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ABOUT THE KYIV SCHOOL MATHEMATICIANS CONTRIBUTION TO THE EXTREMA OF FUNCTIONS OF MANY VARIABLES THEORY

The ***subject*** *of this article is the study of Kyiv school mathematicians' contribution to the many variable extrema function theory. The* ***purpose*** *of this article is to study the work of Kyiv mathematician Professor M. Ye. Vashchenko-Zakharchenko. Task: for the first time, to carry out a detailed analysis of the results obtained in the article by M. Ye. Vashchenko-Zakharchenko «Signs of the highest and lowest value of functions». The research* ***method*** *is a historical and scientific analysis of the original source, which allows the scientific results obtained in the late 19th and early 20th centuries to be estimated from the viewpoint of modern mathematical analysis. The following* ***results*** *were obtained. Thanks to the analysis of this article, it was found out that here M. Ye. Vashchenko-Zakharchenko first used the methods of linear algebra and applied D. Sylvester's criterion of positive (negative) definiteness of quadratic form to obtain sufficient conditions for the existence of an extremum of functions of many variables. The problem is considered in general for functions of many variables and for special cases of functions of two and three variables.* ***Conclusions****. Comparing the results obtained from the Kyiv mathematician, Professor M. Ye. Vashchenko-Zakharchenko with the presentation of this topic in modern textbooks on higher mathematics and mathematical analysis, we can conclude that they are included in these textbooks in virtually the same form. This is what determines the relevance of this topic: methods for solving problems for the extremum of functions of many variables, obtained at the turn of the 19th and 20th centuries, are used in modern optimization problems. The breadth of the scientific erudition of M. Ye. Vashchenko-Zakharchenko allowed him to immediately perceive new methods obtained by foreign scientists and immediately find their application to obtain sufficient conditions for the existence of an extremum of functions of many variables. This shows the high scientific level of the state of mathematics at Kyiv University in the second half of the 19th century.* ***Further research*** *should focus on the contribution of other mathematicians of the Kyiv school to the development of the theory of extrema of functions of many variables.*

Keywords: *optimization; extremum function of several variables; sufficient conditions; positive (negative) definiteness of the quadratic form.*

Introduction

Motivation. The relevance of the topic is determined by the fact that the methods for solving problems for the extremum of several variables functions, obtained in the late 19th - early 20th centuries by O. Cauchy, J.-L. Lagrange, K. Weierstrass, are used in modern optimization problems [1], also, for example, in problems that are reduced to problems of mathematical programming.

The same refers to the design optimization and aircraft construction task [2], and the safety challenge of various systems [3], and in the method of information technology for structure analysis of urban network fire-rescue units [4].

Thus, the optimization problem of determining the structure and composition of technological safety territorial systems (TSTS), which corresponds to the criterion optimal value for the system efficiency:

$$(s^*, t^*) = \arg\max_{s\in D} \Im(\Re(t), P(s,t), \omega(t), Z(s), s, t),$$

where D is the set of admissible options for the TSTS composition, determined by the resource and other constraints of the problem [3].

Here the functional

$$\Im(\Re(t), P(s,t), \omega(t), Z(s), s, t)$$

is a dynamic vector assessment of the TSTS effectiveness.

In general, the problem of mathematical programming for solving technical problems can be written, as is known, as follows: maximize the objective function $f(x_1, x_2, ..., x_n)$ on an admissible set G, where G is given by the system

$$g_i(x_1, ..., x_n) \ge 0, \ i = 1, ..., m, \ (x_1, ..., x_n) \in X,$$

where X is a certain subset $\mathbb{R}^n$. Thus, many optimization problems are actually problems on the conditional extremum of a function of several variables.

Therefore, the optimization task is set as a direct problem of maximizing the system functional quality.

This article is a continuation of the author's research on the development of the extrema functions theory of several variables in the 19th and 20th centuries. In this research the contribution of O. Cauchy [5] and K. Weierstrass [6] to the theory of extremal problems was clarified.

It should be noted that extremum problems appeared long before the differential calculus as an integral science emerged. Actually, those extremum problems were one of the harbingers of this science. At first, there were tasks for the extremum of a single variable function. They were developed in a line with the single valid variable function theory. The extension of some theory results to several variables functions faced significant difficulties. Meanwhile, if was the case of several variables that was more important. for the needs of mathematics, natural science and technology.

The particular interest of mathematicians in extremum functions of several variables problems was observed at the end of the 19th century with the appearance of the appropriate algebraic tool. For example, the quadratic forms theory, namely, J. Sylvester's criteria of positive definitivenes of quadratic forms (1852).

The demand for theoretical development of this case apparently explains the fact that the great mathematicians of the 19th century did not pass by this question.

National mathematicians play an important part in this process. The research of Kyiv University professor M. Ye. Vashchenko-Zakharchenko will be discussed in this article.

State of the art

To find out for the first time the contribution of the Kiev mathematician M. Ye. Vashchenko-Zakharchenko to the theory of extrema of functions of many variables. Also, for the first time to carry out for this purpose the study of his article «Signs of the Highest and Lowest Value of Functions», which was not mentioned earlier in the historical-mathematical literature.

The paper is organized as follows. Section 1 defines the purpose of the article and the method of research. Section 2 contains biographical information on Professor M. Ye. Vashchenko-Zakharchenko of the University of Kyiv. Section 3 is devoted to the study of the article «Signs of the Highest and Lowest Value of Functions» by Kiev University Professor M. Ye. Vashchenko-Zakharchenko.

1. Research objective statement

The purpose of this article is to study the work by M. Ye. Vashchenko-Zakharchenko «Signs of the Highest and Lowest Value of Functions» [7] (1867) from the point of view of the quadratic forms theory and to compare its results with the presentation of this issue in modern Advanced mathematics and mathematical analysis books. The research method is a historical and scientific analysis of the original source, which allows the scientific results obtained in the late 19th and early 20th centuries to be estimated from the view of modern mathematical analysis.

2. About M. Ye. Vashchenko-Zakharchenko

Mykhailo Yegorovych Vashchenko-Zakharchenko (1825-1912) graduated with a silver medal from a gymnasium in Kyiv end entered Kiev University at the mathematical department of the Faculty of Philosophy. Two years later he left for Paris, where he continued his studies at the Sorbonne and the College de France. There he was a student at O. Cauchy's, J.A. Serre's and J. Liouville's lectures.

He was one of the first national mathematicians to develop symbolic calculus. His Master's thesis «Symbolic Calculus and its Implementation to the Integration of Linear Differential Equations» is one of the first national researches on operational calculus. M. Ye. Vashchenko-Zakharchenko correctly assessed the importance of the new calculus for the further development of mathematics. At that time, most people thought the symbolic calculus only made it possible to obtain already known results more briefly. In the introduction to his master's thesis, he emphasized that the symbolic calculus doesn't just cuts the time for a braining results, but it is the same powerful method, a tool for research, as other mathematical analysis methods. This work still occupies an important place in mathematics.

He received a professorship in 1867, shortly after defending his doctoral thesis «The Riemannian Theory of Functions of a Composite Variable». His thesis was also one of the first studies of this issue in this country.

His work on the translation of Euclid's «Beginnings» with «explanations and interpretations» (1880) was also important.

His other works on the math history were devoted to elementary geometry, the analytic geometry history, the history of mathematics of the Chaldeans, Hindus, and others. He also wrote a general course of math history (1883), as well as a number of manuals on mathematics («Elementary Geometry in the Volume of a Four-year Course», «A Short Course on the Theory of Determinants», «Analytic Geometry and Algebraic Analysis»).

M. Ye. Vashchenko-Zakharchenko worked at Kiev University from 1868 to 1902. He is one of the founders of the Physics and Mathematics Society in Kyiv. Among

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his students were famous national mathematicians: such as B. Y. Bukreyev, and V. P. Yermakov, and others.

One to his works and pedagogical activity, M. Ye. Vashchenko-Zakharchenko had a great influence on the development of the national mathematical culture.

3. Study of the work of M. Ye. Vashchenko-Zakharchenko «Signs of the Highest and Lowest Value of Functions»

The article «Signs of the Highest and Lowest Value of Functions» (written in 1867, published in 1868) begins with a presentation of the basic concepts of the quadratic forms theory. First comes the definition of the quadratic form, first of two, and then of *n* variables, which he calls «kind» or type. The author explains this name by the fact that the term «form» is not convenient, as it is often used in a different meaning. Otherwise, his definitions do not differ from modern ones [8].

Then M. Ye. Vashchenko-Zakharchenko claims that any kind of

$$\begin{aligned} &A_{11}t_1t_1 + A_{12}t_1t_2 + \ldots + A_{1n}t_1t_n + \\ &+A_{21}t_2t_1 + A_{22}t_2t_2 + \ldots + A_{2n}t_2t_n + \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &+A_{n1}t_nt_1 + A_{n2}t_nt_2 + \ldots + A_{nn}t_nt_n, \end{aligned} \tag{1}$$

where $A_{rs} = A_{sr}$ $(s = 1, 2, ..., n;\ r = 1, 2, ..., n)$, can always be given the following form

$$E_1X_1^2 + E_2X_2^2 + E_3X_3^2 + \ldots + E_nX_n^2,$$

where E_1, E_2, ..., E_n equal ± 1, which is called the «canonical form» of this type [7, p. 219]. In modern terminology, the normal type of a given quadratic form with real coefficients [9].

M. Ye. Vashchenko-Zakharchenko calls the following formula as discriminant or determinant of this type

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \ldots & A_{1n} \\ A_{21} & A_{22} & A_{23} \ldots & A_{2n} \\ A_{31} & A_{32} & A_{33} \ldots & A_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & A_{n3} \ldots & A_{nn} \end{vmatrix}. \tag{2}$$

«A feature of the type is its invariant, that is, such a function of the coefficients of the form that is equal to the same function, compose of the form coefficients obtained from the given one by a linear transformation, or is equal to the same function, multiplied by the coefficients, independent of the coefficients of the type» [7, p. 219].

Research in this direction is, as it's known, from F. Gauss. In «Arithmetical Studies» (1801), F. Gauss considered the quadratic form

$$P = ax^2 + 2bxy + cy^2$$

in linear transformations of the unknowns x and y

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y' .$$

With this transformation, the form P will take the kind

$$P' = a'x'^2 + 2b'x'y' + c'y'^2 .$$

Gauss finds such quantities that would not change after transformation or change in a certain way, and comes to the determinant first of all,

$$D' = b'^2 - a'c' = r^2 D ,$$

where $r = \alpha\delta - \beta\gamma$, $D = b^2 - ac$

The same calculations could be found in M. Ye. Vashchenko-Zakharchenko's research, when proving that the formula

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2 ,$$

is a constant $ax^2 + 2bxy + cy^2$. Indeed, in a linear transformation of this type

$$x = \lambda X + \mu Y, \quad y = \lambda_1 X + \mu_1 Y$$

we get the form

$$AX^2 + 2BXY + CY^2 , \tag{3}$$

where

$$A = a\lambda^2 + 2b\lambda\lambda_1 + c\lambda_1^2, \quad C = a\mu^2 + 2b\mu\mu_1 + c\mu_1^2,$$

$$B = a\lambda\mu + b(\lambda\mu_1 + \lambda_1\mu) + c\lambda_1\mu_1 .$$

If these quantities are substituted into the attribute value of the form (3) $AC - B^2$, we will get

$$AC - B^2 = (\lambda\mu_1 - \mu\lambda_1)^2 (ac - b^2) .$$

Further, he found conditions under which the kind or the type (1) for all possible values of the variables $t_1, t_2, ..., t_n$ will always have positive or negative value (that is, positively or negatively defined quadratic form). This will happen when in the canonical notation of this form all the coefficients E_k $(k = 1, 2, ..., n)$ are $+1$ or -1. In defining these conditions, only the determinant (2) is the matter, or rather, its main minors Ivanchenko calls the following expressions as: minors of the first order

$$A_{11},\ A_{22},\ A_{33}, ...,\ A_{nn};$$

second order minors

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix},\quad \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix},\quad ...,$$

$$\begin{vmatrix} A_{11} & A_{1n} \\ A_{n1} & A_{nn} \end{vmatrix},\quad \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix},\quad ...;$$

third order minors

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix},\quad \begin{vmatrix} A_{11} & A_{12} & A_{14} \\ A_{21} & A_{22} & A_{24} \\ A_{41} & A_{42} & A_{44} \end{vmatrix}, ...$$

etc. Finally, the determinant (2) itself will be the minor of the n-th order.

From this, M. Ye. Vashchenko-Zakharchenko implies the following necessary and sufficient conditions for $E_k = 1$ $(k = 1, 2, ..., n)$: all minors of the first, second, etc. orders must be positive; so that $E_k = -1$ $(k = 1, 2, ..., n)$ – all minors of even order must be positive, and all minors of odd order must be negative.

After that, the author proceeds to consider the extrema of the function $F(z_1, z_2, ..., z_n)$ of n variables:

$$F(z_1, z_2, ..., z_n) \to extr?$$

$x_1, x_2, ..., x_n$ is the systems that satisfies the equations

$$\frac{\partial F}{\partial z_1} = 0,\quad \frac{\partial F}{\partial z_2} = 0, ...,\quad \frac{\partial F}{\partial z_n} = 0 .$$

«In order for the function $F(z_1, z_2, ..., z_n)$ to have the largest or smallest value for these values, it is necessary that the type

$$\frac{d^2F}{dx_1dx_1}t_1t_1 + \frac{d^2F}{dx_1dx_2}t_1t_2 + \frac{d^2F}{dx_1dx_3}t_1t_3 + ...$$
$$... + \frac{d^2F}{dx_1dx_n}t_1t_n +$$
$$+\frac{d^2F}{dx_2dx_1}t_2t_1 + \frac{d^2F}{dx_2dx_2}t_2t_2 + \frac{d^2F}{dx_2dx_3}t_2t_3 +$$
$$... + \frac{d^2F}{dx_2dx_n}t_2t_n +$$

. . . .

$$+\frac{d^2F}{dx_ndx_1}t_nt_1 + \frac{d^2F}{dx_ndx_2}t_nt_2 + \frac{d^2F}{dx_ndx_3}t_nt_3 + ..$$
$$... + \frac{d^2F}{dx_ndx_n}t_nt_n ,$$

where $t_1, t_2, ..., t_n$ are the increments of particular values $x_1, x_2, ..., x_n$ accordingly, had a negative or positive value for all possible values of the variables $t_1, t_2, ..., t_n$». [7, p. 224].

(According to M. Ye. Vashchenko-Zakharchenko values

$$\frac{d^2F}{dx_idx_j} \quad (i = 1, 2, ..., n;\ j = 1, 2, ..., n)$$

mean the corresponding partial derivatives.)

The conditions for this are given by the determinant

$$\begin{vmatrix} \frac{\partial^2F}{\partial x_1\partial x_1} & \frac{\partial^2F}{\partial x_1\partial x_2} & ... & \frac{\partial^2F}{\partial x_1\partial x_n} \\ \frac{\partial^2F}{\partial x_2\partial x_1} & \frac{\partial^2F}{\partial x_2\partial x_2} & ... & \frac{\partial^2F}{\partial x_2\partial x_n} \\ . & . & . & . \\ \frac{\partial^2F}{\partial x_n\partial x_1} & \frac{\partial^2F}{\partial x_n\partial x_2} & ... & \frac{\partial^2F}{\partial x_n\partial x_n} \end{vmatrix}$$

and its main minors, where the derivatives are taken at the point $(x_1, x_2, ..., x_n)$.

At the end of his article, M. Ye. Vashchenko-Zakharchenko shows the application of the obtained criterion using examples of functions of two and three variables. For functions of two variables $F(z_1, z_2)$, there will be a maximum at the point (x_1, x_2) if

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$$\frac{\partial^2 F(x_1, x_2)}{\partial x_1^2} < 0, \quad \frac{\partial^2 F(x_1, x_2)}{\partial x_2^2} < 0,$$

$$\begin{vmatrix} \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 F(x_1, x_2)}{\partial x_2 \partial x_2} \end{vmatrix} > 0,$$

minimum if

$$\frac{\partial^2 F(x_1, x_2)}{\partial x_1^2} > 0, \quad \frac{\partial^2 F(x_1, x_2)}{\partial x_2^2} > 0,$$

$$\begin{vmatrix} \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 F(x_1, x_2)}{\partial x_2 \partial x_2} \end{vmatrix} > 0.$$

For the function of three variables $F(z_1, z_2, z_3)$ «all principal minors of the first order

$$\frac{d^2 F(x_1, x_2, x_3)}{dx_1^2}, \quad \frac{d^2 F(x_1, x_2, x_3)}{dx_2^2}, \quad \frac{d^2 F(x_1, x_2, x_3)}{dx_3^2}$$

must be below zero when $F(z_1, z_2, z_3)$ has the maximum value, and above zero if this function has the minimum value. Second-order principal minors are

$$\begin{vmatrix} \frac{d^2 F(x_1, x_2, x_3)}{dx_1^2} & \frac{d^2 F(x_1, x_2, x_3)}{dx_1 dx_2} \\ \frac{d^2 F(x_1, x_2, x_3)}{dx_2 dx_1} & \frac{d^2 F(x_1, x_2, x_3)}{dx_2 dx_2} \end{vmatrix},$$

$$\begin{vmatrix} \frac{d^2 F(x_1, x_2, x_3)}{dx_1^2} & \frac{d^2 F(x_1, x_2, x_3)}{dx_1 dx_3} \\ \frac{d^2 F(x_1, x_2, x_3)}{dx_3 dx_1} & \frac{d^2 F(x_1, x_2, x_3)}{dx_3 dx_3} \end{vmatrix},$$

$$\begin{vmatrix} \frac{d^2 F(x_1, x_2, x_3)}{dx_2^2} & \frac{d^2 F(x_1, x_2, x_3)}{dx_2 dx_3} \\ \frac{d^2 F(x_1, x_2, x_3)}{dx_3 dx_2} & \frac{d^2 F(x_1, x_2, x_3)}{dx_3 dx_3} \end{vmatrix}$$

they must all be above zero, both at the maximum and the minimum value of the function $F(z_1, z_2, z_3)$. Finally, the sign itself should be below at the maximum and above zero at the minimum value of the function» [7, p.226].

Therefore, all concepts considered by M. Ye. Vashchenko-Zakharchenko in this article are completely identical to modern ones: linear algebra [8], higher algebra and geometry [9], higher mathematics and mathematical analysis [10].

Conclusions

For the first time the researched article by M. Ye. Vashchenko-Zakharchenko «Signs of the Highest and Lowest Value of Functions» has not been mentioned in the historical and mathematical literature before. It has been established that, using the ideas and methods of algebra, M. Ye. Vashchenko-Zakharchenko was the first to formulate sufficient conditions for the existence of function extremum of several variables using the Sylvester's criterion proper of positive (negative) definitiveness of quadratic forms.

Along with the other mathematicians' researches, these studies contributed to the further development and improvement of methods for finding extrema of functions of many variables, as an integral part of mathematical analysis.

Extremum problems are widely used in mathematical models of various practical problems. For example, in information-extreme cluster-analysis of input data during functional diagnosis [11], when applying evolutionary methods of solving optimization problems of compressors of gas turbine engines [12], problems of segmentation of a two-dimensional signal, which is an image of the product ordered by the consumer [13].

Therefore, such a section of mathematical analysis as the theory of extrema of functions of many variables is always present in textbooks of higher mathematics and is an integral part of the course of higher mathematics in higher educational institutions for students of various specialties for example, linear algebra [8], higher algebra and geometry [9], higher mathematics and mathematical analysis [10].

Further research should focus on the contribution of other mathematicians of the Kyiv school to the development of the theory of extrema of functions of many variables.

Contributions of authors: formulation of the purpose and tasks of research, development of conceptual provisions and methodology of research – **Olha Prokhorova;** review and analysis of references, analysis of research results, formulation of conclusions **– Nataliia Kalchuk.**

Conflict of Interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, author ship or otherwise, that could affect the research and its results presented in this paper.

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This study was conducted without financial support.

Data Availability

The work has no associated data in the data repository.

Use of Artificial Intelligence

The authors confirm that they did not use artificial intelligence methods while creating the presented work.

All the authors have read and agreed to the published version of this manuscript.

References (GOST 7.1:2006)

1. Kononenko, I. Method for Solving the Multi-Criteria Non-Markov Problem of Project Portfolio Optimization [Text] / I. Kononenko, A. Korchakova // Радіоелектронні і комп'ютерні системи. – 2022. – № 1(101). – С. 67-82. DOI: 10.32620/reks.2022.1.05.

2. Титов, М. Ю. Оптимизация магнитной системы СПД-70 путем максимизации градиента индукции магнитного поля [Текст] / М. Ю. Титов, О. В. Лоян // Питання проектування і виробництва літальних апаратів : зб. наук. тр. Нац. аерокосм. ун-ту ім. М. Є. Жуковського «ХАІ». – Вип. 1(89). – Харків, 2017. – С. 83-85. – Режим доступу: http://nti.khai.edu/ojs/index.php/vpkla/article/view/265. – 12.12.2023.

3. Попов, В. М. Механизм адаптивного управления программами развития территориальных систем техногенной безопасности [Текст] / В. М. Попов // Радіоелектронні та комп'ютерні системи. – 2015. – № 3(83). – С. 101-105. – Режим доступу: http://nti.khai.edu/csp/nauchportal/Arhiv/REKS/2015/REKS315/PopovVM.pdf. – 12.12.2023.

4. Danshyna, S. Method of Information Technology for Structure Analysis of Urban Network Fire-Rescue Units [Text] / S. Danshyna, A. Nechausov // Радіоелектронні і комп'ютерні системи. – 2023. – № 4(108). – С. 171-183. DOI: 10.32620/reks.2023.4.12.

5. Прохорова, О. М. Теория экстремумов функций многих переменных в учебнике О. Коши по дифференциальному исчислению [Текст] / О. М. Прохорова // Вісник ХНАУ. Серія: Економічні науки. – 2014. – № 7. – С. 164-168. – Режим доступу: http://nbuv.gov.ua/UJRN/Vkhnau_ekon_2014_7_24. – 12.12.2023.

6. Прохорова, О. М. О вкладе К. Вейерштрасса в теорию экстремумов функций многих переменных [Текст] / О. М. Прохорова // Радіоелектронні та комп'ютерні системи. – 2015. – № 3(73). – С. 125-128. – Режим доступу: http://nti.khai.edu/csp/nauchportal/Arhiv/REKS/2015/REKS315/Prohorova.pdf. – 12.12.2023.

7. Ващенко-Захарченко, М. Е. Признаки наибольшего и наименьшего значения функций [Текст] / М. Е. Ващенко-Захарченко // Математический сборник. – Киев : Типография Императорского Университета Св. Владимира, 1868. – Т. 3. – С. 217 – 226.

8. Чарін, В. С. Лінійна алгебра [Текст] / В. С. Чарін. – 2-ге вид., стер. – Київ : Техніка, 2005. – 416 с. – Режим доступу: http://pdf.lib.vntu.edu.ua/books/2015/Charin_2005_416.pdf. – 12.12.2023.

9. Ніколаєв, О. Г. Алгебра і геометрія [Текст] / О. Г. Ніколаєв. – Харків : Нац. аерокосм. ун-т ім. М. Є. Жуковського «Харків. авіац. ін.-т», 2017. – 368 с. – Режим доступу: http://library.khai.edu/library/fulltexts/metod/Nikolaev_Algebra_I_Geometrija.pdf. – 12.12.2023.

10. Дубовік, В. П. Вища математика. У трьох частинах. Ч. 2 [Текст] / В. П. Дубовик, І. І. Юрик. – Харків : Веста, 2008. – 240 с. – Режим доступу: http://pdf.lib.vntu.edu.ua/books/Dubovik_P2_2008_240.pdf. – 12.12.2023.

11. Зимовець, В. І. Інформаційно-екстремальний кластер-аналіз вхідних даних при функціональному діагностуванні [Текст] / В. І. Зимовець, О. С. Приходченко, М. І. Мироненко // Радіоелектронні і комп'ютерні системи. – 2019. – № 4(92). – С. 105-115. DOI: 10.32620/reks.2019.4.12.

12. Меняйлов, А. В. Применение эволюционных методов для решения задач оптимизации компрессоров газотурбинных двигателей [Текст] / А. В. Меняйлов, А. А. Трончук, Е. М. Угрюмова // Авіаційно-космічна техніка і технології. – 2008. – № 5(52). – С. 59-65. – Режим доступу: https://nti.khai.edu/csp/nauchportal/Arhiv/AKTT/2008/AKTT508/Menyal.pdf. – 12.12.2023.

13. Патрушев, В. О. Сегментація двовимірного сигналу, що являє собою зображення товару, який замовляє споживач [Текст] / В. О. Патрушев, О. І. Патрушева // Радіоелектронні і комп'ютерні системи – 2019. – № 1(89). – С. 37-43. DOI: 10.32620/reks.2019.1.04.

References (BSI)

1. Kononenko, I., & Korchakova, A**.** Method for Solving the Multi-Criteria Non-Markov Problem of Project Portfolio Optimization. *Radioelektronni i*

ISSN 1814-4225 (print)
ISSN 2663-2012 (online)

komp'uterni sistemi – Radioelectronic and computer systems, 2022, no. 1(101), pp. 67-82. DOI: 10.32620/reks.2022.1.05.

2. Tytov, M. Yu. Optimizatsiya magnitnoy sistemy SPD-70 putem maksimizatsii gradiyenta induktsii magnitnogo polya [Optimization of the SPD-70 magnetic system by maximizing the magnetic field induction gradient]. *Pytannya proektuvannya i vyrobnytstva lital'nykh aparativ : zb. nauk. tr. Nats. aerokosm. un-tu im. M. Ye. Zhukovs'koho «KhAI» – Issues of design and manufacture of flying vehicles*, 2017, no. 1 (89), pp. 83-85. Available at:: http://nti.khai.edu/ojs/index.php/vpkla/article/view/265. (accessed 12.12.2023). (in Russian).

3. Popov, V. M. Mekhanizm adaptivnogo upravleniya programmami razvitiya territorial'nykh sistem tekhnogennoy bezopasnosti [Program adaptive management in development of territorial technological safety systems]. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2015, no. 3(83), pp. 101-105. Available at: http://nti.khai.edu/csp/nauchportal/Arhiv/REKS/2015/REKS315/PopovVM.pdf. (accessed 12.12.2023). (in Russian).

4. Danshyna, S., & Nechausov, A. Method of information technology for structure analysis of urban network fire-rescue units. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2023, no. 4, pp. 171-183. DOI: 10.32620/reks.2023.4.12.

5. Prokhorova, O. M. *Teoriya ekstremumov funktsiy mnogikh peremennykh v uchebnike O. Koshi po differentsial'nomu ischisleniyu* [Theory of extremums of the functions of many variable values]. *Visnyk KHNAU. Seriya : Ekonomichni nauky – KHNAU Bulletin. Series: Economic sciences*, 2014, no. 7, pp. 164-168. Available at: http://nbuv.gov.ua/UJRN/Vkhnau_ekon_2014_7_24. (accessed 12.12.2023) (in Russian).

6. Prokhorova, O. M. O vklade K. Veyyershtrassa v teoriyu ekstremumov funktsiy mnogikh peremennykh [Weierstrass`s contribution to the theory of functions of several variables extrema]. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2015, no. 3(73), pp. 125-128. Available at: http://nti.khai.edu/csp/nauchportal/Arhiv/REKS/2015/REKS315/Prohorova.pdf (accessed 12.12.2023). (in Russian).

7. Vashchenko-Zakharchenko, M. Ye. Priznaki naibol'shego i naimen'shego znacheniya funktsiy [Signs of the largest and smallest values of functions]. *Matematiceskij sbornik – Mathematics*. Kyiv, Tipografiya Imperatorskogo Universiteta Sv. Vladimira Publ., 1868, vol. 3, pp. 217–226. (in Russian).

8. Charin, V. S. *Liniyna alhebra* [Linear algebra] 2-he vyd., ster. Kyiv, Tekhnika Publ., 2005. 416 p. Available at: http://pdf.lib.vntu.edu.ua/books/2015/Charin_2005_416.pdf. (accessed 12.12.2023) (in Ukrainian).

9. Nikolayev, O. H. *Alhebra i heometriya* [Algebra and geometry] pidruch. Kharkiv: Nats. aerokosm. un-t im. M. Ye. Zhukovs'koho «Kharkiv. aviats. in.-t», 2017. 368 p. Available at: http://library.khai.edu/library/fulltexts/metod/Nikolaev_Algebra_I_Geometrija.pdf. (accessed 12.12.2023) (in Ukrainian).

10. Dubovik, V. P., & Yuryk, I. I. *Vyshcha matematyka U tr'okh chastynakh. Ch. 2* [Higher mathematics]. Kharkiv, Vesta Publ., 2008. 240 p. Available at: http://pdf.lib.vntu.edu.ua/books/Dubovik_P2_2008_240.pdf. (accessed 12.12.2023) (in Ukrainian).

11. Zymovets', V. I., Prykhodchenko, O. S., & Myronenko, M. I. Informatsiyno-ekstremal'nyy klaster- analiz vkhidnykh danykh pry funktsional'nomu diahnostuvanni [Information-extreme cluster analysis of input data in functional diagnostics]. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2019, no. 4(92), pp. 105-115. DOI: 10.32620/reks.2019.4.12. (in Ukrainian).

12. Menjailov, A. V., Tronchuk, A. A., & Ugrjumova, E. M. Primeneniye evolyutsionnykh metodov dlya resheniya zadach optimizatsii kompressorov gazoturbinnykh dvigateley [Application of evolutional methods to the solution of a turbojet engines compressors optimization tasks]. *Aviacijno-kosmicna tehnika i tehnologia – Aerospace technic and technology*, 2008, no. 5 (52), pp. 59-65. Available at: https://nti.khai.edu/csp/nauchportal/Arhiv/AKTT/2008/AKTT508/Menyal.pdf. (accessed 12.12.2023) (in Russian).

13. Patrushev, V. O., & Patrusheva, O. I. Segmentatcija dvovymirnogo signalu, jakiy predstavljaie sobou zobragennij tovaru, sho zamovljayet'sija spogivachem [Sigmentation of a Bidden Signal, Which Represents a Picture of Ordered Goods by the Consumer]. *Radioelektronni i komp'uterni sistemi – Radioelectronic and computer systems*, 2019, no. 1(89), pp. 37-43. DOI: 10.32620/reks.2019.1.04. (in Ukrainian).

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ПРО ВНЕСОК МАТЕМАТИКІВ КИЇВСЬКОЇ ШКОЛИ В ТЕОРІЮ ЕКСТРЕМУМІВ ФУНКЦІЙ БАГАТЬОХ ЗМІННИХ

О. М. Прохорова, Н. Л. Кальчук

Предметом вивчення статті є дослідження внеску математиків київської школи математиків у теорію екстремумів функцій багатьох змінних. **Метою** є вивчення доробку київського математика професора

М. Є. Ващенко-Захарченка. **Завдання**: вперше провести детальний аналіз результатів, отриманих у статті М. Є. Ващенка-Захарченка «Ознаки найбільшого та найменшого значення функцій». Використовуваними **методами** є: історико-науковий аналіз першоджерел, який дозволяє наукові результати, отримані наприкінці 19 – на початку 20 століть, оцінити з точки зору сучасного математичного аналізу. Отримані такі **результати**. Завдяки проведеному вперше аналізу цієї статті з'ясовано, що тут М. Є. Ващенко-Захарченком вперше використано методи лінійної алгебри та застосовано власне критерій Д. Сильвестра додатної (від'ємної) визначеності квадратичної форми для отримання достатніх умов існування екстремуму функцій багатьох змінних. Задача розглядається в загальному вигляді для функцій багатьох змінних і для окремих випадків функцій двох і трьох змінних. **Висновки.** Порівнюючи отримані київським математиком професором М. Є. Ващенко-Захарченком результати з викладом даної тематики в сучасних підручниках з вищої математики та математичного аналізу, можна зробити висновок, що вони входять до цих підручників фактично в тому самому вигляді. Саме цим і визначається актуальність даної теми: методи розв'язування задач на екстремум функцій багатьох змінних, отримані на зламі 19 – 20 століть застосовуються в сучасних задачах оптимізації. Широта наукової ерудиції професора М. Є. Ващенко-Захарченка дала йому змогу одразу сприйняти нові методи, отримані закордонними вченими, і одразу знайти їм застосування для отримання достатніх умов існування екстремуму функцій багатьох змінних. Це показує високий науковий рівень стану математики в Київському університеті в другій половині 19 століття. Подальші дослідження слід спрямувати на вивчення внеску інших математиків київської школи в розвиток теорії екстремумів функцій багатьох змінних.

Ключові слова: оптимізація; екстремум функції багатьох змінних; достатні умови; додатньо (від'ємно) визначена квадратична форма.

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