow{W} = \overrightarrow{F}\,, \tag{2.1}$$

where $\overrightarrow{W}$ is the acceleration of the particle in the fixed frame of reference (absolute acceleration);

$\vec{F}$ is the resultant of the force system applied to the particle.

The absolute acceleration is

$$\overrightarrow{W} = \overrightarrow{W}^{e} + \overrightarrow{W}^{r} + \overrightarrow{W}^{cor}\,, \tag{2.2}$$

where $\overrightarrow{W}^{e}$ is bulk or transport acceleration;

$\overrightarrow{W}^{r}$ is relative acceleration;$\overrightarrow{W}^{cor}$ is Coriolis acceleration.

So

$$m(\overrightarrow{W}^{e} + \overrightarrow{W}^{r} + \overrightarrow{W}^{cor}) = \overrightarrow{F}\,. \tag{2.3}$$

If we are interested in relative motion of the particle we need to rewrite equation (2.3) in the following form

$$m\overrightarrow{W}^{r} = \overrightarrow{F} - m\overrightarrow{W}^{e} - m\overrightarrow{W}^{cor}\,. \tag{2.4}$$

We denote terms in the right side of the equation as:

$$\overrightarrow{J}^{e} = -m\overrightarrow{W}^{e}\,, \tag{2.5}$$

$\overrightarrow{J}^{e}$ is bulk force of inertia or force of inertia of moving space;

$$\overrightarrow{J}^{cor} = -m\overrightarrow{W}^{cor}\,, \tag{2.6}$$

$\overrightarrow{J}^{Cor}$ is the Coriolis force.

So for moving frame of reference differential equation of free particle motion is

$$\boxed{m\vec{W}^{r} = \vec{F} + \vec{J}^{e} + \vec{J}^{cor}\,.} \tag{2.7}$$

The inertial force features.

- They are not real Newtonian forces because there are not bodies that produce these forces. So forces of inertia are fictive.
- The magnitudes of inertial forces are functions of mass of the particle under consideration.
- Forces of inertia are determined by kinematical characteristics of the moving frame of reference.

Differential equation of constraint particle motion in non-inertial frme is

$$\boxed{\boldsymbol{m\overrightarrow{W}^{r} = \overrightarrow{F} + \vec{N} + \overrightarrow{J}^{e} + \overrightarrow{J}^{cor}\,,}} \tag{2.8}$$

where is resultant of reactions acting on the particle.

2.8. Classical mechanics relativity principle

First we analyze the conditions under which the forces of inertia are zero. To do this we have to remember the conditions under which the appropriate accelerations are zero.

For the bulk acceleration of a particle M, that moves in moving reference $Axyz$ we have

$$\vec{W}^e = \vec{W}_A + \vec{W}_{MA}^{en} + \vec{W}_{MA}^{e\tau} =$$

$$= \vec{W}_A + \vec{\omega}^e \times (\vec{\omega}^e \times \overrightarrow{AM}) + \vec{\varepsilon}^e \times \overrightarrow{AM} =$$

$$= \underbrace{\vec{W}_A^{\tau} + \vec{W}_A^{n}}_{\substack{\text{acceleration of} \\ \text{moving system origin A}}} + \underbrace{\vec{\omega}^e \times (\vec{\omega}^e \times \overrightarrow{AM}) + \vec{\varepsilon}^e \times \overrightarrow{AM}}_{\substack{\text{terms that characterize rotation of} \\ \text{moving reference about its origin A}}} . \tag{2.9}$$

The first two terms of equation (2.9) characterize acceleration of moving system origin A. The acceleration components are zero if origin of the moving reference has uniform rectilinear motion.

The second two terms characterize rotation of moving reference about its origin A. The terms are zero if the moving reference has translation motion.

Conclusion: the moving space force of inertia is zero if moving reference has translational uniform rectilinear motion.

For the Coriolis acceleration we have

$$\vec{W}^{cor} = 2\vec{\omega}^e \times \vec{v}^r . \tag{2.10}$$

The Coriolis acceleration is zero if :

1) moving reference has translation motion ($\vec{\omega}^e = 0$);
2) the relative velocity of the particle is parallel to the axis of rotation (instantaneous or fixed) of moving reference ($\vec{\omega}^e \parallel \vec{v}^r$).

The two cases are realized if moving reference has translational uniform rectilinear motion.

So, if moving reference has translational uniform rectilinear motion, the both forces of inertia are zero and therefore equation (2.8) coincides with the equation (2.1) written for inertial (fixed) frame of reference. You can see that the second Newton's low holds in such moving reference and so the moving reference is inertial one.

Principle of relativity: any frame of reference will be inertial if it is in uniform rectilinear translational motion in relation to inertial frame of reference.

Or from *Albert Einstein: The Foundation of the General Theory of Relativity, Part A, § 1.*

Special principle of relativity: If a system of coordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the *same* laws hold good in relation to any other system of coordinates K' moving in uniform translation relatively to K.

Galileo Galilei first described this principle in 1632 in his “Dialogue Concerning the Two Chief World Systems” using the example of a ship traveling at constant speed, without rocking, on a smooth sea; any observer doing experiments below the deck would not be able to tell whether the ship was moving or stationary. Today one can make the same observations while travelling in an aeroplane with constant velocity. The fact that the earth on which we stand orbits around the sun at approximately 30 km/s offers a somewhat more dramatic example.

2.9. The relative resting conditions

The particle is in the rest relative to a moving frame of reference if its relative velocity is zero during some time interval, $\vec{v}^r = 0$. So, the Coriolis force is zero. The equation (2.8) in this case has the following view

$$\boxed{\vec{F} + \vec{J}^e = 0.} \tag{2.11}$$

Eq.(2.11) is the relative resting conditions.

2.10. Examples

2.10.1. Law of relative motion

Example 1. The projectile moves along a flat trajectory at latitude 60°. Its velocity is 900 m/s. The range of flight is 18 km. Determine the projectile deflection from the aim as a result of Earth rotation. Neglect air resistance.

Solution

Motion of the projectile is considered in two frames of references:

- fixed $Oxyz$ with origin in the center of the Earth (Fig. 2.1), motion of the Earth about the Sun is neglected;

- and moving reference $Ax_1y_1z_1$ with the origin at the initial position of the projectile, reference $Ax_1y_1z_1$ is rigidly connect with Earth, so the reference rotates about the Earth axis of rotation, the moving reference angular velocity vector $\vec{\boldsymbol{\omega}}^{e}$ points toward North Pole, and has magnitude $7.27 \cdot 10^{-5}\ \dfrac{rad}{s}$, angular velocity of Earth rotation is

$$\omega_{Earth} = \frac{2\pi}{24 hours \cdot 3600 s} = 7.27 \cdot 10^{-5} \frac{rad}{s},$$

axis Ay_1 coincides with direction of the projectile motion.

Equation of projectile motion relative the moving reference is

$$m\overrightarrow{W}^{r} = \overrightarrow{F} + \overrightarrow{J}^{e} + \overrightarrow{J}^{cor}\ , \tag{2.12}$$

where Coriolis force is $\overrightarrow{J}^{e} = -m\vec{W}^{Cor}$.

The Coriolis acceleration of the projectile is

$$\vec{W}^{cor} = 2\vec{\omega}^{e} \times \vec{v}^{r}.$$

In the problem x_1-component of velocity is neglected because it is substantially smaller then velocity y_1-component. And in the problem it is assumed that the trajectory is flat (z_1-component of velocity is zero), therefore action of gravity and bulk force of inertia $\overrightarrow{J}^{e}$ is neglected.

So

$$W^{cor} = 2\omega^{e} v^{r} Sin(\widehat{\overrightarrow{\omega^{e}}, \overrightarrow{v^{r}}}) = 2\omega^{e} v_{y} Sin 60^{\circ} =$$

$$= 2 \cdot 7,275 \cdot 10^{-5} \cdot 900 \cdot \sin 60 = 0.113\ m/s^{2}. \tag{2.13}$$

Coriolis acceleration points out from the paper perpendicular to the plane of sketch (see Fig. 2.1, a), so Coriolis force points into the paper along the negative direction of axis Ax_1(see Fig. 2.1, a). Fig. 2.1, b characterizes view in B-direction.

The projection of equation (2.12) onto the axis Ax_1 is

$$m\ddot{x}_1 = mW^{cor}. \tag{2.14}$$

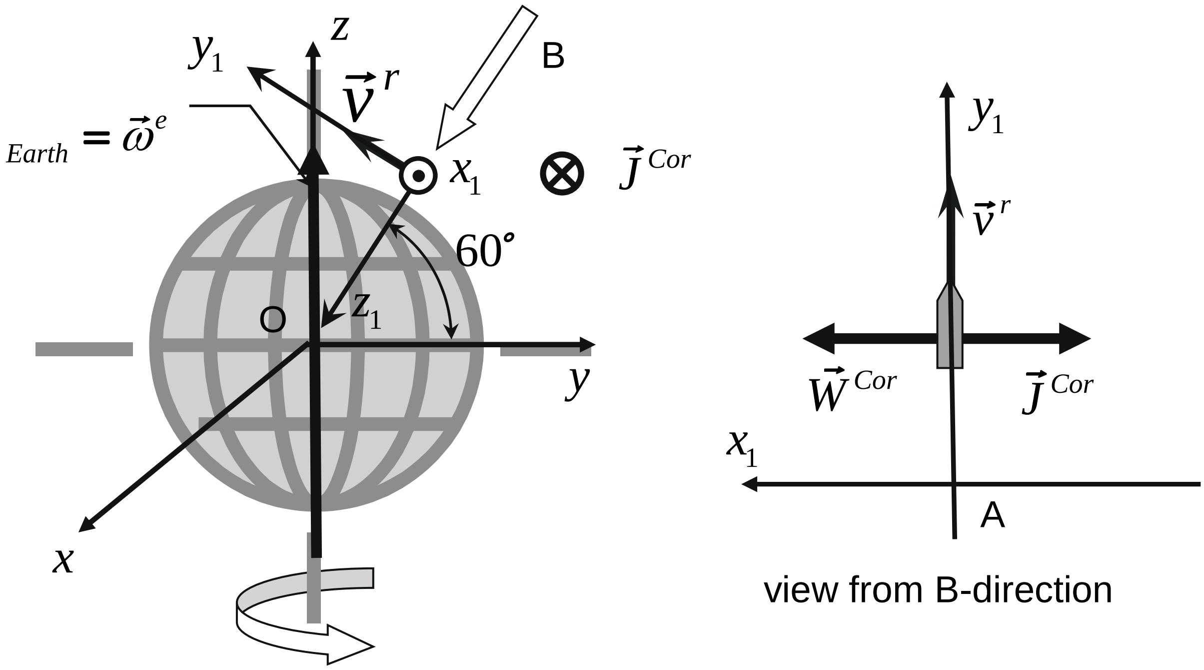


Fig. 2.1

Integrating twice we have

$$x_1 = \frac{W^{cor} t^2}{2} + C_1 t + C_0 . \tag{2.15}$$

When $t = 0$ we have $x_1 = 0$ and $\dot{x}_1 = 0$ so

$$C_1 = C_0 = 0 .$$

The duration of motion can be determined from the equation of the missile uniform motion along the trajectory (along axis Ay_1)

$$\dot{y}_1 = v_0 = 900 \frac{m}{s},$$

integrating we get

$$y_1 = 900t,\ m,$$

for t equals duration of fly

$$t = \frac{y_1}{v_0} = 18000 / 900 = 20\ s .$$

Substituting in the equation (2.15) we get deflection 22,68 m.

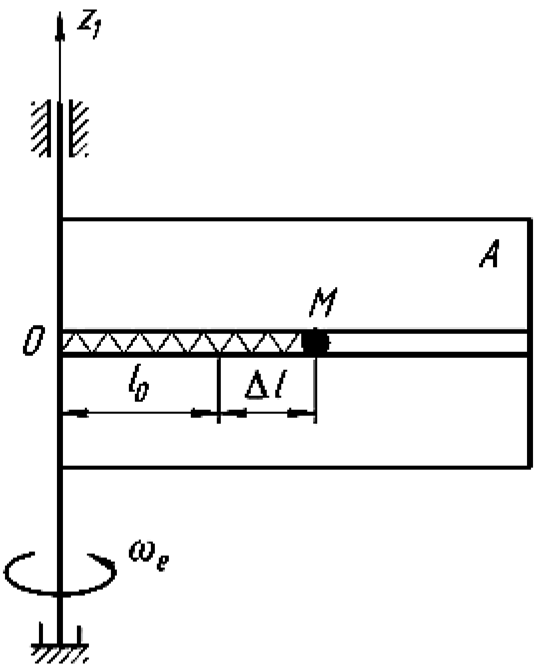


Fig. 2.2

Example 2. The body A uniformly rotates about fixed vertical axis z_1 with a constant angular velocity $\omega_e = 8\pi\ rad/\sec$ (Fig. 2.2). A ball M of a mass $m = 0.02\,kg$ moves in cylindrical channel of the body A. The ball is attached to the end of horizontal spring, the coefficient of stiffness of which is $k = 20\,N/m$. At initial moment the ball was on the distance $OM = 0.2\,m$ and had the initial velocity $V_0 = 2\,m/\sec$.

Find an equation of a relative motion of the ball x(t) and a normal reaction of the channel фt $t = 0.2\sec$. The length of unstretched spring is $l_0 = 0.1\,m$. Neglect a friction force.

Solution

1. We are asked about law of the particle relative motion that is motion with respect to the rotating rectilinear channel. So the moving frame of reference Oxyz we connect with the channel. Motion of moving reference is charectirised in fixed (inertial) reference $O_1x_1y_1z_1$ The motion is rotation about fixed axis O_1z_1, so all points of the moving frame have acceleration and the moving frame is non-inertial.

Conclusion: relative motion is rectilinear along Oxx in $Oxyz$, bulk motion is rotation of the reference $Oxyz$ about O_1z_1, in $O_1x_1y_1z_1$.

2. The channel is a constraint for the ball so we use main equation of motion in non-inertial frame of reference for constrained particle (2.8):

$$m\overline{W^r} = \overline{F} + \overline{N} + \overline{J^e} + \overline{J^{Cor}}.$$

Prepare FBD for the particle at time t (Fig. 2.3), assume that the spring is elongated. Applied forces are: gravity force $m\vec{g}$ directed down, force of spring $\vec{F}_{spr}$ is directed opposite elongation of the spring, according to Hook's law:

$$F_{spr} = k\Delta l = k\left(OM - l_0\right),$$

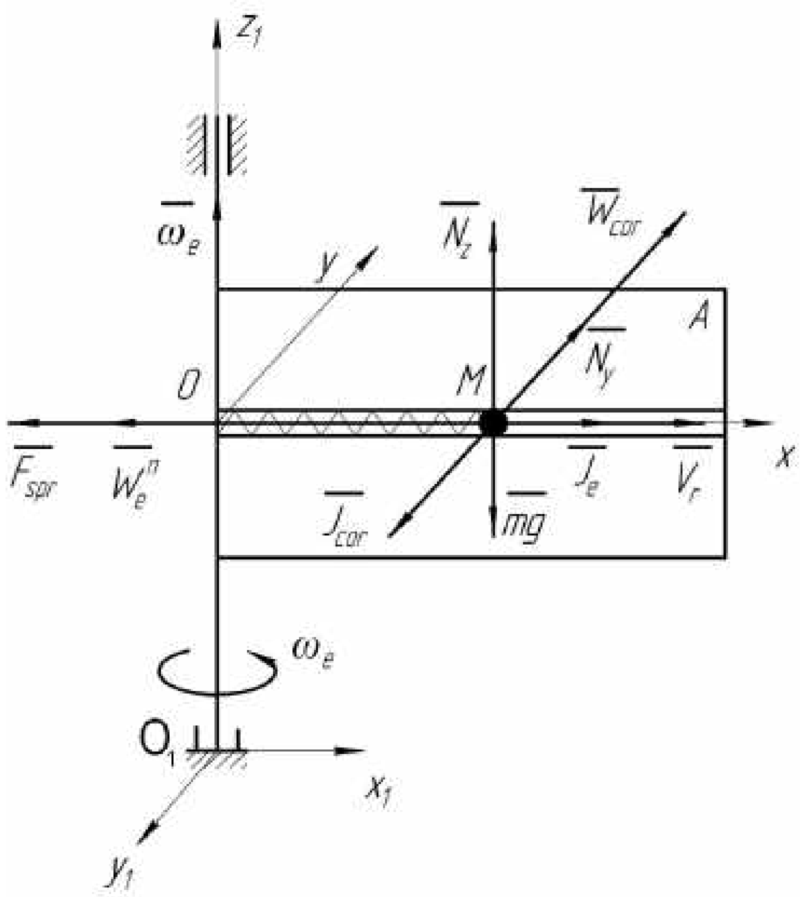


Fig. 2.3

where OM is the shortest distance between the particle and axis of rotation and for the problem OM can be expressed in terms coordinate x as

$$OM = x,$$

so

$$F_{spr} = k\Delta l = k\left(x - l_0\right).$$

Let's analyze normal reaction. When the body A doesn't move (Fig. 2.4, a), normal reaction N is determined by equilibrium equation

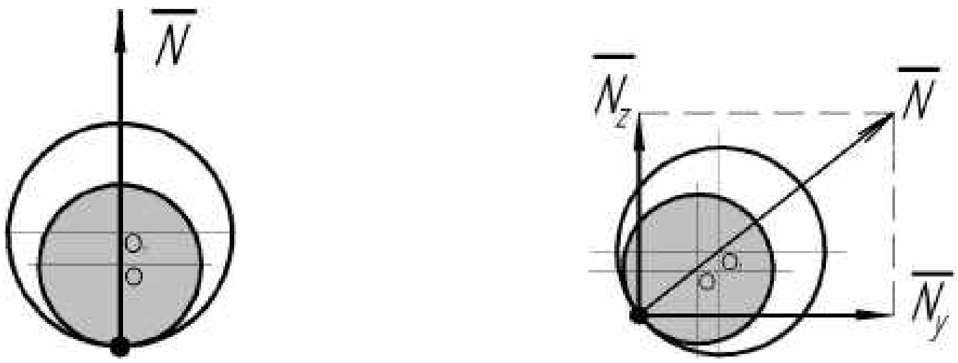
$$0 = \vec{G} + \vec{N}, \tag{2.16}$$

so $\vec{N} = -\vec{G}$, it is vertical an so is applied in the lowest point of that is point of contact.

When the body A moves with the channel (Fig. 2.4 b) we don't know direction of reaction and its application point. So we resolve the normal force into two components: $\vec{N}_y$ and $\vec{N}_z$:

$$\vec{N} = \vec{N}_y + \vec{N}_z,$$

$$N = \sqrt{N_y^2 + N_z^2}.$$



a b

Fig. 2.4

Bulk and Coriolis forces of inertia must be applied.

In accordance with eq. (2.5) Bulk force of inertia is:

$$\overline{J^e} = -m\overline{W^e},$$

where $\overline{W_e}$ is bulk acceleration that is for rotating moving reference:

$$\vec{W}^e = \vec{W}_O + \vec{W}_{MO}^{en} + \vec{W}_{MO}^{e\tau}.$$

Acceleration of the origin O $\overline{W_o} = 0$, because O is on the fixed axis of rotation.

Bulk normal acceleration is calculated according to the formula

$$W_{MO}^{en} = \omega^{e2} \cdot OM = \omega^{e2} x.$$

Bulk tangential acceleration is $W_{MO}^{e\tau} = \varepsilon^e \cdot OM$. Bulk angular acceleration $\varepsilon^e = 0$, because angular velocity $\omega^e = const$. So the total bulk acceleration is equal only to normal component:

$$W^e = W_{MO}^{en} = \omega^{e2} \cdot x,$$

$$J^e = m \cdot \omega^{e2} \cdot x.$$

Coriolis force of inertia is calculated according to the formula (2.6):

$$\overline{J_{cor}} = -m\overline{W_{cor}},$$

where $\overline{W_{cor}}$ is Coriolis acceleration:

$$\overline{W_{cor}} = 2\overline{\omega_e} \times \overline{V_r}.$$

According to right-hand rule Coriolis acceleration is directed perpendicular to the plane of Fig. 2.3 and points into the paper. Coriolis force of inertia is opposite, so it points out of the paper.

The magnitude of Coriolis acceleration is

$$W_{cor} = 2\omega_e \cdot V_r \cdot \sin\left(\widehat{\overline{\omega}_e, \overline{V}_r}\right).$$

An angle between bulk angular velocity (along axis of rotation O_1z_1) and relative velocity is 90°, so $\sin\left(\widehat{\overline{\omega}_e, \overline{V}_r}\right) = 1$ and

$$W_{cor} = 2\omega_e \cdot V_r = 2\omega_e \cdot \dot{x},$$

so value of Coriolis force is

$$J_{cor} = 2m\omega_e \cdot \dot{x}.$$

3. Rewrite equation (2.8) for for the problem in vector form views as:

$$m\overline{W^r} = m\overline{g} + \overline{F_{spr}} + \overline{N_y} + \overline{N_z} + \overline{J^e} + \overline{J^{Cor}}.$$

Projecting this vector equation on axes of moving reference $Oxyz$ we get:

$$\begin{cases} x)\ m\ddot{x} = -F_{spr} + J^e, \\ y)\ m\ddot{y} = N_y - J^{Cor}, \\ z)\ m\ddot{z} = N_z - mg. \end{cases}$$

There are 5 unknown values: $\ddot{x}, \ddot{y}, \ddot{z}, N_y, N_z$ and only 3 equations. We should add equations of constraints to be the system closed:

$$y = const \;\Rightarrow \ddot{y} = 0,$$
$$z = const \;\Rightarrow \;\ddot{z} = 0.$$

Rewriting equations

$$\begin{cases} x)\; m\ddot{x} = -k(x - l_0) + m\omega_e^2 x, \\ y)\; 0 = N_y - 2m\omega_e \dot{x}, \\ z)\; 0 = N_z - mg. \end{cases}$$

From the 2nd equation of the system we get $N_y = 2m\omega_e \dot{x}$.

From the 3d equation we get $N_z = mg$.

Considering 1st equation:

$$m\ddot{x} = -k(x - l_0) + m\omega_e^2 x,$$
$$m\ddot{x} = -kx + kl_0 + m\omega_e^2 x,$$

rewrite

$$\ddot{x} + \left(\frac{k}{m} - \omega_e^2\right) x = \frac{kl_0}{m}. \tag{2.17}$$

This is differential equation of the second order, linear, inhomogeneous. The general solution will be a sum of homogeneous and inhomogeneous part:

$$x = x^h + x^{inh}.$$

Present solution of the homogeneous equation as

$$x^h = ce^{\lambda t},$$

then first and second differentials are

$$\dot{x}^h = c\lambda e^{\lambda t},$$
$$\ddot{x}^h = c\lambda^2 e^{\lambda t}.$$

Substituting these expressions to homogeneous part of the equation (2.15) we have:

$$\ddot{x}+\left(\frac{k}{m}-\omega_e^2\right)x=0,$$

$$c\lambda^2 e^{\lambda t}+\left(\frac{k}{m}-\omega_e^2\right)ce^{\lambda t}=0.$$

Characteristic equation is:

$$\lambda^2+\frac{k}{m}-\omega_e^2=0,$$

determine λ:

$$\lambda^2=\omega_e^2-\frac{k}{m},$$

$$\lambda_{1,2}=\pm\sqrt{\omega_e^2-\frac{k}{m}}.$$

Let's find a magnitude of a constant

$$\sqrt{\omega_e^2-\frac{k}{m}}=\sqrt{(8\pi)^2-\frac{20}{0.02}}=19.2\bar{i}.$$

The result is virtual so the solution of homogeneous part will be

$$x^h=c_1\cos(19.2t)+c_2\sin(19.2t).$$

Present solution of inhomogeneous equation as:

$$x^{inh}=A \quad so \quad \ddot{x}^{inh}=0.$$

Then we substitute $x^{inh}=A$ into equation (2.15):

$$\left(\frac{k}{m}-\omega_e^2\right)A=\frac{kl_0}{m},$$

$$A=\frac{kl_0}{k-\omega_e^2 m}.$$

So the total solution of differential equation is:

$$x = x^h + x^{inh} = c_1 \cos(19.2t) + c_2 \sin(19.2t) + \frac{kl_0}{k - \omega_e^2 m}. \qquad (2.18)$$

To find unknown constants of integration the initial conditions are used:
When $t = 0$:

$$x_0 = OM_0 = 0.2\ m,$$
$$\dot{x}_0 = V_0 = 2\ m/\sec.$$

Substituting $t = 0$ to equation (2.18) we have:

$$x(0) = c_1 + \frac{kl_0}{k - \omega_e^2 m} = x_0,$$
$$c_1 = x_0 - \frac{kl_0}{k - \omega_e^2 m} = 0.2 - \frac{20 \cdot 0.1}{20 - (8\pi)^2 \cdot 0.02} = -0.07\ m.$$

Differentiating equation (2.18) we have:

$$\dot{x} = -19.2 c_1 \sin(19.2t) + 19.2 c_2 \cos(19.2t). \qquad (2.19)$$

Substituting $t = 0$ to equation (2.19):

$$\dot{x}(0) = 19.2 c_2 = \dot{x}_0,$$

$$c_2 = \frac{\dot{x}_0}{19.2} = \frac{2}{19.2} = 0.1\, \frac{m}{s}.$$

So equation of relative motion is

$$x = -0.07 \cos(19.2t) + 0.1 \sin(19.2t) + 0.27,\ m.$$

For relative velocity we get

$$\dot{x} = 0.07 \cdot 19.2 \cdot \sin(19.2t) + 0.1 \cdot 19.2 \cdot \cos(19.2t),\ \frac{m}{s}.$$

At $t = 0.2$ sec:

$$x = -0.07 \cos(19.2t) + 0.1 \sin(19.2t) + 0.27 = 0.24\ (m),$$

$$\dot{x} = 0.07 \cdot 19.2 \cdot \sin(19.2t) + 0.1 \cdot 19.2 \cdot \cos(19.2t) = -2.3\ (m/s).$$

So the reactions will be:

$$N_y = 2m\omega_e \dot{x} = 2 \cdot 0.02 \cdot 8\pi(-2.3) = -2.3\,(N),$$

$$N_z = mg = 0.02 \cdot 9.8 = 0.2\,(N),$$

$$N = \sqrt{N_y^2 + N_z^2} = \sqrt{(-2.3)^2 + (0.2)^2} = 2.3(N).$$

Answer: $x = -0.07\cos(19.2t) + 0.1\sin(19.2t) + 0.27$, $N = 2.3(N)$.

2.10.2. Problems on relative resting condition

Example 3. A small object A is held against the vertical side of the rotating cylindrical container of radius r due to centrifugal action (Fig. 2.5). If the coefficient of static friction between the object and the container is μ_s, determine the expression for the minimum rotational rate $\dot{\theta} = \omega$ of the container which will keep the object from slipping down the vertical side.

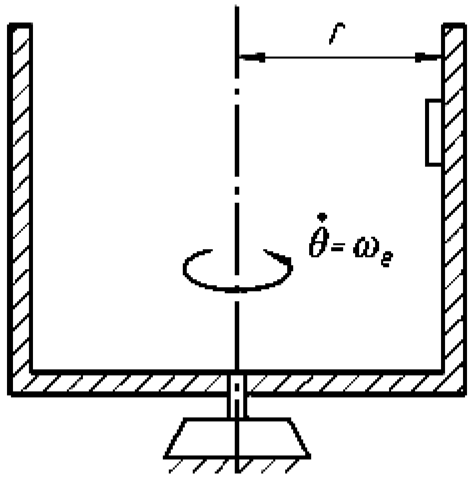


Fig. 2.5

Solution

In this problem relative rest of the body A is considered. So an equation of relative rest of a particle is used:

$$\overline{F} + \overline{N} + \overline{J^e} = 0.$$

Coordinate axes of moving reference system are connected with rotating container (Fig. 2.6). Gravity force $m\vec{g}$ is active force and directed down. Friction force F_f is directed opposite movement of the object A. It slips down, so friction force is up. Normal force is directed perpendicular to the wall of container.

Bulk force of inertia is calculated according to the formula

$$\overline{J_e} = -m\overline{W_e},$$

where bulk acceleration is $\overline{W_e} = \overline{W_e^n} + \overline{W_e^\tau}$.

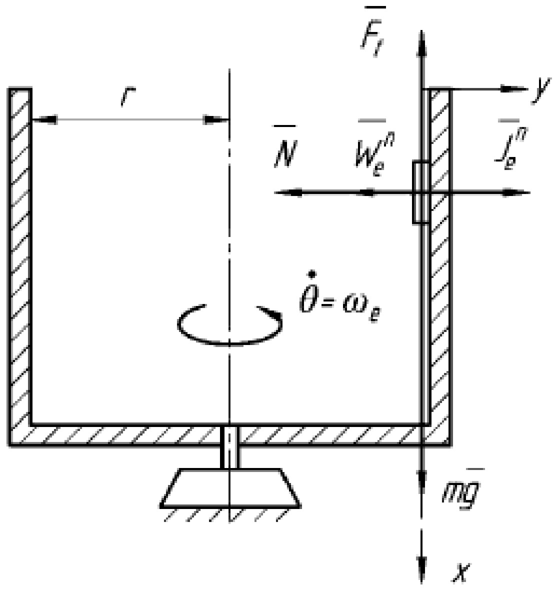


Fig. 2.6

Bulk force of inertia is calculated according to the formula

$$\overline{J_e} = -m\overline{W_e},$$

where bulk acceleration is $\overline{W_e} = \overline{W_e^n} + \overline{W_e^\tau}$.

Normal component of bulk acceleration is

$$W_e^n = \omega^2 r.$$

Tangential component of bulk acceleration is zero ($W_e^\tau = \varepsilon \cdot r$, $\varepsilon = \dot{\omega} = 0$).

So

$$J_e = J_e^n = mW_e^n = m \cdot \omega^2 \cdot r.$$

Vector equation of relative rest for the problem is:

$$m\overline{g} + \overline{F_f} + \overline{N} + \overline{J^e} = 0. \tag{2.18}$$

Projecting equation (2.18) on axes x, y of moving reference system:

$$x)\ \ mg - F_f = 0;$$

$$y)\ J_e^n - N = 0.$$

Rewrite using relation for friction force $F_f = \mu_s \cdot N$:

$$x)\ mg - \mu_s \cdot N = 0; \qquad (2.19)$$

$$y)\ m\omega^2 r = N. \qquad (2.20)$$

Substituting eq. (2.20) into eq. (2.19) we get:

$$mg - \mu_s \cdot m\omega^2 r = 0,$$

$$\omega = \sqrt{\frac{g}{\mu_s \cdot r}}.$$

So if the container rotates with this angular velocity, the body A will be in the state of rest.

Answer: $\omega = \sqrt{\dfrac{g}{\mu_s \cdot r}}$.

Example 4. The small object is placed on the inner surface of the conical dish at the radius shown (Fig. 2.7). If the coefficient of static friction between the object and the conical surface is 0.30, for what range of angular velocities co about the vertical axis will the block remain on the dish without slipping? Assume that speed changes are made slowly so that any angular acceleration may be neglected.

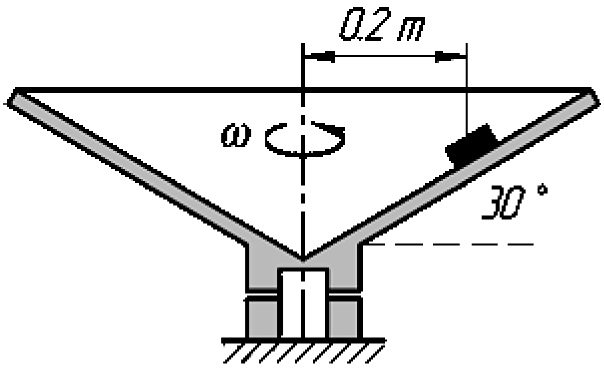


Fig. 2.7

Solution

The problem is about relative resting conditions. Bulk motion for the particle is rotation with the dish. Let us consider the two limit cases: the first is case when particle tends to slip down, in Fig. 2.8 it is presented by direction of impending velocity, and the second when particle tends to slip up (Fig. 2.9).

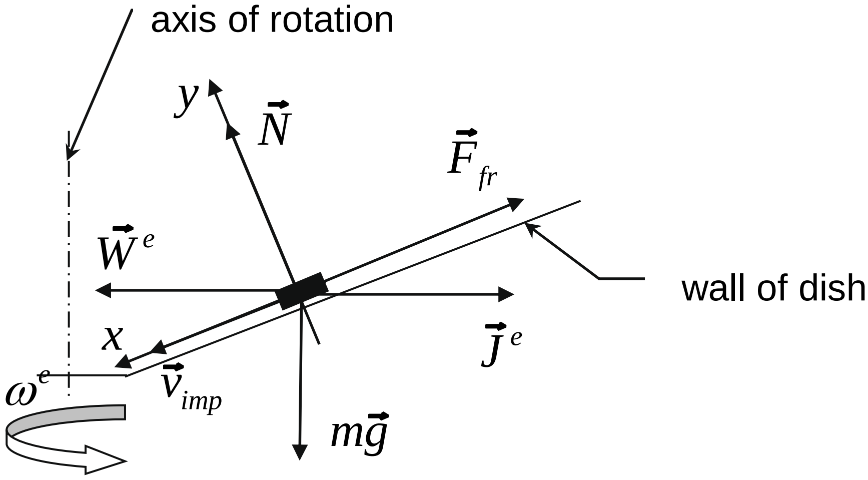


Fig. 2.8

Present relative resting conditions (eq. (2.11)) for the first case

$$m\vec{g} + \overrightarrow{F_{fr}} + \vec{N} + \vec{J}^e = 0 \ ,$$

where $\overrightarrow{F_{fr}}$ is friction force:

$$F_{fr} = \mu N \ ,$$

$\vec{N}$ is normal reaction, $\vec{J}^e$ is bulk force of inertia that is

$$J^e = m\left(\omega_1^e\right)^2 r \ .$$

Projecting onto the axes x, y we get :

$$x)\ mg\sin 30 - F_{fr} - J^e\cos 30 = 0,$$

$$y)\ -mg\cos 30 + N - J^e\sin 30 = 0,$$

or

$$x)\ mg\sin 30 - \mu N - m\left(\omega_1^e\right)^2 r\cos 30 = 0,$$

$$y)\ -mg\cos 30 + N - m\left(\omega_1^e\right)^2 r\sin 30 = 0.$$

Normal force N and angular velocity of the dish ω_1^e are unknowns in the system. Solving with respect to angular velocity, we obtain

$$\omega_1^e = \sqrt{\frac{g\left(Sin30^\circ - \mu Cos30^\circ\right)}{rCos30^\circ + \mu rSin30^\circ}} = 3.405\ \frac{rad}{s}.$$

In the second case (Fig. 2.9) we have

$$m\vec{g} + \overrightarrow{F_{fr}} + N + \vec{J}^e = 0 \ ,$$

$$J^e = m\left(\omega_2^e\right)^2 r \ ,$$
$$F_{fr} = \mu N.$$

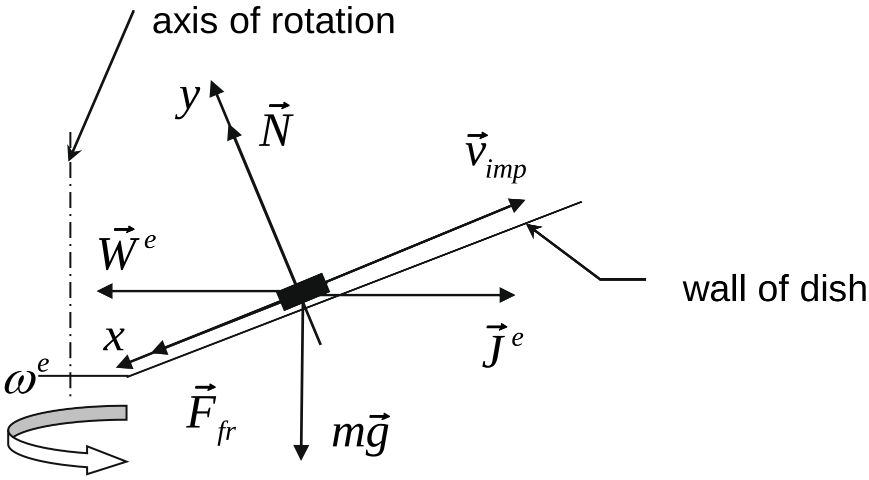


Fig. 2.9

Projecting onto the axes x, y we get :

$$x)\ mg\sin 30 + \mu N - m\left(\omega_{\max}^e\right)^2 r\cos 30 = 0,$$
$$y)\ -mg\cos 30 + N - m\left(\omega_{max}^e\right)^2 r\sin 30 = 0.$$

Solving with respect to angular velocity, we obtain

$$\omega_2^e = \sqrt{\frac{g\left(Sin30^\circ + \mu Cos30^\circ\right)}{rCos30^\circ - \mu rSin30^\circ}} = 7.213\ \frac{rad}{s}.$$

Answer: block remains on the dish without slipping if $\omega_1^e \le \omega^e \le \omega_2^e$ so $3.405 \le \omega^e \le 7.213\,\frac{1}{s}$.

2.11. Problems for self solution

Problem 1. In the design of a space station to operate outside the earth's gravitational field (Fig. 2.10), it is desired to give the structure an angular velocity ω which will simulate the effect of the earth's gravity for members of the crew. If the centers of the crew's quarters are to be located 12 m from the axis of rotation, calculate the necessary angular velocity ω of the space station.

Answer: $\omega = 0.9037\frac{1}{s}$.

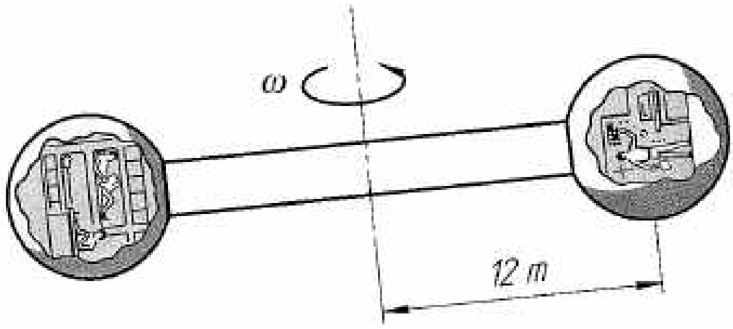


Fig. 2.10

Problem 2. The hollow tube is pivoted about a horizontal axis through point O (Fig. 2.11) and is made to rotate in the vertical plane with a constant counterclockwise angular velocity $\dot{\theta}$ = 3 rad/sec. If a 0.2-lb particle is sliding in the tube toward O with a velocity of 4 ft/sec relative to the tube when the position $\dot{\theta}$ = 30° is passed, calculate the magnitude N of the normal force exerted by the wall of the tube on the particle at this instant.

Answer: N = 0.024 lb.

Problem 3. The barrel of a rifle (Fig. 2.12) is rotating in a horizontal plane about the vertical z-axis at the constant angular rate $\dot{\theta}$ = 0.5 rad/s when a 60-g bullet is fired. If the velocity of the bullet relative to the barrel is 600 m/s just before it reaches the muzzle A, determine the resultant horizontal side thrust P exerted by the barrel on the bullet just before it emerges from A. On which side of the barrel does P act?

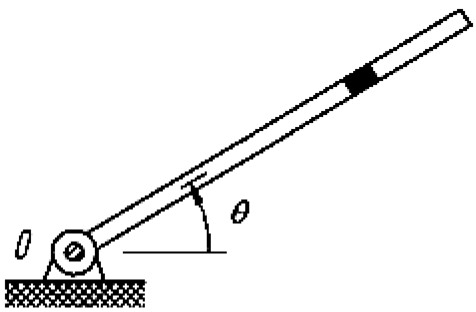


Fig. 2.11

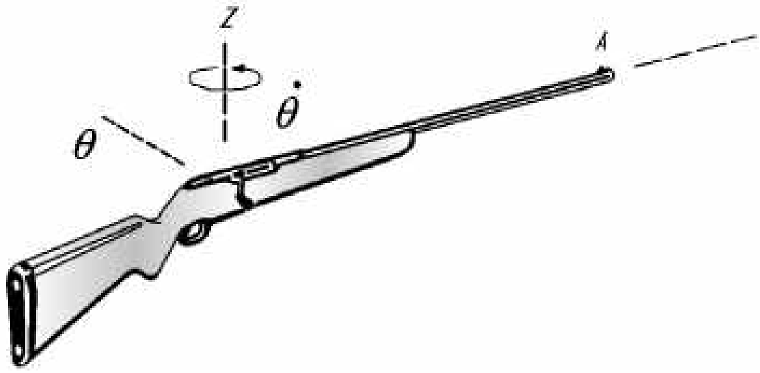


Fig. 2.12

Problem 4. The slotted arm rotates about its center in a horizontal plane at the constant angular rate $\dot{\theta}$ = 10 rad/sec and carries a 3.22-lb spring-mounted slider which oscillates freely in the slot (Fig. 2.13). If the slider has a speed of 24 in./sec relative to the slot as it crosses the center, calculate the horizontal side thrust P exerted by the slotted arm on the slider at this instant. Determine which side, A or B, of the slot is in contact with the slider.

Answer: P = 4 lb, side A.

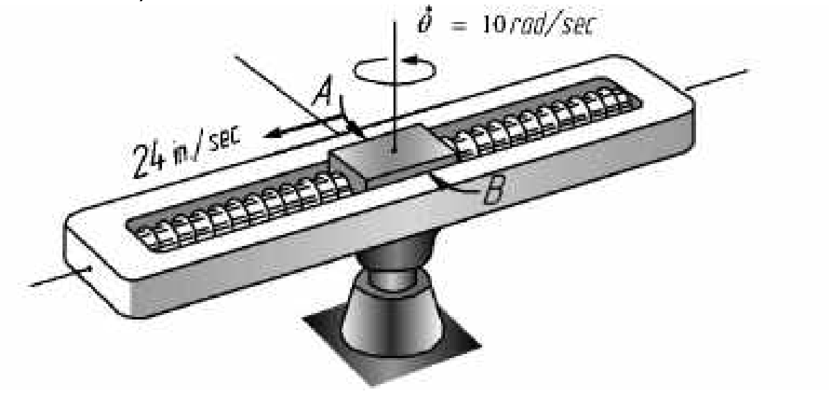


Fig. 2.13

Problem 5. The slotted arm revolves in the horizontal plane about the fixed vertical axis through point O (Fig. 2.14). The 3-lb slider C is drawn toward 0 at the constant rate of 2 in./sec by pulling the cord S. At the instant for which r = 9 in., the arm has a counterclockwise angular velocity ω= 6 rad/sec and is slowing down at the rate of 2 rad/sec^2. For this instant, determine the tension T in the cord and the magnitude N of the force exerted on the slider by the sides of the smooth radial slot. Indicate which side, A or B, of the slot contacts the slider.

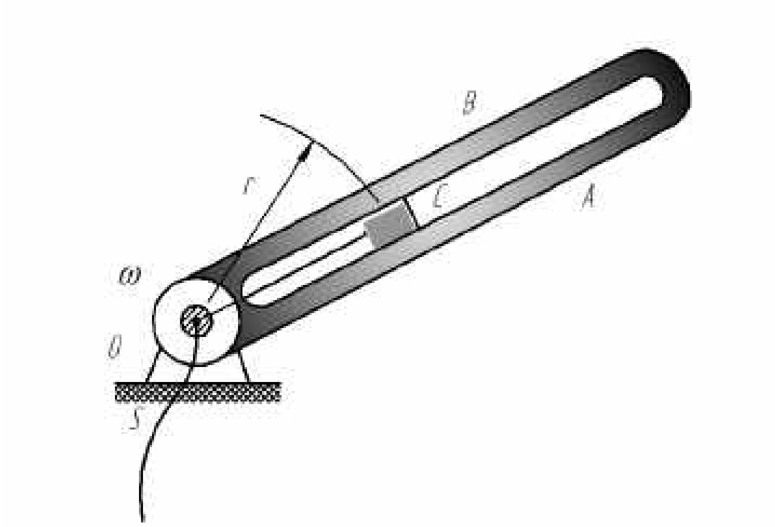


Fig. 2.14

Problem 6. The particle P is released at time t = *0* from the position r = r_0 inside the smooth tube with no velocity relative to the tube, which is driven at the constant angular velocity ω_0 about a vertical axis (Fig. 2.15). Determine the radial velocity v_r, the radial position r, and the transverse velocity v_θ as functions of time t. Explain why the radial velocity increases with time in the absence of radial forces. Plot the absolute path of the particle during the time it is inside the tube for $r_0 = 0.1$ m, $l = 1$ m, and $\omega_0 = 1$ rad/s.

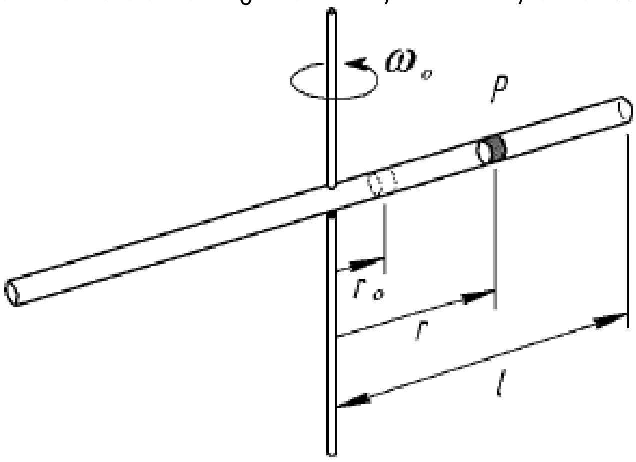


Fig. 2.15

Answer: $v_r = \frac{r_0\omega_0}{2}\left[e^{\omega_0 t} - e^{-\omega_0 t}\right] = r_0\omega_0 \sinh \omega_0 t;$

$$r = \frac{r_0}{2}\left[e^{-\omega_0 t} + e^{\omega_0 t}\right] = r_0 \cosh \omega_0 t;$$

$$v_\theta = \frac{r_0\omega_0}{2}\left[e^{-\omega_0 t} + e^{\omega_0 t}\right] = r_0\omega_0 \cosh \omega_0 t.$$

Problem 7. A hollow tube rotates about the horizontal axis through point O with constant angular velocity ω_0 (Fig. 2.16). A particle of mass m is introduced with zero relative velocity at r = 0 when $\theta = 0$ and slides outward through the smooth tube. Determine r as a function of θ.

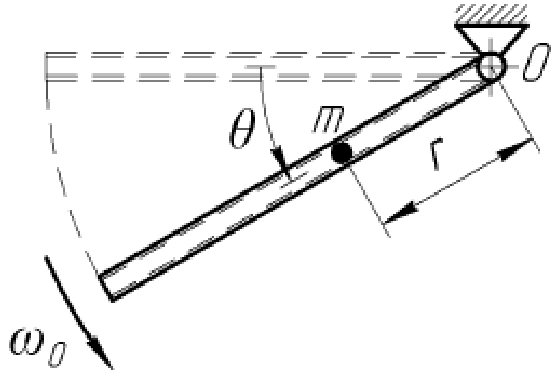


Fig. 2.16

Problem 8. The slotted arm rotates in a horizontal plane around the fixed cam with a constant counterclockwise velocity ω = 20 rad/s (Fig. 2.17). The spring has a stiffness of 5.4 kN/m and is uncompressed with $\theta = 0$. The cam has the shape $r = b - c \cos\theta$. If b = 100 mm, c = 75 mm, and the smooth roller A has a mass of 0.5 kg, find the force P exerted on A by the smooth sides of the slot for the position in which $\theta = 60°$.

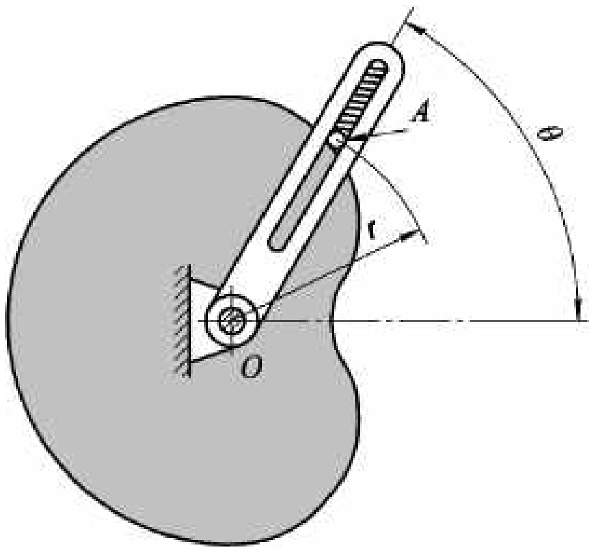


Fig. 2.17

Answer: P= 231 N.

Problem 9. In a mathematical pendulum of a length l the particle suspended moves along the vertical with a uniform acceleration. Determine the period T of small oscillations of the pendulum under two conditions: 1) when the acceleration of the particle is directed upwards and has any value p; 2) when this acceleration is directed downwards and its value is p<g.

Answer: (1) $T = 2\pi\sqrt{\dfrac{l}{p+g}}$; (2) $T = 2\pi\sqrt{\dfrac{l}{g-p}}$.

Problem 10. A particle falls freely from d height of 500 m to the earth in the northern hemisphere. Taking into consideration the rotation of the earth about its axis and neglecting the air resistance, determine the magnitude of the deviation of the falling particle in the east direction before it strikes the ground. The geographical latitude of the place is 60.

Answer: The deviation is 12 cm.

Problem 11. The car runs along a straight horizontal track. A pendulum which is installed in a railway car performs small harmonic oscillations. Its central position is deviated 6° from the vertical. 1) Determine the acceleration ω of the car; 2) find the difference between two oscillation periods of the pendulum: T, when the car is at rest, and T_1, for the present case.

Answer: (1) ω=103 cm/sec^2; T-T_1=0.0028T.

Problem 12. Fig. 2.18 shows a pipe AB which rotates about a vertical axis CD with a constant angular velocity ω. The angle between AB and CD is always 45°. A small heavy ball is placed in the pipe. Determine the motion of the ball, assuming that its initial velocity is zero and the initial distance between the ball and a point O equals a. Neglect friction.

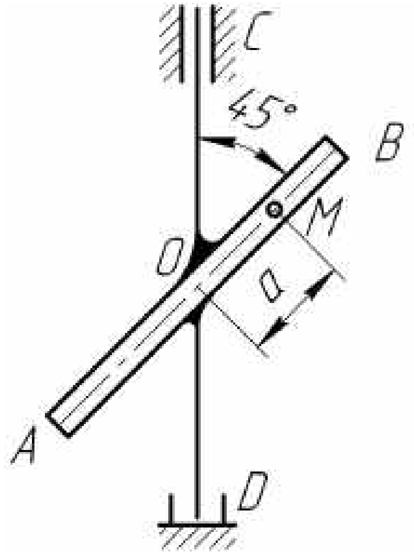


Fig. 2.18

Answer: $OM = \frac{1}{2}\left(a - \frac{g\sqrt{2}}{\omega^2}\right)\left(e^{+0.5\omega t\sqrt{2}} + e^{-0.5\omega t\sqrt{2}}\right) + \frac{g\sqrt{2}}{\omega^2}$.

Problem 13. Determine how the acceleration due to gravity changes in relation to the latitude of the place φ, considering that the earth rotates about its axis. The radius of the earth is R=6370 km.

Answer: If we neglect the term in ω^4 due to its smallness then $g_1 = g\left(1 - \frac{\cos^2\varphi}{289}\right)$, where g is the acceleration of gravity at the pole, φ is the geographical latitude of the place.

Problem 14. How many times should the angular velocity of rotation of the earth about its axis be increased to make a heavy particle at the surface of the earth at the equator completely weightless? The radius of the earth is $\mathrm{R} = 6370$ km.

Answer: 17 times.

Problem 15. A pendulum, suspended from a long thread, is given a small initial velocity in the north-south plane. Assuming that the deviation ot the pendulum is negligible compared to the length of the thread and, taking into consideration the earth's rotation about its axis, determine the time elapsed when the plane of pendulum rotations coincides with that of west-east. The pendulum is located in latitude 60° north.

Answer: T = 13.86 (0.5+K) hours, where K = 0, 1, 2, 3,….

Problem 16. A small bead of mass m is carried by a circular hoop of radius r which rotates about a fixed vertical axis (Fig. 2.19). Show how one might determine the angular speed ω of the hoop by observing the angle θ which locates the bead. Neglect friction in your analysis, but assume that a small amount of friction is present to damp out any motion of the bead relative to the hoop once a constant angular speed has been established. Note any restrictions on your solution.

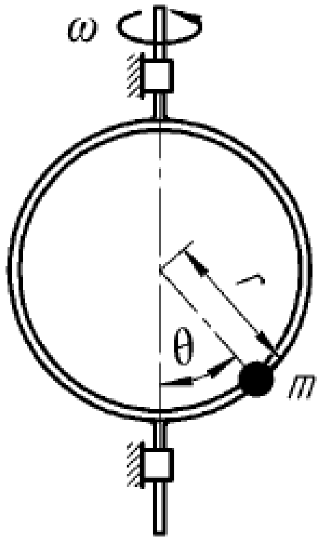


Fig. 2.19

2.12. Short problems

Problem 1. A locomotive of a mass $m = 8 \cdot 10^4\ kg$ moves on rails along equator from the east to the west with a velocity 20 m/sec. Determine a magnitude of Coriolis force of inertia of the locomotive, if angular velocity of the Earth is $\omega = 0.0000729\ rad/\sec$. The locomotive is considered as a particle.

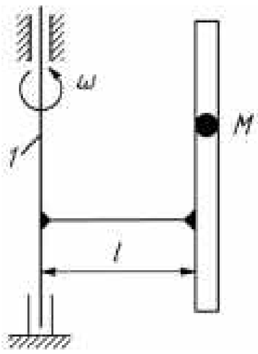


Fig. 2.20

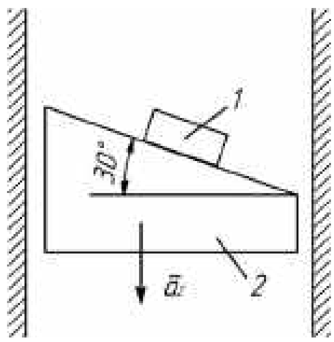


Fig. 2.21

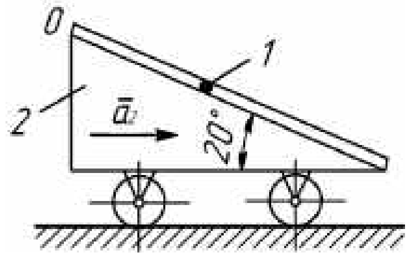


Fig. 2.22

Problem 2. A ball M of a mass $m = 0.2\,kg$ moves with a velocity $v = 19.62\,m/\sec$ relative to a vertical tube, which is attached to a vertical shaft 1 on a distance $l = 0.5\,m$ (Fig. 2.20). The shaft rotates with a constant angular velocity $\omega = 5\ rad/\sec$. Determine a bulk force of inertia of the ball.

Problem 3. A load 1 of a mass $m_1 = 1\,kg$ declines on an incline plane of a body 2 (Fig. 2.21). The body 2 moves in vertical guide down with an acceleration $a_2 = 2\,m/\sec$. Determine a pressure force of the load 1 on the body 2.

Problem 4. A ball 1 of a mass m_1 moves from a state of relative rest at appoint O along a smooth cylindrical channel of a body 2 (Fig. 2.22). The body 2 moves along horizontal plane with a constant acceleration $a_2 = 3.5\,m/s^2$. Determine a relative velocity of the ball at time $t = 5\sec$.

Problem 5. A tube rotates about axis O according to the law $\varphi = t^2$ (Fig. 2.23). A ball M of a mass $m = 0.1\,kg$ moves in a tube according to the law $OM = 0.2\,t^3$. Determine a magnitude Coriolis force of inertia of the ball at time $t = 1\sec$.

Problem 6. An elevator car 2 moves up with an acceleration $a_2 = 0.5g$ (Fig. 2.24). Determine spring tension, if suspended load 1 of a weight 100 N is at a state of relative rest.

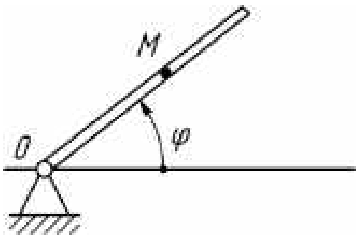


Fig. 2.23

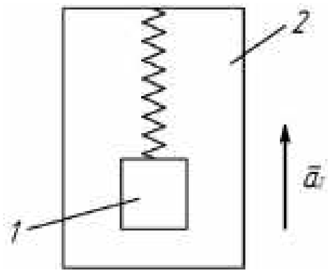


Fig. 2.24

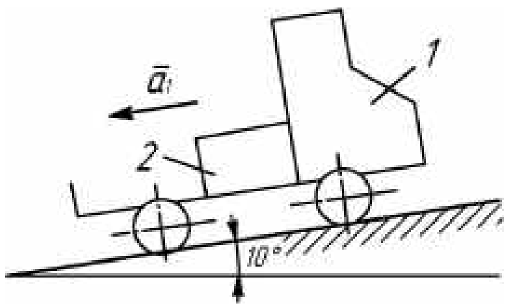


Fig. 2.25

Problem 7. An auto truck 1 (Fig. 2.25) moves up with a constant deceleration $a_1 = 2\,m/s^2$. Determine pressure force of the load 2 of a mass 200 kg on a front wall of the truck body.

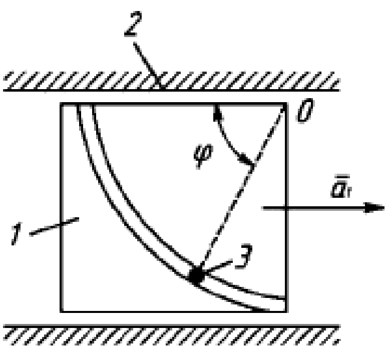


Fig. 2.26

Problem 8. A body 1 moves along rectilinear guide 2 (Fig. 2.26). Inside the body there is a channel in a shape of an arc. A ball 3 of a mass **m** moves along this channel. Determine an acceleration a_1 of the body 1, if at angle $\varphi = 60°$ the ball is at a state of relative rest.

Problem 9. A support with a mathematical pendulum moves along inclined plane down (Fig. 2.27) with an acceleration $a = g\sin\alpha$. Determine an angle β at a state of relative rest of the ball, if $\alpha = 10°$.

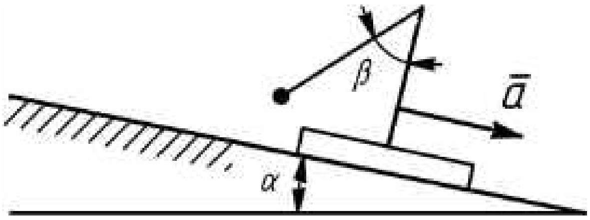


Fig. 2.27

LECTURE 3
3. DYNAMICS OF SYSTEM OF PARTICLES
3.1. Obstacles in analysis of particle system motion

As it is clear from previous consideration analysis of a particle motion includes integrating of the system of three second order differential equations (see the first lecture 1, eq. (1.22)). In general case it is not easy. For a system consists of n particles () we have to integrate system $3n$ equations in which forces are functions of positions and velocities of all particles:

$$
\begin{cases}
m\ddot{x}_1 = f_1\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right), \\
m\ddot{y}_1 = f_2\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right), \\
m\ddot{z}_1 = f_3\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
m\ddot{x}_n = f_{3(n-1)+1}\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right), \\
m\ddot{y}_n = f_{3(n-1)+2}\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right), \\
m\ddot{z}_n = f_{3n}\left(t, x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1, ..., x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n\right).
\end{cases}
$$

Solution of the system is very difficult problem.

But very often we need to determine only some total characteristics of the system of particles motion. These total characteristics are called **measures of system of particles motion**:

– total linear momentum;
– total angular momentum;
– total kinetic energy.

The general principles of dynamics describe relations between time rate of change of these measures and action of forces applied on the system (Table 3.1).

Table 3.1

Measure of a particle motion	Equation	Measure of particle system motion	Equation	Effect of forces	General principle
Linear momentum (vector, $\left[kg \cdot \frac{m}{s}\right]$)	$\vec{q} = m\vec{v}$	Total Linear momentum (vector, $\left[kg \cdot \frac{m}{s}\right]$)	$\vec{Q} = \sum_{k=1}^{n} \vec{q}_k$	Total vector of external forces	Force-Linear momentum principle
Angular momentum (vector, $\left[\frac{kg \cdot m^2}{s}\right]$)	$\vec{l}_O = \vec{r} \times m\vec{v}$	Angular momentum (vector, $\left[\frac{kg \cdot m^2}{s}\right]$)	$\vec{L}_O = \sum_{k=1}^{n} \vec{r} \times \vec{q}_k$	Total moment of external forces about the center O	Moment-Angular momentum principle
Kinetic energy (scalar, $[J]$)	$T = \frac{mv^2}{2}$	Kinetic energy (scalar, $[J]$)	$T = \sum_{k=1}^{n} \frac{m_k v_k^2}{2}$	Total work done by external and internal forces	Work-Energy principle

First we analyze classification and features of forces.

3.2. Force classification

3.2.1. External and internal forces

A **system of particles** is a set of particles (bodies) whose motions are interconnected. Position and kinematical characteristics of each particle of the system are functions of the same parameters of other particles. In engineering practice material systems are more often called **structures**.

There are two types of forces acting on structures: external forces and internal forces:

1. **External forces** represent the action of **other** bodies on the **structures** under consideration

2. **Internal forces** are the result of interaction between the parts of the **structures** under consideration. The internal forces hold together the various parts of the structure**.**

The features of internal forces acting on particles for which third Newton’s low (action equals reaction) applies.

1. **Total vector of internal forces is equal to nil**

$$\boxed{\sum_{k=1}^{n} \vec{F}_k^{(i)} = 0.} \qquad (3.1)$$

Where $\vec{F}_k^{(i)}$ is the resultant vector of internal forces acting on the particle with number k (superscript *i* denotes that force is internal).

Proof: Consider system of n particles. Force of interaction between particle k and particle j is internal force $\vec{F}_{kj}^{(i)}$ (Fig. 3.1). Force of interaction between particle j and particle k is internal force $\vec{F}_{jk}^{(i)}$. The forces $\vec{F}_{kj}^{(i)}$ and $\vec{F}_{jk}^{(i)}$ obey the third Newton's law, they are equal in magnitude, opposite in direction, therefore their sum is zero:

$$\vec{F}_{kj}^{(i)} + \vec{F}_{kj}^{(i)} = 0.$$

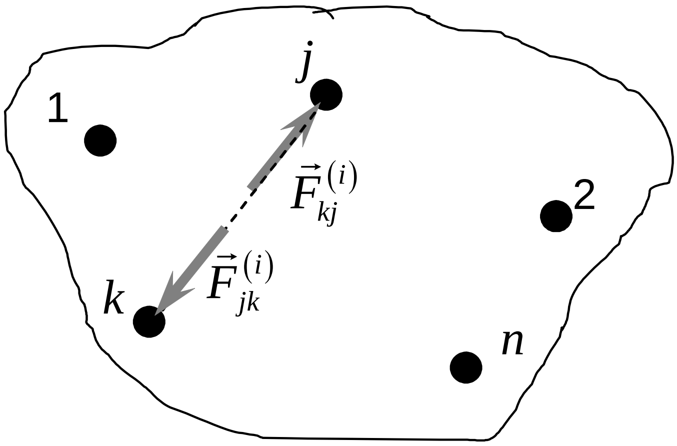


Fig. 3.1

Internal force with indexes $k = j$ does not exist because particle can not interact with itself.

Carrying out the double summation (after all internal forces of the particle system) we get

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \vec{F}_{kj}^{(i)} = \left(\vec{F}_{11}^{(i)} + \vec{F}_{12}^{(i)} + \ldots + \vec{F}_{1n}^{(i)}\right) + \left(\vec{F}_{21}^{(i)} + \vec{F}_{22}^{(i)} + \ldots + \vec{F}_{2n}^{(i)}\right) +$$

$$+ \ldots + \left(\vec{F}_{n1}^{(i)} + \vec{F}_{n2}^{(i)} + \ldots + \vec{F}_{nn}^{(i)}\right) = \left|\begin{array}{l} \vec{F}_{11}^{(i)} = 0 \\ \vec{F}_{jj}^{(i)} = 0 \\ \vec{F}_{nn}^{(i)} = 0 \end{array}\right| =$$

$$= \underbrace{\left(\vec{F}_{12}^{(i)} + \vec{F}_{21}^{(i)}\right)}_{0} + \underbrace{\left(\vec{F}_{13}^{(i)} + \vec{F}_{31}^{(i)}\right)}_{0} + \ldots + \underbrace{\left(\vec{F}_{n\,n-1}^{(i)} + \vec{F}_{n-1\,n}^{(i)}\right)}_{0} = 0. \qquad (3.2)$$

In equation (3.2) the sum $\sum_{j=1}^{n} \vec{F}_{kj}^{(i)}$ is resultant vector of internal forces acting on the particle k $\vec{F}_k^{(i)}$. So equation (3.2) can be rewritten as

$$\sum_{k=1}^{n} \vec{F}_k^{(i)} = 0. \tag{3.3}$$

This completes the proof of the first feature.

2. The total moment of internal forces about a center is equal to nil

$$\boxed{\sum_{k=1}^{n} \overrightarrow{M}_O\left(\overrightarrow{F}_k^{(i)}\right) = \sum_{k=1}^{n} \vec{r}_k \times \overrightarrow{F}_k^{(i)} = 0.} \tag{3.4}$$

Proof: consider the system from the previous proofing. Moment of the force $\vec{F}_{kj}^{(i)}$ about fixed center O (Fig. 3.2, a) is

$$\vec{M}_O\left(\vec{F}_{kj}^{(i)}\right) = \vec{r}_j \times \vec{F}_{kj}^{(i)},$$

moment of the force $\vec{F}_{jk}^{(i)}$ is

$$\vec{M}_O\left(\vec{F}_{jk}^{(i)}\right) = \vec{r}_k \times \vec{F}_{jk}^{(i)}.$$

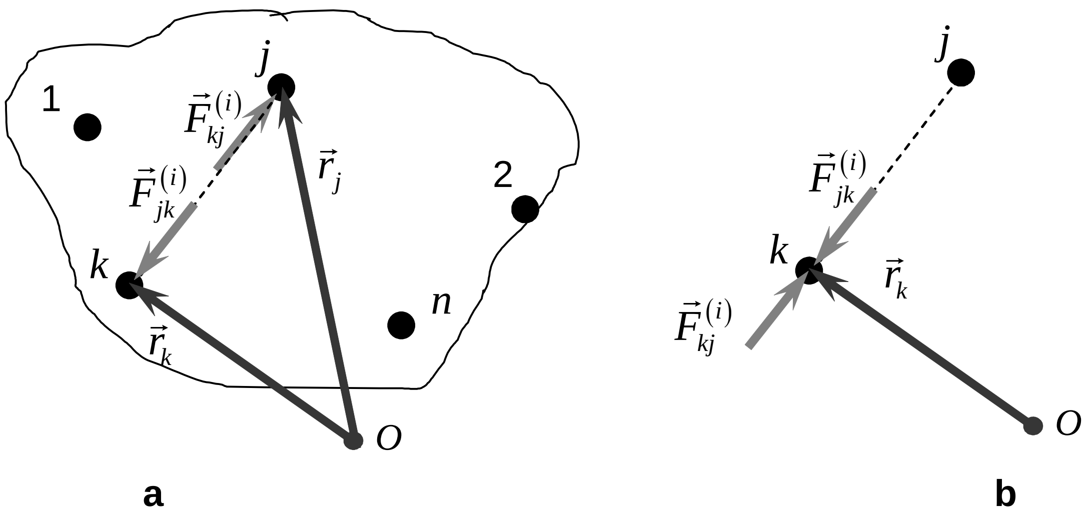


Fig. 3.2

Analogy with the double summation for forces we get

$$\sum_{k=1}^{n}\sum_{j=1}^{n} \vec{M}_O\left(\vec{F}_{kj}^{(i)}\right) = \left(\vec{M}_O\left(\vec{F}_{12}^{(i)}\right) + \vec{M}_O\left(\vec{F}_{21}^{(i)}\right)\right) + \ldots +$$

$$+\left(\vec{M}_O\left(\vec{F}_{n-1n}^{(i)}\right) + \vec{M}_O\left(\vec{F}_{n\,n-1}^{(i)}\right)\right) = 0. \tag{3.5}$$

The forces $\vec{F}_{kj}^{(i)}$ and $\vec{F}_{jk}^{(i)}$ are equal in magnitude, opposite in direction and act along the same line (see Fig. 3.2, b). So

$$\vec{M}_O\left(\vec{F}_{kj}^{(i)}\right) = \vec{r}_k \times \vec{F}_{jk}^{(i)} = -\left(\vec{r}_j \times \vec{F}_{kj}^{(i)}\right) = -\vec{M}_O\left(\vec{F}_{jk}^{(i)}\right).$$

In equation (3.5) the sum $\sum_{j=1}^{n} \vec{M}_O\left(\vec{F}_{kj}^{(i)}\right)$ is resultant moment of internal forces acting on the particle k about center O. So equation (3.5) can be rewritten as

$$\sum_{k=1}^{n} \overrightarrow{M}_O\left(\vec{F}_k^{(i)}\right) = 0. \qquad (3.6)$$

This completes the proof of the second feature.

3.2.2. Active (applied) forces and reactions

If motion of a particle (body) is not restricted such body is called ***free***. If not, body is ***constrained***. Constraints (supports or connections) restrict the body motion in some direction.

The forces exerted on the body by the constraints are known as ***reactions.***

Other forces acting on the body and which are independent of the constraints are called ***active (applied) forces***.

3.3. Force-Linear Momentum principle

3.3.1. A particle linear momentum

A **particle linear momentum** is a vector value (Fig. 3.3), which is equal to the product of the particle mass and its velocity vector

$$\boxed{\vec{q} = m\vec{v}.} \qquad (3.7)$$

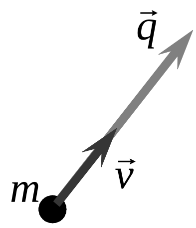


Fig. 3.3

Linear momentum direction is the same as that of the velocity (see Fig. 3.3).

3.3.2. Force-momentum principle for a particle

Now we may write the equation of motion for the particle in inertial frame of reference as

$$m\overrightarrow{W} = \vec{F}, \quad m\overrightarrow{W} = m\frac{d\vec{v}}{dt} = \left|m = const\right| = \frac{d\left(m\vec{v}\right)}{dt} = \frac{d\vec{q}}{dt},$$

$$\boxed{\frac{d\vec{q}}{dt} = \vec{F}.} \tag{3.8}$$

Force-momentum principle for a particle. Differential form. **The resultant of all forces acting on a particle equals its time rate of change of linear momentum.**

This relationship is valid as long as mass of the particle not changing with time.

In scalar form the force-momentum principle is the system of three equations:

$$\begin{cases} \dot{q}_x = F_x, \\ \dot{q}_y = F_y, \\ \dot{q}_z = F_z. \end{cases} \tag{3.9}$$

If we want describe the effect of the resultant force on the linear momentum of the particle over a finite period of time we may integrate the equation (3.8) with respect to time t from time t_1 to t_2. The product of force and elementary time is defined as **elementary linear impulse of the force.** The integral $\int_{t_1}^{t_2} \vec{F} dt$ total linear impulse of the force. Than integrating (3.8) we get

$$\boxed{\vec{q}_2 - \vec{q}_1 = \int_{t_1}^{t_2} \vec{F} dt.} \tag{3.10}$$

Force-momentum principle for a particle. Integral form. **The total linear impulse of force acting on a particle equals the corresponding change in linear momentum of the particle**.

3.3.3. Force-momentum principle for a particle system

Let us consider a system of n particles.

Let denote $\vec{Q}$ the vector sum of the linear momenta of all particles of the system. $\vec{Q}$ is **total linear momentum of the particle system**

$$\boxed{\vec{Q} = \sum_{k=1}^{n} \vec{q}_k.} \tag{3.11}$$

From Equation (3.8) for the *k-th* particle we have:

$$\frac{d\vec{q}_k}{dt} = \vec{F}_k^{(e)} + \vec{F}_k^{(i)}, \tag{3.12}$$

where $\vec{F}_k^{(e)}$ - external force system resultant acting on the particle number k; $\vec{F}_k^{(i)}$ - internal forces resultant.

Summing the equations over k (k runs from1 to n), we obtain

$$\sum_{k=1}^{n} \frac{d\vec{q}_k}{dt} = \sum_{k=1}^{n} \vec{F}_k^{(e)} + \sum_{k=1}^{n} \vec{F}_k^{(i)} . \tag{3.13}$$

According with the first feature of internal forces

$$\sum_{k=1}^{n} \vec{F}_k^{(i)} = 0, \tag{3.14}$$

therefore using denotation $\vec{F}^{(e)} = \sum_{k=1}^{n} \vec{F}_k^{(e)}$ where $\vec{F}^{(e)}$ is total vector of external forces acting on the system we get

$$\boxed{\frac{d\vec{Q}}{dt} = \vec{F}^{(e)}.} \tag{3.15}$$

Force-momentum principle for a particle system. Differential form. **The total external force on a particle system equals the time rate of change of the total linear momentum of the particle system.**

This principle does not apply to the system whose mass changes with time.

The scalar view of equation (3.15) is:

$$\begin{cases} \dfrac{dQ_x}{dt} = \sum_{k=1}^{n} F_{x\,k}^{e}; \\ \dfrac{dQ_y}{dt} = \sum_{k=1}^{n} F_{y\,k}^{e}; \\ \dfrac{dQ_z}{dt} = \sum_{k=1}^{n} F_{z\,k}^{e}. \end{cases} \tag{3.16}$$

3.3.4. Three Corollaries of the Force-momentum principle for a system of particles

The first corollary (principle of conservation of linear momentum).

Let us examine the equation (3.15) under condition when the total vector of external forces is zero

$$\sum_{k=1}^{n} \overrightarrow{F^e}_k = 0 .$$

Then

$$\frac{d\vec{Q}}{dt} = 0; \vec{Q} = const. \qquad (3.17)$$

If the total external force on the system of particles during a time interval equals zero than the total linear momentum of the system is unchanged during the time interval.

The second corollary.

Let us examine the projection of the expression (3.17) on an axis

$$\frac{d}{dt} Q_u = V_u = 0,$$

$$Q_u = const. \qquad (3.18)$$

If the projection on an axis of the total external force on the system of particles during a time interval equals zero than the projection on the axis of the total linear-momentum of the system is unchanged during the time interval.

There is **third corollary** of the force-momentum principle:

the internal forces cannot change the total linear momentum of a system of particles.

3.3.5. Principle of system mass-center motion

It is special form of force-momentum principle.

We remember now the **concept of center of mass**.

The point is called **center of mass** if its position vector is determined by expression

$$\boxed{\vec{r}_C = \frac{1}{M}\sum_{k=1}^{n} m_k \vec{r}_k.} \tag{3.19}$$

where $\vec{r}_k$ is position vector of particle k, m_k is mass of particle k, it is assumed as constant value.

In scalar form we have

$$\begin{cases} x_C = \dfrac{1}{M}\sum\limits_{k=1}^{n} m_k x_k, \\ y_C = \dfrac{1}{M}\sum\limits_{k=1}^{n} m_k y_k, \\ z_C = \dfrac{1}{M}\sum\limits_{k=1}^{n} m_k z_k. \end{cases} \tag{3.20}$$

For the homogeneous rigid body (the body that has uniform mass per unit volume γ) the Equation (3.19) may be rewritten as

$$\begin{cases} x_C = \dfrac{1}{M}\int\limits_V x \cdot \gamma dV, \\ y_C = \dfrac{1}{M}\int\limits_V y \cdot \gamma dV, \\ z_C = \dfrac{1}{M}\int\limits_V z \cdot \gamma dV \end{cases} \tag{3.21}$$

or

$$\left.\begin{aligned} x_C &= \frac{1}{V}\iiint\limits_{(V)} x \cdot dxdydz, \\ y_C &= \frac{1}{V}\iiint\limits_{(V)} y \cdot dxdydz, \\ z_C &= \frac{1}{V}\iiint\limits_{(V)} z \cdot dxdydz. \end{aligned}\right\} \tag{3.22}$$

By differentiating equation (3.19) for the position vector of mass center with respect to time we obtain

$$\vec{v}_C = \frac{d\vec{r}_C}{dt} = \frac{1}{M}\sum_{k=1}^{n} m_k \vec{v}_k = \frac{1}{M}\vec{Q},$$

$$\boxed{\vec{Q} = M\vec{v}_c\,.} \tag{3.23}$$

The total linear momentum of a particle system equals the linear momentum of a particle that has mass of the system and moves with velocity of the mass center.

Than equation (3.15) may be written as

$$\boxed{M\frac{d\vec{v}_C}{dt} = \vec{F}^{(e)}} \tag{3.24}$$

or

$$\boxed{M\vec{W}_C = \vec{F}^{(e)}.} \tag{3.25}$$

A particle system mass-center moves as point acted on by force. Mass of the point is equal to mass of the system. The force is equal to the total vector of external forces applied to the system.

If the total external force on a system of particles is zero, it is clear from the previous discussion that there can be no change in the linear momentum of the system. This is the principle of **conservation of linear momentum**, which means, furthermore, that with a zero external force on a system of particles, there can be no change in the velocity of the mass center.

3.3.6. Examples

Example 1. A man in a boat 1 pushes away a log 2 of a mass $m_2 = 200\ kg$. a mass of the boat with the man is $m_1 = 160\ kg$. Determine a boat velocity after a push, if at initial moment of time the boat and the log were at rest, and after a push the log has a velocity $V_2 = 0.5\ m/\sec$. Neglect water resistance.

Solution

Show the system at initial moment t_0 , before a push (Fig. 3.4, a) and final moment t_1 , after a push (Fig. 3.4, b).

Show external forces: gravity forces $m_1\overline{g}, m_2\overline{g}$ and Arhimed's forces $\overline{F_1}, \overline{F_2}$.

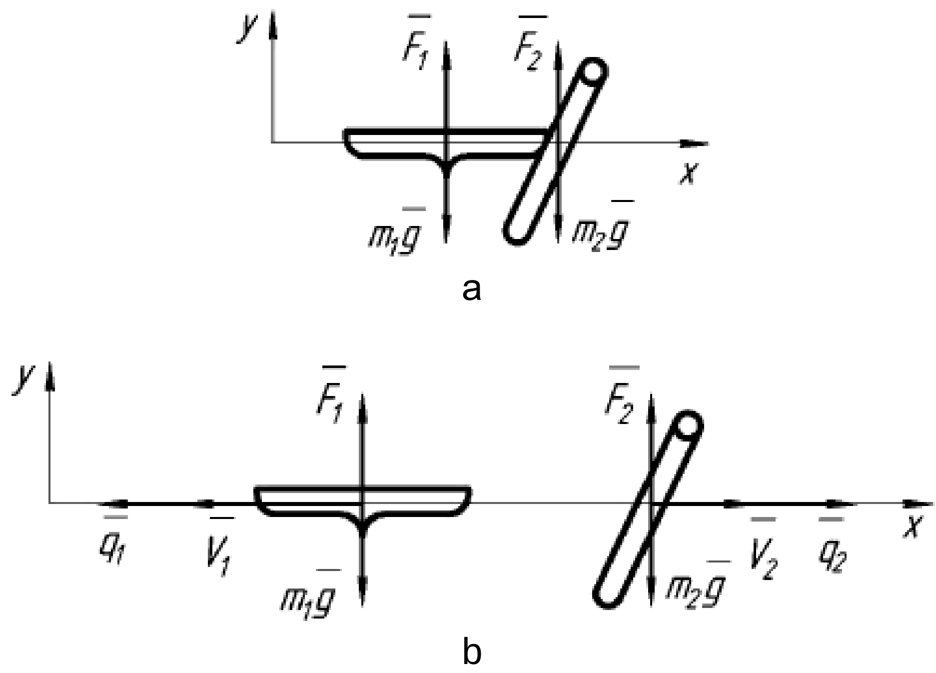


Fig. 3.4

Direct x-axis along water surface and write force-momentum principle in projection on x-axis:

$$\frac{dQ_x}{dt} = \sum_{k=1}^{n} F_{kx}^e.$$

Projections of all external forces on x-axis are equal to zero, i.e.

$$\sum_{k=1}^{n} F_{kx}^e = 0 \Rightarrow \frac{dQ_x}{dt} = 0.$$

So we have linear momentum projection on axis x conservation :$Q_x = const$, i.e.

$$Q_{0x} = Q_{1x}, \tag{3.26}$$

where Q_{0x} and Q_{1x} are projections of linear momentum of the system on x-axis at initial moment t_0 and final moment t_1 correspondingly.

The system consists of the boat 1 and the log 2. So the total linear momentum is:

$$\overline{Q} = \overline{q_1} + \overline{q_2}\,.$$

At initial moment, the system was at rest. That's why $Q_{0x} = 0$. At final moment

$$Q_{1x} = q_{1x} + q_{2x} = -m_1 v_1 + m_2 v_2\,.$$

Substituting Q_{0x}, Q_{1x} in equation (3.26):

$$0 = -m_1 v_1 + m_2 v_2\,,$$

$$v_1 = \frac{m_2 v_2}{m_1} = \frac{200 \cdot 0.5}{160} = 0.625\left(\frac{m}{\text{sec}}\right).$$

Answer: After a push the boat will move in opposite side to the log motion with a velocity $v_1 = 0.625\left(\frac{m}{\text{sec}}\right)$.

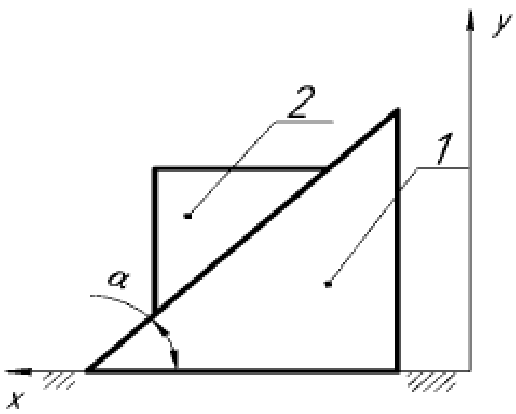


Fig. 3.5

Example 2. A prism 2 of a mass m_2 slides down a smooth side of another prism 1 of a mass m_1 $(m_1 = 2m_2)$, as shown in Fig. 3.5. The edge makes an angle $\alpha = 45°$ with the horizontal. Determine a displacement of the 1st prism, when the 2nd prism moves down on $h_2 = 0.4\ m$. At initial moment the system was at rest. Neglect friction between the prism 1 and the horizontal plane.

Solution

Analyze external forces acting on the system: gravity forces $m_1\overline{g}, m_2\overline{g}$ and normal reaction of horizontal surface $\overline{N}$ (Fig. 3.6). In this problem the principle of a system mass-center motion will be used. As displacement of the first prism is unknown x_1^*, only projection of equation (3.25) on x-axis is necessary:

$$M\ \ddot{x}_c = F_x^e,$$

where M is a system mass, $\ddot{x}_c$ is projection of a mass center acceleration on x-axis, F_x^e is projection of all external forces on x-axis.

Projections of all forces on x-axis are zero, i.e. $F_x^e = 0$. Then

$$M\ \ddot{x}_c = 0, \quad \ddot{x}_c = 0,$$

$$\dot{x}_c = c_1, \qquad (3.27)$$

$$x_c = c_1 t + c_2. \qquad (3.28)$$

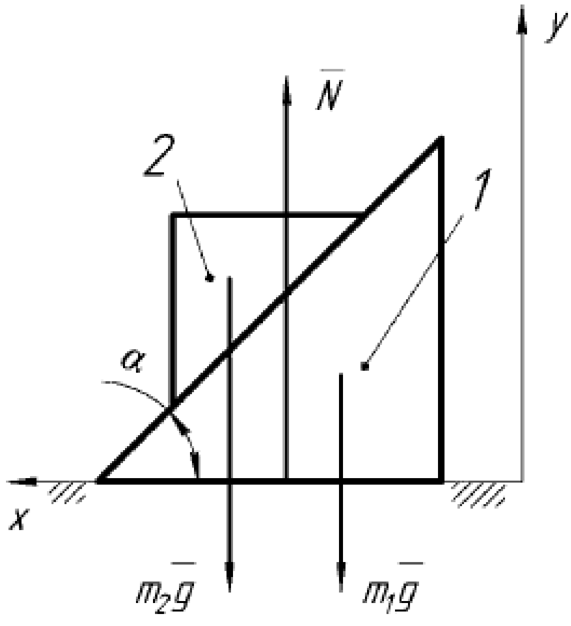


Fig. 3.6

To find unknown constants of integration the initial conditions are used. According to condition of the problem at initial moment of time the system was at rest, i.e. $\dot{x}_{c0} = 0$. The coordinate x_{c0} depends on chose of position of origin of x-axis. We can assume that it passes through the system mass center: $x_{c0} = 0$. Then substituting $t = 0$ to the equations (3.27), (3.28) and taking into account initial conditions we obtain constants of integration: $c_1 = c_2 = 0$. So $x_c = 0$. It means that the system mass center doesn't move.

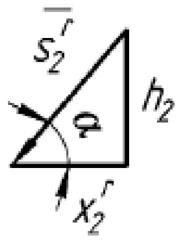
Then we get:

$$x_c = \frac{1}{M}\left(m_1 x_1 + m_2 x_2\right) = 0,$$

$$m_1 x_1 + m_2 x_2 = 0, \qquad (3.29)$$

where x_1 and x_2 are absolute coordinates of the mass centers of the 1st and the 2nd prisms.

The absolute coordinate of the mass center of the 1st is equal to: $x_1 = x_1^* + x_{1_0}$. The 2nd prism has compound motion. It moves with the 1st prism (bulk motion) and slides down the side of the 1st prism (relative motion with displacement $\vec{s}_2^{\,r}$, fig. 3.7), i. e.



$$x_2 = x_2^a = x_2^r + x_2^e = h_2 \cdot c\tan\alpha + \left(x_1^* + x_{2_0}\right).$$

Substituting these expressions to the equation (3.29):

$$m_1 \cdot \left(x_1^* + x_{1_0}\right) + m_2 \cdot \left(h_2 \cdot c\tan\alpha + \left(x_1^* + x_{2_0}\right)\right) = 0.$$

Fig. 3.7

In accordance with (3.29) at initial moment $m_1 x_{1_0} + m_2 x_{2_0} = 0$. Then

$$x_1^* = -h_2 \cdot c\tan\alpha \frac{m_2}{m_2 + 2m_2} = -0.4 \cdot c\tan\alpha \cdot \frac{1}{3} = -0.13\,(m).$$

Answer: the 1st prism moves in negative direction of x-axis on $0.13\ m$.

Example 3. The rotor of the electric motor rotates clockwise with angular velocity $n = 980\ \dfrac{rev}{\min}$ (Fig. 3.8). The weight of the motor is $P_1 = 700\ N$, and the weight of the rotor is $P_2 = 300\ N$. The gravity center of the rotor is shifted from axis of rotation on a distance $l = 0.05\ m$. Determine horizontal shearing force acting on the bolts and vertical pressure on supporting plane.

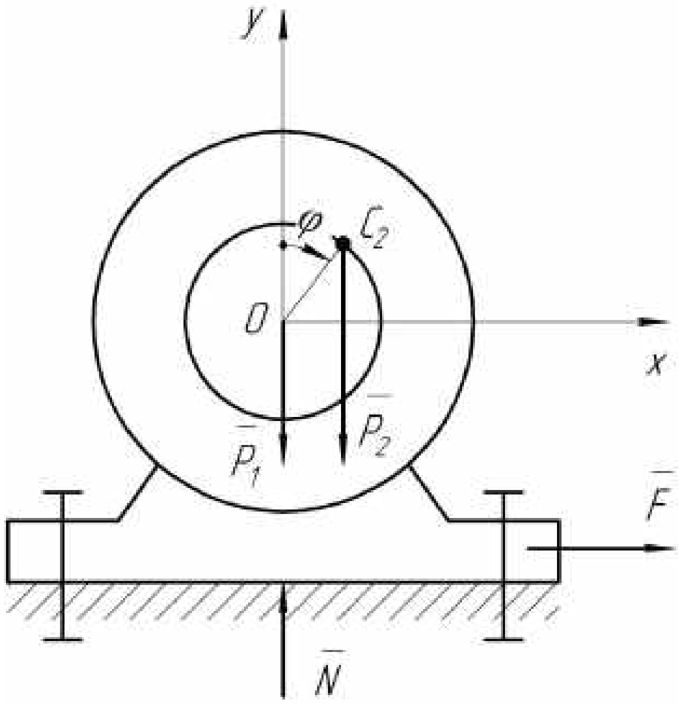


Fig. 3.8

Solution

Assume, that at $t = 0$ the gravity center of the rotor C_2 was on y-axis. Then at a moment *t* the coordinates of the rotor mass-center are:

$$x_2 = l\sin\varphi = l\sin(\omega t),$$
$$y_2 = l\cos\varphi = l\cos(\omega t),$$

where $\omega = \frac{\pi n}{30} = \pi\frac{980}{30}\left(\frac{rad}{\sec}\right)$ is angular velocity.

External forces (see Fig. 3.8) are: gravity forces $\overline{P_1}, \overline{P_2}$, reaction of the plane $\overline{N}$ and reaction of the bolts $\overline{F}$. To solve this problem a principle of system mass-center motion is used:

$$M\frac{d\overline{v_c}}{dt} = \vec{F}^{(e)} = \sum_{k=1}^{n}\vec{F}_k^e.$$

Projecting on the axes *x* and *y*:

$$\begin{cases} x)\ M\ \ddot{x}_c = F, \\ y)\ M\ \ddot{y}_c = N - P_1 - P_2, \end{cases} \tag{3.30}$$

where the total mass of the system is $M = m_1 + m_2 = \frac{P_1 + P_2}{g}$.

Let's determine the coordinate of the mass center:

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}; \qquad y_c = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}.$$

Taking into account, that the mass center of the motor is fixed point, i.e. $x_1 = 0,\ y_1 = 0$:

$$x_c = \frac{m_2x_2}{m_1 + m_2} = \frac{P_2 l\sin(\omega t)}{P_1 + P_2};\ \ y_c = \frac{m_2y_2}{m_1 + m_2} = \frac{P_2 l\cos(\omega t)}{P_1 + P_2}.$$

Let's find the derivatives:

$$\ddot{x}_c = -\frac{P_2 l\omega^2}{P_1 + P_2}\sin(\omega t); \quad \ddot{y}_c = -\frac{P_2 l\omega^2}{P_1 + P_2}\cos(\omega t).$$

Substituting these expressions to the system (3.30):

$$\begin{cases} x) \dfrac{P_1 + P_2}{g}\left(-\dfrac{P_2 l\omega^2}{P_1 + P_2}\sin(\omega t)\right) = F, \\ y) \dfrac{P_1 + P_2}{g}\left(-\dfrac{P_2 l\omega^2}{P_1 + P_2}\cos(\omega t)\right) = N - P_1 - P_2. \end{cases}$$

From these equations we can find unknown values:

$$F = -\frac{P_2 l\omega^2}{g}\sin(\omega t),$$

$$N = P_1 + P_2 - \frac{P_2 l\omega^2}{g}\cos(\omega t).$$

The force acting on the bolts is $\vec{F}' = -\vec{F}$, and the pressure of the motor on the supporting plane is $\vec{N}' = -\vec{N}$.

The maximum magnitudes of the forces are:

If $\left|\sin(\omega t)\right| = 1$: $F'_{\max} = \dfrac{P_2 l\omega^2}{g} = \dfrac{300 \cdot 0.05 \cdot 98^2 \cdot \pi^2}{9.8 \cdot 3^2} = 16.1(kN)$;

If $\cos(\omega t) = -1$:

$$N'_{\max} = P_1 + P_2 + \frac{P_2 l\omega^2}{g} = 700 + 300 + \frac{300 \cdot 0.05 \cdot 98^2 \cdot \pi^2}{9.8 \cdot 3^2} = 17.2(kN).$$

If there are no bolts, the motor can hop (jump). The condition of hopping impending is $N = 0$ (absence of interaction with support). There is such angular velocity $\omega*$ of the rotor rotation, at which the motor is not hoping, but there is no pressure on the plane for some orientation of rotor. If angular velocity exceeds $\omega*$, the motor will begin hopping.

Answer: $F'_{\max} = 16.1(kN)$, $N'_{\max} = 17.2(kN)$.

3.3.7. Problems for self-solution

Problem 1. The man of mass m_1 and the woman of mass m_2 are standing on opposite ends of the platform of mass m_0 which moves with negligible friction and is initially at rest with $s = 0$ (Fig. 3.9). The man and woman begin to approach each other. Derive an expression for the displacement s of the platform when the two meet in terms of the displacement x_1 of the man relative to the platform.

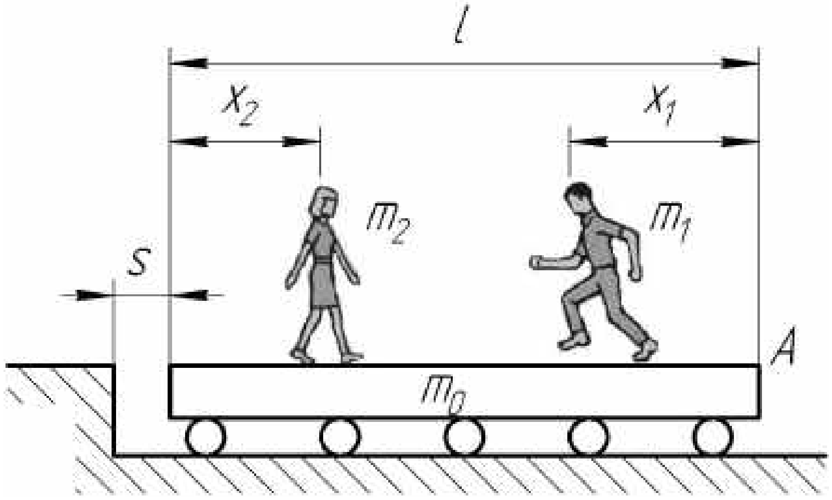


Fig. 3.9

Problem 2. The small car which has a mass of $20\ kg$ rolls freely on the horizontal track and carries the $5-kg$ sphere mounted on the light rotating rod with $r = 0.4\ m$ (Fig. 3.10). A geared motor drive maintains a constant angular speed $q = 4 rad / s$ of the rod. If the car has a velocity $v = 0.6\ m/s$ when $q = 0$, calculate v when $q = 60°$. Neglect the mass of the wheels and any friction.

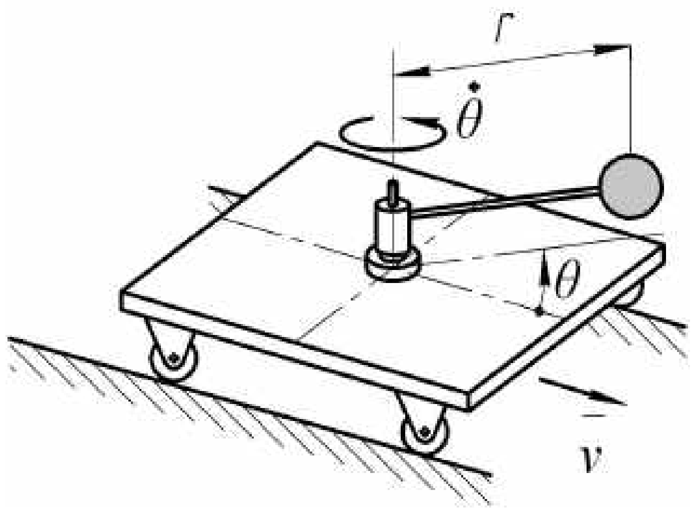


Fig. 3.10

Problem 3. The 50,000-lb flatcar supports a 15,000-lb vehicle on a 5° ramp built on the flatcar (Fig. 3.11). If the vehicle is released from rest with the flatcar also at rest, determine the velocity v of the flatcar when the vehicle has rolled $s = 40\ ft$ down the ramp just before hitting the stop at B. Neglect all friction and treat the vehicle and the flatcar as particles.

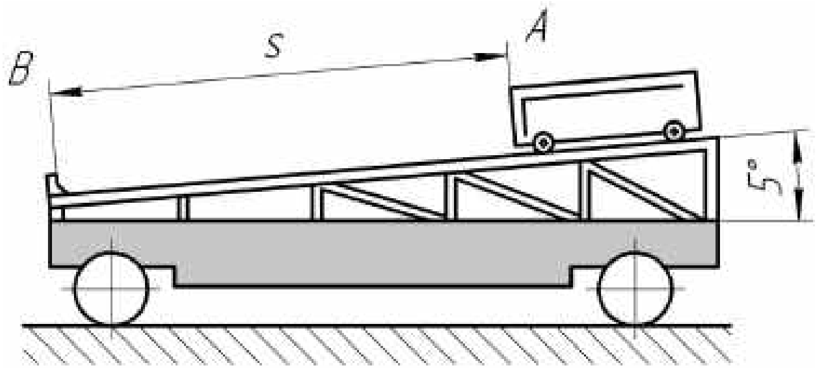


Fig. 3.11

Problem 4. A horizontal bar of mass m_1 and small diameter is suspended by two wires of length l from a carriage of mass m_2, which is free to roll along the horizontal rails (Fig. 3.12). If the bar and carriage are released from rest with the wires making an angle θ with the vertical, determine the velocity $v_{b/c}$ of the bar relative to the carriage and the velocity v_c of the carriage at the instant when $\theta = 0$. Neglect all friction and treat the carriage and the bar as particles in the vertical plane of motion.

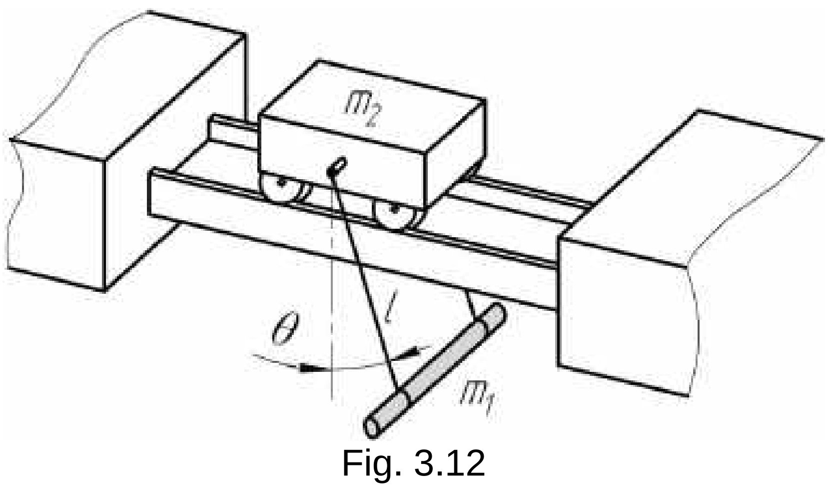


Fig. 3.12

Problem 5. A test firing of two projectiles each weighing $20\ lb$ takes place from the vehicle which weighs $2000\ lb$ and is moving with an initial velocity $v_0 = 4\ ft/\sec$ in the direction opposite to the firing (Fig. 3.13). The muzzle velocity of each projectile (relative to the barrel) is $v_r = 800\ ft/\sec$. Calculate the velocity v' of the vehicle after the projectiles have been fired (a) simultaneously or (b) in sequence. Neglect the friction and mass of the wheels.

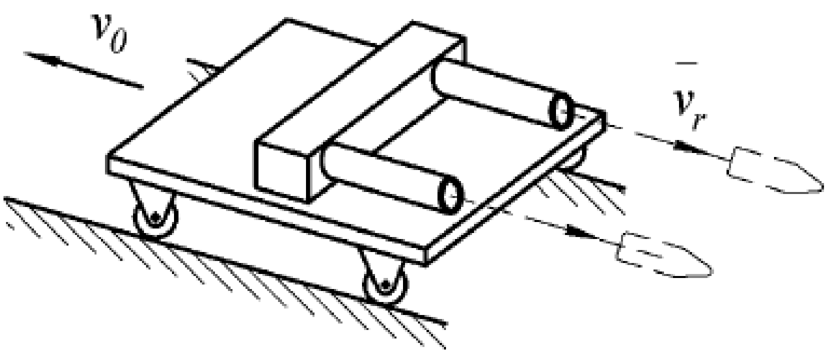


Fig. 3.13

3.3.8. Short problems

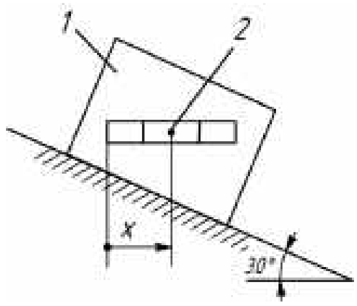


Fig. 3.14

Problem 1. Determine an acceleration of a body 1 (Fig. 3.14), sliding on a smooth inclined plane, if in horizontal guides relative to it under action of internal forces of the system the body 2 moves according to the equation $x = t^2$. Masses of the bodies are: $m_1 = m_2 = 1\,kg$. The bodies have translational motion.

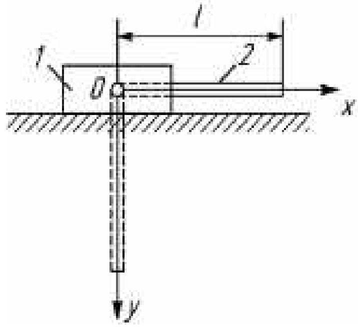


Fig. 3.15

Problem 2. A body 1 with a mass $4\,kg$ can move along a horizontal guide (Fig. 3.15). On which distance will the body 1 move when a homogeneous rod *2* with a mass $2\,kg$ and length $l = 0,6m$, going down under an action of a gravity force, has vertical position. At initial moment the system was at rest.

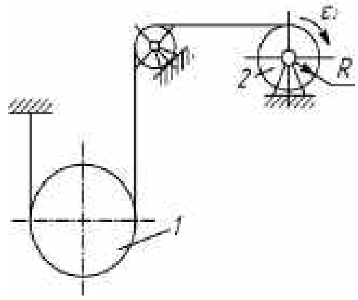
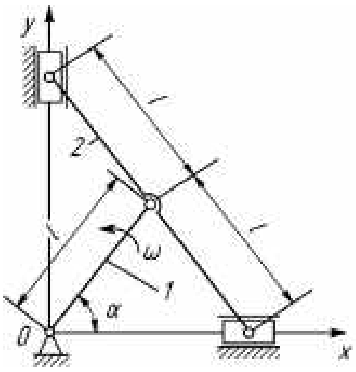


Fig. 3.16

Problem 3. The pulley 2 with a radius $R = 0,2m$ rotating with angular acceleration $\varepsilon_2 = 10\,rad/s^2$, lifts a homogeneous cylinder 1, a mass of which is $m = 50\,kg$ (Fig. 3.16). Determine a magnitude of a resultant vector of external forces acting on the cylinder.

Problem 4. Determine a projection on *Oy*-axis of a linear momentum vector of a homogeneous rod 2 (Fig. 3.17) of a mass $m = 4\ kg$ at a moment when the crank 1 rotates with angular velocity $\omega = 10\ rad/s$ and angle is $\alpha = 60°$. The length is $l = 0.2\ m$.



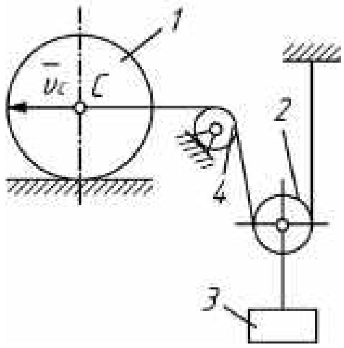
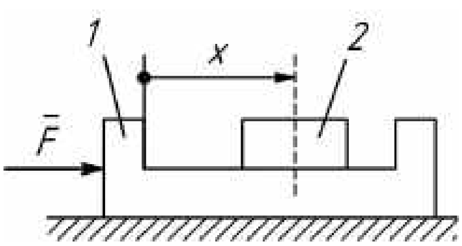


Fig. 3.17

Fig. 3.18

Problem 5. Determine a magnitude of a linear momentum of a mechanical system (Fig. 3.18), if the mass center C of the cylinder 1 moves with velocity $v_C = 4 m/s$, and masses of the bodies 1, 2 and 3 are equal correspondingly to $m_1 = 40\ kg$, $m_2 = 10\ kg$, $m_3 = 12 kg$. The bodies 2 and 4 are homogeneous disks.

Problem 6. The link 1 with length $OA = 1m$ of parallel link mechanism *OABO₁* rotates with angular velocity $\omega = 20\ rad/s$ (Fig. 3.19). Determine a magnitude of a linear momentum of the mechanism in the indicated position. The links1, 2 and 3 are homogeneous rods with masses $m_1 = m_2 = m_3 = 4\ kg$.



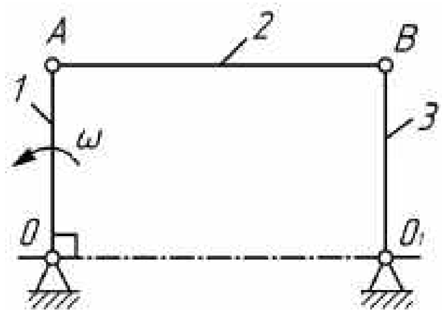


Fig. 3.19

Fig. 3.20

Problem 7. On the body 1 (Fig. 3.20) a constant force $F = 10N$ acts. Determine an acceleration of this body at a moment $t = 0.5 s$, if relative to it under action of internal forces of the system the body 2 moves according to an equation $x = \cos\pi t$. Masses of the bodies are: $m_1 = 4\ kg$, $m_2 = 1kg$. The bodies have translational motion.

Problem 8. Determine a projection of the mass center C acceleration (Fig.3.21) of the mechanical system on *Oy*-axis at a moment, when a coordinate is $y_C = 0{,}8m$, if a mass of the system is $m = 10\ kg$, and a resultant vector of applied external forces is $\overline{R} = 3\cdot\overline{i} + 6t\cdot\overline{j}$.At initial moment of time the mass center of the system was at the point O at rest.

Problem 9. The slider A moves under action of a force $\overline{F}$ with a constant velocity $\vec{v}_A$ (Fig. 3.22). Determine the reaction of a guide on the slider A at that moment of time, when an acceleration of the slider B is equal to $W_B = 4\ m/s^2$ if a mass of the homogeneous rod AB is equal to $5\ kg$. The masses of the sliders are neglected.

Problem 10. A homogeneous equilateral triangle OAB with a mass $m = 5\ kg$ rotates uniformly about a fixed axis (Fig. 3.23). Determine its angular velocity ω if a resultant vector of external forces acting on it is equal to $300\ N$ and a length is $l = 0{,}4m$.

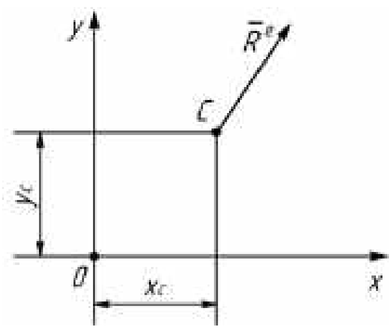


Fig. 3.21

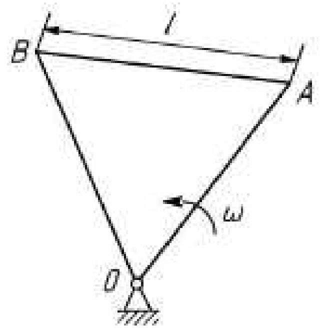


Fig. 3.22

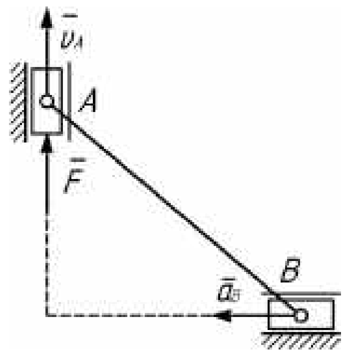


Fig. 3.23

LECTURE 4

3.4. Moment-Angular Momentum principle

3.4.1. A particle linear momentum

Consider a particle of mass m moving along a curve in space. The particle is located by its position vector $\vec{r}$ with respect to a convenient origin of inertial coordinate system $Oxyz$. The velocity of the particle is $\vec{v}$ and particle linear momentum is $\vec{q} = m\vec{v}$.

The **angular momentum (moment of momentum) of a particle of mass m about the origin of the inertial reference** is given by cross-product:

$$\vec{l}_o = \overrightarrow{M}_o(\vec{q}) = \vec{r} \times m\vec{v}. \qquad (3.31)$$

The direction is given by the right hand ruler (Fig. 3.24). That is to say,

1. the angular momentum is perpendicular to the plane of the velocity and the particle position vector, from the end of the vector of angular momentum rotation of the velocity about the origin is viewed counterclockwise.
2. the angular momentum is applied at the origin O.
3. the magnitude of the angular momentum can be determined as

$$l_o = mvr\,sin(\vec{r},\vec{v}\,) = mvh. \qquad (3.32)$$

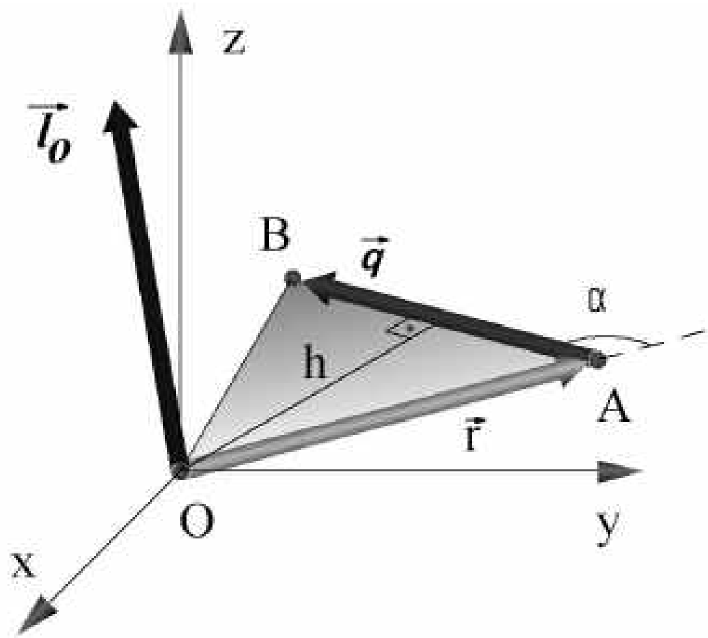


Fig. 3.24

The scalar components l_x, l_y, l_z of the angular momentum about the origin may be obtained from the following expression

$$\vec{l}_0 = \vec{r} \times m\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} =$$

$$= \vec{i}\,(ym\dot{z} - zm\dot{y}) + \vec{j}\,(zm\dot{x} - xm\dot{z}) + \vec{k}\,(xm\dot{y} - ym\dot{x}),$$

$$\begin{cases} l_x = m\left(y\dot{z} - z\dot{y}\right), \\ l_y = m\left(z\dot{x} - x\dot{z}\right), \\ l_z = m\left(x\dot{y} - y\dot{x}\right). \end{cases} \qquad (3.33)$$

3.4.2. Angular momentum of a system of particles

A system of n particles in inertial frame of reference is shown (Fig. 3.25).

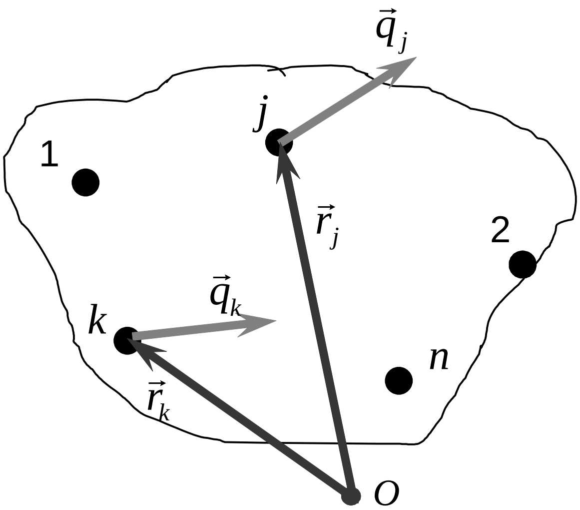


Fig. 3.25

The ***total angular momentum*** **about origin** O **fixed in an inertial reference** is the vector sum of the angular momenta of the particles of the system

$$\vec{L}_O = \sum_{k=1}^{n} \vec{l}_{Ok} = \sum_{k=1}^{n} \vec{r}_k \times \vec{q}_k = \sum_{k=1}^{n} \vec{r}_k \times m_k \vec{v}_k \ , \qquad (3.34)$$

or in scalar form

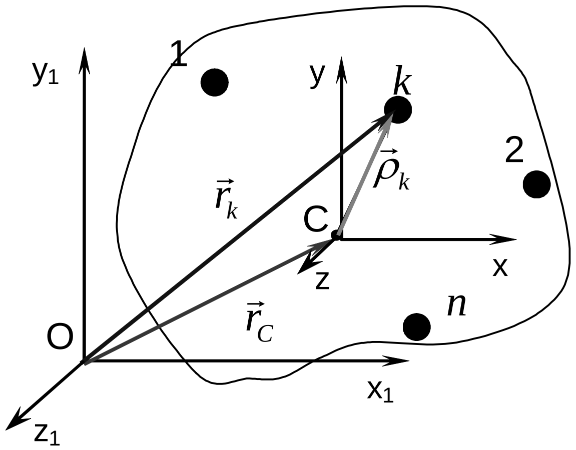


Fig. 3.26

$$\begin{cases} L_x = \sum_{k=1}^{n} l_{xk} = \sum_{k=1}^{n} m_k (y_k \dot{z}_k - z_k \dot{y}_k); \\ L_y = \sum_{k=1}^{n} l_{yk} = \sum_{k=1}^{n} m_k (z_k \dot{x}_k - x_k \dot{z}_k); \\ L_{Oz} = \sum_{k=1}^{n} l_{zk} = \sum_{k=1}^{n} m_k (x_k \dot{y}_k - y_k \dot{x}_k). \end{cases} \tag{3.35}$$

3.4.3. The total angular momentum with respect to a chosen center O under the condition of compound motion of the particles

Consider the system of n particles. The motion of the particles is described in inertial $\left(Ox_1y_1z_1\right)$ and noninertial $\left(Cxyz\right)$ references (Fig. 3.26). The noninetial reference has translational motion relative to the inertial one and its origin coincides with the centerof mass of the particlesystem. The reference is called **Koenig's system of reference**.

The vector position of any particle (see Fig. 3.26) is

$$\vec{r}_k = \vec{r}_C + \vec{\rho}_k , \tag{3.36}$$

where $\vec{\rho}_k$ is vector position of the particle relative moving reference $\vec{r}_C$ is the center mass vector position relative inertial reference:

$$\vec{v}_k = \vec{v}_k^{\,e} + \vec{v}_k^{\,r} = \vec{v}_C + \vec{\omega}^{e} \times \vec{\rho}_k + \vec{v}_k^{\,r} = \vec{v}_C + \vec{v}_k^{\,r} . \tag{3.37}$$

Putting the (3.36) and (3.37) in the (3.34) we have

$$\vec{L}_O = \sum_{k=1}^{n} \left(\vec{r}_C + \vec{\rho}_k\right) \times m_k \left(\vec{v}_C + \vec{v}_k^{\,r}\right). \tag{3.38}$$

Carry out the cross product and extract the $\vec{r}_C$ from summation

$$\vec{L}_O = \vec{r}_C \times M\vec{v}_c + \left(\sum_{k=1}^{n} m_k \vec{\rho}_k\right) \times \vec{v}_C + \sum_{k=1}^{n} \vec{\rho}_k \times m_k \vec{v}_k^{\,r},$$

but $\sum_{k=1}^{n} m_k \vec{\rho}_k = M\,\vec{\rho}_C = 0$ by definition of the position vector of the mass center with respect to the noninertial (Koenig's) reference and denoting $\vec{L}_C^r = \sum_{i=1}^{n} \vec{\rho}_k \times m_k \vec{v}_k^{\,r}$ we get

$$\vec{L}_O = \vec{r}_C \times M\vec{v}_c + \vec{L}_C^r. \tag{3.39}$$

The angular momentum of an aggregate of particles about a fixed point can be given as the angular momentum of the center of mass about the fixed point plus the angular momentum of the particles relative to the center of mass $\vec{L}_C^r$.

3.4.4. Moment-Angular momentum principle for a system of particles in inertial frame of reference

Let us consider the system of *n* particles (Fig. 3.27). The momentum equation for the k'th particle written about the origin of the inertial reference is

$$\frac{d\vec{q}_k}{dt} = \vec{F}_k^{(e)} + \vec{F}_k^{(i)}, \tag{3.40}$$

where $\vec{F}_k^{(e)}$ is resultant of external forces acting on the particle number k; $\vec{F}_k^{(i)}$ is internal forces resultant.

Using cross-product we have

$$\vec{r}_k \times \frac{d\vec{q}_k}{dt} = \vec{r}_k \times \vec{F}_k^{(e)} + \vec{r}_k \times \vec{F}_k^{(i)}$$

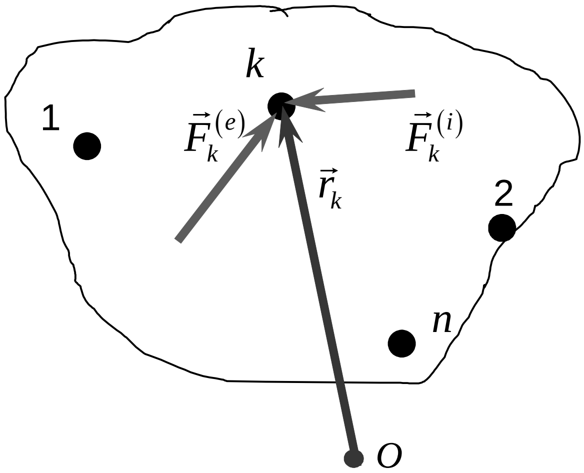


Fig. 3.27

or

$$\frac{d\vec{l}_{Ok}}{dt} = \vec{M}_O\left(\vec{F}_k^{(e)}\right) + \vec{M}_O\left(\vec{F}_k^{(i)}\right).$$

We now sum this equation for all n particles

$$\sum_{k=1}^{n} \frac{d\vec{l}_{Ok}}{dt} = \frac{d\vec{L}_O}{dt} = \sum_{i=1}^{n} \vec{M}_O\left(\vec{F}_k^{(e)}\right) + \sum_{i=1}^{n} \vec{M}_O\left(\vec{F}_k^{(i)}\right). \qquad (3.41)$$

But when we sum the moments of internal forces about point O, they cancel one other and their moments add up to zero (it is termed as internal forces second feature)

$$\sum_{k=1}^{n} \overrightarrow{M}_O(\vec{F}_k^{(i)}) = \sum_{k=1}^{n} \vec{r}_k \times \vec{F}_k^{(i)} = 0,$$

so

$$\boxed{\frac{d\vec{L}_O}{dt} = \vec{M}_O^{(e)}.} \qquad (3.42)$$

The total moment $\vec{M}_O^{(e)}$ of external forces acting on a system of particles about a point O fixed in an inertial reference equals the time rate of change of the total angular momentum relative to the point O.

There are three corollaries from this principle.

The first corollary. If the total moment of external forces acting on a system of particles about a point O has no component in any direction, the total angular momentum about the point O remains constant. This is known as the **principle of conservation of angular momentum**

$$\frac{d\vec{L}_O}{dt} = 0; \; \vec{L}_O = const. \tag{3.43}$$

The second corollary. If the total moment L_u of external forces acting on a system of particles about an axis u equals zero during finite time interval, the total angular momentum about the axis remains constant during this time interval

$$\frac{d}{dt} L_u = 0,$$
$$L_u = const.$$

The third corollary. The internal forces do not have an influence on the changing of the total angular momentum.

3.4.5. Angular momentum principle for a system of particles in noninertial frame of reference

Let us remember that in inertial frame of reference

$$\frac{d\vec{L}_O}{dt} = \vec{M}_O^{(e)} \; . \tag{3.44}$$

Represent $\vec{M}_O^{(e)}$ as (see Fig. 3.26)

$$\vec{M}_O^{(e)} = \sum_{k=1}^{n} \vec{r}_k \times \vec{F}_k^{(e)} = \left| \vec{r}_k = \vec{r}_C + \vec{\rho}_k \right| = \sum_{k=1}^{n} \left(\vec{r}_C + \vec{\rho}_k \right) \times \vec{F}_k^{(e)} =$$

$$= \sum_{k=1}^{n} \vec{r}_C \times \vec{F}_k^{(e)} + \sum_{k=1}^{n} \vec{\rho}_k \times \vec{F}_k^{(e)} = \vec{r}_C \times \sum_{k=1}^{n} \vec{F}_k^{(e)} + \vec{M}_C^{(e)} \, . \tag{3.45}$$

Rewrite $\dfrac{d\vec{L}_O}{dt}$ using equation (3.39) and (3.24)

$$\frac{d\vec{L}_O}{dt} = \frac{d}{dt}\left(\vec{r}_C \times M\vec{v}_C + \vec{L}_C^r\right) = \frac{d\vec{r}_C}{dt} \times M\vec{v}_C + \vec{r}_C \times \frac{d(M\vec{v}_C)}{dt} + \frac{d\vec{L}_C^r}{dt} =$$

$$= \left| \begin{array}{l} \dfrac{d\vec{r}_C}{dt} = \vec{v}_C ;\ \dfrac{d\vec{r}_C}{dt} \times M\vec{v}_C = 0, \\ \text{because } \dfrac{d\vec{r}_C}{dt} = \vec{v}_C \text{ and } \uparrow\uparrow M\vec{v}_C , \\ \dfrac{d(M\vec{v}_C)}{dt} = \sum\limits_{k=1}^{n} \vec{F}_k^{\,(e)} \text{ is eq.3.24} \\ \text{law of mass-center motion} \end{array} \right| = \vec{r}_C \times \sum_{k=1}^{n} \vec{F}_k^{\,(e)} + \frac{d\vec{L}_C^r}{dt}. \tag{3.46}$$

Putting (3.45) and (3.46) in the (3.44) we get

$$\vec{r}_C \times \sum_{k=1}^{n} \vec{F}_k^{\,(e)} + \frac{d\vec{L}_C^r}{dt} = \vec{r}_C \times \sum_{k=1}^{n} \vec{F}_k^{\,(e)} + \vec{M}_C^{\,(e)},$$

cancel the same terms at the left and right parts we obtain

$$\boxed{\frac{d\vec{L}_C^r}{dt} = \vec{M}_C^{\,(e)}.} \tag{3.47}$$

The total moment $\vec{M}_C^{\,(e)}$ of external forces acting on particle system about the mass-center as origin of translationally moving reference equals the time rate of change of the total angular momentum of the system in the relative motion taken about the mass-center $\vec{L}_C^r$.

We thus get the similar formulation for the center of mass as for a fixed point in inertial space. Please note that $\vec{L}_C^r$ is the total moment about the center of mass of the linear momenta as seen from the center of mass but that the time derivative is as seen from inertial reference.

LECTURE 5

3.4.6. Angular momentum of a rigid body rotating about fixed axis

Consider a single particle M of mass m attached to a rod rotating about fixed axis Oz with angular velocity ω_z (Fig. 3.28). Linear momentum of the particle is in the plane perpendicular to the axis

$$\vec{q} = m\vec{v}, \tag{3.48}$$

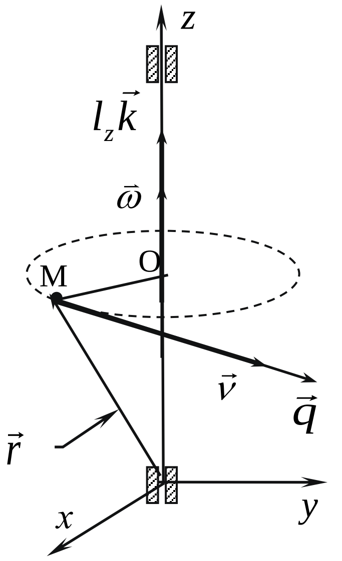


Fig. 3.28

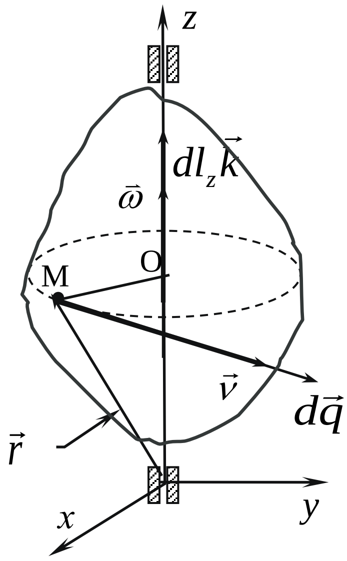


Fig. 3.29

where velocity magnitude is $v = \omega_z \rho,$ $\rho = OM$ is perpendicular distance between the particle and the axis.

Angular momentum of the particle about the axis of is

$$l_z = (\vec{r} \times m\vec{v})_z = mv\rho = m\rho^2\omega_z. \tag{3.49}$$

It is positive for counterclockwise direction of angular velocity.

Value $m\rho^2$ is called **particle moment of inertia about axis Oz .**

Consider a rigid body of mass m rotating about a fixed axis Oz (Fig. 3.29). Rigid body can be presented as system of particles of mass dm . Angular momentum of elementary mass dm about the axis of is

$$dl_z = (\vec{r} \times d\vec{q})_z = (\vec{r} \times dm\vec{v})_z = v\rho dm = \rho^2\omega_z dm\ . \tag{3.50}$$

Total angular momentum of the body about the axis is sum of elementary momenta about the axis

$$L_z = \int\limits_{(m)} \rho^2 \omega_z dm = \omega_z \int\limits_{(m)} \rho^2 dm, \tag{3.51}$$

value

$$I_z = \int\limits_{(m)} \rho^2 dm = \int\limits_{(m)} \left(x^2 + y^2\right) dm. \tag{3.52}$$

is **mass moment of inertia about the axis** Oz **(or second moment of inertia)** of the rigid body. It is independent of kinematical characteristics of the body and characterized the distribution of the body mass relative the axis Oz

If density γ is constant (body is uniform) through the body we get

$$dm = \gamma dV,$$

$$I_z = \int\limits_{(m)} \rho^2 dm = \gamma \int\limits_{(V)} \left(x^2 + y^2\right) dV. \tag{3.53}$$

For uniform body moment of inertia about the axis Oz characterizes purely geometry of the body.

Finally total angular momentum of the body about the axis is

$$L_z = I_z \omega_z. \tag{3.54}$$

3.4.7. Equation (or law) of a rigid body rotation about a fixed axis

Scalar form of Moment angular momentum principle (3.42) for a body rotating about fixed axis Oz is:

$$\frac{dL_z}{dt} = M_z^{(e)},$$

put Eq. (3.54) into the principle:

$$\frac{dL_z}{dt} = \frac{d\left(I_z \omega_z\right)}{dt} = I_z \frac{d\omega_z}{dt} = I_z \varepsilon_z,$$

$$\boxed{I_z \varepsilon_z = M_z^{(e)}.} \tag{3.55}$$

Eq. (3.55) is **equation or law of a rigid body rotation about a fixed axis**.

Substituting ε_z for $\ddot{\varphi}$ we get differential equation of a rigid body rotation about a fixed axis:

$$\boxed{I_z\ddot{\varphi} = M_z^{(e)}.} \tag{3.56}$$

The equation (3.56) is second order differential equation, it needs two initial conditions for solution. Result of this solution is law of rigid body rotation about fixed axis

$$\varphi = f(t).$$

3.4.7.1. Mass moment of inertia about an axis

From the equation (3.55) it is clear that mass moment of inertia about an axis is very important notion for cases of rigid body motion with angular acceleration. The mass moment of inertia about an axis is measure of rigid body inertial features in rotation. It characterizes the body resistance to change in angular acceleration due to radial distribution of mass around the axis of rotation. So all engineers have to be familiar with methods of mass moment of inertia about an axis calculation.

3.4.7.2. About mass moment of inertia about an axis

Consider two references: the first is $Cxyz$ with origin in the body mass center C and the second is $Ox_1y_1z_1$ displaced under a translation (no rotation) from reference $Cxyz$ (Fig. 3.30).

Find relation between axial mass-moment of inertia I_{z_1} determined in reference $Ox_1y_1z_1$ and axial mass-moment of inertia I_{Cz} determined in reference $Oxyz$. The centroidal moment of inertia I_{Cz} is presumed known $I_{Cz} = \int\limits_{(M)} \left(x^2 + y^2\right) dm$.

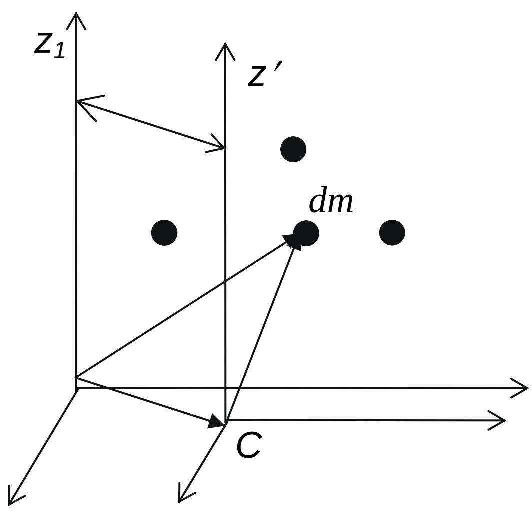


Fig. 3.30

From Fig. 3.30 we get

$$\vec{r} = \vec{r}_c + \vec{\rho},$$

or in scalar form

$$x_1 = a + x,$$
$$y_1 = b + y.$$

We can write eq. (3.53) in the following view

$$I_{z_1} = \int\limits_{(M)} \left(x_1^2 + y_1^2\right)dm = \int\limits_{(M)} \left(\left(x+a\right)^2 + \left(y+b\right)^2\right)dm =$$
$$= \int\limits_{(M)} \left(x^2 + y^2\right)dm + 2a\int\limits_{(M)} xdm + 2b\int\limits_{(M)} ydm + \left(a^2+b^2\right)\int\limits_{(M)} dm =$$

$$= \left| \begin{array}{l} Mx_C = \int\limits_{(M)} x\,dm = 0, \\ My_C = \int\limits_{(M)} y\,dm = 0, \\ \int\limits_{(M)} \left(x^2+y^2\right)dm = I_{Cz} \end{array} \right| = I_{Cz} + \left(a^2+b^2\right)M = I_{Cz} + d^2M,$$

$$\boxed{I_{z_1} = I_{Cz} + d^2M.} \qquad (3.57)$$

Guygens-Steiner Theorem or parallel axis theorem

The total moment of inertia of a body about any axis equals the moment of inertia of the body about the parallel axis that goes through the center of mass plus the total mass times the perpendicular distance between the axes squared.

Consequence: The mass-moment of inertia about the axis that goes through the body’s center of mass is smaller than moment of inertia about any parallel axis.

3.4.7.3. The simplest bodies second moments

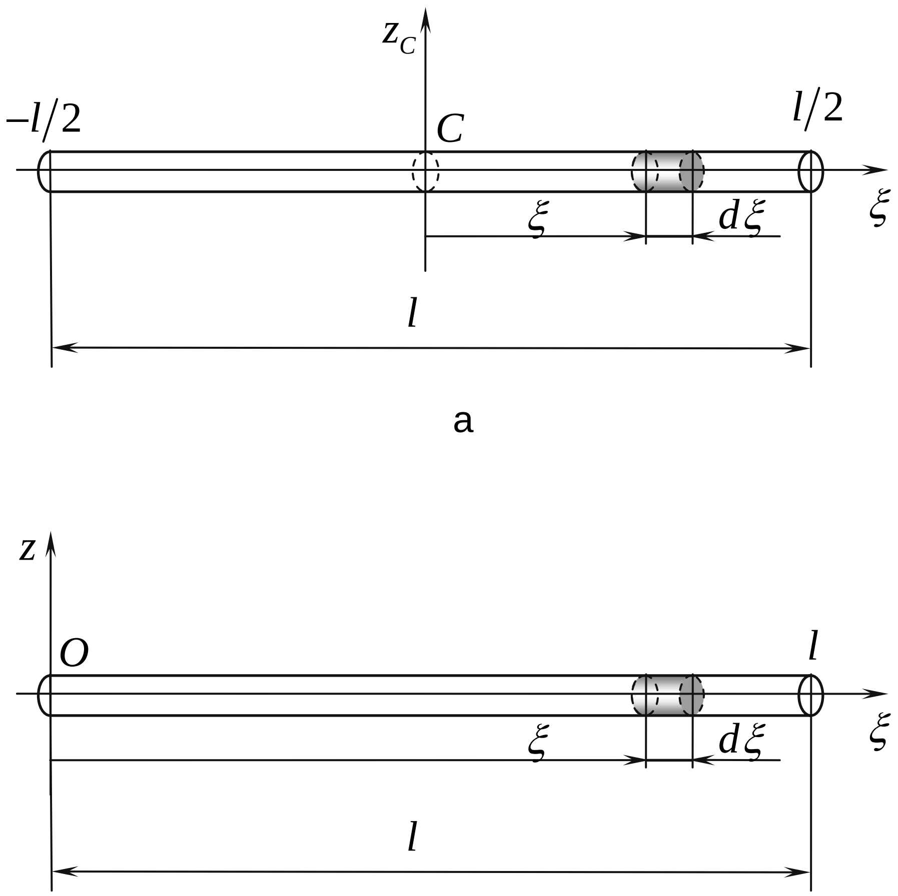
Thin uniform rod moment of inertia about the axis through the mass center

The mass element dm can be expressed in terms of a length element $d\xi$ along the rod (Fig. 3.31, a), the mass of the rod unit length is $\gamma = \dfrac{M}{l}$:

$$dm = \frac{M}{l}d\xi,$$

$$\rho^2=\xi^2,$$

$$I_{z_C}=\frac{M}{l}\int_{-l/2}^{l/2}\xi^2 d\xi=\frac{M}{l}\left(\frac{l^3}{3\cdot 8}-\frac{(-l)^3}{3\cdot 8}\right)=\frac{Ml^2}{12}.$$



a

b

Fig. 3.31

Thin uniform rod moment of inertia about the axis through end of the rod (Fig. 3.31, b)

$$I_z=\int_M \rho^2 dm=\left|\begin{array}{l} dm=\dfrac{M}{l}d\xi, \\ \rho^2=\xi^2 \end{array}\right|= .$$

$$=\frac{M}{l}\int_0^l \xi^2 d\xi=\frac{M}{l}\left(\frac{l^3}{3}-0\right)=\frac{Ml^2}{3}.$$

Thin uniform disk moment of Inertia about the axis through the mass center:

a) the axis z_C is perpendicular to the plane of the disk (Fig. 3.32), the mass element dm can be expressed in terms of an area element ds (ds is curvilinear trapezoid with height $d\rho$ and midline $\rho d\varphi$) and mass of the disk unit area $\gamma = \dfrac{M}{\pi R^2}$,

$$I_{z_C} = \int_M \rho^2 dm = \begin{vmatrix} dm = \dfrac{M}{\pi R^2} ds, \\ ds = \rho d\varphi d\rho \end{vmatrix} = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R \rho^3 d\varphi d\rho = \frac{MR^2}{2};$$

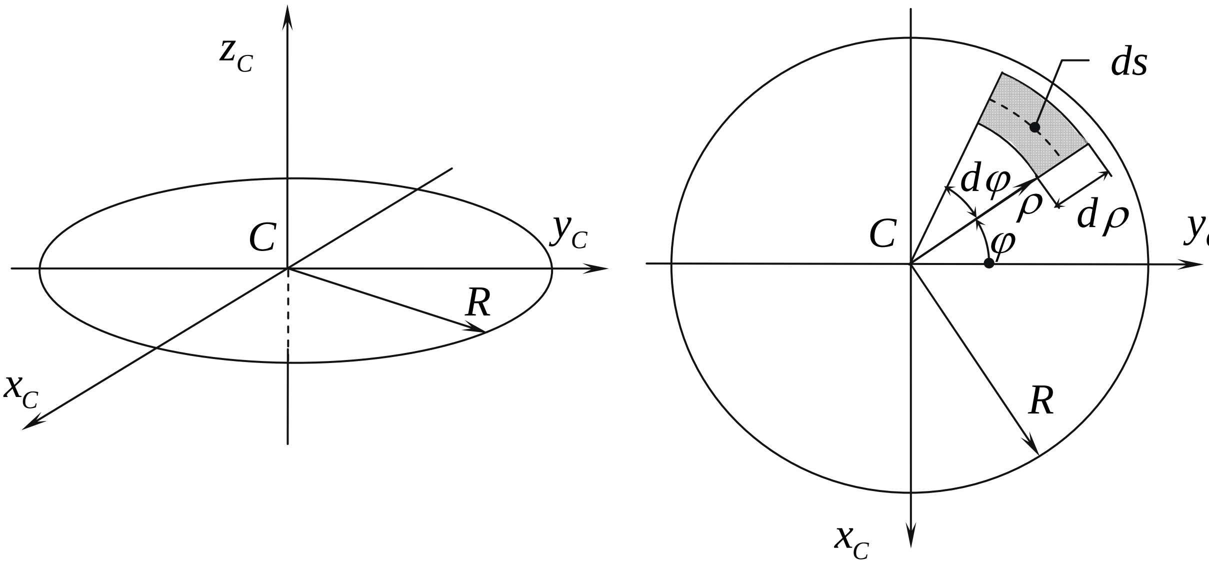


Fig. 3.32

b) the axis x_C is in the plane of the disk, distance between the aria mass center and axis x_C is $\rho \sin\varphi$ (first and second order infinitesimal are neglected):

$$I_{x_C} = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R \rho^2 \sin^2\varphi \cdot \rho d\phi d\rho = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R \rho^3 \left(\frac{1-\cos 2\varphi}{2} \right) \cdot d\phi d\rho = \frac{MR^2}{4}.$$

3.4.7.4. Radius of gyration

Using radius of gyration i_z second moment of inertia of a body of mass m can be determined as moment of inertia of a particle of the same mass:

$$I_z = i_z^2 m,$$

$$i_z^2 = \frac{I_z}{m}.$$

3.4.8. Examples

Example 1. Disc H of a mass m_1=40 kg and radius R=1 m rotates about vertical axis z with an angular velocity $\omega_{z0} = 2\, rad/\sec$ (Fig. 3.33). A particle K of a mass m_2=10 kg is at the point A.

It is needed to analyze the two periods of the system motion.

The fist. At some moment of time (t=0) a couple with a moment $M_z = 120t$ begins acting on the system. At $t = \tau = 1\sec$ this action is stopped. Determine angular velocity ω_τ of disc H at moment $t = \tau$.

The second. Disk H obtained angular velocity ω_τ and continues to rotate due to inertia. At some moment t_1=0 (it is new time reference) the particle K begins relative motion from the point A along the channel AB by the law $AK = s(t_1) = 0.5\ t_1$. Determine an angular velocity ω_T of the body H at $t_1 = T = 3\ sec$.

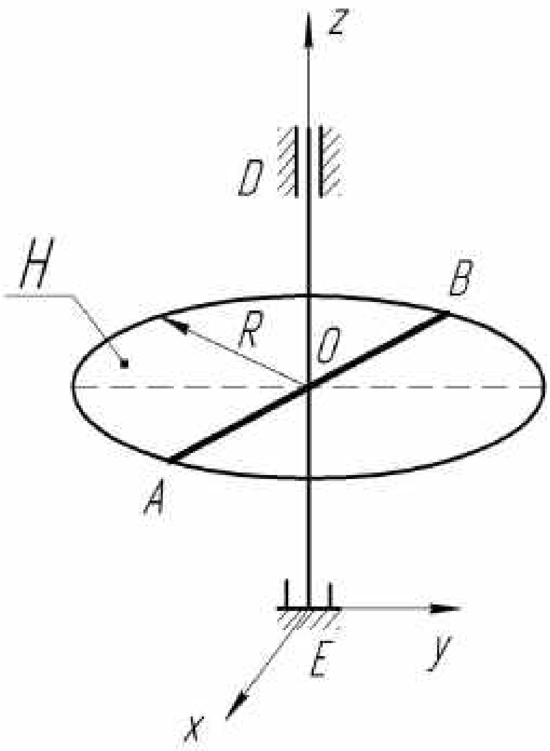


Fig. 3.33

Solution

To solve this problem the moment angular momentum principle in projection on z-axis is used:

$$\frac{dL_z}{dt} = \sum M_z^e. \tag{3.58}$$

The first period. During the time from $t = 0$ to $t = \tau$ there are forces acting on the system (Fig. 3.34): gravity forces $m_1\overline{g}, m_2\overline{g}$, a couple with a moment M_z, and reactions of bearing $\overline{X_D}, \overline{Y_D}$ and trust bearing $\overline{X_E}, \overline{Y_E}, \overline{Z_E}$. As ω_{z0} and M_z are positive values, they are directed counterclockwise.

System consists of two bodies: disk H and particle K. The total angular momentum of the system is

$$L_z = l_{z1} + l_{z2}.$$

Angular momentum of the disk as rotating body is:

$$l_{z1} = I_z \cdot \omega,$$

where $I_z = \dfrac{m_1 R^2}{2}$ is moment of inertia.

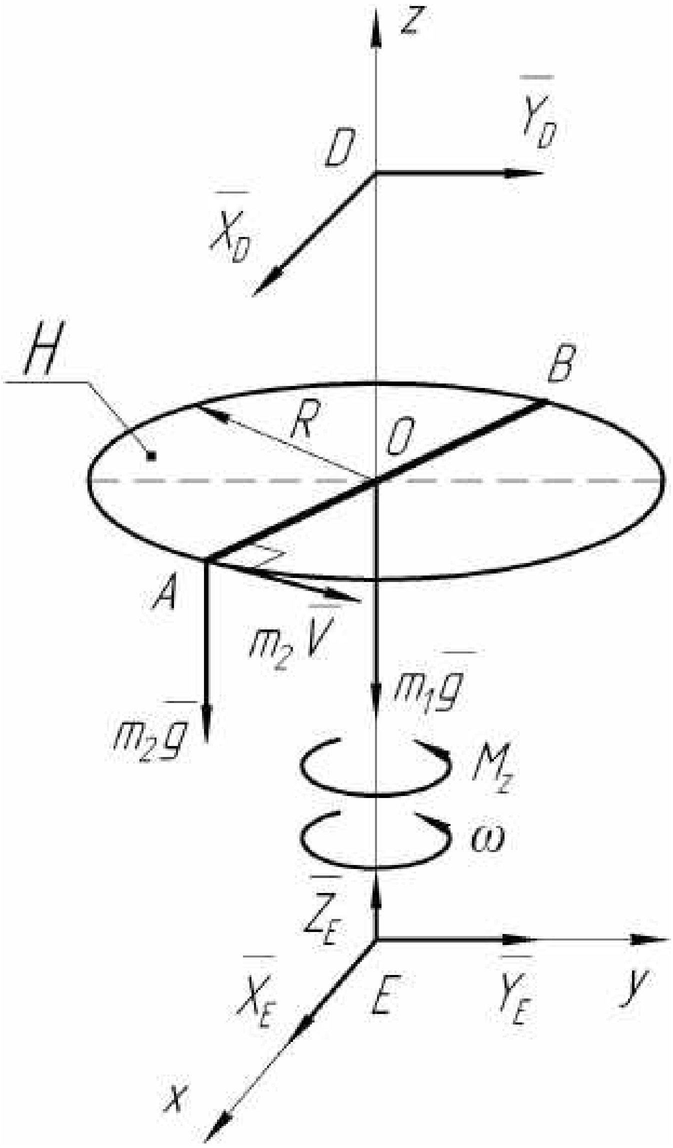


Fig. 3.34

So

$$l_{z1} = \frac{m_1 R^2}{2} \cdot \omega.$$

Angular momentum of the particle is:

$$l_{z2} = m_2 v_2 R,$$

where $v_2 = \omega R$ is particle velocity.

So

$$l_{z2} = m_2 R^2 \omega.$$

So the total angular momentum is:

$$L_z = \frac{m_1 R^2}{2} \cdot \omega + m_2 R^2 \omega = R^2 \omega \left(\frac{1}{2} m_1 + m_2 \right). \tag{3.59}$$

All forces except moment M_z don't create moments about z-axis, so

$$\sum M_z^e = M_z = 120t. \tag{3.60}$$

Substituting the expressions(3.59) and (3.60) in the equation (3.58):

$$\frac{d}{dt}\left(R^2 \omega \left(\frac{1}{2} m_1 + m_2 \right) \right) = 120\, t.$$

Separating the variables:

$$R^2 \left(\frac{1}{2} m_1 + m_2 \right) d\omega = 120\, t dt,$$

$$R^2 \left(\frac{1}{2} m_1 + m_2 \right) \int_{\omega_0}^{\omega_\tau} d\omega = 120 \int_0^\tau t dt,$$

$$R^2 \left(\frac{1}{2} m_1 + m_2 \right) \omega \Big|_{\omega_0}^{\omega_\tau} = 120 \frac{t^2}{2} \Big|_0^\tau,$$

$$R^2 \left(\frac{1}{2} m_1 + m_2 \right) \left(\omega_\tau - \omega_0 \right) = 120 \frac{\tau^2}{2}.$$

From the last equation angular velocity at time $t = \tau = 1 \sec$:

$$\omega_\tau = \frac{120 \frac{\tau^2}{2}}{R^2 \left(\frac{1}{2} m_1 + m_2 \right)} + \omega_0 = \frac{120 \frac{1^2}{2}}{1^2 \left(\frac{1}{2} \cdot 40 + 10 \right)} + 2 = 4 \left(\frac{rad}{\sec} \right).$$

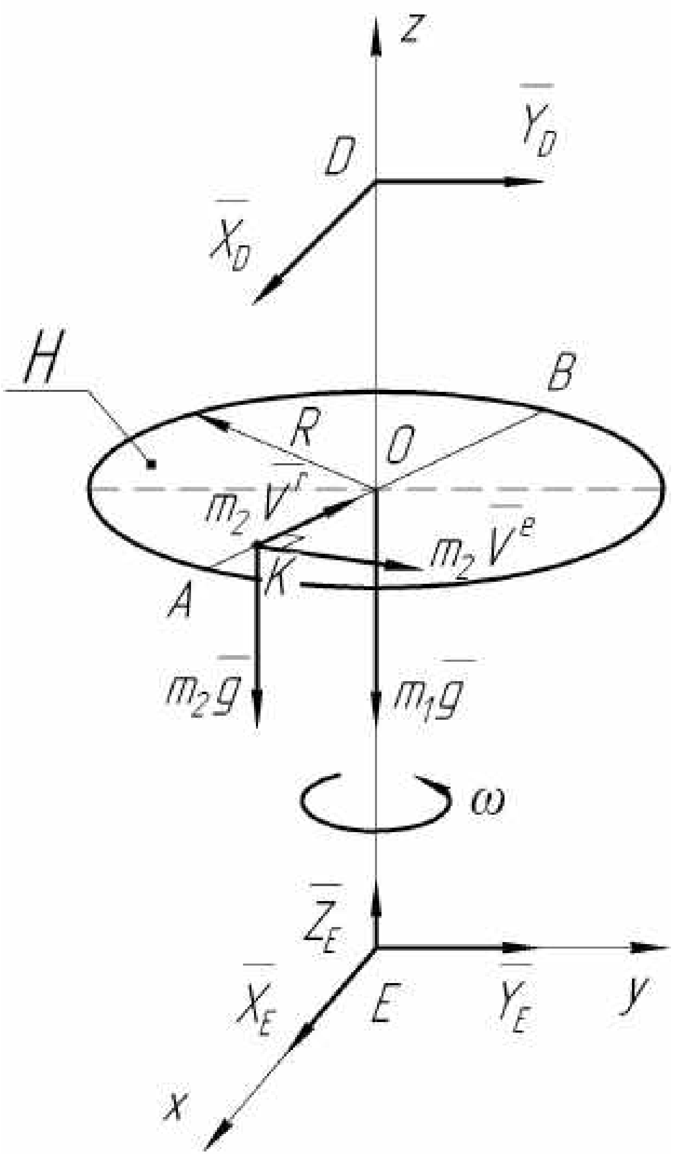


Fig. 3.35

The second period. When the moment M_z stops its action, disk H rotates by inertia with an angular velocity ω_τ. Gravity forces $m_1\overline{g}, m_2\overline{g}$ and reactions of bearing $\overline{X_D}, \overline{Y_D}$ and trust bearing $\overline{X_E}, \overline{Y_E}, \overline{Z_E}$ are applied to the system (Fig. 3.35). These forces act t_1=0 to t_1=T and the particle K begins its motion along the channel.

In this part of the problem the angular momentum principle in projection on z-axis is also used:

$$\frac{dL_z}{dt} = \sum M_z^e .$$

But in this case all forces don't have moments about z-axis, so

$$\sum M_z^e = 0.$$

Then

$$\frac{dL_z}{dt} = 0 \Rightarrow L_z = const\,.$$

It is the second corollary from moment angular momentum principle. So the angular momenta at the moment $t_1 = 0$ and $t_1 = T$ are equal, i. e.

$$L_{z0} = L_{zT}\,. \tag{3.61}$$

The total angular momentum at initial moment of time $t_1 = 0$ is:

$$L_{z0} = l_{z0_1} + l_{z0_2}\,,$$

where

$$l_{z0_1} = I_z \cdot \omega_\tau\,,\;\; I_z = \frac{m_1 R^2}{2}\,,$$

so

$$l_{z0_1} = \frac{m_1 R^2}{2} \cdot \omega_\tau\,,$$

and

$$l_{z0_2} = m_2 V_2 R = m_2 R^2 \omega_\tau\,.$$

Then

$$L_{z0} = \frac{m_1 R^2}{2} \cdot \omega_\tau + m_2 R^2 \omega_\tau = \omega_\tau R^2 \left(\frac{1}{2} m_1 + m_2 \right).$$

The total angular momentum of the system at final moment of time is:

$$L_{zT} = l_{zT_1} + l_{zT_2}\,. \tag{3.62}$$

Angular momentum of disk:

$$l_{zT_1} = I_z \cdot \omega_T = \frac{m_1 R^2}{2} \cdot \omega_T\,. \tag{3.63}$$

The particle has compound motion, so angular momentum consists of two parts:

$$l_{zT_2} = l^e_{zT_2} + l^r_{zT_2}\,,$$

where $l^e_{zT_2}$ is moment of linear momentum of the particle in bulk motion about z-axis;

$l_{zT_2}^r$ is moment of linear momentum of the particle in relative motion;

$l_{zT_2}^r = \left(\vec{r} \times \vec{q}^{\,r}\right)_z = \left(\vec{r} \times m\vec{v}^{\,r}\right)_z = 0$, because the linear momentum crosses the z-axis, so it doesn't have moment about the axis.

Now we have

$$l_{zT_2} = l_{zT_2}^e = \left(\vec{r} \times \vec{q}^{\,e}\right)_z = \left(\vec{r} \times m\vec{v}^{\,e}\right)_z = m_2 v^e KO,$$

where $v^e = \omega_T KO$ is bulk velocity of the particle,

$$KO = AO - AK = R - AK = R - 0.5t_1 = R - 0.5T\,.$$

Then

$$l_{zT_2} = l_{zT_2}^e = m_2\omega_T KO^2 = m_2\omega_T\left(R - 0.5T\right)^2. \tag{3.64}$$

Substituting expressions (3.62) and (3.63) in equation (3.64):

$$L_{zT} = \frac{m_1R^2}{2}\cdot\omega_T + m_2\omega_T\left(R-0.5T\right)^2 = \omega_T\left(\frac{m_1R^2}{2} + m_2\left(R-0.5T\right)^2\right). \tag{3.65}$$

Using equation (3.61) we have:

$$\omega_\tau R^2\left(\frac{1}{2}m_1 + m_2\right) = \omega_T\left(\frac{m_1R^2}{2} + m_2\left(R-0.5T\right)^2\right).$$

From here:

$$\omega_T = \frac{\omega_\tau R^2\left(\frac{1}{2}m_1 + m_2\right)}{\frac{m_1R^2}{2} + m_2\left(R-0.5T\right)^2} = \frac{4\cdot 1^2\left(\frac{1}{2}\cdot 40 + 10\right)}{\frac{40\cdot 1^2}{2} + 10\left(1 - 0.5\cdot 3\right)^2} = 5.3\left(\frac{rad}{\sec}\right).$$

So if the particle approaches to the axis of the body rotation, the angular velocity will increase.

Answer:

$$\omega_\tau = 4\left(\frac{rad}{\sec}\right),\ \omega_T = 5.3\left(\frac{rad}{\sec}\right).$$

Example 2. A circular plate of mass ***M*** and radius ***R*** rotates about an axis z, parallel to the central axis of the plate (Fig. 3.36). There is a particle of mass ***m*** at point A on the disk. If the system initially at rest, determine the angular velocity of the disk, when particle is at point B and has the relative velocity ***u***.

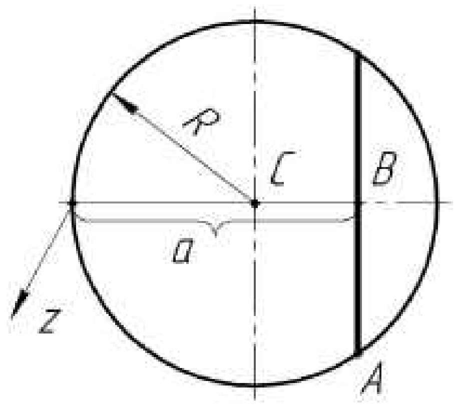


Fig. 3.36

Solution

Forces acting on the system are gravity forces and reactions of supports. The gravity forces are parallel to z-axis so they don't have moment about this axis. Reactions of supports are applied on the axis so hey don't have moment about this axis too.
That's why principle of angular momentum conservation is used:

$$L_z = const\,.$$

So the angular momenta at the initial moment t=0 and final t are equal, i.e.

$$L_{z0} = L_z\,. \tag{3.66}$$

The angular momentum at initial moment of time is zero, because the system was at rest, i. e.:

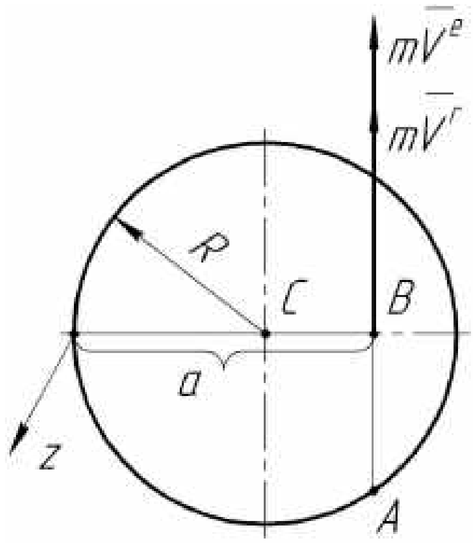


Fig. 3.37

$$L_{z0} = 0.$$

The angular momentum at final moment of time when the particle is at point B (Fig. 3.37) is:

$$L_z = l_{zp} + l_{zd}\,,$$

where l_{zp} is angular momentum of the particle at final moment;

l_{zd} is angular momentum of the disk at final moment in accordance to eq.(3.54)

$$l_{zd} = I_z \omega.$$

To find moment of inertia of the disk parallel-axes theorem is used, because axis of rotation doesn't coincide with central axis of the disk:

$$I_z = I_{z_C} + MR^2, \quad I_{z_C} = \frac{MR^2}{2}.$$

Then

$$l_{zd}=\left(I_{z_C}+MR^2\right)\cdot\omega=\left(\frac{MR^2}{2}+MR^2\right)\cdot\omega=\frac{3}{2}MR^2\omega.$$

The particle has a compound motion, that's why

$$l_{zp}=l_{zp}^e+l_{zp}^r.$$

A moment of linear momentum of particle in bulk motion is

$$l_{zp}^e=mv^e a=m\omega a^2.$$

A moment of linear momentum of particle in relative motion along the line AB is:

$$l_{zp}^r=mv^r a=mua.$$

Then

$$l_{zp}=m\omega a^2+mua.$$

So using equation (3.66) we have:

$$0=\frac{3}{2}MR^2\omega+m\omega a^2+mua,$$

$$\omega=\frac{-mua}{\frac{3}{2}MR^2+ma^2}.$$

Answer: $\omega=\dfrac{-mua}{\frac{3}{2}MR^2+ma^2}$.

Example 3. A heavy ball is attached to the end of weightless rod of length l. The rod rotates about a vertical axis z in an oil bath (Fig. 3.38). Oil force of resistance is proportional to the angular velocity of the rod $F_r=\alpha m\omega$. α is a constant coefficient, *m* is the ball mass. The initial angular velocity of the rod is ω_0.

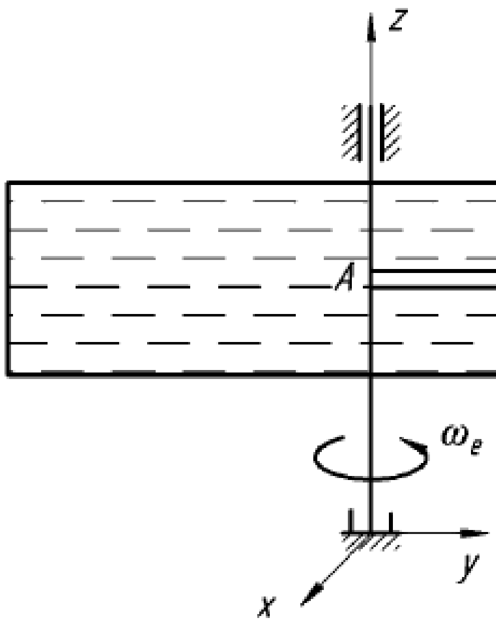


Fig. 3.38

Determine the time ***t*** required for the rod to halve its angular velocity and the number of revolution of the rod correspondingly.

Solution

In this problem angular momentum principle in projection on z-axis is used.

$$\frac{dL_z}{dt} = \sum M_z^e. \qquad (3.67)$$

External forces are (Fig. 3.39): gravity force of the ball $m\bar{g}$, force of resistance $F_r = \alpha m\omega$ and reactions of bearing $\overline{R_K}$ and trust bearing $\overline{R_L}$.

Angular momentum L_z is angular momentum of the ball, the rod is weightless and its angular momentum is zero:

$$L_z = l_{zb} = mV_B l = m\omega l^2.$$

Only force of oil resistance has

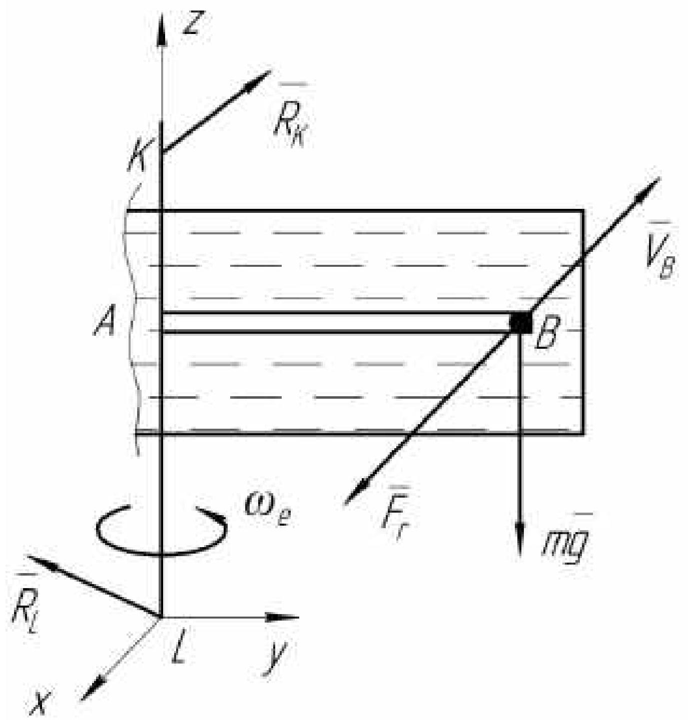


Fig. 3.39

moment about z-axis, so

$$\sum M_z^{(e)} = -F_r \cdot l = -\alpha m\omega l.$$

Substituting these expressions to the equation (3.67), we obtain differential equation:

$$\frac{d\left(m\omega^2 l\right)}{dt} = -\alpha m\omega l.$$

After some transformations and separating the variables:

$$\frac{d\omega}{\omega} = -\frac{\alpha}{l}dt.$$

Integrate right and left parts using as limits of integration conditions given in the problem:

$$\int_{\omega_0}^{\omega_0/2} \frac{d\omega}{\omega} = -\frac{\alpha}{l}\int_0^{\tau} dt, \quad \ln\omega\Big|_{\omega_0}^{\omega_0/2} = -\frac{\alpha}{l}t\Big|_0^{\tau}.$$

Finally we get

$$\tau = \frac{l}{\alpha}\ln 2.$$

Answer: $\tau = \dfrac{l}{\alpha}\ln 2.$

3.4.9. Short problems

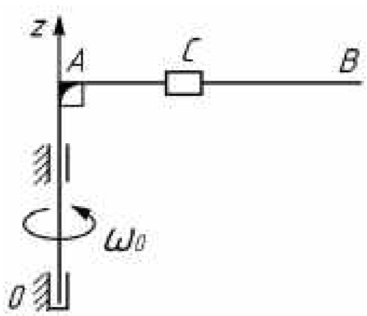


Fig. 3.40

Problem 1. Along the rod *AB* (Fig. 3.40) a sliding bar C moves according to the law *AC* = 0.2 + 1.2 *t.* The sliding bar is considered as a material point with the mass *m* = 1 kg. The moment of inertia of the shaft *OA* with the rod is I_z = 2,5kg•m^2. Determine an angular velocity of the shaft at the moment *t* = 1 s if the initial angular velocity is ω_0 = 10 rad/s.

Problem 2. Under the action of internal forces the 20 kg flywheel 2 (Fig. 3.41) untwists to the relative angular velocity $\omega_r = 40\ rad/s$. The flywheel central moment of inertia is $I_{z1} = 1\ kg \cdot m^2$. Determine angular velocity ω of the holder 1 if moment of inertia is $I_z = 4 kg \cdot m^2$, the length is $l = 1m$.

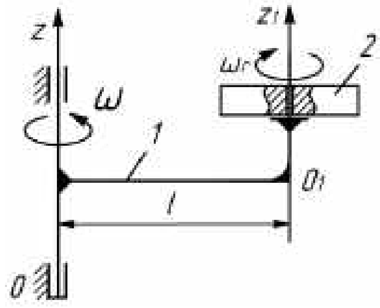
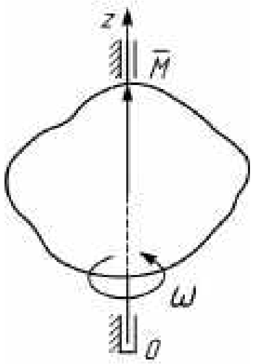
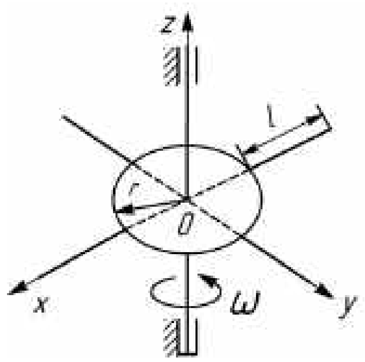


Fig. 3.41

Problem 3. A body rotates about vertical axis *Oz* under action of a couple with moment *M = 16 t , Nm* (Fig. 3.42). Determine a moment of inertia about axis *Oz* if it is known that at the moment *t* = 3 s an angular velocity is ω = 2 rad/s. At *t*= 0 the body is at rest.





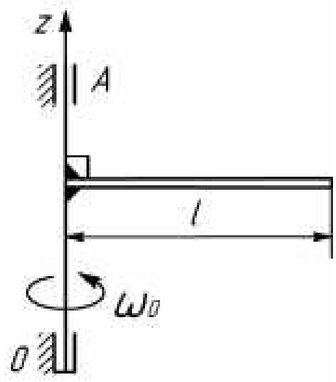


Fig. 3.42 Fig. 3.43 Fig. 3.44

Problem 4. A body rotates about vertical axis *Oz* under action of two couples with moments $\overline{M}_1 = 3\bar{i} + 4\bar{j} + 5\bar{k}$ and $\overline{M}_2 = 4\bar{i} + 6\bar{j} + 4\bar{k}$. The moment of inertia about axis *Oz* is equal 3 kg•m^2. Determine angular velocity of the body at the moment of time *t* = 2 s if at initial moment the body did not rotate.

Problem 5. The homogeneous disk (Fig. 3.43) with the radius *r* = 0.1 m and the mass 5 kg is connected with four rods. Every one of them has length *l*=0.5 m and mass 1 kg. The system begins rotating under action of external forces with angular velocity ω = 3t*.* Determine a moment of external forces about axis Oz.

Problem 6. A homogeneous rod (Fig. 3.44) with the mass *m*=3 kg and the length *l* = 1 m rotates about a vertical axis *Oz* with angular velocity ω_0 = 24 rad/s. A constant moment of braking forces is applied to the shaft *OA*. Determine a magnitude of this moment if the rod stops in 4 s after braking beginning.

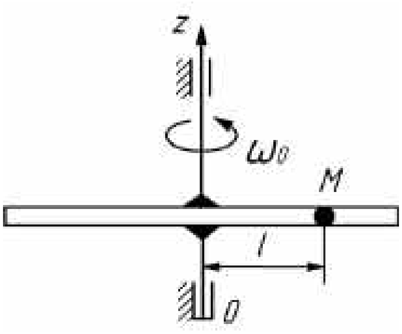


Fig. 3.45

Problem 7. A tube (Fig. 3.45) rotates about vertical axis *Oz,* its moment of inertia is I_z = 0,075 kg • m^2. Along the tube under action of internal forces of the system the ball *M* with a mass *m* = 0,1 kg moves. When the ball is on axis *Oz,* angle velocity isω_0 = 4 rad/s. At what distance ***l*** is angle velocity equal to 3 rad/s?

LECTURE 6

3.5. Plane Motion Dynamics

3.5.1. Differential Equations of a Rigid Body Plane Motion

Let us analyze what are conditions for realization of rigid body plane motion.

Remember that plane motion is motion in which each point of the moving body remains at a constant distance from a fixed plane. Each point of the body moves in a plane that is called the *plane of the motion.*

Rigid body is in plane motion if

- the rigid body mass is be distributed symmetric with respect to the plane that is parallel to the plane of motion (such bodies we shall call slablike bodies); this plane will be denoted as xOy;
- all forces acting on the body are in the xOy plane.

As we know from kinematics plane motion can be represented as combination of the two simplest motions: translation with the body mass-center and rotation about the axis Cz passes through the mass center perpendicular to the plane of motion. The translation is characterized by the principle of body mass center motion in vector form (3.24):

$$M\frac{d\vec{v}_c}{dt} = \vec{F}^{(e)},$$

or in scalar form

$$\begin{cases} M\ddot{x}_C = F_x^{(e)}, \\ M\ddot{y}_C = F_y^{(e)}. \end{cases} \qquad (3.68)$$

The rotation is characterized by angular momentum principle for a system of particles with respect to the noninertial reference because rotation is about moving axis *Cz* through the body mass center

$$\frac{d\vec{L}_C^r}{dt} = \sum_{k=1}^{n} \vec{M}_C \left(\vec{F}_k^e \right) , \qquad (3.69)$$

projecting onto the axis of rotation *Cz* we get

$$L_{Cz}^r = I_{Cz}\omega_z , \qquad (3.70)$$

where $I_{Cz} = const$, and

$$\frac{d\left(I_{Cz}\omega_z\right)}{dt} = I_{Cz}\varepsilon_z = \sum_{k=1}^{n} M_{Cz}\left(F_k^e\right) . \qquad (3.71)$$

So equations (3.68) and equation (3.71) together form the system of differential equations of a rigid body plane motion:

$$\begin{cases} M\ddot{x}_C = F_x^{(e)}, \\ M\ddot{y}_C = F_y^{(e)}, \\ I_{Cz}\varepsilon_z = \sum\limits_{k=1}^{n} M_{Cz}\left(F_k^e\right). \end{cases} \qquad (3.72)$$

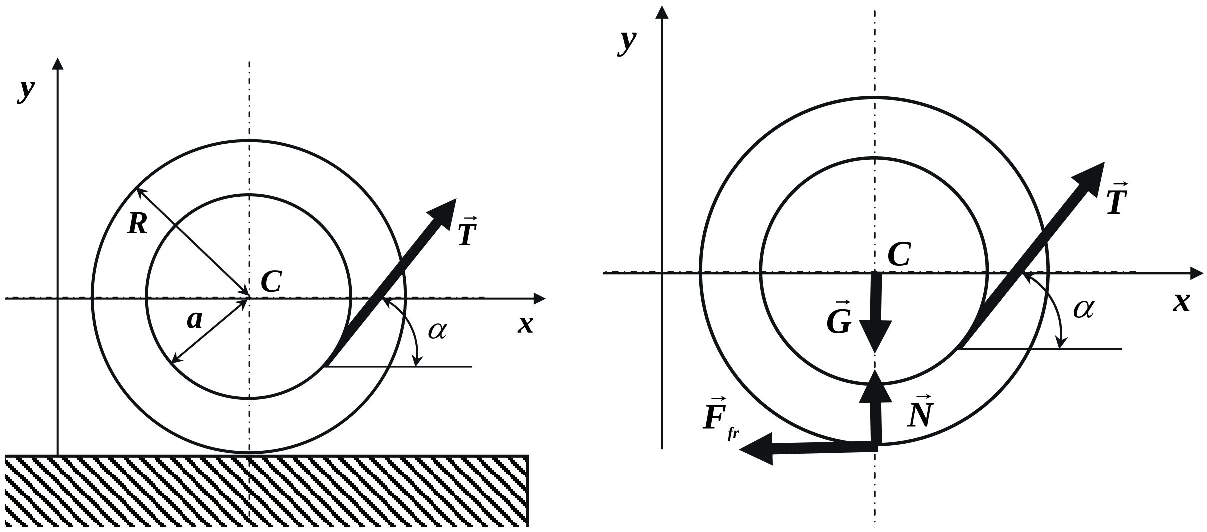
Let us analyze the rolling of slablike bodies such as cylinders, spheres, or plane gears along straight line on the fixed plane.

The first case is rolling of a body without slipping.

A roller has the weight G and radius $\boldsymbol{R}$. It is pulled along a rough horizontal floor by a force *T* applied to the end of a string wound round the drum, as shown in Fig. 3.46 a. The force *T* is applied at an angle α to the horizontal. The radius of the drum is a and the radius of gyration of the roller is ρ. Find the equation of motion of the axis *C* of the roller.

In the Fig.3.46 b the free body diagram of the cylinder is represented. Force of gravity $\vec{\boldsymbol{G}}$ and tension $\vec{\boldsymbol{T}}$ are applied forces, normal force $\vec{\boldsymbol{N}}$ and force of friction $\vec{\boldsymbol{F}}_{fr}$ are reactions.

We apply the equations of plane motion (3.72):



a b

Fig. 3.46

$$M\ddot{x}_c = \sum_{k=1}^{n} F_{ix}^{e} = T\cos\alpha - F_{fr} \ , \tag{3.73}$$

$$M\ddot{y}_c = \sum_{k=1}^{n} F_{iy}^{e} = T\sin\alpha - G + N \ , \tag{3.74}$$

$$I_{Cz}\varepsilon_z = \sum_{k=1}^{n} M_{Cz}\left(F_k^e\right) = Ta - F_{fr}R \ . \tag{3.75}$$

We note that for body that rolls without slipping the friction force between the wheel and supporting surface is generally less then it maximum value determined by Coulomb's low $F_{fr\max} = f_{st}N$. Therefore the value of friction force must be determined form the solution of the system of equations.

There are five unknowns in this system: $\ddot{x}_C$, $\ddot{y}_c$, $\varepsilon_z = \ddot{\phi}_z$, F_{fr} , N . Therefore to be solvable the system of equation must be added by two additional equations (equations of constraints).

If a rigid body (cylinder) rolls without slipping on a fixed surface, the point that is in contact with the surface has zero velocity. So the relation between the velocity of the body mass center and angular velocity may be written as the following

$$\left|v_C\right| = \left|v_{Cx}\right| = \left|\omega_z\right| CP = \left|\omega_z\right| R \ , \tag{3.76}$$

or in coordinate form

$$\dot{x}_C = -\dot{\phi}CP = -\dot{\phi}R \,. \tag{3.77}$$

The equation (3.77) means that if the cylinder angular velocity is negative (in clockwise direction) then the projection onto the axis Ox of the mass center velocity is positive. And by differentiation we get

$$\ddot{x}_C = -\ddot{\phi}\,CP = -\ddot{\phi}R\,. \tag{3.78}$$

The equation (3.78) gives us the relation between unknown linear acceleration of the mass center and unknown angular acceleration.

The motion of the cylinder mass center in direction normal to the fixed surface is constrained and it is possible to describe this restriction as the following

$$y_C = const,\ \ddot{y}_C = 0\,. \tag{3.79}$$

As result we have the system of equations

$$\begin{cases} M\ddot{x}_c = T\cos\alpha - F_{fr}\,, \\ 0 = T\sin\alpha - G + N\,, \\ I_{Cz}\ddot{\phi}_z = Ta - F_{fr}R, \\ \ddot{x}_c = -\ddot{\phi}_z R. \end{cases} \tag{3.80}$$

From which taking into consideration that $I_{Cz} = \rho^2 M$ we get

$$\ddot{x}_C = \frac{T}{M}\frac{R\left(R\cos\alpha - a\right)}{\left(\rho^2 + R^2\right)}, \tag{3.81}$$

$$F_{fr} = T\,\frac{\rho^2\cos\alpha + Ra}{\left(\rho^2 + R^2\right)}. \tag{3.82}$$

It is evident from the equation (3.81) that if $R\cos\alpha > a$ the cylinder mass center acceleration is directed to the right and the cylinder will move to the right, if $R\cos\alpha < a$ the cylinder will move t the left.

The second case is rolling of a body with slipping.

A homogeneous cylinder, with horizontal axis (Fig. 3.47), rolls sliding down an inclined plane by virtue of its weight. The coefficient of sliding friction is f. Determine the angle of inclination between the plane and the horizontal and the acceleration of the axis of the cylinder.

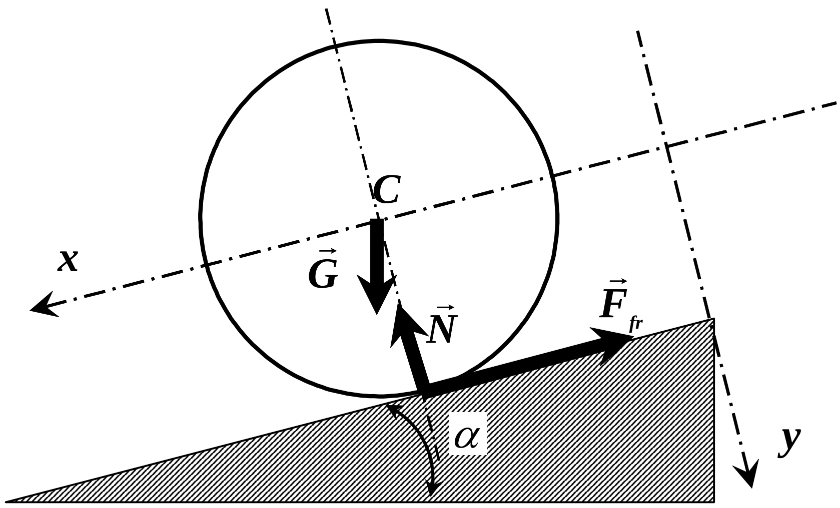


Fig. 3.47

In this case we may use for solution the system of equations (3.72) with equation of constrain (3.79):

$$\begin{cases} M\ddot{x}_c = G\sin\alpha - F_{fr}, \\ 0 = -G\cos\alpha + N, \\ I_{Cz}\ddot{\varphi}_z = F_{fr}R. \end{cases} \tag{3.83}$$

At the same time the relation between unknown linear acceleration of the mass center and unknown angular acceleration (3.78) is not valid in this case, because the point of contact of cylinder with the fixed surface has nonzero velocity. In order to close the system of equations instead of equation (3.78) we have to use the Coulomb's law for the determination of friction force that acts on the cylinder during the rolling with slipping (we neglect the difference between the static f_{st} and kinetic f_k coefficients of friction, so we suppose that $f_{st} = f_k = f$)

$$F_{fr} = fN. \tag{3.84}$$

Solving the system (3.83) with the equation (3.84) we get

$$N = G\cos\alpha, \tag{3.85}$$

$$\ddot{x}_c = g\left(\sin\alpha - f\cos\alpha\right). \tag{3.86}$$

To determine the angle of plate inclination we have to consider the case when motion is without slipping yet slipping impends (motion is on the verge of slipping). It means that the relation between unknown mass center linear acceleration and unknown angular acceleration (3.78) is valid and force of friction amount to the limiting value $\boldsymbol{F}_{fr\,max} = \boldsymbol{fN}$. We get the system

$$\begin{cases} M\ddot{x}_c = G\sin\alpha_{verge} - F_{fr}, \\ 0 = -G\cos\alpha_{verge} + N, \\ I_{Cz}\ddot{\varphi}_z = F_{fr}R, \\ \ddot{x}_c = \ddot{\varphi}_z R. \end{cases} \tag{3.87}$$

From which we determine the required angle α:

$$\alpha_{verge} = a\,tan\left(3f\right). \tag{3.88}$$

So if $\alpha_{verge} \le a\,tan\left(3f\right)$ rolling is without slipping, if $\alpha_{verge} > a\,tan\left(3f\right)$ rolling is with slipping.

In general it is possible that we don't know beforehand is the motion without slipping or not, but at the same time the values of the static f_{st} and kinetic f_k coefficients of friction are given. The algorithm of such problem analyze is the following:

Step 1. Draw the free body diagram.

Step 2. Form the system of plan motion differential equations (3.72) and additional equations of constraints (3.78) and (3.79) with the initial assumption that the body rolls without slipping determine force of friction and compare with its limiting value $F_{fr\max} = f_{st}N$. If condition $F_{fr} \le F_{fr\,\max} = f_{st}N$ holds it means our assumption is true, in opposite case we conclude that our assumption of rolling without slipping was wrong. Therefore the body slips as it rolls and the constraint equations do not hold.

Step 3. Resolve problem for $F_{fr} = f_{kin}N$ and $\ddot{x}_C \ne \pm\ddot{\varphi}CP$.

3.5.2. Examples

Example 1. A homogeneous rod of a mass *m=3 kg* is allowed to fall from rest, sliding on rough horizontal plane (Fig. 3.48). At angle $\varphi = 60°$ determine the projection of the acceleration of the mass center C on x-axis, if a normal reaction is *N=18.17 N* and coefficient of friction is *f=0.1*.

Solution

FBD is presented in Fig. 3.49. The rod has plane motion, so we can write 3 differential equations of motion, but to solve this problem we need only one equation because we are asked about x component of acceleration only:

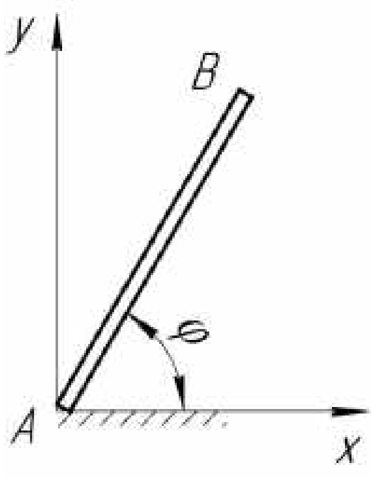


Fig. 3.48

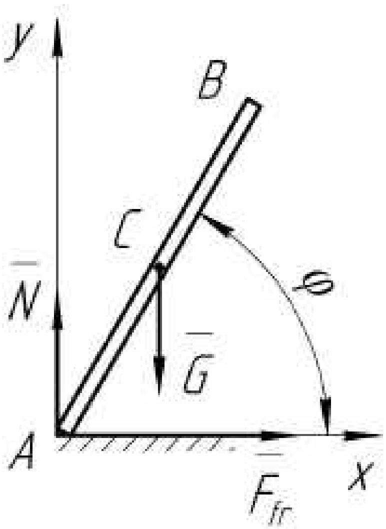


Fig. 3.49

$$m \cdot \ddot{x}_C = \sum_{k=1}^{n} F_{x\,k}^{e} = F_{fr}. \tag{3.89}$$

As the rod slides we can find friction force according to Column law:

$$F_{fr} = f \cdot N. \tag{3.90}$$

Substituting the equation (3.90) into (3.89):

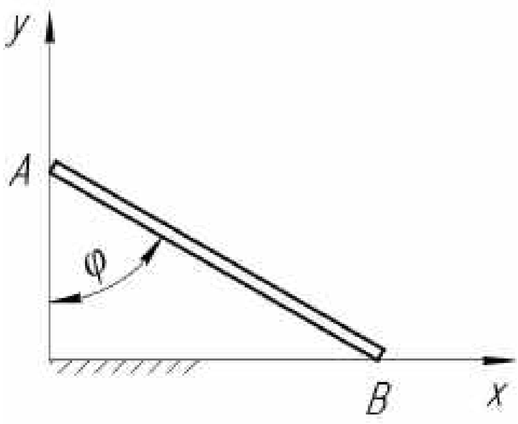
$$m \cdot \ddot{x}_C = f \cdot N.$$

From here:

$$\ddot{x}_C = \frac{f \cdot N}{M} = \frac{0.1 \cdot 18.17}{3} = 0.606\left(\frac{m}{\sec^2}\right).$$

Answer: $\ddot{x}_C = 0.606\left(\dfrac{m}{\sec^2}\right)$.

Example 2. A rod of mass m=3kg is placed in a vertical plane at angle $\varphi = 60°$ in such a way that one end A leans against a smooth vertical wall while the other end B rests to a smooth horizontal floor (Fig. 3.50). The rod starts to fall with the acceleration of the mass center $\overline{W_C} = \bar{i} - 5.5\bar{j}$. Determine normal reaction at point A.



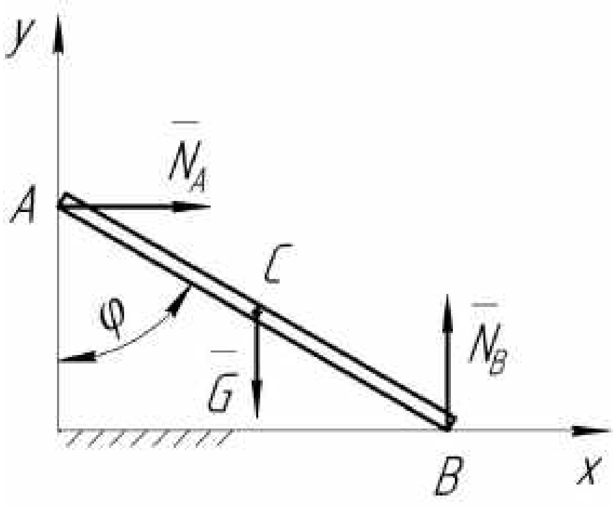


Fig. 3.50 Fig. 3.51

Solution

FBD is presented in Fig. 3.51. The rod has plane motion, so we can write 3 differential equations of motion, but to solve this problem we need only one eq. (3.73):

$$m \cdot \ddot{x}_C = \sum_{k=1}^{n} F_{x\,k}^{e} = N_A .$$

According to the statement of the problem a projection of the acceleration of point C on x-axis is:

$$\ddot{x}_C = 1.$$

So

$$N_A = m \cdot \ddot{x}_C = 3 \cdot 1 = 3\left(N\right).$$

Answer: $N_A = 3\left(N\right).$

Example 3. A rod AB of a mass 2 kg, sliding along rough horizontal plane, begins to fall in vertical plane (Fig. 3.52). At angle $\varphi = 45°$ determine a normal reaction N, if projection of mass center acceleration on y-axis is $\ddot{y}_C = -5.64\left(\dfrac{m}{\sec^2}\right).$

Solution

FBD is presented in Fig. 3.53. The rod has plane motion, so we can write 3 differential equations of motion, but to solve this problem we need only one equation because we are asked about y component of acceleration only

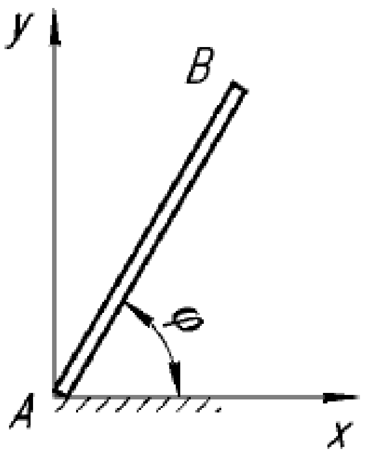


Fig. 3.52

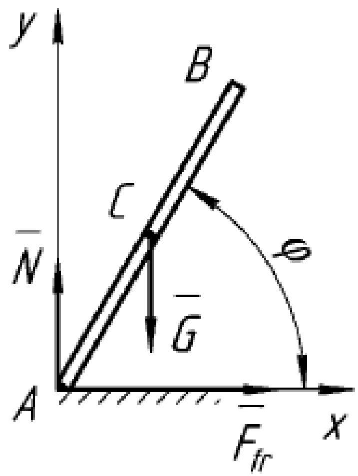


Fig. 3.53

$$m \cdot \ddot{y}_C = \sum_{k=1}^{n} F_{y\,k}^{e} = N - mg\ .$$

From this equation the normal reaction:

$$N = m \cdot \ddot{y}_C + mg = 2 \cdot \left(-5.64\right) + 2 \cdot 9.8 = 8.32\left(N\right).$$

Answer: $N = 8.32\left(N\right)$.

Example 4. A metal hoop with a radius $\mathrm{r} = 0.6$ m is released from rest on the 20° incline (Fig. 3.54). If the coefficients of static and kinetic friction are $f_{st} = 0.15$ and $f_{kin} = 0.12$, determine the angular acceleration W of the hoop and the time *t* for the hoop to move a distance of 10 m down the incline.

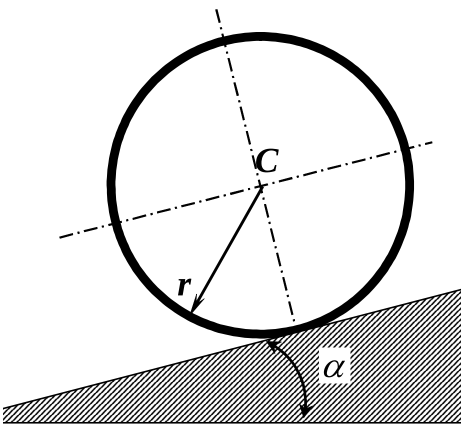


Fig. 3.54

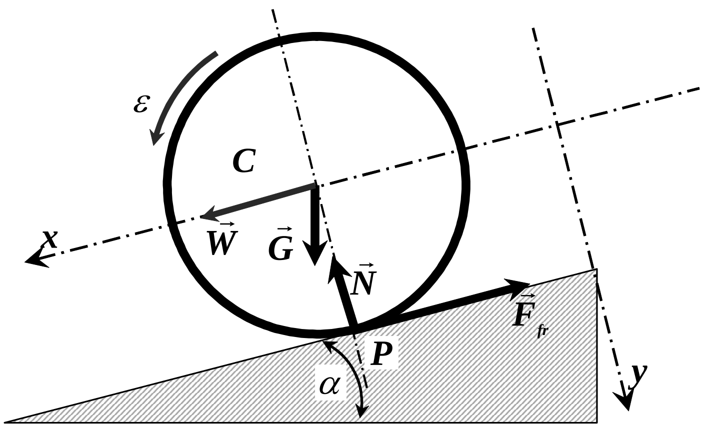


Fig. 3.55

Solution. The FBD (Fig. 3.55) shows the unspecified weight mg, the normal force *N,* and the friction force $\vec{F}_{fr}$ acting on the hoop at the contact point

P with the incline. The acceleration W through C in the positive direction axis x and the angular acceleration ε are shown in Fig. 3.55 too. The counterclockwise angular acceleration requires a counterclockwise moment about C, so $\vec{\boldsymbol{F}}_{fr}$ must be up the incline. Assume that the hoop rolls without slipping, so that

$$\ddot{x}_C = \ddot{\varphi} CP, \tag{3.91}$$

$$\ddot{y}_C = 0. \tag{3.92}$$

Application of equations (3.72) gives

$$m\ddot{x}_c = mg \cdot \sin 20^\circ - F_{fr}, \tag{3.93}$$

$$m\ddot{y}_c = -mg \cdot \cos 20^\circ + N, \tag{3.94}$$

$$I_{Cz}\ddot{\varphi}_z = F_{fr} r,\ I_{Cz} = mr^2. \tag{3.95}$$

Elimination of F_{fr} between the first and third equations and substitution of the constraint equation (3.91) give

$$\ddot{x}_c = \frac{g}{2} \cdot \sin 20^\circ. \tag{3.96}$$

Note that $\ddot{x}_c$ is independent of both m and *r.*

To check our assumption of no slipping, we calculate F_{fr} and N and compare F_{fr} with its limiting value. From the above equations

$$F_{fr} = mg \cdot \sin 20^\circ - m\ddot{x}_c = 0.1710mg, \tag{3.97}$$

$$N = mg \cdot \cos 20^\circ = 0.9397mg. \tag{3.98}$$

But the maximum possible friction force is

$$F_{fr\_\max} = \mathrm{f_{st}}(0.9397\mathrm{mg}) = 0.1410\mathrm{mg}. \tag{3.99}$$

Because our calculated value of $0.1710mg$ exceeds the limiting value of $0.1410\mathrm{mg}$, we conclude that our assumption of pure rolling was wrong.

Therefore the hoop slips as it rolls and the constraint equation does not hold. The friction force then becomes the kinetic value

$$F_{fr\_\max} = f_{kin}(0.9397mg) = 0.1128mg. \tag{3.100}$$

The motion equations now give

$$m\ddot{x}_c = mg \cdot \sin 20^\circ - 0.1128mg\,, \tag{3.101}$$

$$\ddot{x}_c = g \cdot \sin 20^\circ - 0.1128g\,, \tag{3.102}$$

$$I_{Cz}\ddot{\varphi}_z = F_{fr}r\,, \tag{3.103}$$

$$mr^2\ddot{\varphi}_z = 0.11128mg \cdot r\,, \tag{3.104}$$

$$\ddot{\varphi}_z = \frac{0.11128g}{r}. \tag{3.105}$$

The time required for the center C of the hoop to move 10 m from rest with constant acceleration is

$$t = \sqrt{\frac{2x}{\ddot{x}_c}} \tag{3.106}$$

Example 5. The wheel and its hub (Fig. 3.56) have a mass of 30 kg with a radius of gyration about the center $i_z = 450\ mm$. A cord, wrapped securely around its hub, is attached to the fixed support, and the wheel is released from rest on the incline. If the static and kinetic coefficients of friction between the wheel and the incline are 0.4 and 0.3, respectively, calculate the acceleration of the center of the wheel. First prove that the wheel slips.

Solution

Let's prove that the wheel slips. At point P (Fig. 3.57) we have instantaneous center of zero velocity (ICZV), because the cord is fixed. Assume, that we don't have slipping at point A. So we have ICZV at point A too. In such case the wheel will not move. So our assumption is wrong. So we have slipping of the wheel.

Analyze forces acting on the body: gravity force $m_1\overline{g}$, normal reaction $\overline{N}$ of the incline, tension force $\overline{T}$ and friction force $\overline{F_{fr}}$. We have ICZV at point

P and the body moves down. So the velocity of the point A is also downwards. Thus $\overline{F_{fr}}$ has opposite direction.

The wheel has plane motion. So we can write differential equations of motion for the wheel (3.72). For the problem equations (3.72) have view:

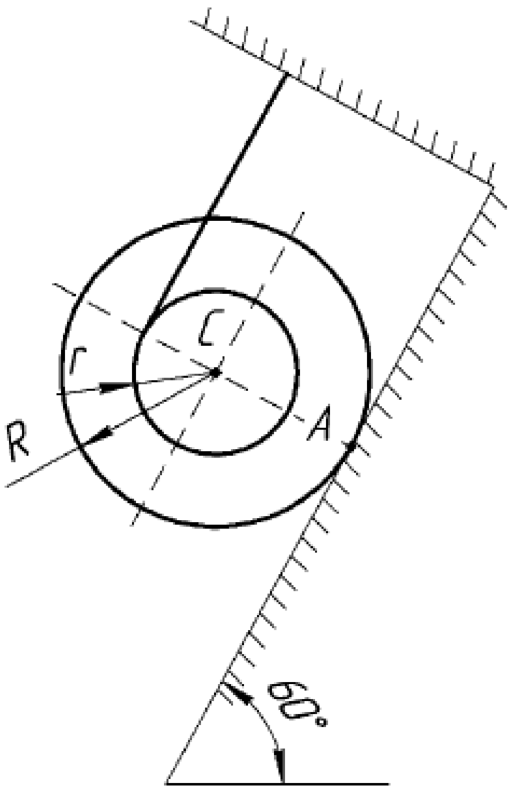


Fig. 3.56

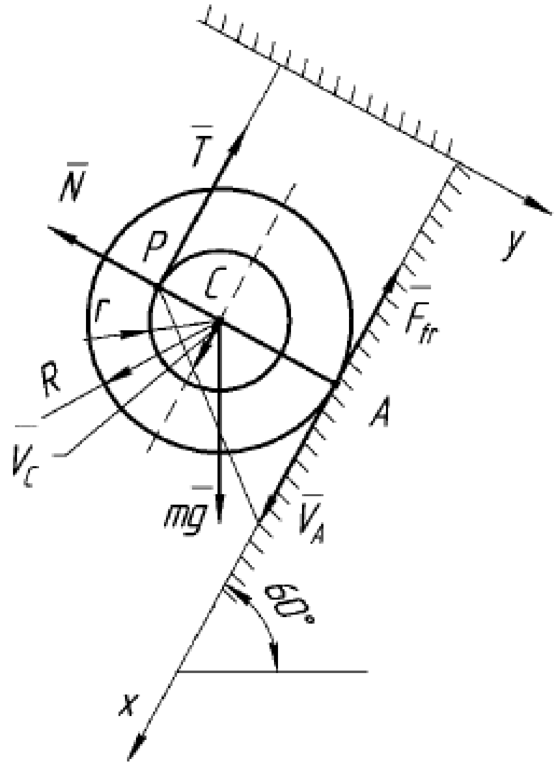


Fig. 3.57

$$\begin{cases} M\ddot{x}_C = -T - F_{fr} + mg\sin 60°; \\ M\ddot{y}_C = -N + mg\cos 60°; \\ I_{C_z}\ddot{\varphi}_z = -T \cdot r + F_{fr} \cdot R. \end{cases} \tag{3.107}$$

There are six unknowns in this system: $\ddot{x}_c$, $\ddot{y}_c$, $\ddot{\varphi}_z$, T, N, F_{fr}. Therefore to be the system of equations solvable we must add three additional equations (equations of constraints).

The motion of the wheel mass center in direction normal to the fixed surface is constrained and it is possible to describe this restriction as following

$$y_C = const \Rightarrow \ddot{y}_C = 0.$$

If a rigid body (wheel) rolls with slipping on a fixed surface, we can find friction force using kinetic coefficient of friction:

$$F_{fr} = k_k \cdot N .$$

As there is ICZV at point P, we can add one more equation:

$$\left|V_C\right| = \left|V_{C\,x}\right| = \omega \cdot CP = \omega \cdot r$$

or in coordinate form

$$\dot{x}_C = -\dot{\varphi}_z \cdot CP = -\dot{\varphi}_z \cdot r.$$

This equation means that if the wheel angular velocity is negative (in clockwise direction) then the projection on the axis Ox of the mass center velocity is positive. And by differentiation we get

$$\ddot{x}_C = -\ddot{\varphi}_z \cdot CP = -\ddot{\varphi}_z \cdot r .$$

As a result we have the system of equations

$$\begin{cases} M\ddot{x}_C = -T - F_{fr} + mg\sin 60°; & (3.108) \\ M\ddot{y}_C = -N + mg\cos 60°; & (3.109) \\ I_{C_z}\ddot{\varphi}_z = -T \cdot r + F_{fr}R; & (3.110) \\ \ddot{y}_C = 0; & (3.111) \\ F_{fr} = k_k \cdot N; & (3.112) \\ \ddot{x}_C = -\ddot{\varphi}_z \cdot r. & (3.113) \end{cases}$$

Substituting equation (3.111) to (3.109):

$$N = mg\cos 60° .$$

Then friction force will be:

$$F_{fr} = k_k mg\cos 60° .$$

Taking into account that moment of inertia is

$$I_{C_z} = m \cdot i_z^2$$

and considering equations (3.108), (3.110) and (3.113) we get:

$$\begin{cases} M\ddot{x}_C = -T - k_k mg\cos 60° + mg\sin 60°; \\ m \cdot R_g^2 \cdot \left(-\dfrac{\ddot{x}_C}{r} \right) = -T \cdot r + Rk_k mg\cos 60°. \end{cases}$$

From here the acceleration of the wheel mass center is:

$$\ddot{x}_C = \frac{g\sin 60° - k_k g\cos 60° - \dfrac{k_k g\cos 60° R}{r}}{1 + \dfrac{R_g^2}{r^2}} = 1.255\left(\frac{m}{\sec^2} \right).$$

Answer: $\ddot{x}_C = 1.255\left(\dfrac{m}{\sec^2} \right)$.

3.5.3. Problems for self-solution

Problem 1. A heavy circular cylinder of mass m is suspended by a cord, one end of which is wound round the middle part of the cylinder while the other end is fixed at B (Fig. 3.58). the cylinder is allowed to fall from rest so that the cord unwinds. Determine the velocity of the cylinder axis after it has fallen a distance h. Also find the tension T in the cord.

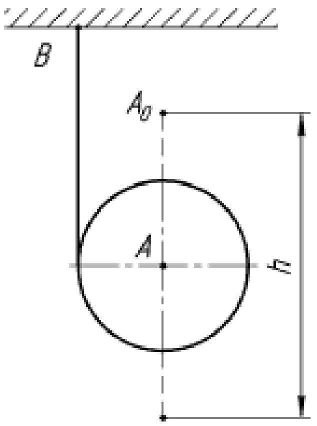


Fig. 3.58

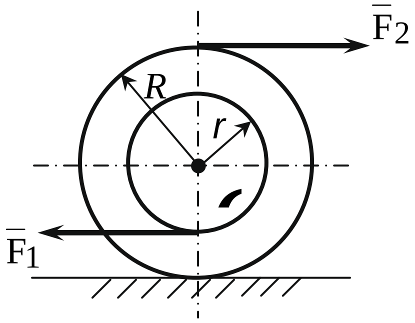


Fig. 3.59

Problem 2. Roller of mass m and central moment of inertia $\rho = r\sqrt{2}$ rolls without slipping along the horizontal rail under the action of constant forces F_1 and F_2 (Fig. 3.59). For known R and r, determine $\overline{W}_c$ and friction force.

3.5.4. Short problems

Problem 1. The homogeneous rod AB of length1 m and mass m=2 kg from a state of rest at angle φ = 45° to a vertical begins sliding on a smooth wall and a smooth floor (Fig. 3.60). Determine angular acceleration of the rod if at the points A and B normal reactions are N_A = 7,3 N and N_B= 12,2 N.

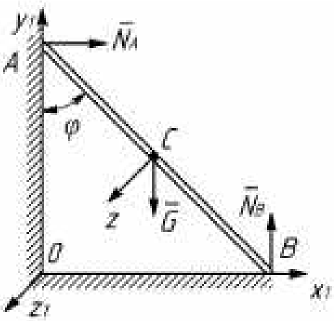


Fig. 3.60

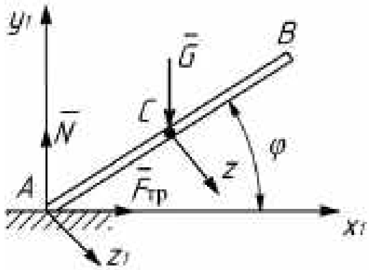


Fig. 3.61

Problem 2. At motion of the rod AB (Fig. 3.61) of the length 0,5 m in the plane Ax_1y_1 at the given moment of time the angle is φ = 30°, normal reaction is N=12 N, friction force is F_{fr}= 1,2 N. Determine a magnitude of angular acceleration if the moment of inertia is I_{Cz} = 0.08 kg•m^2.

Problem 3. A homogeneous cylinder of a mass m=6kg and a radius R = 0.08 m falls in vertical plane unwinding a cord (Fig. 3.62), tension of which is T = 19,6N. Determine an angle velocity ω of the cylinder at the moment of time t = 0.4 s, if at t_0 = 0 an angle velocity is equal to zero.

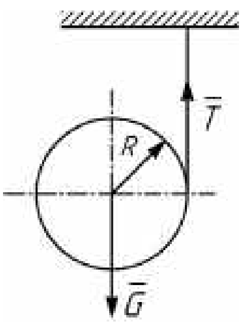


Fig. 3.62

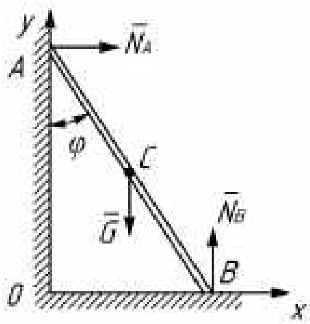


Fig. 3.63

Problem 4. A rod AB of a length 1 m and a mass 2 kg, leaning on a vertical smooth wall at an angle φ=30°, begins sliding (Fig. 3.63). Determine a normal reaction N_B at the point B, if a projection of mass center acceleration C on Oy-axis has the value $\ddot{y}_C$ = -1,84 m/s^2.

LECTURE 7

3.6. Work-energy principle

3.6.1. Work

3.6.1.1. Elementary work (work done by a force during an infinitesimal displacement)

Elementary work or work done by a force $\vec{F}$ during an infinitesimal displacement $d\vec{r}$ of its point of application M (Fig. 3.63) is the dot (scalar) product of the force and a radius vector differential:

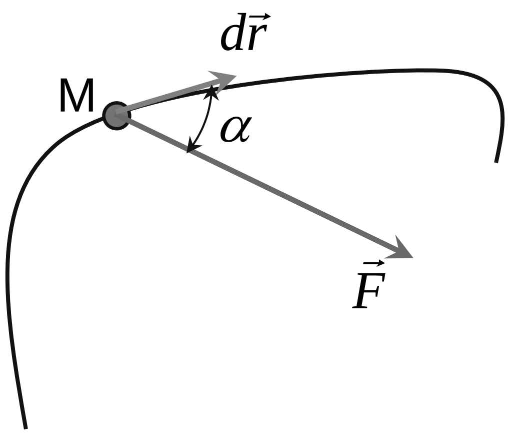


Fig. 3.63

$$\boxed{d'A = \vec{F} d\vec{r} = F\, dr\, Cos\alpha.} \qquad (3.114)$$

Dot product result is scalar value so elementary work is a scalar quantity.

Work units of measuring expressed in J (joule), $[J] = Nm$.

It is possible to express the infinitesimal displacement $d\vec{r}$ in term of infinitesimal path

$$d\vec{r} = ds\vec{\tau},$$
$$dr = ds,$$

so

$$d'A = \vec{F}\, ds\vec{\tau} = Fds\cos\alpha. \qquad (3.115)$$

If the force forms an acute angle (see Fig. 3.63) with the direction of the displacement, the work done by the force is positive; if the force forms an obtuse angle (Fig. 3.64) with the direction of the displacement, the work done by the force is negative; if the force forms right angle (Fig. 3.65) with the direction of the displacement, the work done by the force is zero :

$$d'A \begin{cases} > 0 \ if \ \ 0 \le \alpha < \pi/2, \\ = 0 \ if \ \ \alpha = \pi/2, \\ < 0 \ if \ \ \pi/2 < \alpha \le \pi. \end{cases}$$

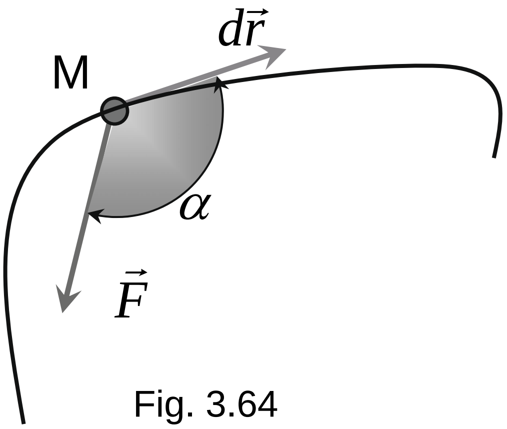


Fig. 3.64

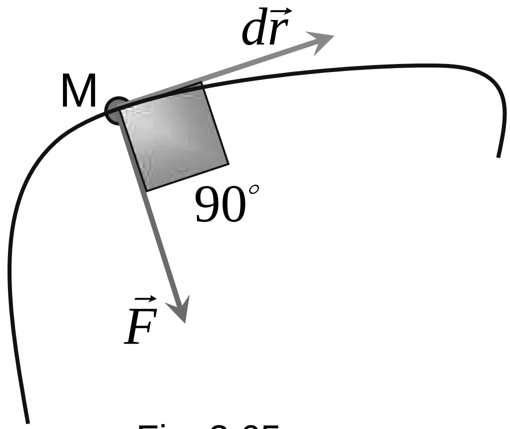


Fig. 3.65

Another form of elementary work:

a) if we denote product of $F\,cos\alpha$ as F_τ, we obtain next

$$d'A = F_\tau ds, \tag{3.116}$$

where F_τ is the projection of the force on the axis which is tangential to the trajectory of the particle;

b) if we determine radius vector differential as

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}dt,$$

then

$$d'A = \vec{F}\vec{v}dt, \tag{3.117}$$

where $\vec{v}$ is the velocity of the particle.

From (3.117) follows that if force is applied at the instantaneous center of zero velocity of the body in plane motion (Fig. 3.66) its elementary work is equal nil. For example from Fig 3.66 we have

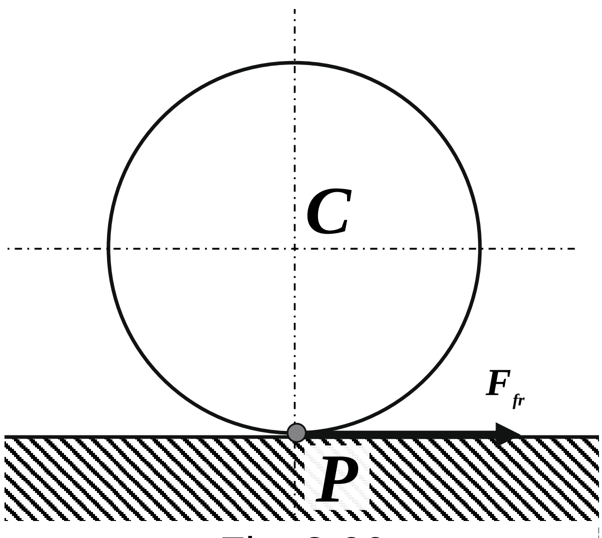


Fig. 3.66

$$d'A(\vec{F}_{fr}) = \vec{F}_{fr}\vec{v}_P dt = 0;$$

c) if we determine radius vector differential by its projection on the coordinate axes

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \ ,$$

we can get cordinate form of the equation (3.114):

$$\boxed{d'A = \vec{F}(dx\vec{i} + dy\vec{j} + dz\vec{k}) = F_x dx + F_y dy + F_z dz.} \qquad (3.118)$$

3.6.1.2. Total work (work done by a force during an finite displacement)

The work done by the force during some finite displacement $\overset{\frown}{M_{in}M_{fin}}$ (Fig. 3.67) is integral of the elementary work over the trajectory of point of application of the force

$$\boxed{A = \int\limits_{M_{in}M_{fin}} \vec{F}d\vec{r} = \int\limits_{M_{in}M_{fin}} FdrCos\alpha.} \qquad (3.119)$$

For the general case work depends on the form of trajectory.

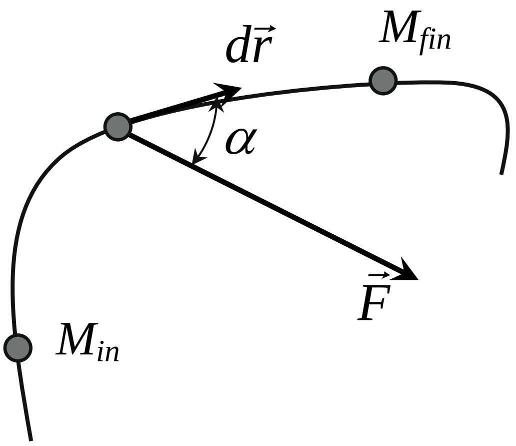


Fig. 3.67

3.6.1.3. Theorems about work

Theorem 1. If a number of forces act at the same particle, the work done by the resultant force is the algebraic sum of work done by a separate force

$$\mathrm{d'A} = \vec{\mathrm{R}}\mathrm{d}\vec{\mathrm{r}} = \left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \vec{\mathrm{F}}_{\mathrm{i}}\right)\mathrm{d}\vec{\mathrm{r}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \vec{\mathrm{F}}_{\mathrm{i}}\mathrm{d}\vec{\mathrm{r}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{d'A_i}.$$

Theorem 2. The elementary work done by a force during compound infinitesimal displacement is equal to the algebraic sum of the elementary works during the components of displacement

$$d'A = \vec{F}d\vec{r} = \left| d\vec{r} = d\vec{r}_1 + d\vec{r}_2 \right| = \vec{F}\left(d\vec{r}_1 + d\vec{r}_2\right) = \vec{F}d\vec{r}_1 + \vec{F}d\vec{r}_2 \ .$$

3.6.1.4. Sum of elementary works of the forces applied to the rigid body

Let us assume that some forces are applied to the rigid body. The body can move free in space. For purposes of dynamic it is convenient to select auxiliary frame of reference $O_1x_1y_1z_1$ such as has origin in any point of the body and moves transnationally (Fig. 3.68). Now any displacement of force $\vec{F}_k$ application point M_k can be regarded as a combined translation of auxiliary frame of reference and rotation about its origin with angular velocity $\vec{\omega}$:

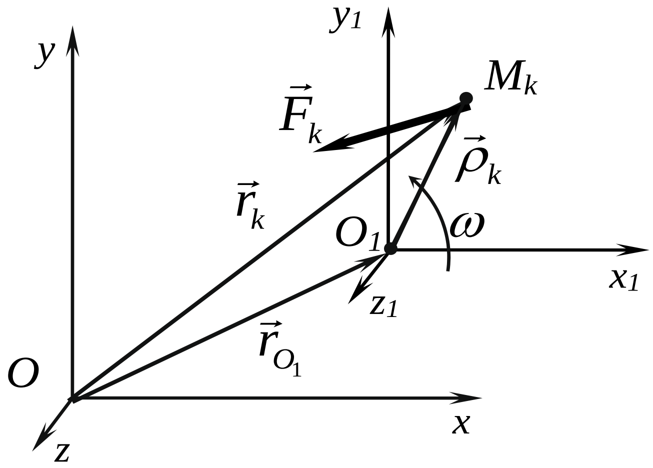


Fig. 3.68

$$\vec{r}_k = \vec{r}_{O_1} + \vec{\rho}_k,$$
$$d\vec{r}_k = d\vec{r}_{O_1} + d\vec{\rho}_k,$$

from kinematics of rigid body general motion we have

$$\frac{d\vec{\rho}_k}{dt} = \vec{\omega} \times \vec{\rho}_k,$$
$$d\vec{r}_k = d\vec{r}_{O_1} + \left(\vec{\omega} \times \vec{\rho}_k\right)dt = d\vec{r}_{O_1} + \left(\vec{\omega}dt \times \vec{\rho}_k\right).$$

Sum of elementary works of forces is

$$\sum_{k=1}^{n} d'A = \sum_{k=1}^{n} \vec{F}_k d\vec{r}_k =$$

$$= \sum_{k=1}^{n} \vec{F}_k d\vec{r}_{O_1} + \sum_{k=1}^{n} \vec{F}_k \cdot \left(\vec{\omega} dt \times \vec{\rho}_k\right).$$

In the second term we have mixed product, it can be rewritten as

$$\vec{F}_k \cdot \left(\vec{\omega} dt \times \vec{\rho}_k\right) = \left(\vec{\rho}_k \times \vec{F}_k\right) \cdot \vec{\omega} dt.$$

Now the elementary work is

$$\sum_{k=1}^{n} d'A = \sum_{k=1}^{n} \vec{F}_k d\vec{r}_{O_1} + \sum_{k=1}^{n} \left(\vec{\rho}_k \times \vec{F}_k\right) \vec{\omega} dt.$$

In the second sum in brackets we have moment of force $\vec{F}_k$ about origin of auxiliary reference O_1:

$$\overrightarrow{M}_{O_1}\left(\vec{F}_k\right) = \vec{\rho}_k \times \vec{F}_k,$$

and determine $\vec{\omega} dt$ as vector of elementary angular displacement $\vec{\theta}$

$$\vec{\omega} dt = \vec{\theta},$$

so

$$\sum_{k=1}^{n} d'A = \sum_{k=1}^{n} \vec{F}_k d\vec{r}_{O_1} + \sum_{k=1}^{n} \overrightarrow{M}_{O_1}\left(\vec{F}_k\right) \vec{\theta}.$$

Factors $d\vec{r}_{O_1}$ and $\vec{\theta}$ are independent of indexes of summation so can be factored out from the sums:

$$\sum_{k=1}^{n} d'A = \left(\sum_{k=1}^{n} \vec{F}_k\right) d\vec{r}_{O_1} + \left(\sum_{k=1}^{n} \overrightarrow{M}_{O_1}\left(\vec{F}_k\right)\right) \vec{\theta}.$$

Sums in brackets are total force $\vec{F}^*$ and total moment $\overrightarrow{M}_{O_1}^*$ of the force system about origin O_1:

$$\vec{F}^* = \sum_{k=1}^{n} \vec{F}_k,$$

$$\overrightarrow{M}_{O_1}^* = \sum_{k=1}^{n} \overrightarrow{M}_{O_1}\left(\vec{F}_k\right).$$

Finally we get

$$\boxed{\sum_{k=1}^{n} d'A = \vec{F}^* d\vec{r}_{O_1} + \overrightarrow{M}_{O_1}^* \vec{\theta}.} \qquad (3.120)$$

The algebraic sum of the elementary works done by forces applied to the body is equal to work done by the total vector of the force system during the elementary displacement of origin and work done by the total moment about this origin during the elementary angular displacement.

3.6.1.5. Some case of work determination

1. Work of couple of forces $\overrightarrow{M}$ applied to a rigid body rotating about fixed axis O_1z axis

$$d'A = \overrightarrow{F}^{*} d\vec{r}_{O_1} + \overrightarrow{M}^{*}_{O_1}\vec{\theta} = \begin{vmatrix} d\vec{r}_{O_1} = 0, \\ \overrightarrow{M}^{*}_{O_1} = \overrightarrow{M} \end{vmatrix} = \overrightarrow{M}\vec{\theta} = \left(\overrightarrow{M}\right)_z d\varphi = M_z d\varphi,$$

$$\boxed{d'A\left(\overrightarrow{M}\right) = M_z d\varphi,} \tag{3.121}$$

where $d\varphi$ is elementary angular displacement about axis of rotation, it is positive for counterclockwise rotation.

From eq. (3.121) clear that elementary work of couple may be positive for the same directions of couple and body rotation (Fig.3.69, a) or negative for opposite directions (Fig.3.69, b):

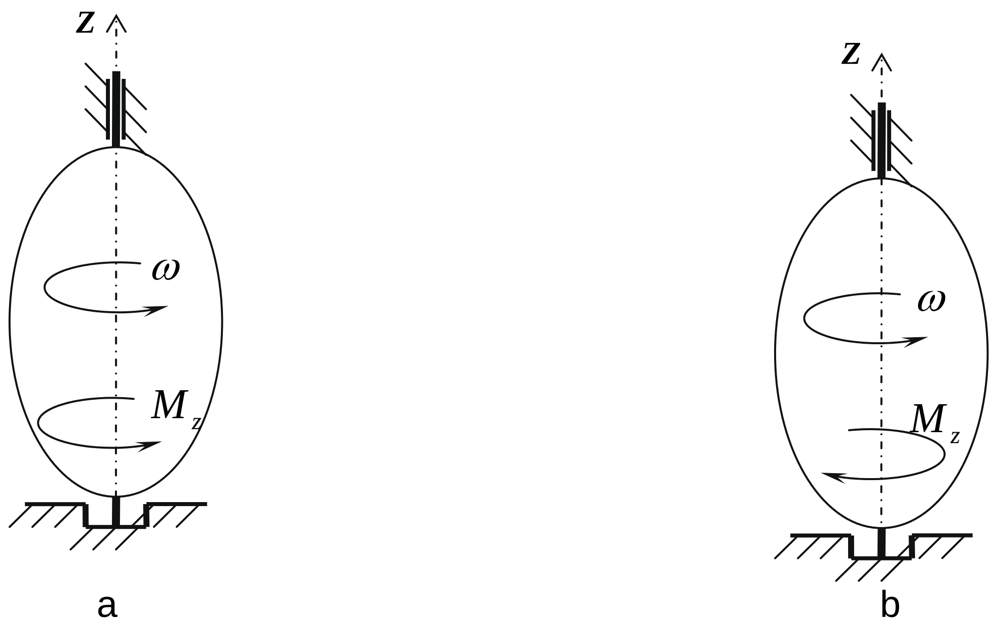


Fig. 3.69

For case of constant couple total work done by couple during finite turning on angle $\Delta\varphi = \varphi_{fin-}\varphi_{in}$ is

$$\boxed{A\left(\overline{M}\right) = M_z\left(\varphi_{fin-}\varphi_{in}\right).} \tag{3.122}$$

2. Work done by gravity.

Let us analyze work done by gravity of particle $\vec{G}$ during its displacement dz (Fig. 3.70) assuming that positive direction of axis Oz is vertically up, it means opposite with direction of the gravity. Using coordinate form of elementary work (3.118) we get

$$d'A(\vec{G}) = G_x dx + G_y dy + G_z dz.$$

But $G_x = 0$ and $G_y = 0$ so

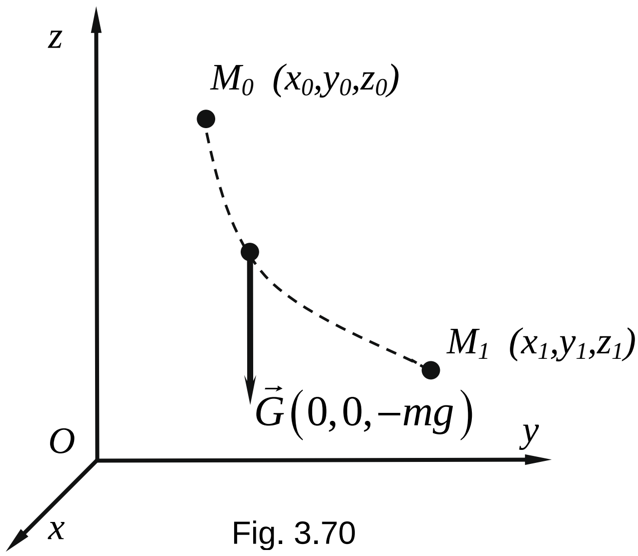


Fig. 3.70

$$\boxed{d'A = -mgdz.} \tag{3.123}$$

Using the last equation we can obtain total work done by gravity:

$$A(\vec{G}) = \int_{z_0}^{z_1} -gmdz = -gm\int_{z_0}^{z_1} dz = -gm(z_1 - z_0). \tag{3.124}$$

Denoting $h = |z_0 - z_1|$ we get another form

$$\boxed{A(\vec{G}) = \pm gmh.} \tag{3.125}$$

where sign "+" corresponds motion of the particle downwards, sign "-" corresponds motion of the particle upwards.

3. Work done by elastic force.

Consider a spring constrained to move along axis Ox, coincide the origin O of the axis with the end of undeformed spring, l_0 is natural

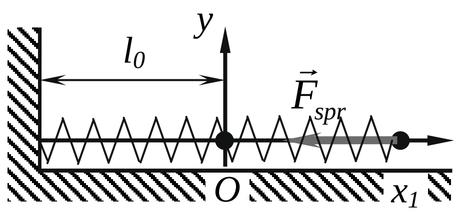


Fig. 3.71

length of the spring, c, $\left[c\right]=\dfrac{N}{m}$, is spring constant or stiffness coefficient. Particle is attached to the spring free end. Using equation (3.114) for elementary work done by the spring force during positive infinitesimal displacement of the particle we have

$$\dot{d}'A(\vec{F}_{spr})=\vec{F}_{spr}d\vec{r}=\begin{vmatrix} F_x=-cx \\ F_y=F_z=0 \end{vmatrix}=-cxdx. \tag{3.126}$$

Total work done by spring force during finite displacement x_1 is

$$A(\vec{F}_{spr})=\int_0^{x_1}-cxdx=-\frac{1}{2}cx_1^{\;2}. \tag{3.127}$$

In three dimensional case for linear spring we have

$$A(\vec{F}_{spr})=\int_{r_{st}}^{r_{fn}}-c\vec{r}d\vec{r}=-\frac{1}{2}c(r_{fn}^{\;2}-r_{st}^{\;2}). \tag{3.128}$$

The same can be repeated for a spiral spring with stiffness coefficient c, $\left[c\right]=Nm$, position $\varphi=0$ corresponds untwisted spring:

$$\dot{d}'A(\overrightarrow{M}_{spr})=M_{spr\_z}dz=-c\varphi d\varphi, \tag{3.129}$$

$$A(\overrightarrow{M}_{spr})=\int_0^{\varphi_1}-c\varphi d\varphi=-\frac{1}{2}c\varphi_1^{\;2}. \tag{3.130}$$

4. The work $A^{(i)}$ done by internal force of a rigid body.

The total work $A^{(i)}$ done by internal force of a rigid body during any displacement is equal to nil.

3.6.1.6. Power

Power is the rate at which work is performed. Power can be obtained as differential of work with respect to time:

for power of force

$$N(\vec{F})=\frac{\dot{d}'A(\vec{F})}{dt}=\frac{\vec{F}d\vec{r}}{dt}=\vec{F}\vec{v},$$

$$\boxed{N(\overrightarrow{F}) = \overrightarrow{F}\vec{v}.} \tag{3.131}$$

for power of couple of forces using equation (3.121) we get

$$N(\overrightarrow{M}) = \frac{\dot{d}'A(\overrightarrow{M})}{dt} = \frac{M_z d\varphi}{dt} = M_z \omega_z,$$

$$\boxed{N(\overrightarrow{M}) = M_z \omega_z.} \tag{3.132}$$

Power is scalar value, it is measured in Watt.

LECTURE 8

3.6.2. Kinetic energy

The kinetic energy of a particle is equal to half of the product of the mass of particle and the square of its velocity:

$$\boxed{T = \frac{mv^2}{2}.} \tag{3.133}$$

Kinetic energy is a scalar. The units are the same as for work (i.e. Joules, J)

The kinetic energy of particle system is equal to the sum of kinetic energies of all particles:

$$\boxed{T = \sum_{i=1}^{n} \frac{m_k {v_k}^2}{2}.} \tag{3.134}$$

From (3.134) it is clear that the kinetic energy is always positive. The kinetic energy of particle system is equal to nil only if all particles of the system are at rest.

3.6.2.1. Koenig's theorem

Assume that $T = \sum_{i=1}^{n} \frac{m_k {v_k}^2}{2}$ is kinetic energy of a system determined in inertial frame of reference $Oxyz$; denote as $T' = \sum_{i=1}^{n} \frac{m_k \left({v_k}^r\right)^2}{2}$ kinetic energy of a system of particles determined in noninertial frame of reference $Cx_1y_1z_1$, where C is the center of mass of the system, $Cx_1y_1z_1$ moves translationally with respect to fixed reference $Oxyz$ (Fig. 3.72).

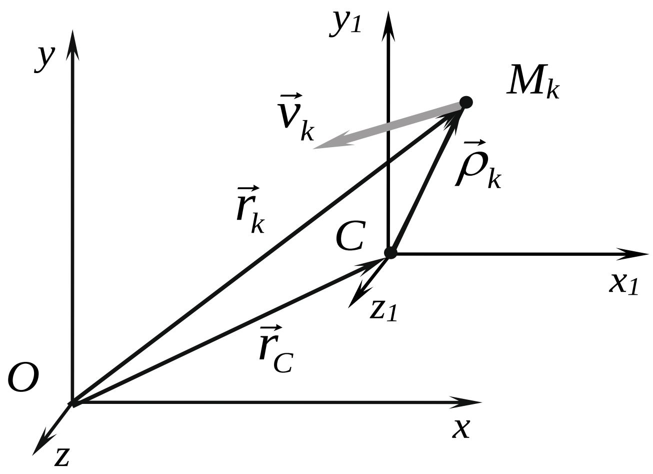


Fig. 3.72

Absolute velocity $\vec{v}_k^{\,a}$ of k-th particle is

$$\vec{v}_k^{\,a} = \vec{v}_k^{\,e} + \vec{v}_k^{\,r}, \tag{3.135}$$

where $\vec{v}_k^{\,e}$ is bulk velocity, $\vec{v}_k^{\,r}$ is relative velocity.

For translational motion of moving reference $Cx_1y_1z_1$ we have

$$\vec{v}_k^{\,e} = \vec{v}_C. \tag{3.136}$$

Put equations (3.135) and (3.136) in to equation (3.134)

$$T = \sum_{k=1}^{n} \frac{m_k v_k^{a\,2}}{2} = \sum_{k=1}^{n} \frac{m_k \left(\vec{v}_C + \vec{v}_k^{\,r}\right)^2}{2} = \sum_{k=1}^{n} \frac{m_k v_C^{\,2}}{2} + 2\sum_{k=1}^{n} \frac{m_k \vec{v}_C \cdot \vec{v}_k^{\,r}}{2} + \sum_{k=1}^{n} \frac{m_k v_k^{r\,2}}{2}. \tag{3.137}$$

In the first sum v_C is independent of index k so the sum can be factorized:

$$\sum_{k=1}^{n} \frac{m_k v_C^{\,2}}{2} = \left(\sum_{k=1}^{n} \frac{m_k}{2}\right) v_C^{\,2} = \frac{M v_C^{\,2}}{2},$$

where $M = \sum\limits_{k=1}^{n} m_k$ is total mass of the particle system.

The second sum can be transformed as

$$2\sum_{k=1}^{n} \frac{m_k \vec{v}_C \cdot \vec{v}_k^{\,r}}{2} = \vec{v}_C \cdot \sum_{k=1}^{n} m_k \vec{v}_k^{\,r} = \left| \vec{v}_k^{\,r} = \frac{d\vec{\rho}_k}{dt} \right| = \vec{v}_C \cdot \sum_{k=1}^{n} m_k \frac{d\vec{\rho}_k}{dt} = \vec{v}_C \cdot \frac{d}{dt}\left(\sum_{k=1}^{n} m_k \vec{\rho}_k\right).$$

Comparing the last brackets with the equation for mass center position (3.19), and remembering that C is origin of moving reference ($\vec{\rho}_C = 0$) we obtain

$$\sum_{k=1}^{n} m_k \vec{\rho}_k = M \vec{\rho}_C = 0,$$

so

$$2\sum_{k=1}^{n} \frac{m_k \vec{v}_C \cdot \vec{v}_k^{\,r}}{2} = \vec{v}_C \cdot \frac{d}{dt}\left(\sum_{k=1}^{n} m_k \vec{\rho}_k\right) = 0.$$

The third sum in equation (3.135) is kinetic energy of a system of particles determined in noninertial frame of reference $Cx_1y_1z_1$: $\sum_{i=1}^{n} \frac{m_k {v_k^r}^2}{2} = T'$.

Now equation (3.135) has view

$$\boxed{T = \frac{M{v_C}^2}{2} + T'.} \tag{3.138}$$

König's theorem:

The kinetic energy of an aggregate of particles relative to any reference is the sum of two parts:

– fisrt is the kinetic energy of a hypothetical particle which has the mass equal to the total mass of the system and moves with velocity of particle system mass-center $\frac{M(v_C)^2}{2}$;

– and second is the kinetic energy of a particle system with respect to the auxiliary frame of reference which has motion of translation and origin in the particle system mass-center (König's reference system).

3.6.2.2. Kinetic energy of a rigid body

Rigid body can be presented as an unchangeable system of particles, then kinetic energy of the rigid body is

$$\boldsymbol{T} = \sum_{i=1}^{\infty} \frac{m_i {v_i}^2}{2} = \frac{1}{2}\int_M v^2 dm. \tag{3.139}$$

Consider a **translating rigid body** of a mass $\boldsymbol{M}$. All of its points have a common velocity $\boldsymbol{v}$, so the body kinetic energy is

$$\boldsymbol{T} = \sum_{i=1}^{\infty} \frac{m_i v_i^{\,2}}{2} = \frac{1}{2}\int_M v^2 dm = \frac{Mv^2}{2},$$

$$\boxed{\boldsymbol{T} = \frac{Mv^2}{2}.} \tag{3.140}$$

The equation shows that kinetic energy of translating rigid body has the same view as kinetic energy of a particle.

Consider a **rigid body rotating about a fixed axis** z through O with angular velocity ω . Its kinetic energy is

$$\boldsymbol{T} = \frac{1}{2}\int_M v^2 dm = \left| v = \omega\rho \right| = \frac{1}{2}\int_M \left(\omega\rho\right)^2 dm = \frac{I_{Oz}\omega^2}{2},$$

$$\boxed{\boldsymbol{T} = \frac{I_{Oz}\omega^2}{2},} \tag{3.141}$$

where ρ is the shortest distance between the axis of rotation and arbitrary point of the rigid body, $I_{Oz} = \int_M \rho^2 dm$ is the body moment of inertia about the fixed axis Oz, $I_{Oz} = const$ for the rigid body.

Consider **a rigid boy in plane motion**. Present plane motion as translation with mass center C and rotation with angular velocity ω about axis Cz. In accordance with Konig’s theorem (see Fig. 3.72) we get

$$\boldsymbol{T} = \frac{M\left(\vec{v}_C\right)^2}{2} + \int_M \frac{\left(\vec{v}_i^{\,r}\right)^2 dm}{2} = \left|\vec{v}_i^{\,r} = \vec{\omega}\times\vec{\rho}_i\right| =$$

$$= \frac{M\left(\vec{v}_C\right)^2}{2} + \int_M \frac{\left(\omega\rho\right)^2 dm}{2} = \frac{M\left(\vec{v}_C\right)^2}{2} + \frac{I_{z_C}\omega^2}{2},$$

$$\boxed{\boldsymbol{T} = \frac{M\left(\vec{v}_C\right)^2}{2} + \frac{I_{z_C}\omega^2}{2},} \tag{3.142}$$

where $I_{Cz} = \int_M \rho^2 dm$ is the body moment of inertia about the axis through the body mass center, $I_{Cz} = const$ for the rigid body.

Another form of kinetic energy equation for a rigid body in the plane motion is obtained for ICZV as pole of translation:

$$\boldsymbol{T} = \int_{\boldsymbol{M}} \frac{(\vec{\boldsymbol{v}})^2 \boldsymbol{dm}}{2} = \left| \vec{\boldsymbol{v}}_i = \vec{\omega} \times \vec{\rho}_i \right| = \int_{\boldsymbol{M}} \frac{(\omega\rho)^2 \boldsymbol{dm}}{2} =$$

$$= \frac{\omega}{2} \int_{\boldsymbol{M}} \rho^2 \boldsymbol{dm} = \frac{\boldsymbol{I}_{\boldsymbol{Pz}}(\boldsymbol{t})\omega^2}{2},$$

$$\boxed{\boldsymbol{T} = \frac{\boldsymbol{I}_{\boldsymbol{Pz}}(\boldsymbol{t})\omega^2}{2},} \tag{3.143}$$

where $\boldsymbol{I}_{\boldsymbol{Pz}}(\boldsymbol{t}) = \int_{\boldsymbol{M}} \rho^2 \boldsymbol{dm}$ is the body moment of inertia about the axis through the ICZV, $\boldsymbol{I}_{\boldsymbol{Pz}}(\boldsymbol{t})$ is function of time because the ICZV position changes with time.

3.6.3. Work-energy principle for a particle

Remember and write the Newton’s second low for constrained particle (1.26):

$$m\vec{W} = \vec{F} + \vec{N}, \tag{3.144}$$

Multiplying the equation (3.144) by elementary displacement $d\vec{r}$ of a particle we obtain

$$m\vec{W}d\vec{r} = \left(\vec{F} + \vec{N}\right)d\vec{r}, \tag{3.145}$$

rewrite the left side

$$m\vec{W}d\vec{r} = m\frac{d\vec{v}}{dt}d\vec{r} = m\frac{d\vec{r}}{dt}d\vec{v} = m\vec{v}d\vec{v} = d\left(\frac{m\vec{v}^2}{2}\right) = dT,$$

rewrite the right side

$$\left(\vec{F} + \vec{N}\right)d\vec{r} = d'A.$$

So equation (3.145) has view

$$\boxed{dT = d'A.} \tag{3.146}$$

Principle of work and kinetic energy for a particle in differential form 1: increment of the kinetic energy of a particle during its infinitesimal displacement is equal to elementary work done by all active forces and reactions during the same displacement.

Dividing the equation (3.146) by the elementary interval of time dt we obtain another form of (3.146) :

$$\frac{d}{dt}\left(\frac{m\vec{v}^2}{2}\right) = \frac{dT}{dt},$$

$$\frac{d'A}{dt} = \frac{\left(\vec{F} + \vec{N}\right)d\vec{r}}{dt} = N(\vec{F}^*),$$

$$\boxed{\frac{dT}{dt} = N(\vec{F}^*).} \tag{3.147}$$

Principle of work and kinetic energy for a particle in differential form 2: the derivative of the kinetic energy of particle with respect to the time is equal to the power of the resultant force.

Let the particle move from position M_{in} to position M_{fin} (finite displacement). Integrating the equation (3.146) between limits M_{in} and M_{fin} we obtain the following:

$$\int_{M_{in}}^{M_{fin}} dT = \int_{M_{in}}^{M_{fin}} d'A(\vec{F}^*),$$

$$\boxed{T_{fin} - T_{in} = A(\vec{F}) + A(\vec{N}).} \tag{3.148}$$

Principle of work and kinetic energy for a particle in integral form: change in kinetic energy of a particle during the finite movement equals the work done on the particle by active and reactive forces during the movement.

3.6.4. Work-energy principle for a particle system

For mechanical system consisting of *n* particles the forces acting on each particle may be divided into two classes: those exerted by fields or bodies outside the system $\vec{F}_k^{(e)}$ (external forces) and those exerted by other particle of the system $\vec{F}_k^{(i)}$ (internal forces).

For *k-th* we can say

$$dT_k = d'A(\vec{F}_k^{(e)}) + d'A(\vec{F}_k^{(i)}), \tag{3.149}$$

where $\vec{F}_k^{(e)}$ - resultant vector of external forces acting on *k-th* particle ;

$\vec{F}_k^{\,i}$ - resultant vector of internal forces acting on *k-th* particle.

Summing over *k, k=1,…,n,* we have

$$\boxed{dT = \sum_{k=1}^{n} d'A(\vec{F}_k^{(e)}) + \sum_{k=1}^{n} d'A(\vec{F}_k^{\,i}).} \tag{3.150}$$

Principle of work and kinetic energy for a system of particles in differential form 1: increment of the kinetic energy of system of particles is equal to sum of elementary works done by all external and internal forces during the infinitesimal displacements of its points of application.

Sometimes it is useful to use another form of work-energy principle:

$$\boxed{\frac{dT}{dt} = \sum_{k=1}^{n} N(\vec{F}_k^{(e)}) + \sum_{k=1}^{n} N(\vec{F}_k^{\,i}).} \tag{3.151}$$

Principle of work and kinetic energy for a system of particles in differential form 2: Differential with respect to time of system of particles kinetic energy of is equal to sum of powers all external and internal forces.

Integrating the equation (3.150) between initial and final configurations of the system, we obtain follow

$$\boxed{T_{fn} - T_{st} = \sum_{k=1}^{n} A(\vec{F}_k^{(e)}) + \sum_{k=1}^{n} A(\vec{F}_k^{i}).} \tag{3.152}$$

Principle of work and kinetic energy for a particle system in integral form: Change of the kinetic energy of particle system during the finite displacements of particles is equal to total work done by all external and internal forces during the same displacements.

Differential form of work-energy principle is used for determination of accelerations, integral form for determination of velocities.

LECTURE 9

3.6.5. Force field

A part of space in which any particle is acted on by force is **force field**. In general the force may be function of coordinates of the particle, its velocity and time

$$\vec{F}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t). \quad (3.153)$$

If the force function explicitly depends on time the force field is called **nonstationary or nonstedy**. Otherwise it is called stationary.

3.6.5.1. Conservative force fields

A force field is called **conservative** if the following two conditions hold:

1. Field force depend on position of a particle only

$$\vec{F}(x, y, z). \quad (3.154)$$

2. The work done by the field force does not depend on the path, but depends only on the position of the end points of the path.

This statement is **the first definition** of a conservative force field.

These two conditions eliminate from the considerations all resistance forces (such as air and water resistance), forces of friction.

Earth's gravitational field is example of conservative force field. Let us consider a body acted on only by gravity G as an active force (i.e. as force that can do work) and moving along frictionless path from position 1 to position 2. The elementary work done by gravity is then

$$d'A(\vec{G}) = -mgdz. \quad (3.155)$$

The total work is

$$A(\vec{G}) = \int_1^2 -mgdz = -mg\left(z_2 - z_1\right) = mgz_1 - mgz_2. \quad (3.156)$$

So the work done by force of gravity does not depend on the path, but depends only on the initial and final position of the particle.

The necessary and sufficient conditions of force field work independence of the path and dependence on the position of the end points of the path only is existing of single valued function of coordinates such as its partial derivatives with respect to coordinates x,y,z are equal to the negative of the force rectangular components

$$F_x = -\frac{dP}{dx};\ F_y = -\frac{dP}{dy};\ F_z = -\frac{dP}{dz}, \qquad (3.157)$$

where $P(x,y,z)$ is the function of coordinates called the potential energy function.

In other words

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_k\vec{k} = -(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}) = -grad\,P = -\nabla P. \qquad (3.158)$$

The operator $grad$ or ∇ that we have introduced is called gradient operator. Conservative force field must be a function of position and expressible as the gradient of a scalar function. The inverse to this statement is also valid. That is, **if a force of a field is a function of position and the gradient of a scalar function, the force must then be a conservative force field**. This statement is **the second definition** of a conservative force field.

The work done by the field force is

$$A_{12} = \int_{M_1}^{M_2} d'A = \int_{M_1}^{M_2}\left(F_x dx + F_y dy + F_z dz\right) =$$

$$= \int_{M_1}^{M_2}\left(-\frac{dP}{dx}dx - \frac{dP}{dy}dy - \frac{dP}{dz}dz\right).$$

The expression in the integral is exact differential, so

$$A_{12} = -\int_{M_1(x_1,y_1,z_1)}^{M_2(x_2,y_2,z_2)} dP = P(x_1,y_1,z_1) - P(x_2,y_2,z_2) = P_1 - P_2 = -\Delta P. \qquad (3.159)$$

Note that the potential energy, $P(x,y,z)$, depends on the reference $Oxyz$ used or the datum used. However, the change in potential energy is independent of the datum used. Changing the position of O but keeping the same direction of xyz axes (changing the datum) does not affect the value of difference of potential energy. Since for work calculation we shall be using the change in potential energy, the datum is arbitrary and is chosen for convenience.

From equation (3.159) we can say that the change in potential energy of a conservative force field is the negative of the work done by this conservative force field on a particle in going from position 1 to position 2 along any path.

3.6.5.2. Physical meaning of the potential energy function at the given point of the space

Let us suppose that at the initial position of a particle the potential energy function equals zero $P_0 = 0$. The work done by force of the field along any path between position 1 and 0 is

$$A_{10} = P_1.$$

So it is evident that the potential energy function at the given point of the space is equal to the work that field force would do to move the particle from the given position to the position where potential energy function equals zero.

From equation (3.159) we can say also that if a particle travels in a closed loop, the net work done by a conservative force is zero.

$$\oint \vec{F} d\vec{r} = 0. \tag{3.160}$$

This is **the third** way to define a conservative force field.

3.6.5.3. Mathematical criterion of force field conservatism

To provide path independence of the work the function in the integral must be exact differential. It is possible if

$$\boldsymbol{rot}\vec{\boldsymbol{F}} = \nabla \times \vec{\boldsymbol{F}} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ \dfrac{\partial}{\partial \boldsymbol{x}} & \dfrac{\partial}{\partial \boldsymbol{y}} & \dfrac{\partial}{\partial \boldsymbol{z}} \\ \boldsymbol{F}_x & \boldsymbol{F}_y & \boldsymbol{F}_z \end{vmatrix} = 0,$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}; \ \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}; \ \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}. \tag{3.161}$$

The first equivalence can be rewritten in such way

$$\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right). \tag{3.162}$$

We know that mixed partial derivatives are independent of the order of differentiation, so we can say that

$$\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial y \partial x}.$$

3.6.5.4. Conservative field equipotential surfaces

It is convenient to visualize a Conservative field with help of equipotential surfaces that are similar to level surfaces at the map.

A locus of points that have equal level of potential energy is called **equipotential surface.** Equation of a equipotential surface is

$$P(x, y, z) = const\,. \qquad (3.163)$$

Features of the equipotential surfaces:

1. There is an equipotential surface at any point of space. Different equipotential surfaces do not have common points. A total work done by the field force during finite movement of a particle along the equipotential surface equals zero.
2. The field force is directed along the equipotential surfaces normal towards the potential energy decreasing.
3. A total work done by the field force during finite movement of a particle from one equipotential surface to another is equal to the difference of potential energy at the initial and at the finite equipotential surfaces.

3.6.5.5. Constant Force field

If the force field is constant in all position, it can always be expressed as gradient of a scalar function of the form

$$P(x, y, z) = ax + by + cz + C\ .$$

Where *a,b,c* are constants. The constant force field, then,

$$\vec{F} = a\vec{i} + b\vec{j} + c\vec{k}.$$

In limited changes of position near the earth's surface, we can consider the gravitational force on a particle of mass m as a constant force field given by $\vec{G} = -mg\vec{k}$. Thus, the constant for the general force field given about are a=b=0 *and* c=-mg. Cleary potential energy of Gravity field

$$P = mgz + C. \qquad (3.164)$$

3.6.5.6. Force proportional to linear displacement

Consider a body limited by constrains to move along a straight line. A force directly proportional to the displacement of the body from some position 0 at x=0 along line is developed. This force is always directed toward point O; it is termed a restoring force (восстанавливающая сила). We can give this force as

$$F_x = -cx,\ F_y = F_z = 0,$$
$$\vec{F} = -cx\vec{i}.$$

where x is the displacement from point 0.

An example of this force is that of linear spring of stiffness *c* with units of $\frac{N}{m}$. The potential energy of this force field is given as follows wherein x is measured from the **undeformed** geometry of the spring

$$P = \frac{cx^2}{2}. \tag{3.165}$$

If the deformation of a spring increases from x_1 to x_2 during the motion interval then the change in the potential energy of the spring is its final value minus its initial value

$$\Delta P = \frac{cx_2^{\,2}}{2} - \frac{cx_1^{\,2}}{2} = \frac{c}{2}\left(x_2^{\,2} - x_1^{\,2}\right). \tag{3.166}$$

We can repeat the same reasoning for spiral spring. Torsion springs obey an angular form of Hooke's law:

$$M_{el\,z} = -c\phi, \tag{3.167}$$

where M_{el} is the moment of elasticity or torque exerted by the spring in $N \cdot m$, and φ is the angle of twist from its equilibrium position in radians. *c* is a constant with units of $\frac{N \cdot m}{rad}$, variously called the spring's **torsion coefficient**, **torsion elastic modulus** or **spring constant**, equal to the torque required to twist the spring through an angle of 1 radian. It is analogous to the linear spring stiffness.

The potential energy of this force field is given as

$$P = \frac{c\phi^2}{2}, \tag{3.168}$$

$$\Delta P = \frac{c}{2}\left(\phi_2^{\,2} - \phi_1^{\,2}\right). \tag{3.169}$$

3.6.6. Law of mechanical energy conservation

Let us consider a motion of a particle system which is acted on conservative forces only. The change of kinetic energy of system is equal to work of the conservative forces and therefore is equal to the change in potential energy of the system

$$T_{fin} - T_{in} = P_{in} - P_{fin} \; , \qquad (3.170)$$

or

$$T_{fin} + P_{fin} = P_{in} + T_{in} \; ,$$

or

$$\boxed{E = P + T = const.} \qquad (3.171)$$

The sum of kinetic energy and potential energy for a system remains constant at all time during the motion of the system in conservative force field.

3.6.7. Examples

Example 1. A mechanical system (Fig. 3.73) initially at rest comes into operation by gravity. The initial position of the system is represented in the figure. Take into consideration dry friction and rolling resistance for the body 3 motion. Neglect other resistance forces. The body 3 rolls without slipping, cords are ideally flexible inextensible weightless.

Determine the velocity of the body 1 when the distance traveled is $s_1 = 2.4\,m$, if $m_1 = m, m_2 = 1/4\,m, m_3 = 1/8\,m; R_3 = 35\,cm = 0.35\,m;$ $\alpha = 30°, \beta = 30°,\; f = 0.2, \delta = 0.2\,cm$.

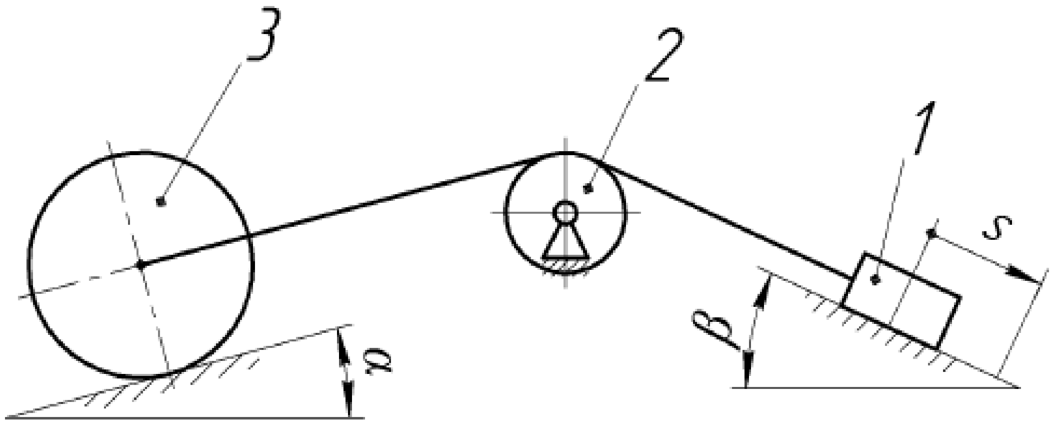


Fig. 3.73

Solution

This mechanical system consists of the block 1 that has translation motion, the pulley 2 that rotates about fixed axis and the cylinder 3 that has plane motion.

To solve this problem we apply work-energy principle:

$$T_f - T_i = \sum_{k=1}^{n} A\left(\overline{F_k^{(e)}}\right) + \sum_{k=1}^{n} A\left(\overline{F_k^{(i)}}\right),$$

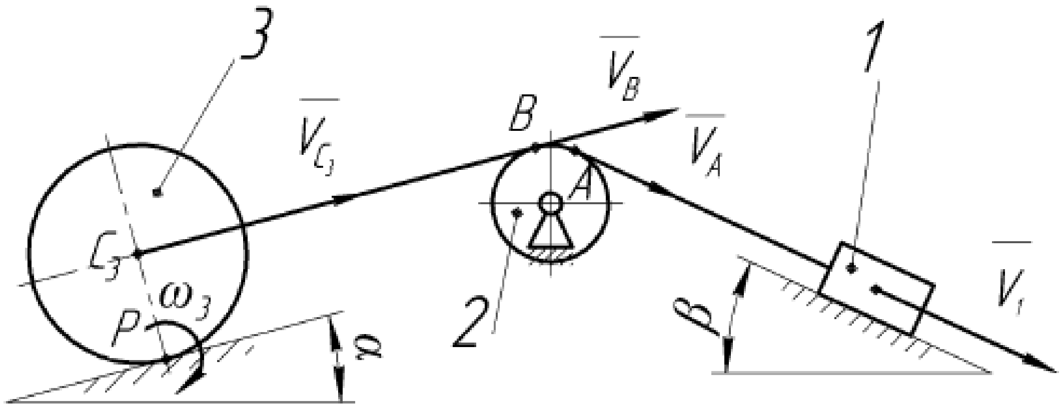


Fig. 3.74

where T_f is total final kinetic energy of the system;

T_i is total initial kinetic energy of the system;

$\sum_{k=1}^{n} A\left(\overline{F_k^{(e)}}\right)$ is total work done by external forces during the finite movement of the system;

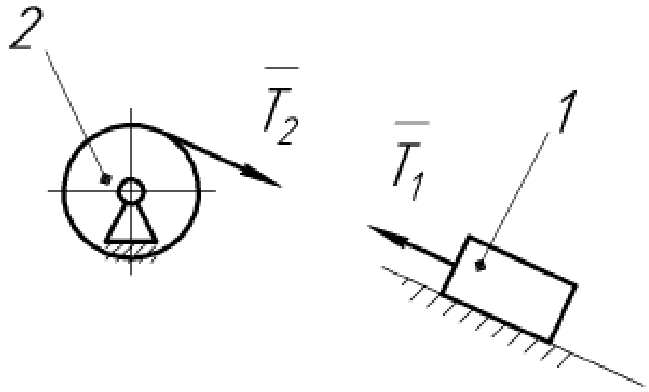


Fig. 3.75

$\sum_{k=1}^{n} A\left(\overline{F_k^{(i)}}\right)$ is total work done by internal forces during the finite movement of the system.

For our system of rigid bodies joined by ideal flexible inextensible cords no work is done by internal forces. If we break the cord we obtain to tension forces (Fig. 3.75) that are equal by a magnitude and have opposite direction: $\overline{T_1} = -\overline{T_2}$. Separately these forces have work, but as they have different directions, the total work is zero.

So

$$\sum_{k=1}^{n} A\left(\overline{F_k^{(i)}}\right) = 0.$$

The system is at rest at initial moment of time, that's why kinetic energy at initial moment is zero:

$$T_i = 0.$$

Finally work-energy principle for this problem is:

$$T_f = \sum_{k=1}^{n} A\left(\overline{F_k^{(e)}}\right). \qquad (3.172)$$

The total kinetic energy is the sum of kinetic energies of the three bodies:

$$T_f = T_1 + T_2 + T_3. \qquad (3.173)$$

For the 1st body kinetic energy is:

$$T_1 = \frac{1}{2} m_1 V_1^2. \qquad (3.174)$$

Taking into consideration that the 2nd body has rotational motion, so:

$$T_2 = \frac{1}{2} I_2 \omega_2^2. \qquad (3.175)$$

The 3rd body has plane motion. Then kinetic energy is:

$$T_3 = \frac{1}{2} m_3 V_{C_3}^2 + \frac{1}{2} I_3 \omega_3^2. \qquad (3.176)$$

We have to express all velocities of the points and angular velocities in terms of the velocity of the 1st body. Bodies are connected by ideal cord, so:

$$V_{C3} = V_B = V_A = V_1,$$

$$V_A = \omega_2 R_2 \;\Rightarrow\; \omega_2 = \frac{V_A}{R_2} = \frac{V_1}{R_2},$$

$$V_{C3} = \omega_3 R_3 \;\Rightarrow\; \omega_3 = \frac{V_{C3}}{R_3} = \frac{V_1}{R_3}.$$

Moments of inertia of the 2nd and the 3rd bodies:

$$I_2 = \frac{1}{2} m_2 R_2^2,\; I_3 = \frac{1}{2} m_3 R_3^2.$$

Substituting expressions in equations (3.175) and (3.176) we can rewrite them:

$$T_2 = \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} \cdot \frac{1}{2} m_2 R_2^2 \left(\frac{V_1}{R_2} \right)^2 = \frac{1}{4} m_2 V_1^2, \tag{3.177}$$

$$T_3 = \frac{1}{2} m_3 V_1^2 + \frac{1}{2} \cdot \frac{1}{2} m_3 R_3^2 \left(\frac{V_1}{R_3} \right)^2 = \frac{3}{4} m_3 V_1^2. \tag{3.178}$$

Substituting equations (3.174), (3.177) and (3.178) in (3.173) we obtain the total kinetic energy system:

$$T_f = \frac{1}{2} m_1 V_1^2 + \frac{1}{4} m_2 V_1^2 + \frac{3}{4} m_3 V_1^2 = \frac{21}{32} m V_1^2. \tag{3.179}$$

Now we shall analyze forces (Fig. 3.76), acting on the system. The work done by external forces is:

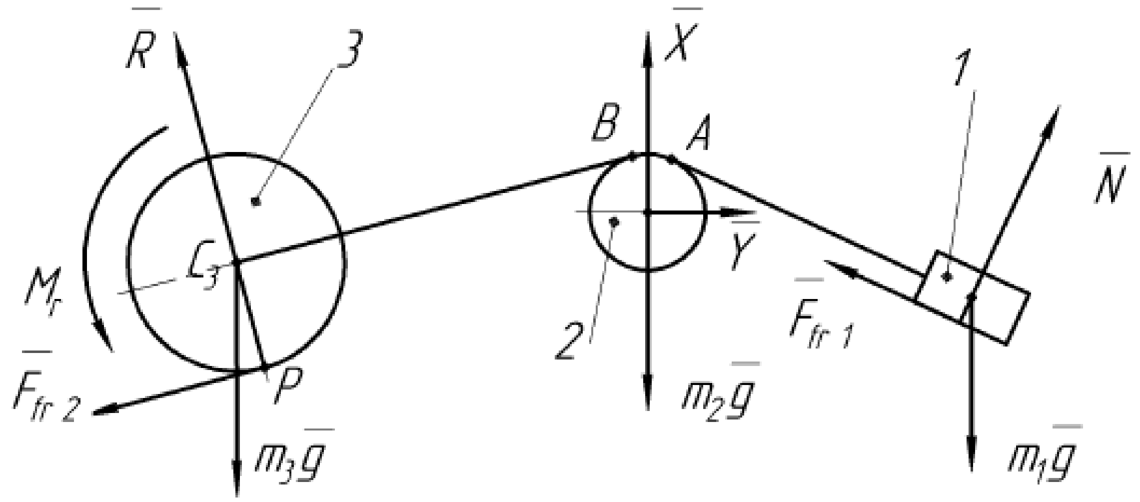


Fig.3.76

$$\sum_{k=1}^{n} A\left(\overline{F_k^e}\right) = A\left(\overline{N}\right) + A\left(\overline{R}\right) + A\left(\overline{X}\right) + A\left(\overline{Y}\right) + A\left(m_1 \overline{g}\right) + A\left(m_2 \overline{g}\right) +$$
$$+ A\left(m_3 \overline{g}\right) + A\left(\overline{F_{fr1}}\right) + A\left(\overline{F_{fr2}}\right) + A\left(M_r\right).$$

The works done by the reactive forces $\overline{X}, \overline{Y}$, gravity force of the 2nd body $m_2 \overline{g}$ are zero, because they are applied in fixed point, and work done by the reactive force $\overline{R}$ and the friction force $\overline{F_{fr2}}$ are zero too because they are applied at ICZV.

The work done by the gravity (equation(3.125)) force of the first body is positive, because the body moves down (Fig. 3.77):

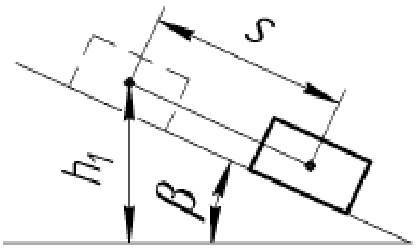


Fig. 3.77

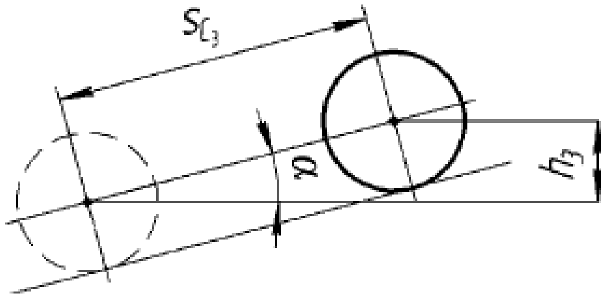


Fig. 3.78

$$A\left(m_1\overline{g}\right) = m_1 g h_1 = m_1 g s_1 \sin\beta.$$

The height h1 is found according to the figure:

$$h_1 = s_1 \sin\beta\,.$$

The work done by the gravity force of the 3rd body is negative, because the body moves up:

$$A\left(m_3\overline{g}\right) = -m_3 g h_{C_3}\,.$$

The height h_{C_3} is found according to the Fig. 3.78:

$$h_{C_3} = s_{C_3} \sin\alpha\,.$$

So

$$A\left(m_3\overline{g}\right) = -m_3 g h_{C_3} = -m_3 g s_{C_3} \sin\alpha\,,$$

where s_{C3} is the displacement of the mass center the 3rd body.

We know the relation between velocity of the first body and velocity of the mass center of the third body:

$$V_{C_3} = V_1.$$

A velocity is a derivative of the displacement by the time, so we can rewrite this equation:

$$\frac{ds_{C_3}}{dt} = \frac{ds_1}{dt}\,,$$

$$\int_0^{s_{C_3}} ds_{C_3} = \int_0^{s} ds_1,$$

$$s_{C_3} = s_1.$$

So

$$A\left(m_3\overline{g}\right) = -m_3 g s_1 \sin\alpha .$$

The work done by the reactive force $\overline{N}$ (Fig. 3.79) is zero because an angle between the force vector and displacement of a point of application is 90°:

$$A\left(\overline{N}\right) = N s_1 \cos 90° = 0.$$

The work done by the friction force between the first body and the incline:

$$A\left(\overline{F_{fr1}}\right) = F_{fr1} \cdot s_1 \cdot \cos 180° = -F_{fr1} \cdot s_1 .$$

Friction force can be found according to Column law:

$$F_{fr1} = fN ,$$

where f is friction coefficient, N is reaction.

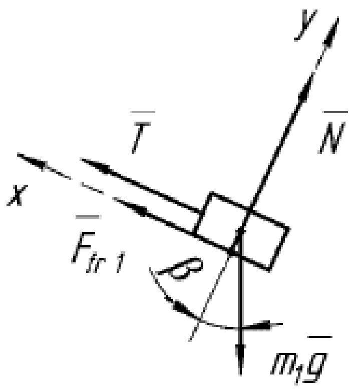


Fig. 3.79

To find the reaction $\overline{N}$ we consider the first body separately and write equation of motion for this body:

$$m_1 \ddot{y}_C = N - m_1 g \cos\beta .$$

As equation of constraint we use the fact that the coordinate

$$y_C = const,$$

so

$$\ddot{y}_C = 0.$$

Then

$$0 = N - m_1 g \cos\beta \Rightarrow N = m_1 g \cos\beta,$$

$$F_{fr1} = f m_1 g \cos\beta.$$

Therefore

$$A\left(\overline{F_{fr1}}\right) = -f m_1 g \cos\beta \cdot s.$$

The work done by the moment of rolling resistance is negative because the moment and the angle have different directions:

$$A(M_r) = -M_r \varphi_3.$$

To find the unknown angle φ_3 we use the relation

$$\omega_3 = \frac{V_1}{R_3},$$

$$\frac{d\varphi_3}{dt} = \frac{ds_1}{dt} \Big/ R_3,$$

$$\int_0^{\varphi_3} d\varphi_3 = \frac{1}{R_3}\int_0^{s_1} ds_1,$$

$$\varphi_3 = \frac{s_1}{R_3}.$$

The moment of rolling resistance (Fig. 3.80) is a product of coefficient of rolling resistance and reactive force:

$$M_r = \delta \cdot R.$$

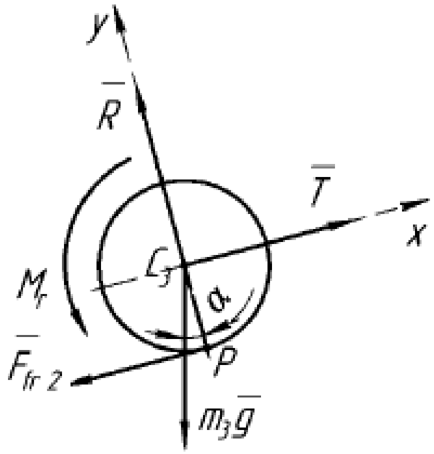


Fig. 3.80

To find the reactive force we consider the 3rd body separately and write one equation of motion for this body:

$$m_3 \ddot{y}_C = R - m_3 g \cos\alpha .$$

Taking into account that the coordinate of the mass center of the third body

$$y_C = const$$

and that's why

$$\ddot{y}_C = 0,$$

we can rewrite the previous equation:

$$0 = R - m_3 g \cos\alpha,$$
$$R = m_3 g \cos\alpha.$$

Therefore the moment of rolling resistance is:

$$M_r = \delta \cdot m_3 g \cos\alpha ,$$

$$A(M_r) = -\delta \cdot m_3 g \cos\alpha \frac{s_1}{R_3}.$$

So the sum of external forces is:

$$\sum_{k=1}^{n} A\left(\overline{F_k^e}\right) = m_1 g s_1 \sin\beta - m_3 g s_1 \sin\alpha - f m_1 g \cos\beta \cdot s - \delta \cdot m_3 g \cos\alpha \frac{s_1}{R_3}.$$

Making some transformations we have:

$$\sum_{k=1}^{n} A\left(\overline{F_k^e}\right) = 2.87 s_1 m. \qquad (3.180)$$

Substituting equations (3.180) and (3.179) into (3.172):

$$\frac{21}{32} m V_1^2 = 2.87 s_1 m.$$

From here the desired velocity of the 1st body is:

$$V_1 = \sqrt{4.37 s_1} = \sqrt{4.37 \cdot 2.4} = 3.24 \left(\frac{m}{\sec}\right).$$

Answer: $V_1 = 3.24 \left(\frac{m}{\sec}\right)$.

Example 2. A gear mechanism is represented in Fig. 3.81. The carrier OC_4 is rigidly connected with the wheel 2. Determine the velocity of the body 1 when the distance traveled is $s_1 = 0.05\pi\, m$, if $m_1 = m, m_2 = 1/10\, m,$ $m_3 = 1/20\, m,\ m_4 = 1/10\, m, R_2 = 0.1\, m, R_3 = 0.12\, m, OC_4 = 6R_3, R_4 = 2R_3.$

Solution

The system consists of 4 bodies.

The 1st body is the load that moves translationally with the velocity V_1. The 2nd wheel rotates about fixed axis passing through its mass center. The 3rd wheel and the 4th have plane motion. Also the planetary carrier belongs to the system, but we neglect its mass, so it doesn't have kinetic energy.

To solve this problem we apply work-energy principle:

$$T_f - T_i = \sum_{k=1}^{n} A\left(\overline{F_k^e}\right) + \sum_{k=1}^{n} A\left(\overline{F_k^i}\right),$$

where T_f is total final kinetic energy of the system; T_i is total initial kinetic energy of the system; $\sum_{k=1}^{n} A\left(\overline{F_k^e}\right)$ is total work done by external forces during the finite movement of the system; $\sum_{k=1}^{n} A\left(\overline{F_k^i}\right)$ is total work done by internal forces during the finite movement of the system.

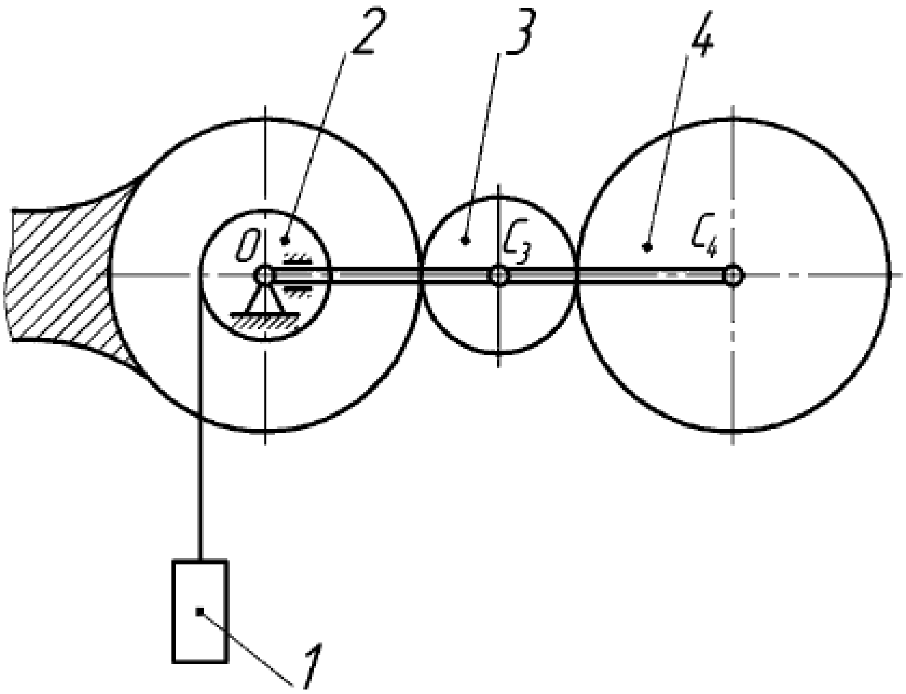


Fig. 3.81

Work done by internal forces is zero, because pins are frictionless, motions of wheels are without slipping and rolling resistance is neglected.

The system is at rest at initial moment of time, that's why kinetic energy at initial moment is zero: $T_i = 0$.

Finally work-energy principle for this problem is:

$$T_f = \sum_{k=1}^{n} A\left(\overline{F_k^e}\right). \tag{3.181}$$

The total kinetic energy is the sum of kinetic energies of the four bodies:

$$T_f = T_1 + T_2 + T_3 + T_4. \tag{3.182}$$

The kinetic energy of the body 1 is:

$$T_1 = \frac{1}{2} m_1 V_1^2. \tag{3.183}$$

Taking into consideration that the body 2 has rotational motion, so:

$$T_2 = \frac{1}{2} I_2 \omega_2^2. \tag{3.184}$$

The 3rd body has plane motion. Then kinetic energy is:

$$T_3 = \frac{1}{2} m_3 V_{C_3}^2 + \frac{1}{2} I_3 \omega_3^2 . \tag{3.185}$$

For the body 4:

$$T_4 = \frac{1}{2} m_4 V_{C_4}^2 + \frac{1}{2} I_4 \omega_4^2 . \tag{3.186}$$

We have to express all velocities of the points and angular velocities in terms of the velocity of the 1st body. All points on the cord have the same velocities, so velocity of the point A is equal to velocity of the first body:

$$V_A = V_1.$$

On the other hand:

$$V_A = \omega_2 R_2 \;\Rightarrow\; \omega_2 = \frac{V_A}{R_2} = \frac{V_1}{R_2}.$$

As the carrier is rigidly attached to the 2nd wheel, angular velocities of these two bodies are equal:

$$\omega_C = \omega_2 = \frac{V_1}{R_2}.$$

Velocities of the points C_3 and C_4 (Fig. 3.82) as points on the carrier are:

$$V_{C_3} = \omega_C OC_3 = \frac{V_1}{R_2} \cdot 3R_3, \; V_{C_4} = \omega_C OC_4 = \frac{V_1}{R_2} \cdot 6R_3.$$

The 3rd wheel has a contact with the fixed wheel at the point P, so there is ICZV at this point and we can write the formula for velocity of the point C_3:

$$V_{C_3} = \omega_3 R_3 \;\Rightarrow\; \omega_3 = \frac{V_{C_3}}{R_3} = \frac{V_1}{R_2} \cdot 3R_3 \Big/ R_3 = \frac{3V_1}{R_2}.$$

Velocity of the point E as a point on the 3rd wheel is:

$$V_E = \omega_3 \cdot 2R_3 = \frac{3V_1}{R_2} \cdot 2R_3 = \frac{6R_3}{R_2} \cdot V_1.$$

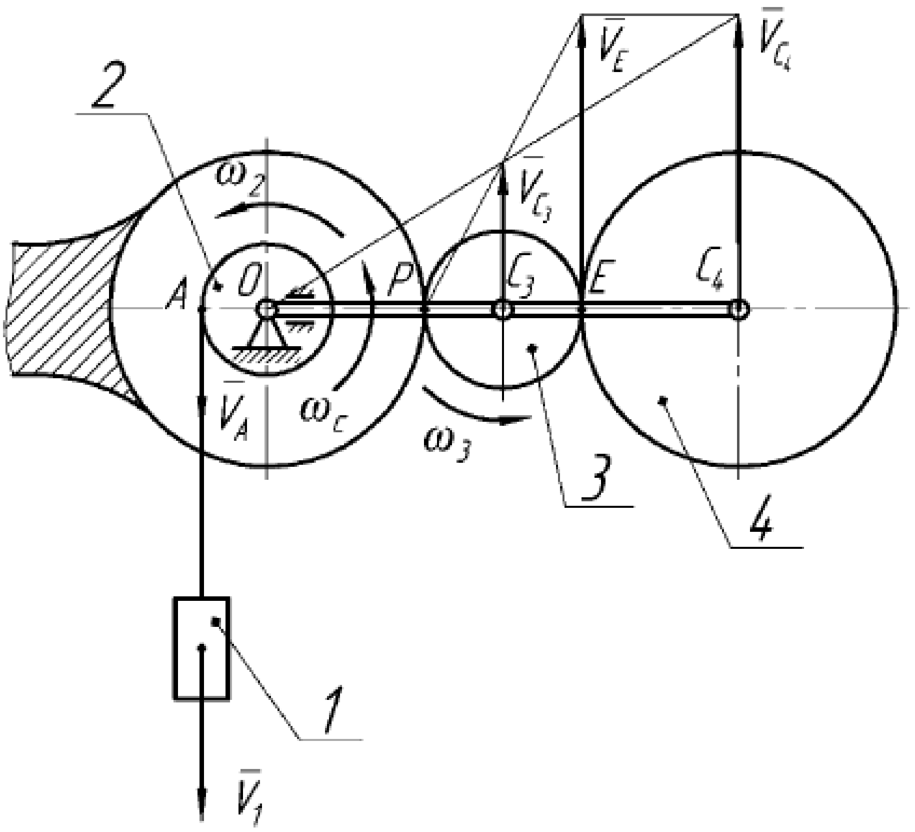


Fig. 3.82

It is easy to see that velocities of the points E and C_4 are equal. So we can make a conclusion that the 4^{th} wheel has instantaneous translational motion. That's why angular velocity of this wheel is zero: $\omega_4 = 0$.

Moments of inertia of the 2^{nd} and the 3^{rd} bodies:

$$I_2 = \frac{1}{2} m_2 R_2^2, \; I_3 = \frac{1}{2} m_3 R_3^2.$$

Substituting all these expressions in equations (4), (5) and (6) we can rewrite them:

$$T_2 = \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} \cdot \frac{1}{2} m_2 R_2^2 \left(\frac{V_1}{R_2} \right)^2 = \frac{1}{4} m_2 V_1^2; \tag{3.187}$$

$$T_3 = \frac{1}{2} m_3 V_{C_3}^2 + \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} m_3 \left(\frac{3R_3}{R_2} \cdot V_1 \right)^2 + \frac{1}{2} \cdot \frac{1}{2} m_3 R_3^2 \left(\frac{3V_1}{R_2} \right)^2 = \frac{27 m_3 R_3^2}{4 R_2^2} V_1^2; \tag{3.188}$$

$$T_4 = \frac{1}{2} m_4 V_{C_4}^2 + \frac{1}{2} I_4 \omega_4^2 = \frac{1}{2} m_4 \left(\frac{6R_3}{R_2} \cdot V_1 \right)^2. \tag{3.189}$$

Substituting equations (3.183), (3.187), (3.188) and (3.189) in (3.182) we obtain the total kinetic energy system:

$$T_f = \frac{1}{2}m_1V_1^2 + \frac{1}{4}m_2V_1^2 + \frac{27m_3R_3^2}{4R_2^2}V_1^2 + \frac{1}{2}m_4\left(\frac{6R_3}{R_2}\cdot V_1\right)^2 = 3.603mV_1^2. \quad (3.190)$$

Final position of the system depends on the position of the carrier. We know that at final moment of time the displacement of the body 1 is $s_1 = 0.05\pi\,(m)$. Angular velocity of the carrier is:

$$\omega_C = \frac{V_1}{R_2}.$$

Rewriting this equation:

$$\frac{d\varphi_C}{dt} = \frac{ds_1}{dt}\Big/R_2,$$

$$\int_0^{\varphi_C} d\varphi_C = \frac{1}{R_2}\int_0^{s_1} ds_1,$$

$$\varphi_C = \frac{s_1}{R_2}.$$

At final moment angle of rotation of carrier is:

$$\varphi_C\Big|_{s=0.05\pi} = \frac{0.05\pi}{0.1} = \frac{\pi}{2}.$$

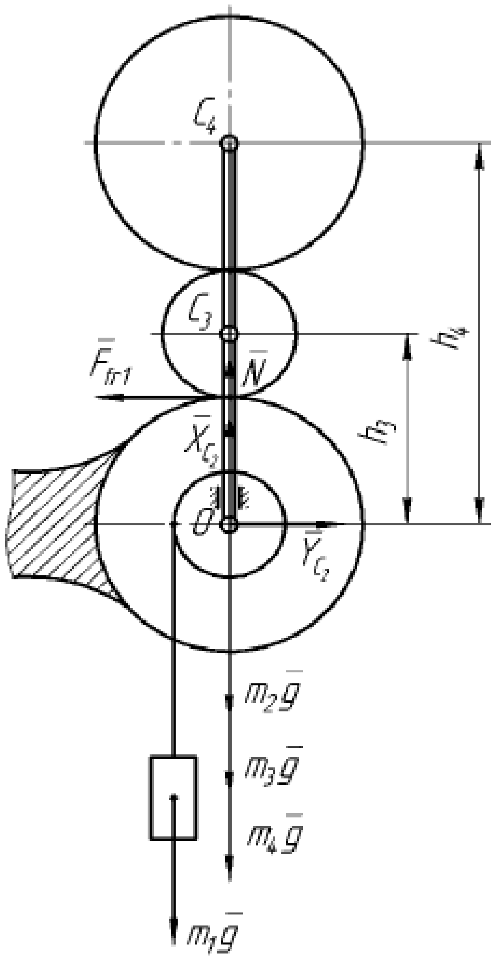


Fig. 3.83

So the carrier rotates on $90°$. Final position of the system is shown in Fig. 3.83.

Let's calculate the work of all external forces acting on the system: gravity forces $m_1\overline{g}, m_2\overline{g}, m_3\overline{g}, m_4\overline{g}$, normal reaction $\overline{N}$, reactive forces $\overline{X_{C_2}}, \overline{Y_{C_2}}$ and friction force $\overline{F_{fr}}$. So

$$\sum_{k=1}^{n} A\left(\overline{F_k^e}\right) = A\left(\overline{N}\right) + A\left(\overline{X_{C_2}}\right) + A\left(\overline{Y_{C_2}}\right) + A\left(m_1\overline{g}\right) + A\left(m_2\overline{g}\right) +$$

$$+A\left(m_3\overline{g}\right) + A\left(m_4\overline{g}\right) + A\left(\overline{F_{fr1}}\right). \quad (3.191)$$

As gravity force of the wheel 2 and reactive forces are applied at fixed point C_2 the works done by these forces are zero:

$$A\left(\overline{X_{C_2}}\right)=A\left(\overline{Y_{C_2}}\right)=A\left(m_2\overline{g}\right)=0.$$

The friction force and normal reaction are applied in ICZV, so the works done by these forces are zero too:

$$A\left(\overline{N}\right)=A\left(\overline{F_{fr1}}\right)=0.$$

Works done by gravity forces are:

$$A\left(m_1\overline{g}\right)=m_1gh_1=m_1gs_1,$$
$$A\left(m_3\overline{g}\right)=-m_3gh_3=-m_3g\cdot 3R_3,$$
$$A\left(m_4\overline{g}\right)=-m_4gh_4=-m_4g\cdot 6R_3.$$

So we can rewrite equation (3.191):

$$\sum_{k=1}^{n}A\left(\overline{F_k^e}\right)=m_1gs_1-m_3g\cdot 3R_3-m_4g\cdot 6R_3=0.067mg. \qquad (3.192)$$

Substituting equations (3.190) and (3.192) in (3.181):

$$3.603mV_1^2=0.067mg.$$

From here velocity of the 1st body is:

$$V_1=\sqrt{\frac{0.067g}{3.603}}=0.43\left(\frac{m}{\sec}\right).$$

Example 3. Make the differential equation of the system (Fig. 3.84) motion, if masses are given: $m_1=m, m_2=m$; the length of unstretched spring is

l_0,coefficient of stiffness is k. The block 2 is a homogeneous disk. Neglect masses of the cord and the spring.

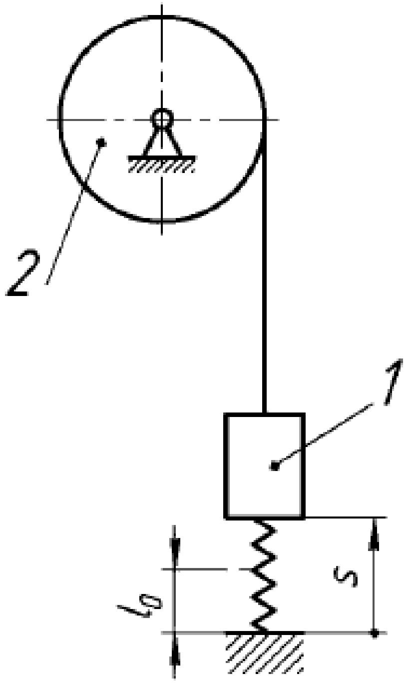


Fig. 3.84

Solution

We assume that the 1st body moves up, so V_1 is directed up and we introduce displacement *s*. to solve this problem we use work-energy principle in differential form:

$$\frac{dT}{dt} = \sum_{k=1}^{n} N\left(\overline{F_k^e}\right) + \sum_{k=1}^{n} N\left(\overline{F_k^i}\right).. \qquad (3.193)$$

Determine kinetic energy of the system:

$$T = T_1 + T_2. \qquad (3.194)$$

The 1st body moves translationally with the velocity V_1, so its kinetic energy is:

$$T_1 = \frac{1}{2} m_1 V_1^2.$$

The 2nd body has rotational motion:

$$T_2 = \frac{1}{2} I_2 \omega_2^2.$$

Velocity of the point A is equal to velocity of the 1st body (Fig. 3.85):

$$V_A = V_1.$$

On the other hand:

$$V_A = \omega_2 R_2 \;\Rightarrow\; \omega_2 = \frac{V_A}{R_2} = \frac{V_1}{R_2}.$$

Moment of inertia of the 2nd body:

$$I_2 = \frac{1}{2} m_2 R_2^2.$$

So kinetic energy of the 2nd body is:

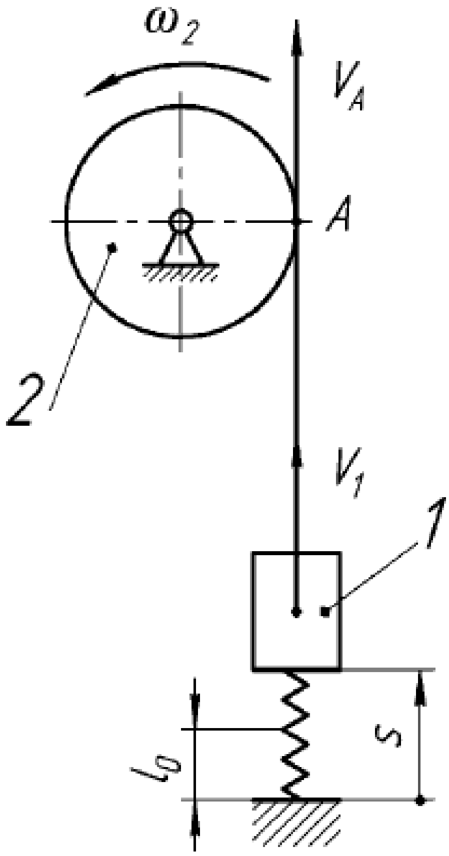


Fig. 3.85

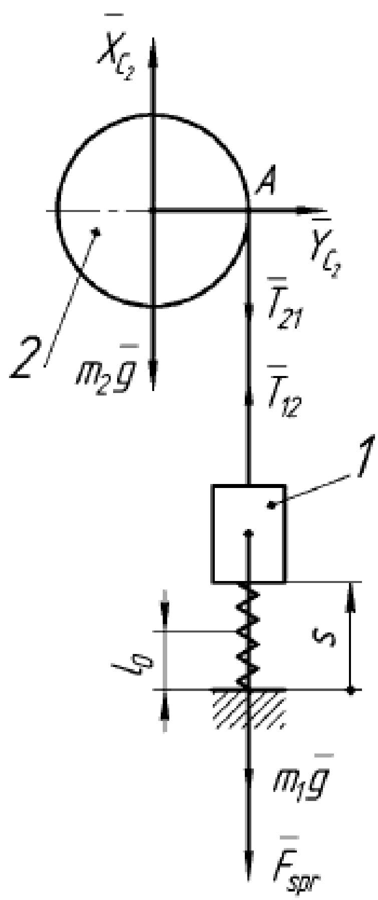


Fig. 3.86

$$T_2 = \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} \cdot \frac{1}{2} m_2 R_2^2 \left(\frac{V_1}{R_2} \right)^2 = \frac{1}{4} m_2 V_1^2 .$$

Therefore the total kinetic energy of the system:

$$T = \frac{1}{2} m_1 V_1^2 + \frac{1}{4} m_2 V_1^2 = \frac{3}{4} m V_1^2 .$$

Let's find the derivative of this expression:

$$\frac{dT}{dt} = \frac{3}{4} m \frac{d}{dt}\left(V_1^2\right) = \frac{3}{4} m V_1 \frac{dV_1}{dt} = \frac{3}{2} m V_1 \cdot W_{1s}, \qquad (3.195)$$

where W_{1s} is a projection of the load 1 acceleration on axis s.

The sum of powers of internal forces acting on the system is zero

$$\sum_{k=1}^{n} N\left(\overline{F_k^i}\right) = 0. \qquad (3.196)$$

Let’s find the sum of powers of external forces acting on the system (see Fig. 3.86). As reactive forces $\overline{X_{C_2}}, \overline{Y_{C_2}}$ and gravity force of the body 2 $m_2\overline{g}$ are applied at fixed point, their powers are zero:

$$N\left(\overline{X_{C_2}}\right) = N\left(\overline{Y_{C_2}}\right) = N\left(m_2\overline{g}\right) = 0\,.$$

Power of the gravity force of the 1st body is:

$$N\left(m_1\overline{g}\right) = m_1 g V_1 \cos 180^\circ = -mgV_1.$$

Power of the force of spring is:

$$N\left(\overline{F_{spr}}\right) = F_{spr}V_1 \cos 180^\circ = -F_{spr}V_1 = -k\left(s - l_0\right)V_1\,.$$

Therefore the total power of all external forces is:

$$\sum_{k=1}^{n} N\left(\overline{F_k^e}\right) = -mgV_1 - k\left(s - l_0\right)V_1. \qquad (3.197)$$

Substituting equations (3.195), (3.196) and (3.197) in (3.193):

$$\frac{3}{2}mV_1 \cdot W_{1s} = -mgV_1 - k\left(s - l_0\right)V_1.$$

Dividing on $mV_1 \neq 0$ we obtain:

$$\frac{3}{2} \cdot W_{1s} = -g - \frac{k}{m}\left(s - l_0\right).$$

Taking into account that at rectilinear motion of the load

$$W_{1s} = \ddot{s},$$

$$\ddot{s} = -\frac{2}{3}\left(g + \frac{k}{m}\left(s - l_0\right)\right). \qquad (3.198)$$

Equation (3.198) is differential equation of the system motion.

Example 4. The load A of mass M_1 moving downwards with the use of a cord, that is threw over the pulley D, lifts up the load B of a mass M_2, that is attached to the axis of the movable pulley C (Fig. 3.87). The pulleys D and C are homogeneous disks. A mass of every disk is M_3. Determine the velocity of the load A at a moment, when it moves down on a height h. Neglect a mass of the cord, slipping on the pulleys and forces of resistance. At initial moment the system was at rest.

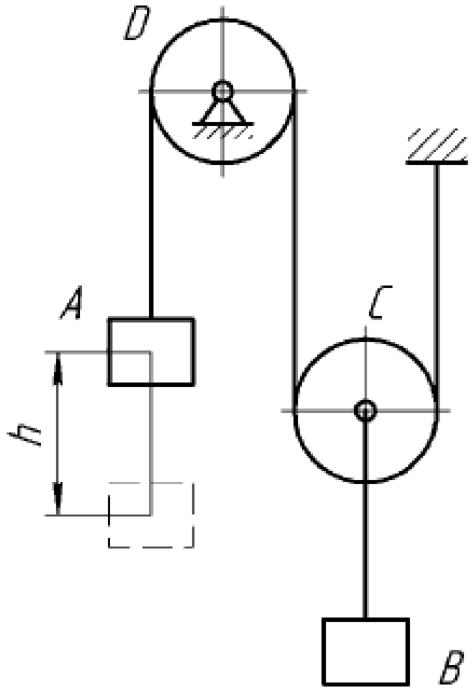


Fig. 3.87

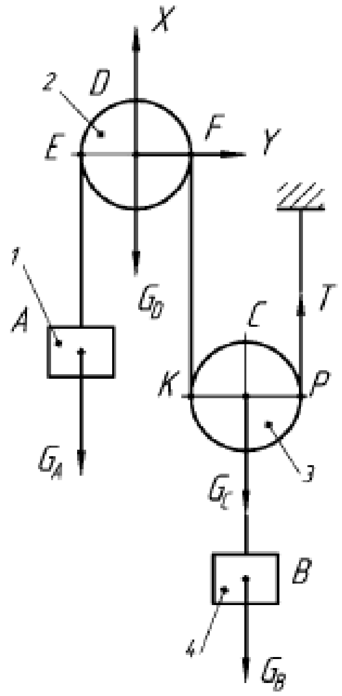


Fig. 3.88

Solution

We consider forces, acting on the system (Fig.3.88). Reactions $\overline{X}, \overline{Y}$ and tension force $\overline{T}$ are nonworking, because reactions are applied at fixed point and $\overline{T}$ is applied in ICZV (point P). Gravity force is conservative force. So we can say that this system is under action of conservative forces and we can apply the law of conservation of mechanical energy:

$$T_i + P_i = T_f + P_f, \tag{3.199}$$

where T_i is kinetic energy at initial moment of time; T_f is kinetic energy at final moment of time; P_i is potential energy at initial moment of time; P_f is potential energy at final moment of time.

The system is at rest, so kinetic energy is zero:

$$T_i = 0.$$

At defining a potential energy of gravity force we assume, that at initial moment of time it is equal to zero:

$$P_i = 0.$$

Determine kinetic energy of the system:

$$T = T_1 + T_2 + T_3 + T_4. \tag{3.200}$$

The body 1 has translational motion, so

$$T_1 = \frac{1}{2} M_1 v_1^2. \tag{3.201}$$

For the body 2 kinetic energy is:

$$T_2 = \frac{1}{2} I_2 \omega_2^2. \tag{3.202}$$

The body 3 has plane motion:

$$T_3 = \frac{1}{2} M_3 v_{C_3}^2 + \frac{1}{2} I_3 \omega_3^2. \tag{3.203}$$

Taking into account translational motion of the body 4:

$$T_4 = \frac{1}{2} M_2 v_4^2. \tag{3.204}$$

The cord is inextensible so every point on it has the same velocity, so

$$v_E = v_1.$$

Angular velocity of the 2nd wheel is:

$$\omega_2 = \frac{v_E}{R_2} = \frac{v_1}{R_2}.$$

Velocity of the point K on the body 3 is the same as velocities of the points F and E:

$$v_K = v_F = v_E = v_1.$$

On the other hand the velocity of the point K is:

$$v_K = \omega_3 \cdot 2R_3 \Rightarrow \omega_3 = \frac{v_1}{2R_3}.$$

Velocity of the center of the disk 3 is:

$$v_{C_3} = \omega_3 R_3 = \frac{v_1}{2R_3} R_3 = \frac{1}{2} v_1.$$

Velocity of the body4 is equal to the velocity of the center of the disk 3:

$$v_4 = v_{C_3} = \frac{1}{2} v_1.$$

Moments of inertia of the disks 2 and 3 are:

$$I_2 = \frac{1}{2} M_3 R_2^2,$$

$$I_3 = \frac{1}{2} M_3 R_3^2.$$

Substituting received expressions to equations (3.202), (3.203), (3.204):

$$T_2 = \frac{1}{2} \cdot \frac{1}{2} M_3 R_2^2 \left(\frac{v_1}{R_2} \right)^2 = \frac{1}{4} M_3 v_1^2;$$

$$T_3 = \frac{1}{2} M_3 \left(\frac{1}{2} v_1 \right)^2 + \frac{1}{2} \cdot \frac{1}{2} M_3 R_3^2 \left(\frac{v_1}{2R_3} \right)^2 = \frac{3}{16} M_3 v_1^2;$$

$$T_4 = \frac{1}{2} M_2 \left(\frac{1}{2} v_1 \right)^2 = \frac{1}{8} M_2 v_1^2.$$

Therefore the total kinetic energy is:

$$T = \frac{1}{2} M_1 v_1^2 + \frac{1}{4} M_3 v_1^2 + \frac{3}{16} M_3 v_1^2 + \frac{1}{8} M_2 v_1^2 = \left(\frac{1}{2} M_1 + \frac{1}{8} M_2 + \frac{7}{16} M_3 \right) v_1^2. \quad (3.205)$$

The potential energy of the system is:

$$P = P\left(\overline{G_A}\right) + P\left(\overline{G_D}\right) + P\left(\overline{G_C}\right) + P\left(\overline{G_B}\right).$$

The potential energy of the force $\overline{G_D}$ is zero, because this force is applied at fixed point:

$$P\left(\overline{G_D}\right) = 0.$$

The potential energy is a work done by the force during displacement from current position to initial, where energy is equal to zero. So

$$P\left(\overline{G_A}\right) = -G_A h_1 = -M_1 g h,$$

$$P\left(\overline{G_C}\right) = G_C h_{C_3} = M_3 g\left(\frac{1}{2}h\right),$$

$$P\left(\overline{G_B}\right) = G_B h_4 = M_2 g\left(\frac{1}{2}h\right).$$

So the total potential energy is:

$$P = -M_1 g h + M_3 g\left(\frac{1}{2}h\right) + M_2 g\left(\frac{1}{2}h\right) = gh\left(-M_1 + \frac{1}{2}M_2 + \frac{1}{2}M_3\right). \quad (3.206)$$

Substituting expressions (3.205) and (3.206) to equation (3.199):

$$\left(\frac{1}{2}M_1 + \frac{1}{8}M_2 + \frac{7}{16}M_3\right)v_1^2 = gh\left(-M_1 + \frac{1}{2}M_2 + \frac{1}{2}M_3\right).$$

From here the velocity of the load A is:

$$v_1^2 = \frac{16gh\left(2M_1 - M_2 - M_3\right)}{2\left(8M_1 + 2M_2 + 7M_3\right)}.$$

Answer: $v_1^2 = \dfrac{16gh\left(2M_1 - M_2 - M_3\right)}{2\left(8M_1 + 2M_2 + 7M_3\right)}$.

Example 5. Fig. 3.89 represents an eccentric mechanism lying in a horizontal plane. The eccentric A sets the roller B and the rod D in a reciprocating motion. A spring E, that is connected with the rod, provides a constant contact between the roller and the eccentric. The weight of the eccentric is **p** and the eccentricity **e** equals half of its radius. The coefficient of stiffness of the spring is **c**. At the extreme left position of the rod the spring is not-compressed. What angular velocity is required for the eccentric to move the rod D from the extreme left to the extreme right position? Neglect the masses of the roller, the rod and the spring. The eccentric is assumed to be a homogeneous disk.

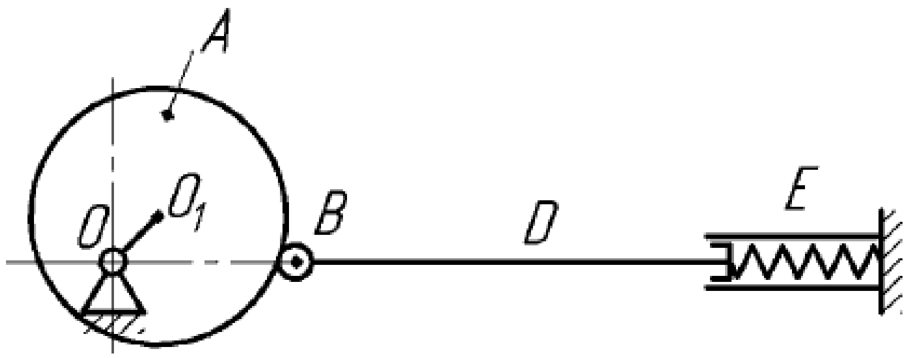


Fig. 3.89

Solution

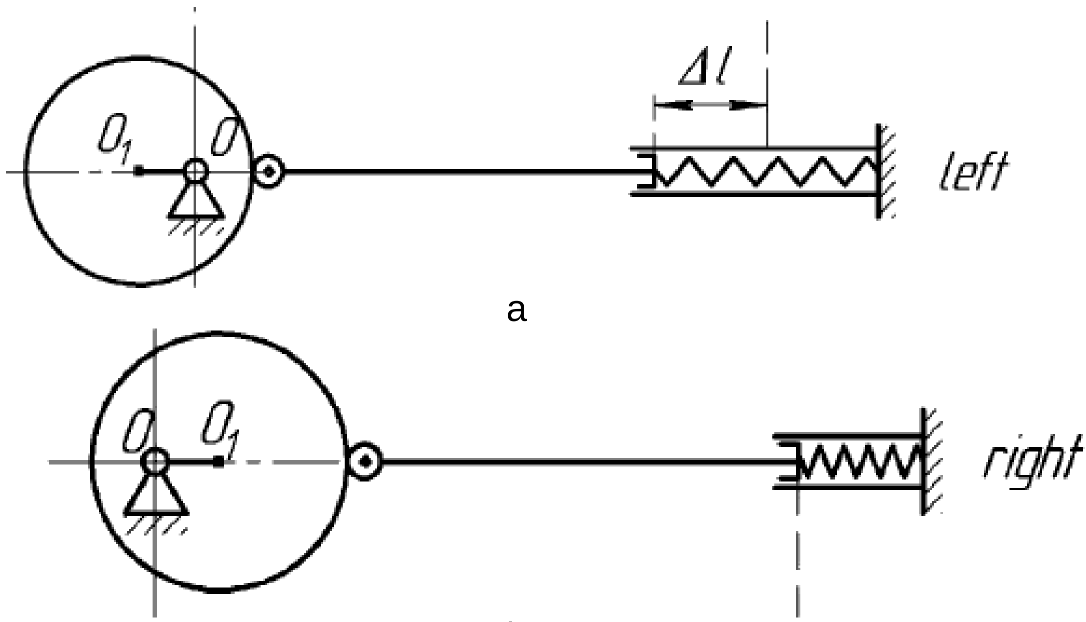


Fig. 3.90

The system is under action of conservative forces. So we can apply the law of conservation of mechanical energy:

$$T_i + P_i = T_f + P_f. \tag{3.207}$$

The spring is not compressed (Fig. 3.90, a) at initial moment of time, so

$$P_i = 0.$$

The kinetic energy of the body at initial moment is

$$T_i = \frac{1}{2} I_O \omega^2 , \quad (3.208)$$

where I_O is moment of inertia about axis of rotation.

We can find it using parallel-axis theorem:

$$I_O = I_{O_1} + m \cdot OO_1^2 = \frac{1}{2} mR^2 + m \cdot \left(\frac{1}{2} R \right)^2 = \frac{3}{4} mR^2 . \quad (3.209)$$

Substituting (3.209) into (3.208):

$$T_i = \frac{1}{2} \cdot \frac{3}{4} mR^2 \omega^2 = \frac{3}{8} \frac{p}{g} R^2 \omega^2 . \quad (3.210)$$

At final moment the kinetic energy is zero

$$T_f = 0 ,$$

because the spring brakes the eccentric at final moment.

Potential energy at final moment (Fig. 3.90, b) is equal to potential energy of the force of spring:

$$P_f = \frac{c \Delta l^2}{2} = \frac{cR^2}{2} . \quad (3.211)$$

Substituting expressions (3.210) and (3.211) into (3.207):

$$\frac{3}{8} \frac{p}{g} R^2 \omega^2 = \frac{cR^2}{2} .$$

From here angular velocity is:

$$\omega = \sqrt{\frac{8cR^2 g}{2 \cdot 3R^2 p}} = \sqrt{\frac{4cg}{3p}} .$$

Answer: $\omega = \sqrt{\frac{4cg}{3p}}$.

Example 6. The Fig. 3.91 shows the cross-section of a uniform 91-kg ventilator door hinged about its upper horizontal edge at O. The door is controlled by the spring-loaded cable which passes over the small pulley at A.

The spring has a stiffness of 219 N/m of stretch and it undeformed, when $\theta = 0$. If the door is released from rest in the horizontal position, determine the maximum angular velocity ω reached by the door and the corresponding angle θ.

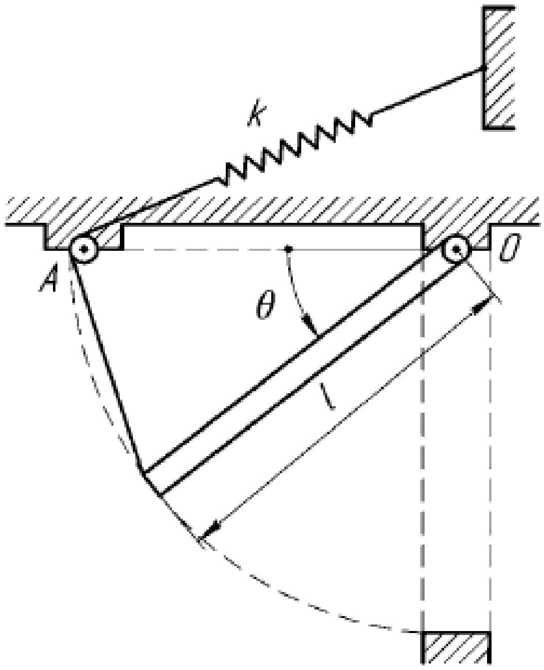


Fig. 3.91

Solution

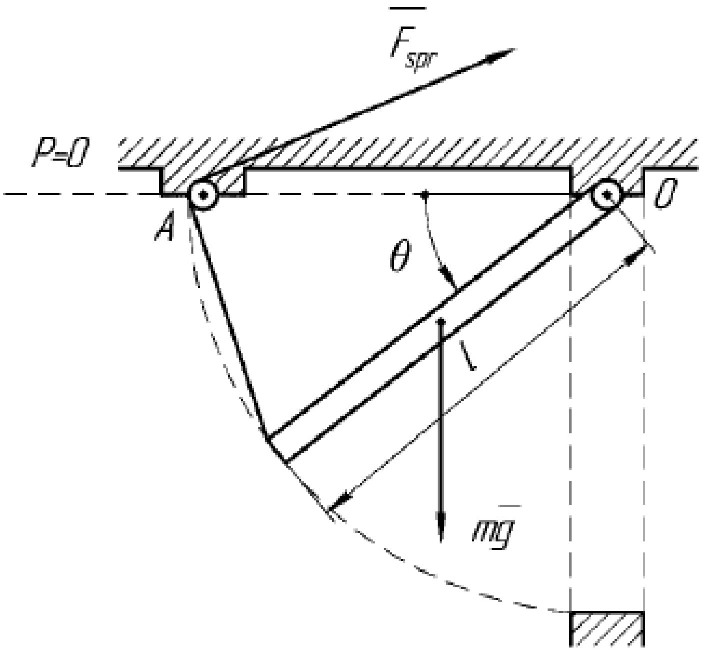


Fig. 3.92

Force of spring and gravity force (Fig. 3.92) are conservative forces, so we can say, that this system is under action of conservative forces and we apply the law of conservation of mechanical energy:

$$T_i + P_i = T_f + P_f , \qquad (3.212)$$

The spring is not stretched at initial moment of time, so

$$P_i = 0$$

and the body doesn't move, so

$$T_i = 0 .$$

At final moment kinetic energy of door is computed as for rotational body:

$$T_f = \frac{1}{2} I \omega^2 . \qquad (3.213)$$

Moment of inertia is:

$$I = \frac{1}{3} m l^2 .$$

Therefore

$$T_f = \frac{1}{2} \cdot \frac{1}{3} m l^2 \omega^2 = \frac{1}{6} m l^2 \omega^2 . \qquad (3.214)$$

Potential energy at final moment of time is equal to a sum of potential energies of the gravity force and force of spring (Fig. 3.93):

$$P_f = P\left(m\overline{g}\right) + P\left(\overline{F_{spr}}\right),$$

$$P\left(m\overline{g}\right) = -mgh = -mg \frac{l}{2} \sin\theta ,$$

$$P\left(\overline{F_{spr}}\right) = \frac{c \Delta l^2}{2} = \frac{c}{2}\left(2l \sin\frac{\theta}{2}\right)^2 = 2cl^2 \sin^2\frac{\theta}{2} .$$

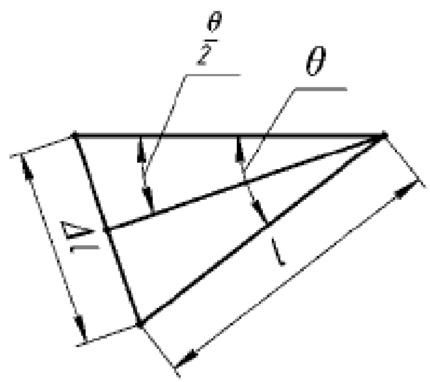


Fig. 3.93

So the total potential energy is:

$$P_f = -mg \frac{l}{2} \sin\theta + 2cl^2 \sin^2\frac{\theta}{2} . \qquad (3.215)$$

Substituting expressions (3.214) and (3.215) into (3.212):

$$\frac{1}{6}ml^2\omega^2 - mg\frac{l}{2}\sin\theta + 2cl^2\sin^2\frac{\theta}{2} = 0. \qquad (3.216)$$

Angular velocity is a derivative of an angle by the time:

$$\omega = \dot{\theta}.$$

Differentiating equation (3.216) by the time:

$$\frac{1}{6}ml^2 2\dot{\theta}\cdot\ddot{\theta} - mg\frac{l}{2}\cos\theta\cdot\dot{\theta} + 2cl^2 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\frac{\dot{\theta}}{2} = 0.$$

Making some transformations:

$$\frac{1}{3}ml^2\cdot\ddot{\theta} - mg\frac{l}{2}\cos\theta + cl^2\sin\theta = 0.$$

Using condition of function extremeness $\dot{\theta} = \omega = \omega_{\max}$, when $\ddot{\theta} = \dot{\omega} = \varepsilon = 0$:

$$-mg\frac{l}{2}\cos\theta + cl^2\sin\theta = 0,$$

$$cl^2\cdot\tan\theta - \frac{l}{2}mg = 0,$$

$$\tan\theta = \frac{lmg}{2cl^2} = \frac{mg}{2cl} = \frac{91\cdot 9.8}{2\cdot 219\cdot 1.52} = 1.34.$$

$$\theta = 53^\circ$$

Substituting the value of the angle θ into the equation (3.216) we can find desired angular velocity:

$$\omega = \sqrt{\frac{mg\frac{l}{2}\sin\theta + 2cl^2\sin^2\frac{\theta}{2}}{\frac{1}{6}ml^2}} = 4.6\left(\frac{rad}{\sec}\right).$$

Answer: $\omega = 4.6\left(\frac{rad}{\sec}\right)$.

3.6.8. Short problems

Problem 1. The homogeneous cylindrical rolls *1* and *2* of the mass 20 kg every are actuated from a state of rest by the constant moment of the couple $M = 2\ \mathrm{N \cdot m}$ (Fig. 3.94). Determine velocity of the roll axes at their displacement on the distance 3 m if radiuses are $R_1 = R_2 = 0{,}2$m.

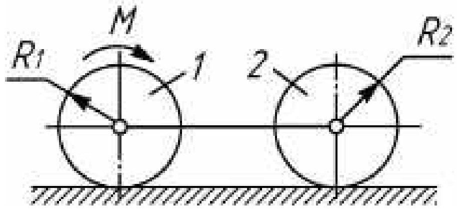


Fig. 3.94

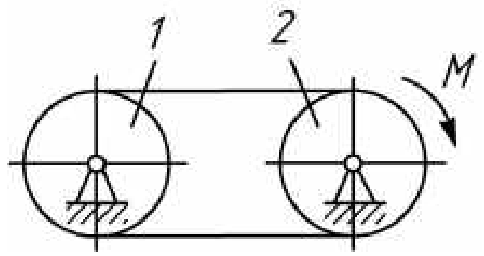


Fig. 3.95

Problem 2. Motion of the pulley *2* of the belt transmission (Fig. 3.95) begins from a state of rest under action of the constant motion M = 0,5N • m. After three revolutions the identical by a mass and dimensions the pulleys *1* and *2* have angular velocity 2 rad/s. Determine moment of inertia of the one of the pulleys about axis of rotation.

Problem 3. The moment of inertia of the gear wheel 1 about axis of rotation is equal to 0,1kg • m^2 (Fig. 3.96). The total mass of the rack *2* and the load *3* is equal to 100 kg. Determine the rack velocity at its displacement on the distance s = 0,2m if at the beginning the system was at rest. The radius of the wheel is r= 0,1m.

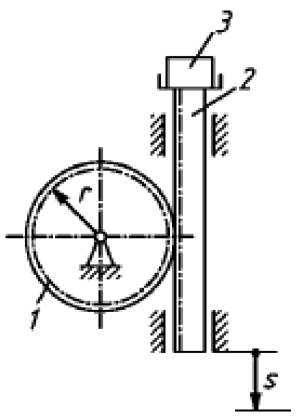


Fig. 3.96

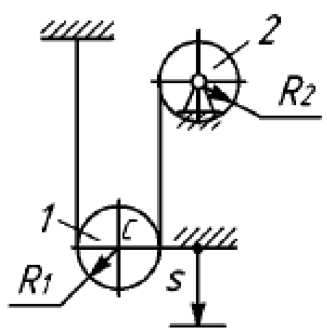


Fig. 3.97

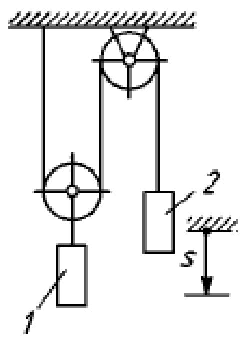


Fig. 3.98

Problem 4. The identical blocks *1* and *2* (Fig. 3.97) of the masses $m_1 = m_2$ and the radiuses $R_1 = R_2$ representing the homogeneous disks begin motion from a state of rest under action of gravitational force. Determine center velocity *C* of the block *1* after a moment when it will sink down on the distance s = 1 m.

Problem 5. Determine the load 2 (Fig. 3.98) velocity at the moment of time when it sink down on the distance s = 4 m if masses of the loads are $m_1 = 2\ kg$, m_2= 4 kg. The system of bodies was at rest at initial time.

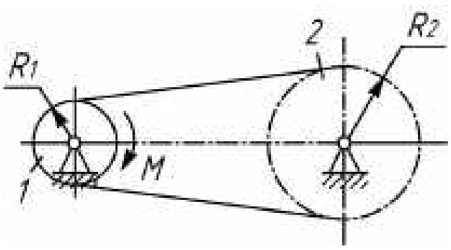


Fig. 3.99

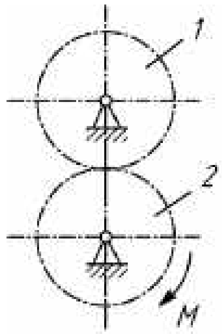


Fig. 3.100

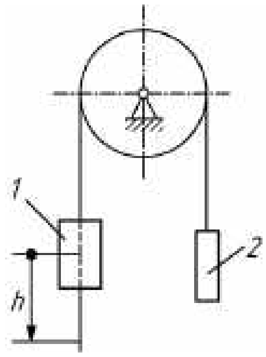


Fig. 3.101

Problem 6. The belt transmission (Fig. 3.99) begins motion from a state of rest under action of the moment of the couple M = 2,5N • m. The moments of inertia of the pulleys about their axes of rotation $I_2 = 2I_1$ = 1 kg • m^2. Determine angular velocity of the pulley 1 after three revolutions if the radiuses of the pulleys are $R_2 = 2R_1$.

Problem 7. The identical gear wheels 1 and 2 (Fig. 3.100) of the mass 2 kg every are actuated from a state of rest by the constant moment of the couple M = 1 N•m. Determine an angular velocity of the wheels after two revolutions if the radius of inertia of every wheel about axis of rotation is equal to 0.2 m.

Problem 8. The loads 1 and 2 (Fig. 3.101) with the masses $m_1=2$ kg and m_2=1 kg are hung to the ends of flexible cord threw over the block. Determine the velocity of the load 1 at the moment when it sinks on the distance $h = 3$ m. Loads motion begins from a state of rest.

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ТЕОРЕТИЧНА МЕХАНІКА.

КІНЕМАТИКА

(Англійською мовою)

Редактор С. В. Опаріна
Технічний редактор Л. О. Кузьменко

Зв. план, 2013

Підписано до друку 17.10.2013

Формат 60×84 1/8. Папір офс. № 2. Офс. друк

Ум. друк. арк. 10. Обл.-вид. арк. 11,5. Наклад 100 пр.

Замовлення 297. Ціна вільна

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Свідоцтво про внесення суб'єкта видавничої справи
до Державного реєстру видавців, виготовлювачів і розповсюджувачів
видавничої продукції сер. ДК № 391 від 30.03.2001