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INTEGRATION: THEORY AND APPLICATIONS

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MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
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«Kharkiv Aviation Institute»

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INTEGRATION: THEORY AND APPLICATIONS

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Наведено теорію невизначених і визначених інтегралів та її застосування до деяких задач геометрії й фізики із прикладами розв'язків. Зміст відповідає програмі курсу „Вища математика” вищих технічних навчальних закладів.

Для студентів першого курсу, які вивчають вищу математику англійською мовою.

Reviewers: Dr. Sc. (Phys.-Math.), Prof. S. S. Zub,
Cand. Sc. (Phys.-Math.), Assoc. Prof. O. O. Shougaylo

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The theory of indefinite and definite integrals and its application to some problems of geometry and physics with examples of solutions are given. The content corresponds to the program of the course "Higher Mathematics" of technical universities.

For first-year students who study higher mathematics in English.

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1. BASIC PROBLEMS OF THE INTEGRAL CALCULUS. THE INDEFINITE INTEGRAL

1.1. The Concept of an Indefinite Integral

One of the basic tasks of the differential calculus is to find the derivative or differential of a given function.

The primary task of the integral calculus consists in the converse — finding the function, given its derivative or differential.

Let the derivative

$$y' = f(x)$$

or differential

$$dy = f(x)dx$$

be given of the unknown function y.

*A function $F(x)$, possessing a given function $f(x)$ as its derivative, or $f(x)dx$ as its differential, is called **a primitive** or **an anti-derivative** of the given function $f(x)$.*

If, for example,

$$f(x) = x^2,$$

a primitive (an anti-derivative) of the function will be $F(x) = x^3$; we have, in fact,

$$\left(\frac{1}{3}x^3\right)' = \frac{1}{3}\cdot 3x^2 = x^2.$$

Suppose that a primitive $F(x)$ of the given function $f(x)$ has been found, so that we have the relationship

$$F'(x) = f(x).$$

Since the derivative of an arbitrary constant C is equal to zero, we also have:

$$\left[F(x) + C\right]' = F'(x) = f(x),$$

i.e. the function $F(x) + C$ is also a primitive of $f(x)$.

Hence it follows that, if the problem of finding a primitive has one solution, it will have an infinity of further solutions, differing from the first by an arbitrary constant. On the other hand, it can be shown that there are no other solutions apart from these, i.e. *if $F(x)$ is any one primitive of a given function $f(x)$, any other primitive has the form* :

$$F(x)+C,$$

where C is an arbitrary constant.

Let $F_1(x)$ be any function, whose derivative is $f(x)$. We have:

$$F_1'(x)=f(x).$$

On the other hand, $F(x)$ possesses the derivative $f(x)$, i.e.

$$F'(x)=f(x).$$

Subtracting this equation from the previous one, we get:

$$F_1'(x)-F'(x)=\left[F_1(x)-F(x)\right]'=0,$$

whence

$$F_1(x)-F(x)=C,$$

where C is a constant: which it was required to prove.

The result we have obtained can also be formulated as: *If the derivatives (or differentials) of two functions are identically equal, the functions themselves differ only by a constant.*

The most general expression for a primitive is also referred to as ***the indefinite integral*** *of the given function* $f(x)$*, or of the given differential* $f(x)dx$*, and is denoted by the symbol*

$$\int f(x)dx,$$

$f(x)$ *being referred to as* ***the integrand****, and* $f(x)dx$ *as* ***the integrand expression***.

Having found one primitive $F(x)$, we can write, by what was shown above:

$$\int f(x)dx=F(x)+C,$$

where C is an arbitrary constant.

Mechanical and geometrical interpretations can be given of the indefinite integral. Suppose we have a law giving an analytic relationship between velocity and time:

$$v=f(t),$$

and we want to express the path s in terms of time. Since the velocity of a point in a given trajectory is the derivative ds/dt of the path with respect to time, the problem reduces to finding a primitive of the function $f(t)$, i.e.

$$s=\int f(t)dt.$$

We get an infinite number of solutions, differing by a constant term. This lack of precision in the answer results from us not fixing the point from which the traversed path s is measured. If, for instance, $v = gt + v_0$, (uniformly accelerated motion), we obtain the expression for s:

$$s = \frac{1}{2}gt^2 + v_0 t + C \tag{1.1}$$

because, as is easily shown, the derivative of (1) with respect to t coincides with the given expression $v = gt + v_0$. If we agree to measure s from the point corresponding to $t = 0$, i.e. if we agree to take $s = 0$ at $t = 0$, we have to put the constant $C = 0$ in (1.1). Of course it is of no significance that we have denoted the independent variable by t in the above discussion, and not by x.

We now pass to the geometrical interpretation of the problem of finding a primitive. The relationship $y' = f(x)$ shows that the graph of any required primitive, or, as we usually say, of any integral curve:

$$y = F(x),$$

is such that the tangent to the curve for any given x has the direction determined by the slope

$$y' = f(x). \tag{1.2}$$

In other words, the direction of the tangent to the curve is given by (1.2) for any given value of the independent variable x; the problem is to find this curve. Having constructed one such integral curve, all the curves obtained by moving this by any amount in a direction parallel to the axis OY will have parallel tangents with the same slope $y' = f(x)$ as in the case of the initial curve, given the same value of x (Fig. 1.1). The parallel shift referred to is equivalent to adding a constant C to the ordinate of the curve; and the general equation of the curves, giving solutions of the problem, will be:

$$y = F(x) + C. \tag{1.3}$$

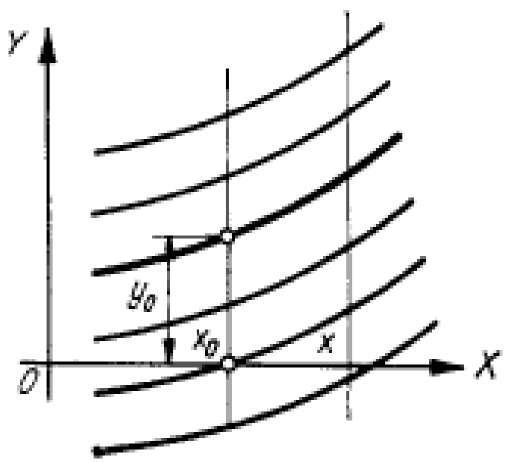

Fig. 1.1

In order to define fully the position of a curve, i.e. to fully define the expression for the primitive, a point must be assigned through which the integral curve must pass. The point assigned may be the point of intersection of the curve with a line

$$x = x_0$$

parallel to the axis OY. This is equivalent to assigning the initial value y_0 that the required function $y = F(x)$ takes for the given value $x = x_0$. We substitute this initial value in equation (1.3), and obtain an equation defining the arbitrary constant C:

$$y_0 = F(x_0) + C,$$

so that finally, the primitive satisfying our initial condition will have the form:

$$y = F(x) + \left[y_0 - F(x_0) \right].$$

Before examining the properties of the indefinite integral, and methods for finding the primitive, we note a second basic problem of the integral calculus, and examine it from the point of view of the problem already stated — namely, that of finding the primitive. A new concept is essential for what follows, this being the concept of a definite integral. We choose a natural approach here by starting from the intuitive idea of area, which also enables us to examine the connection between the concepts of definite integral and primitive. The discussion of the next two articles, based as it is on the intuitive idea of area, cannot be considered as a rigorous proof of new facts. A logically rigorous approach to the fundamentals of the integral calculus is indicated later.

1.2. The Definite Integral as the Limit of a Sum

We take the graph of the function $f(x)$ in the plane XOY, and assume that it consists of a continuous curve, lying wholly above OX, i.e. all ordinates of the graph are assumed positive. We consider the area S_{ab} bounded by OX, the curve, and the two ordinates $x = a$ and $x = b$ (Fig. 1.2), and we try to find the magnitude of this area. We start by dividing the interval (a, b) into n parts by means of the points:

$$a = x_0 < x_1 < x_2 < \ldots < x_{k-1} < x_k < \ldots < x_{n-1} < x_n = b.$$

The area S_{ab} is now divided into n vertical strips, the length of base of the $k-th$ strip being $(x_k - x_{k-1})$. Let m_k and M_k respectively denote the least and greatest values of function $f(x)$ in the interval $(x_{k-1,} x_k)$, i.e. the least and greatest ordinates of our graph in this interval. The area of the strip lies between the areas of the two rectangles of heights m_k and M_k and having the common base $\left(x_k - x_{k-1}\right)$ (Fig. 1.3). These rectangles are the "interior" and "exterior" rectangles for the $k-th$ strip.

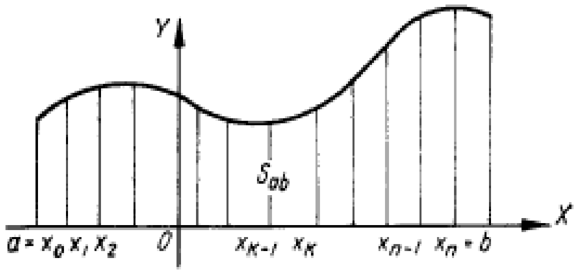

Fig. 1.2

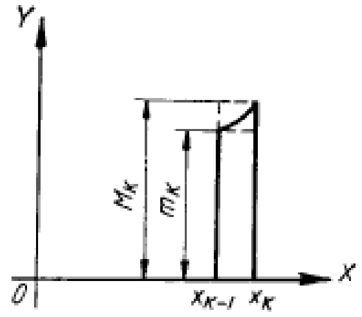

Fig. 1.3

The magnitude of the area of the $k-th$ strip is thus comprised between the areas of the rectangles in question, i.e. between the two numbers:

$$m_k\left(x_k - x_{k-1}\right) \text{ and } M_k\left(x_k - x_{k-1}\right),$$

and hence the total area S_{ab} will lie between the sums of the areas of these interior and exterior rectangles, i.e. S_{ab} lies between the sums:

$$s_n = \sum_{i=1}^{n} m_i\left(x_i - x_{i-1}\right), \quad S_n = \sum_{i=1}^{n} M_i\left(x_i - x_{i-1}\right). \qquad (1.4)$$

We thus have the inequality:

$$s_n \le S_{ab} \le S_n. \qquad (1.5)$$

We now draw a mean rectangle in place of the interior and exterior rectangles for each strip, taking base $\left(x_k - x_{k-1}\right)$ as usual but with the height taken as the ordinate $f\left(\xi_k\right)$ of our curve at any given point ξ_k of the interval $\left(x_{k-1}, x_k\right)$(Fig. 1.4). We consider the sum of the areas of these mean rectangles:

$$\begin{aligned} S_n^{/} = {} & f\left(\xi_1\right)\left(x_1 - x_0\right) + f\left(\xi_2\right)\left(x_2 - x_1\right) + \ldots + \\ & + f\left(\xi_k\right)\left(x_k - x_{k-1}\right) + \ldots + f\left(\xi_{n-1}\right)\left(x_{n-1} - x_{n-2}\right) + f\left(\xi_n\right)\left(x_n - x_{n-1}\right) \end{aligned} \qquad (1.6)$$

This, like the area S_{ab}, will lie between the sums of the areas of the interior and exterior rectangles, i.e. we have

$$s_n \le S_n^{/} \le S_n. \qquad (1.7)$$

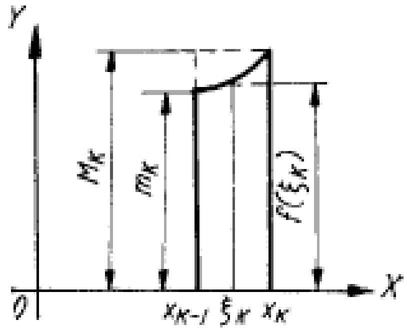

Fig. 1.4

We now indefinitely increase the number n of divisions of the interval (a,b), and so also make the greatest of the differences $(x_k - x_{k-1})$ tend to zero. Since $f(x)$ is continuous by hypothesis, the difference $(M_k - m_k)$ between its greatest and least values in the interval (x_{k-1}, x_k) will tend to zero with indefinite decrease of the length of this interval, irrespective of its position in the fundamental interval (a,b). We thus have ε_n tending to zero on passing to the above-mentioned limit, where ε_n is the greatest of the differences:

$$(M_1 - m_1),\ (M_2 - m_2),\ldots,\ (M_k - m_k),\ldots,\ (M_{n-1} - m_{n-1}),\ (M_n - m_n).$$

We now state the difference between the sums of the areas of the interior and exterior rectangles:

$$S_n - s_n = (M_1 - m_1)(x_1 - x_0) + (M_2 - m_2)(x_2 - x_1) + \ldots +$$
$$+ (M_k - m_k)(x_k - x_{k-1}) + \ldots + (M_n - m_n)(x_n - x_{n-1});$$

on replacing all the $(M_k - m_k)$by the greatest difference ε_n and recalling that all the $(x_k - x_{k-1})$ are positive, we have:

$$S_n - s_n \le \varepsilon_n (x_1 - x_0) + \varepsilon_n (x_2 - x_1) + \ldots +$$
$$+ \varepsilon_n (x_k - x_{k-1}) + \ldots + \varepsilon_n (x_n - x_{n-1}),$$

i.e.

$$S_n - s_n \le \varepsilon_n (x_n - x_0) = \varepsilon_n (b - a).$$

We can thus write:

$$0 \le S_n - s_n \le \varepsilon_n (b - a),$$

i.e.

$$\lim_{n\to\infty} (S_n - s_n) = 0. \qquad (1.8)$$

On the other hand, we had for any n:

$$s_n \le S_{ab} \le S_n, \qquad (1.9)$$

the magnitude of the area S_{ab} being a definite number. It follows directly from (1.8) and (1.9) that the magnitude of area S_{ab} is the common limit of s_n and S_n, i.e. of the areas of the interior and exterior rectangles :

$$\lim s_n = \lim S_n = S_{ab}.$$

But the sum $S_n^/$ of the mean rectangles lies between s_n and S_n, as we have seen, so that this must also tend to the area S_{ab}, i.e.

$$\lim S_n = S_{ab}.$$

The sum $S_n^/$ is more general than s_n or S_n, in as much as we can arbitrarily choose the ξ_k in the interval (x_{k-1}, x_k), and in particular, we can take $f(\xi_k)$ as equal to the least ordinate m_k or the greatest ordinate M_k.

With these choices, the sum $S_n^/$ transforms to s_n or S_n.

The above discussion leads us to the following:

Suppose the function $f(x)$ is continuous in an interval (a,b); suppose that, having divided the interval into n parts by the points

$$a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b,$$

we find the corresponding value of the function $f(\xi_k)$ for any $x = \xi_k$ in the interval (x_{k-1}, x_k), and that we now form the sum :

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{1.10}$$

then this sum tends to a definite limit on indefinite increase of the number n of divisions of the interval and on indefinite decrease of the greatest of the differences $(x_k - x_{k-1})$. This limit is equal to the area bounded by the axis OX, the graph of function $f(x)$, and the two ordinates $x = a$ and $x = b$.

The limit in question is referred to as the definite integral of the function $f(x)$ with respect to the variable x between the lower limit $x = a$ and the upper limit $x = b$; it is denoted by:

$$\int_a^b f(x)\,dx.$$

We note that the existence of a limit I of the sum (1.10) in the case of indefinite decrease of the greatest of $(x_k - x_{k-1})$ amounts to the following assertion: for any given positive ε there exists a positive δ such that

$\left| I - \sum_{k=1}^{n} f\left(\xi_k\right)\left(x_k - x_{k-1}\right) \right| < \varepsilon$ for any chosen point ξ_k in the interval $\left(x_{k-1}, x_k\right)$, provided only that all the (positive) differences $x_k - x_{k-1} < \delta$. This limit I is the definite integral.

We have assumed above that the graph of $f\left(x\right)$ is wholly located above axis OX, i.e. that all the ordinates of the graph are positive. We now take the general case, where part of the graph is above OX, and part below (Fig. 1.5).

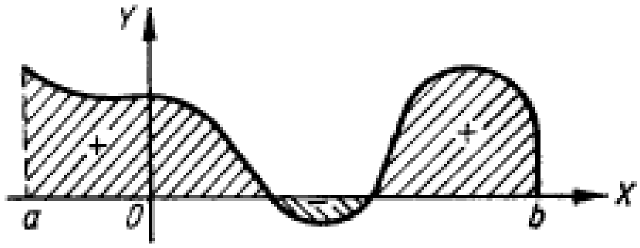

Fig. 1.5

If we form the sum (6) in this case, the terms $f\left(\xi_k\right)\left(x_k - x_{k-1}\right)$ corresponding to parts of the graph below OX will be negative, since the difference $\left(x_k - x_{k-1}\right)$ is positive and the ordinate $f\left(\xi_k\right)$ is negative.

The definite integral obtained on passing to the limit will reckon areas above OX with the (+) sign and areas below with the (—) sign, i.e. in the general case, the definite integral

$$\int_a^b f\left(x\right)dx$$

will give the algebraic sum of the areas included between OX, the graph of $f\left(x\right)$, and the ordinates $x = a$ and $x = b$. Areas above OX are here given the positive sign, and areas below, the negative sign.

As will be seen later, finding the limit of a sum of the form (1.6) is not only involved in calculating areas, but is encountered in a wide variety of scientific problems. We take just one example. Let a certain point M be moving along axis OX from $x = a$ to $x = b$. Let it be acted on by a certain force T, also directed along OX. If the force T is constant, the work done in moving the point from the position $x = a$ to $x = b$ is given by the product $R = T\left(b - a\right)$, i.e. the product of the magnitude of the force and the path traversed by the point. If the force T is variable, the above formula is no longer valid. Suppose that the magnitude of the force depends on the position of the point on OX, i.e. we have $T = f\left(x\right)$.

To find the work done in this case, we subdivide the total path traversed by means of the points

$$a = x_0 < x_1 < x_2 < ... < x_{k-1} < x_k < ... < x_{n-1} < x_n = b,$$

and we take one of the intervals (x_{k-1}, x_k). We can take the force acting on the point as it moves from x_{k-1} to x_k as constant, with an error that is smaller, the shorter the length $(x_k - x_{k-1})$, and we can set its value as $f(\xi_k)$ for some point ξ_k of the interval (x_{k-1}, x_k).

Hence we obtain an approximate expression for the work done in the interval (x_{k-1}, x_k):

$$R_k \approx f(\xi_k)(x_k - x_{k-1}).$$

The total work done will be given approximately by:

$$R \approx \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}).$$

On indefinite increase in the number n of subdivisions and on indefinite decrease of the greatest of the differences $(x_k - x_{k-1})$, we get in the limit a definite integral, accurately expressing the work done:

$$R = \int_a^b f(x)\,dx.$$

Disregarding any possible geometrical or mechanical interpretations, we can now fix the concept of the definite integral of a function $f(x)$ over the interval $a \le x \le b$ as the limit of a sum of the form (1.6). The second basic task of the integral calculus is to study the properties of the definite integral and, above all, to evaluate it. If $f(x)$ is a given function, and $x = a$ and $x = b$ are given numbers, the definite integral

$$\int_a^b f(x)\,dx$$

is a determinate number. The $\int$ sign is a stylized letter S, recalling the summation that gives, in the limit, the magnitude of the definite integral. The expression under the integral, $f(x)dx$, recalls the form of the term in the summation, viz., $f(\xi_k)(x_k - x_{k-1})$. The letter x, standing under the sign of the definite integral, is usually referred to as the variable of integration. We note an important detail as regards this letter. The magnitude of the integral is a determinate number, as already mentioned, and is of course not dependent on the notation x for the variable of integration; any letter can be used to denote the variable of integration in a definite integral. The choice has evidently no influence at all on the magnitude of the integral, which depends only on the ordinates of the graph of $f(x)$ and on the limits of integration a and b. Since the notation for the independent variable plays no part, we have for instance:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

The second task of the integral calculus — that of evaluating the definite integral — consists in forming a sum of the form (1.6) and passing to the limit. This would seem a fairly complicated problem at first sight. We note that the number of terms in this sum increases indefinitely on passing to the limit, whilst each term tends to zero. Apart from this, the second task of the integral calculus would appear to have no connection with the first task, that of finding the primitive of a given function $f(x)$.

We show in the following article that both tasks are intimately related, and that evaluation of the definite integral $\int_a^b f(x)\,dx$ is accomplished very simply, if the primitive of $f(x)$ is known.

1.3. The Relation Between the Definite and Indefinite Integrals

We again consider the area S_{ab} bounded by the axis OX , the graph of function $f(x)$, and ordinates $x = a$ and $x = b$. In addition to this area, we also consider a part of it, bounded by the left-hand ordinate $x = a$ and by a movable ordinate, corresponding to a variable value of x (Fig. 1.6).

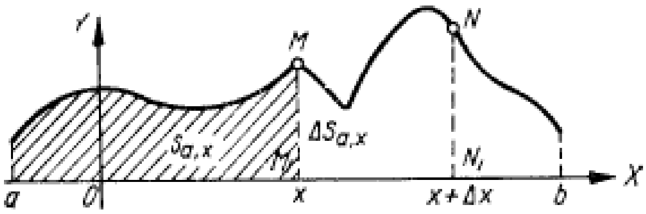

Fig. 1.6

The magnitude S_{ax} of this latter area will evidently depend on where we locate the right-hand ordinate, i.e. it is a function of x, represented by the definite integral of $f(x)$, taken from the lower limit a to the upper limit x. Since the letter x is used to denote the upper limit, we shall avoid confusion by denoting the variable of integration by a second letter, say t. We can thus write:

$$S_{ax} = \int_a^x f(t)\,dt \tag{1.11}$$

We have here a definite integral with a variable upper limit x, and its magnitude is clearly a function of this limit. We show that this function is one of the primitives of $f(x)$. We find the derivative of the function by considering its increment ΔS_{ax} for the increment Δx of the independent variable x. We obviously have (Fig. 1.7) $\Delta S_{ax} = area\left(M_1MNN_1\right)$.

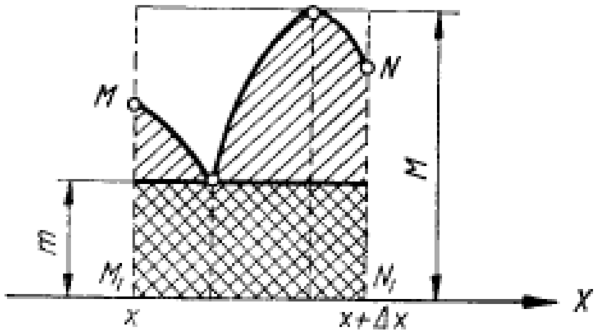

Fig. 1.7

We denote the least and greatest ordinates of the graph of $f(x)$ in the interval $(x, x+\Delta x)$ by m and M respectively. The curvilinear figure M_1MNN_1, drawn on a larger scale in Fig. 1.7, will lie wholly inside the rectangle of height M and base Δx, and will contain the rectangle of height m and the same base; hence

$$m\Delta x \le \Delta S_{xa} \le M\Delta x \ ,$$

or, dividing by Δx:

$$m \le \frac{\Delta S_{xa}}{\Delta x} \le M \ .$$

By the continuity of $f(x)$, both m and M tend to the common limit $M_1M = f(x)$, the ordinate of the curve at the point x, as Δx tends to zero; hence:

$$\lim \frac{\Delta S_{xa}}{\Delta x} = f(x).$$

which is what we wanted to prove. We can formulate the result obtained as follows: a definite integral with a variable upper limit

$$\int_a^x f(t)\,dt$$

is a function of this upper limit, the derivative of which is equal to the integrand $f(x)$ at the upper limit. In other words, the definite integral with variable upper limit is a primitive of the integrand.

Having established the connection between the concepts of definite and indefinite integral, we now show how the definite integral

$$\int_a^b f(x)\,dx$$

can be evaluated, if the primitive $F(x)$ of $f(x)$ is known. The definite integral with variable upper limit is in fact a primitive of $f(x)$, as we have shown, and we can write:

$$\int_a^x f(t)\,dt = F(x) + C, \qquad (1.12)$$

where C is a constant. We determine this constant by noting that the area S_{ax} evidently vanishes if its right-hand ordinate coincides with the left-hand, i.e. if $x = a$; so that the left-hand side of (1.12) vanishes for $x = a$. Putting $x = a$ in (1.12) thus gives us:

$$0 = F(a) + C, \text{ i.e. } C = -F(a).$$

Substituting for C in (1.12), we get:

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

Finally, putting $x = b$ here, we find:

$$\int_a^b f(t)\,dt = F(b) - F(a) \text{ or } \int_a^b f(x)\,dx = F(b) - F(a). \qquad (1.13)$$

We thus arrive at the following fundamental rule, giving the magnitude of a definite integral in terms of values of a primitive: the magnitude of a definite integral is equal to the difference between the values of the primitive of the integrand at the upper and lower limits of integration.

The rule stated shows that finding a primitive, i.e. solving the first problem of the integral calculus, also solves the second problem, that of evaluating the definite integral; so that we do not need to carry out the complicated operations of forming the sum (1.6) and passing to the limit, in order to evaluate a definite integral.

We take as an example the definite integral

$$\int_0^1 x^2\,dx.$$

A primitive of the function x^2 is $x^3/3$.

We have by the rule we deduced :

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3},$$

where the symbol $\varphi(x)\big|_a^b$ denotes the difference $\left[\varphi(b)-\varphi(a)\right]$.

If we were to calculate this definite integral directly from its definition as the limit of a sum, without using the primitive, we would find ourselves with a much more complicated calculation, which is briefly reproduced. We divide the interval $(0,1)$ into n equal parts with the points:

$$0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1.$$

We now have the following n intervals:

$$\left(0,\frac{1}{n}\right),\ \left(\frac{1}{n},\frac{2}{n}\right),\ \left(\frac{2}{n},\frac{3}{n}\right),\ldots,\ \left(\frac{n-1}{n},1\right),$$

the length of each being equal to $1/n$. We form the sum (1.6) with ξ_k taken as the left-hand end of the interval, i.e.

$$\xi_1 = 0,\ \xi_2 = \frac{1}{n},\ \xi_3 = \frac{2}{n},\ldots,\xi_n = \frac{n-1}{n}.$$

We have $x_k - x_{k-1} = 1/n$ for all the differences, and we note that the integrand $f(x) = x^2$ has the values at the left-hand ends of the intervals:

$$f(\xi_1) = 0,\ f(\xi_2) = \frac{1}{n^2},\ f(\xi_3) = \frac{2^2}{n^2},\ldots,\ f(\xi_n) = \frac{(n-1)^2}{n^2}.$$

So we can write:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\frac{1}{n^2}\cdot\frac{1}{n} + \frac{2^2}{n^2}\cdot\frac{1}{n} + \ldots + \frac{(n-1)^2}{n^2}\cdot\frac{1}{n}\right] = \lim_{n\to\infty}\frac{\sum_{k=1}^{n-1}k^2}{n^3}. \qquad (1.14)$$

We find the sum in the numerator by noting the series of obvious equalities :

$$(1+k)^3 = 1 + 3\cdot k + 3\cdot k^2 + k^3 \quad (k = 1,\ldots,n-1).$$

Adding term by term, we get:

$$2^3 + 3^3 + \ldots + n^3 = (n-1) + 3\left[1 + 2 + \ldots + (n-1)\right] +$$
$$+3\left[1^2 + 2^2 + \ldots + (n-1)^2\right] + 1^3 + 2^3 + \ldots + (n-1)^3.$$

Cancelling $2^3 + \ldots + (n-1)^3$, and using the formula for the sum of an arithmetic progression, we can write:

$$n^3=(n-1)+3\frac{n(n-1)}{2}+3\left[1^2+2^2+3^2+\ldots+(n-1)^2\right]+1,$$

whence

$$1^2+2^2+3^2+\ldots+(n-1)^2=\frac{n^3-n}{3}-\frac{n(n-1)}{2}=\frac{n(n-1)(2n-1)}{6}.$$

Substituting the expression obtained in(14), we have:

$$\int_0^1 x^2\,dx=\lim_{n\to\infty}\frac{n(n-1)(2n-1)}{6n^3}=\frac{1}{6}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)=\frac{2}{6}=\frac{1}{3}.$$

We have now explained the basic problems of the integral calculus and their inter-relationship. We devote the next paragraph, to further consideration of the first problem, that of finding and investigating the properties of the indefinite integral.

Our above discussion of the definite integral has been based on purely geometrical concepts, viz., on consideration of the areas S_{ab}, and S_{ax}. In particular, the proof of the basic fact that the sum (1.6) has a limit started from the assumption that there exists a definite area S_{ab} for every continuous curve. This assumption has no sound basis for all its apparent obviousness and the only mathematically rigorous course would be in the opposite direction: to prove the existence of a limit S of the sum

$$\sum_{k=1}^{n} f(\xi_k)(x_k-x_{k-1})$$

by direct analytic means, without regard to geometrical interpretation, then use the limit for defining the area S_{ab}. We give this proof at the end of the present chapter, and at the same time we make more general assumptions regarding $f(x)$ than those of continuity.

We also remark that a geometrical interpretation played an essential part in proving the basic proposition that, given continuity of the integrand, the derivative of the definite integral with respect to its upper limit is equal to the value of the integrand at the upper limit. A rigorous analytic proof of this proposition is given in the next section. Combining this proof with the proof of the existence of a definite integral of a continuous function, we are able to assert that a primitive exists for every continuous function, i.e. an indefinite integral exists. We describe the basic properties of the indefinite integral later, on the assumption that we are only concerned with continuous functions.

We give a rigorous proof of the basic formula (1.13) when we come to describe the properties of definite integrals. Thus, the only unproved fact that remains is the existence of a limit of the sum (1.10) for a continuous function $f(x)$. This is proved at the end of the chapter, as already mentioned.

1.4. Properties of Indefinite Integrals

We saw that any two primitives of a given function can only differ by a constant term. This leads us to the first property of indefinite integrals:

I. If two functions or two differentials are identical, their indefinite integrals can only differ by a constant term.

Conversely, to show that two functions differ by a constant term, it is sufficient to show that their derivatives (or differentials) are identical.

The next properties, II and III, follow immediately from the concept of indefinite integral as a primitive, i.e. from the fact that the indefinite integral

$$\int f(x)\,dx$$

is a function such that its derivative with respect to x is equal to the integrand $f(x)$, or that its differential is equal to the integrand expression $f(x)dx$.

II. The derivative of an indefinite integral is equal to the integrand, whilst its differential is equal to the integrand expression :

$$\left(\int f(x)\,dx\right)'_x = f(x);\quad d\int f(x)\,dx = f(x)\,dx. \qquad (1.15)$$

III. We have, along with (1.15):

$$\int F'(x)\,dx = F(x)+C,$$

and this can be rewritten as:

$$\int dF(x) = F(x)+C, \qquad (1.16)$$

which, combined with property II, gives: the signs d and $\int$ eliminate each other when juxtaposed in any order, provided we agree to neglect the arbitrary constant in the equation for an indefinite integral.

IV. A constant factor can be taken outside the integration sign :

$$\int Af(x)\,dx = A\int f(x)\,dx + C \qquad (1.17)$$

V. The integral of an algebraic sum is equal to the algebraic sum of the integrals of each term :

$$\int (u+v-w+...)\,dx = \int u\,dx + \int v\,dx - \int w\,dx + ... + C. \qquad (1.18)$$

Formulae (1.17) and (1.18) are easily seen to be correct by differentiating both sides and observing the identity of the derivatives obtained. For (1.17), for instance:

$$\left(\int Af(x)\,dx\right)' = Af(x);$$

$$\left(A\int f(x)\,dx + C\right)' = A\left(\int f(x)\,dx\right)' = Af(x).$$

1.5. Table of Elementary Integrals

The rules for differentiation have given us extensive powers over the problem of differentiating given functions. Almost always, however, the inverse problem of integration greatly exceeds it in importance. Hence we must now study the art of integrating given functions.

The results attained by means of our differentiation formulae may be summed up as follows:

Every function which is formed from the elementary functions by means of a "closed expression" can be differentiated, and its derivative is also a closed expression formed from the elementary functions.

By "closed expression" we mean a function which can be built up from the elementary functions by repeated application of the rational operations and the processes of compounding and inversion.

On the other hand, we have not met with any exactly corresponding fact applying to the integration of elementary functions. It will be shown that every elementary function, and, in fact, every continuous function, can be integrated, and we can integrate a large number of elementary functions either directly or by inversion of differentiation formulae and can find their integrals to be expressions involving elementary functions only. But we are not able to find a general solution of the following problem: given a function $f(x)$ which is expressed in terms of the elementary functions by any closed expression, to find an expression for its indefinite integral, $F(x)=\int f(x)dx$, which is itself a closed expression in terms of the elementary functions.

The fact is that this problem is in general insoluble; it is by no means true that every elementary function has an integral which itself is an elementary function. In spite of this, it is extremely important that we should be able actually to carry out such integrations when they are possible, and that we should acquire a certain amount of technical skill in the integration of given functions.

The further sections of this chapter will be devoted to the development of methods useful for this purpose. In this connection we would expressly warn the beginner against merely memorizing the many formulae obtained by using these technical methods. The student should instead direct his efforts towards gaining a clear understanding of the methods of integration and learning how to apply them. Moreover, he should remember that even when integration by these methods is impossible the integral does exist (at least for all continuous functions), and can actually be calculated to as high a degree of accuracy as is desired by means of numerical methods which will be developed later.

In the latter part of the chapter we shall endeavour to deepen and extend our conceptions of integration and integral, quite apart from the problem of the technique of integration.

First of all we repeat that to each of the differentiation formula proved earlier there corresponds an equivalent integration formula. Since these elementary integrals are used time and again as materials for the art of integration, we collect them in the list that also is referred to as the table of elementary integrals :

1. $\int x^m\,dx = \frac{x^{m+1}}{m+1} + C, \text{ if } m \neq -1; \int \frac{1}{x}\,dx = \ln x + C$.

2. $\int e^x\,dx = e^x + C$; $\int a^x\,dx = \frac{a^x}{\ln a} + C$.

3. $\int \cos x\,dx = \sin x + C$.

4. $\int \sin x\,dx = -\cos x + C$.

5. $\int \frac{dx}{\cos^2 x} = \tan x + C$; $\int \frac{dx}{\sin^2 x} = -\cot x + C$.

6. $\int \frac{dx}{1+x^2} = \arctan x + C = -\operatorname{arccot} x + C$.

7. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$.

8. $\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x + \sqrt{x^2+1}\right| + C = \operatorname{arsinh} x + C$.

9. $\pm\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x \pm \sqrt{x^2-1}\right| + C = \operatorname{arcosh} x + C$.

10. $\int \cosh x\,dx = \int \frac{e^x + e^{-x}}{2}\,dx = \frac{e^x - e^{-x}}{2} + C = \sinh x + C$.

11. $\int \sinh x\,dx = \int \frac{e^x - e^{-x}}{2}\,dx = \frac{e^x + e^{-x}}{2} + C = \cosh x + C$.

12. $\int \frac{1}{\sinh^2 x}\,dx = -\coth x + C$.

13. $\int \frac{1}{\cosh^2 x}\,dx = \tanh x + C$.

To check this table, it is sufficient to establish that the derivative of the right-hand side of each equation is identical with the integrand on the left. In general, knowledge of the function, of which a given function $f(x)$ is the derivative, gives at once the indefinite integral of the latter. Usually, however, even in the simplest examples, the given functions are not to be found in the table of derivatives, which makes problems of the integral calculus a good deal more difficult to work out than those of the differential calculus. It is always a case of transforming the given integral to one contained in the table of elementary integrals. These transformations need experience and practice; they are facilitated by use of the basic rules of the integral calculus to be found below. In the following sections we shall attempt to reduce the calculation of integrals of given functions in some way or other to the elementary integrals collected in this table. Apart from devices which the beginner certainly could not acquire systematically, but which, on the contrary, occur only to those with long experience, this reduction is based essentially on two useful methods. Each of these methods enables us to transform a given integral in many ways; the object of such transformations is to reduce the given integral, in one step or in a sequence of steps, to one or more of the elementary integration formulae given above.

1.6. Integration by Parts

If $u,\ v$ are any two functions of x with continuous derivatives, we know that:

$$d(uv) = u\,dv + v\,du, \text{ or } u\,dv = d(uv) - v\,du.$$

This gives us, using properties I, V and III:

$$\int u\,dv = \int \left[d(uv) - v\,du \right] + C = \int d(uv) - \int v\,du + C = uv - \int v\,du + C,$$

leading to the formula for integration by parts :

$$\int u\,dv = uv - \int v\,du + C. \tag{1.19}$$

We use this to pass from evaluating $\int u\,dv$ to evaluating $\int v\,du$, the latter being possibly simpler.

1.7. Rule for Change of Variables

An integral $\int f(x)\,dx$ can often be simplified by introducing a new variable t in place of x, putting

$$x = \varphi(t). \tag{1.20}$$

An indefinite integral can be transformed simply by substituting the new variable in the integrand expression:

$$\int f(x)\,dx = \int f\left[\varphi(t)\right] \varphi'(t)\,dt + C. \tag{1.21}$$

This is proved from property I of 1.4 by showing the identity of the differentials of the left and right-hand sides of (1.21). We have on differentiating:

$$d\left(\int f(x)dx\right) = f(x)dx = f\left[\varphi(t)\right]\varphi'(t)dt\,,$$

$$d\left(\int f\left(\varphi(t)\right)\varphi'(t)dt\right) = f\left[\varphi(t)\right]\varphi'(t)dt\,.$$

The inverse is often used instead of substitution (1.20):

$$t = \psi(x) \text{ and } \psi'(x)dx = dt\,.$$

1.8. Examples

1. $\int (ax+b)^m dx \quad (\text{with } m \neq -1)$

The integral is simplified by substituting:

$$ax+b=t,\; adx=dt,\; dx=\frac{dt}{a}\,.$$

Setting these values in the integral given, we find:

$$\int (ax+b)^m\,dx = \frac{1}{a}\int t^m\,dt = \frac{1}{a}\frac{t^{m+1}}{m+1} + C = \frac{1}{a}\frac{(ax+b)^{m+1}}{m+1} + C\,.$$

Similarly we can prove:

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C\,,$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C\,.$$

2. The following example allows us to prove the rule: If numerator is equal to the derivative of the denominator primitive function is equal to logarithm of the denominator taken in absolute value

$$\int \frac{f'(x)\,dx}{f(x)} = \left| \begin{matrix} t = f(x) \\ dt = f'(x)dx \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C\,,$$

in particular $\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{a\,dx}{ax+b} = \frac{\ln|ax+b|}{a} + C\,.$

3.

$$\int \frac{dx}{a^2+x^2} = \left| \begin{matrix} t = \frac{x}{a}; x = at; \\ dt = \frac{dx}{a} \end{matrix} \right| = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{\arctan t}{a} + C = \frac{\arctan \frac{x}{a}}{a} + C.$$

4.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d\left(x/a\right)}{\sqrt{1-\left(x/a\right)^2}} = \left| \begin{matrix} t = \dfrac{x}{a} \\ dt = \dfrac{dx}{a} \end{matrix} \right| =$$

$$= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin \frac{x}{a} + C.$$

5.

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{d\,\frac{x}{a}}{\sqrt{\frac{x}{a^2}^{\,2} + 1}} = \left| t = \frac{x}{a} \right| = \int \frac{dt}{\sqrt{t^2+1}} =$$

$$= \ln\left|t + \sqrt{t^2+1}\right| + C = \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}+1}\right| + C = \ln\left|x + \sqrt{x^2+a^2}\right| + \underbrace{\ln a^{-1} + C}_{C}.$$

We also can evaluate this integral by using Euler's substitution, about which more will be said below. We introduce the new variable t given by the formula:

$$\sqrt{x^2+a^2} = t - x, \;\; t = x + \sqrt{x^2+a^2}\,.$$

We square both sides to find x and dx:

$$x^2 + a^2 = \left(t-x\right)^2 = t^2 - 2tx + x^2, \; x = \frac{t^2-a^2}{2t} = \frac{1}{2}\left(t - \frac{a^2}{t}\right),$$

$$\sqrt{x^2+a^2} = t - x = t - \frac{t^2-a^2}{2t} = \frac{t^2+a^2}{2t},$$

$$dx = \frac{1}{2}\left(1 + \frac{a^2}{t^2}\right)dt = \frac{1}{2}\frac{t^2+a^2}{t^2}dt\,.$$

On carrying out all these substitutions in the integral given, we have:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{2}\int \frac{2t}{t^2+a^2}\frac{t^2+a^2}{t^2}dt = \int \frac{dt}{t} = \ln t + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C\,.$$

6. The integral

$$\int \frac{dx}{x^2 - a^2}$$

is evaluated by means of a special method, fully described later: that of splitting the integrand into partial fractions.

Having factorized the denominator of the integrand:

$$x^2 - a^2 = (x-a)(x+a),$$

we write the integrand as a sum of simpler fractions:

$$\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}.$$

Constants A and B are found by clearing fractions:

$$1 = A(x+a) + B(x-a) = (A+B)x + a(A-B),$$

which must be true for any x. We thus find A and B from

$$a(A-B) = 1, \;\; A+B = 0, \;\; A = -B = \frac{1}{2a}.$$

We thus have:

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right] =$$

$$= \frac{1}{2a}\left[\ln(x-a) - \ln(x+a)\right] + C = \frac{1}{2a}\ln\frac{x-a}{x+a} + C.$$

7. Integrals of the more general type:

$$\int \frac{mx+n}{x^2+px+q}dx$$

can be reduced to the forms already given by completing the square in the denominator of the integrand. We have:

$$x^2 + px + q = \left(x + \frac{1}{2}p\right)^2 + q - \frac{p^2}{4}.$$

We now put:

$$x + \frac{1}{2}p = t, \;\; x = t - \frac{1}{2}p, \;\; dx = dt,$$

giving

$$mx + n = m\left(t - \frac{1}{2}p\right) + n = At + B,$$

where

$$A = m \text{ and } B = n - \frac{1}{2}mp.$$

We finally put

$$q - \frac{p^2}{4} = \pm a^2,$$

where the (+) or (-) sign must be taken in accordance with the sign of the left-hand side of this equation, a being taken positive; so that we can rewrite the given integral as:

$$\int \frac{mx+n}{x^2+px+q}dx = \int \frac{At+B}{t^2 \pm a^2}dt = A\int \frac{t\,dt}{t^2 \pm a^2} + B\int \frac{dt}{t^2 \pm a^2}.$$

The first of these integrals is evaluated at once by putting:

$$t^2 \pm a^2 = z; \quad 2tdt = dz,$$

which gives

$$\int \frac{t\,dt}{t^2 \pm a^2} = \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln z = \frac{1}{2}\ln\left(t^2 \pm a^2\right).$$

The second integral has the form calculated in Example 3 (for +) or 6 (for -).

8. Integrals of the type

$$\int \frac{mx+n}{\sqrt{x^2+px+q}}dx$$

can be reduced to known forms by the same method of completing the square. Using the notation of Example 7, we can rewrite the integral as

$$\int \frac{mx+n}{\sqrt{x^2+px+q}}dx = \int \frac{At+B}{\sqrt{t^2+b}}dt = A\int \frac{t\,dt}{\sqrt{t^2+b}} + B\int \frac{dt}{\sqrt{t^2+b}},$$

where

$$b = \pm a^2 = q - \frac{p^2}{4}.$$

The first of these integrals is evaluated by substituting

$$t^2 + b = z^2, \quad 2tdt = 2zdz,$$

giving

$$\int \frac{t\,dt}{\sqrt{t^2+b}} = \int \frac{z\,dz}{z} = \int dz = z = \sqrt{t^2+b}.$$

The second integral was worked out in Example 5 and is equal to $\ln\left(t+\sqrt{t^2+b}\right)$.

9. A similar method of completing the square can be used to reduce

$$\int \frac{mx+n}{\sqrt{q+px-x^2}}dx$$

to the form:

$$A_1\int \frac{t\,dt}{\sqrt{a^2-t^2}} + B_1\int \frac{dt}{\sqrt{a^2-t^2}},$$

where we have

$$\int \frac{t\,dt}{\sqrt{a^2 - t^2}} = -\sqrt{a^2 - t^2} + C$$

by using the substitution $a^2 - t^2 = z^2$. The second integral is worked out in Example 4.

10. Let we consider the three integrals

$$\int \sin mx \sin nx\,dx, \quad \int \sin mx \cos nx\,dx, \quad \int \cos mx \cos nx\,dx,$$

where m and n are positive integers. By well-known trigonometrical formulae we can divide each of these integrals into two parts, writing

$$\sin mx \sin nx = \frac{1}{2}\left\{\cos(m-n)x - \cos(m+n)x\right\},$$

$$\sin mx \cos nx = \frac{1}{2}\left\{\sin(m+n)x + \sin(m-n)x\right\},$$

$$\cos mx \cos nx = \frac{1}{2}\left\{\cos(m+n)x + \cos(m-n)x\right\}.$$

If we now make use of the substitutions $t = (m+n)x$ and $t = (m-n)x$ respectively, we directly obtain the following system of formulae:

$$\int \sin mx \sin nx\,dx = \begin{cases} \dfrac{1}{2}\left\{\dfrac{\sin(m-n)x}{m-n} - \dfrac{\sin(m+n)x}{m+n}\right\}, & m \neq n, \\ \dfrac{1}{2}\left(x - \dfrac{\sin 2mx}{2m}\right), & m = n, \end{cases}$$

$$\int \sin mx \cos nx\,dx = \begin{cases} -\dfrac{1}{2}\left\{\dfrac{\cos(m+n)x}{m+n} + \dfrac{\cos(m-n)x}{m-n}\right\}, & m \neq n, \\ -\dfrac{1}{2}\left(\dfrac{\cos 2mx}{2m}\right), & m = n, \end{cases}$$

$$\int \cos mx \cos nx\,dx = \begin{cases} \dfrac{1}{2}\left\{\dfrac{\sin(m+n)x}{m+n} + \dfrac{\sin(m-n)x}{m-n}\right\}, & m \neq n, \\ \dfrac{1}{2}\left(x + \dfrac{\sin 2mx}{2m}\right), & m = n. \end{cases}$$

Particularly

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x \right) + C = \frac{1}{2}\left(x - \sin x \cos x\right) + C;$$

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x \right) + C = \frac{1}{2}\left(x + \sin x \cos x\right) + C.$$

11. $\int \ln x\,dx$.

Here we use integration by parts. Let's suppose

$$u = \ln x, \quad dx = dv,$$

so

$$du = \frac{dx}{x}, \quad v = x,$$

whence by (19):

$$\int \ln x\,dx = x\ln x - \int x\frac{dx}{x} + C = x\ln x - x + C.$$

12. $\int x^2 e^x\,dx = \begin{vmatrix} u = x^2, du = 2xdx \\ v = e^x, dv = e^x dx \end{vmatrix} = e^x x^2 - 2\int xe^x\,dx$, then on integrating by parts again

$$\int xe^x\,dx = \begin{vmatrix} u = x, du = dx \\ dv = e^x dx, v = e^x \end{vmatrix} = e^x x - \int e^x\,dx = e^x x - e^x + C,$$

we get final answer

$$\int x^2 e^x\,dx = e^x\left(x^2 - 2x + 2\right) + C.$$

13. $\int x^3 \sin x\,dx = \left[\begin{array}{l|l} u = x^3 & du = 3x^2 dx \\ dv = \sin xdx & v = -\cos x \end{array}\right] = -x^3\cos x + 3\int x^2 \cos x\,dx$

finding

$$\int x^2\cos x\,dx = \left[\begin{array}{l|l} u = x^2 & du = 2xdx \\ dv = \cos xdx & v = \sin x \end{array}\right] = x^2\sin x - 2\int x\sin x\,dx,$$

and substituting above we get

$$\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx,$$

$$\int x\sin x\,dx = \begin{vmatrix} u = x & du = dx \\ v = -\cos x & dv = \sin xdx \end{vmatrix} =$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x$$

and finally we get

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$$

The method indicated by three previous examples is used generally for evaluating integrals of the type:

$$\int P_m(x)\ln x\,dx,\ \int P_m(x)e^{ax}\,dx,\ \int P_m(x)\sin(bx)\,dx,\ \int P_m(x)\cos(bx)\,dx,$$

where m is any positive integer $P_m(x)$ – is a polynomial of degree m; care is only needed to see that the power of x decreases with each successive transformation, until it reaches zero.

The following example is of a somewhat different nature; here a double application of the method of integration by parts bring us back to the original integral, for which we thus obtain an equation.

14. $\int e^x \cos x\,dx$. Integrating by parts twice, we get:

$$J = \int e^x\cos x\,dx = \left[\begin{array}{l|l} u = e^x & du = e^x dx \\ dv = \cos x dx & v = \sin x \end{array}\right] = e^x \sin x - \int e^x \sin x\,dx =$$

$$= \left[\begin{array}{l|l} u = e^x & du = e^x dx \\ dv = \sin x dx & v = -\cos x \end{array}\right] = \sin x - \left(-e^x\cos x + \int e^x \cos x dx\right) =$$

$$= e^x = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx = e^x\sin x + e^x\cos x - J,$$

solving the equation $J = e^x\sin x + e^x\cos x - J$ relatively to J we finally get:

$$J = \frac{e^x\sin x + e^x\cos x}{2} + C.$$

15. $\int x^\alpha \ln x\,dx \quad (\alpha \neq -1)$.

$$\int x^\alpha \ln x\,dx = \left|\begin{array}{l} v'(x) = x^\alpha, u(x) = \ln x \\ v(x) = \dfrac{x^{\alpha+1}}{\alpha+1},\quad u'(x) = \dfrac{1}{x} \end{array}\right| = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{1}{\alpha+1}\int x^\alpha\,dx =$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C.$$

In the case of $\alpha = -1$ we have obviously:

$$\int \frac{\ln x}{x}dx = \left|\begin{array}{l} t = \ln x \\ dt = \dfrac{dx}{x} \end{array}\right| = \frac{1}{2}(\ln x)^2 + C.$$

16. Integrals of inverse trigonometrical functions also can be taken by integration by parts:

$$\int \arcsin x\,dx = \begin{vmatrix} u(x) = \arcsin x \\ v'(x) = 1 \end{vmatrix} = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx =$$

$$= x\arcsin x + \sqrt{1-x^2} + C;$$

$$\int \arctan x\,dx = \begin{vmatrix} u(x) = \arctan x, v'(x) = 1 \\ u'(x) = \dfrac{1}{1+x^2}, v(x) = x \end{vmatrix} = x\arctan x - \int \frac{x\,dx}{1+x^2} =$$

$$= x\arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

17. Recurrence formulae. In many cases the integrand is a function not only of the independent variable, but also of an integral index n, and on integrating by parts we obtain, instead of the value of the integral, another similar expression in which the index n has a smaller value. We thus arrive after a number of steps at an integral which we can deal with by means of our table of integrals. Such a process is called a recurrence process. The following examples illustrate this: by repeated integration by parts we can calculate the trigonometrical integrals

$$\int \cos^n x\,dx,\quad \int \sin^n x\,dx,\quad \int \sin^m x \cos^n x\,dx$$

provided that m and n are integers. For we find that

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + \left(n-1\right)\int \cos^{n-2} x \sin^2 x\,dx,$$

we can write the right-hand side in the form

$$\cos^{n-1} x \sin x + \left(n-1\right)\int \cos^{n-2} x\,dx - \left(n-1\right)\int \cos^n x\,dx,$$

thus obtaining the recurrence relation

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

This formula enables us to keep on diminishing the index in the integrand until we finally arrive at the integral

$$\int \cos x\,dx = \sin x \text{ or } \int dx = x,$$

according as n is odd or even. In a similar way we obtain the analogous recurrence formulae

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

and

$$\int \sin^m x \cos^n x\,dx = \frac{1}{n+m}\sin^{m+1} x \cos^{n-1} x + \frac{n-1}{n+m}\int \sin^m x \cos^{n-2} x\,dx.$$

In particular, these formulae enable us to calculate the integrals (see also ex. 12)

$$\int \sin^2 x\,dx = \frac{1}{2}\left(x - \sin x \cos x\right), \quad \int \cos^2 x\,dx = \frac{1}{2}\left(x + \sin x \cos x\right).$$

It need hardly be mentioned that the corresponding integrals for the hyperbolic functions can be calculated in exactly the same way.

Further *recurrence formulae* are given by the following transformations:

$$\int (\log x)^m dx = x(\log x)^m - m\int (\log x)^{m-1}\,dx;$$

$$\int x^m e^x\,dx = x^m e^x - m\int x^{m-1} e^x\,dx;$$

$$\int x^m \sin x\,dx = -x^m \cos x + m\int x^{m-1} \cos x\,dx;$$

$$\int x^m \cos x\,dx = x^m \sin x - m\int x^{m-1} \sin x\,dx;$$

$$\int x^\alpha (\ln x)^m\,dx = \frac{x^{\alpha+1}(\ln x)^m}{\alpha+1} - \frac{m}{\alpha+1}\int x^\alpha (\ln x)^{m-1}\,dx \quad (\alpha \neq -1).$$

18. The integral

$$\int \sqrt{x^2 + a}\,dx$$

can be reduced to a known form with the aid of integration by parts:

$$\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} - \int x\,d\left(\sqrt{x^2+a}\right) = x\sqrt{x^2+a} - \int \frac{x^2}{\sqrt{x^2+a}}\,dx \ .$$

Adding and subtracting a in the numerator of the last integrand, we can rewrite the above equation as

$$\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} - \int \sqrt{x^2+a}\,dx + a\int \frac{1}{\sqrt{x^2+a}}\,dx \ ,$$

or

$$2\int \sqrt{x^2+a}\,dx = x\sqrt{x^2+a} + a\int \frac{1}{\sqrt{x^2+a}}\,dx,$$

hence finally

$$\int \sqrt{x^2+a}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a} + a\ln\left(x + \sqrt{x^2+a}\right)\right] + C.$$

19. $$\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{b^2}\int \frac{1}{\frac{a^2}{b^2}\tan^2 x + 1} \cdot \frac{dx}{\cos^2 x} =$$

$$= \left| \begin{matrix} t = \dfrac{a}{b}\tan x \\ dt = \dfrac{a}{b}\dfrac{dx}{\cos^2 x} \end{matrix} \right| = \frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right) + C$$

and

$$\int \frac{dx}{a^2\sin^2 x - b^2\cos^2 x} = \begin{cases} -\dfrac{1}{ab}\operatorname{arth}\left(\dfrac{a}{b}\tan x\right) + C, \\ -\dfrac{1}{ab}\operatorname{arcth}\left(\dfrac{a}{b}\tan x\right) + C. \end{cases}$$

Inverse hyperbolic functions can be reduced to logarithms by formulae

$$\operatorname{arth} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \quad \operatorname{arcth} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right).$$

20. We evaluate the integral

$$\int \frac{dx}{\sin x}$$

by writing $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2}$, and putting $t = \tan\frac{x}{2}$, so that $dt = \frac{1}{2}\sec^2\frac{x}{2}dx$; the integral then becomes

$$\int \frac{dx}{\sin x} = \int \frac{dt}{t} = \ln\left|\tan\frac{x}{2}\right| + C,$$

If we replace x by $x + \pi/2$, this formula becomes

$$\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C.$$

21. By the substitution $x = \cos t$, equivalent to $t = \arccos x$, or, more generally, $x = a\cos t \quad (a \neq 0)$, we can reduce

$$\int \sqrt{1-x^2}\, dx \text{ and } \int \sqrt{a^2 - x^2}\, dx$$

respectively to these formulae. We thus obtain

$$\int \sqrt{a^2 - x^2}\, dx = -\frac{a^2}{2}\arccos\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2}.$$

Similarly, by the substitution $x = a\cosh t$ we obtain the formula

$$\int \sqrt{x^2 - a^2}\, dx = -\frac{a^2}{2}\operatorname{arcosh}\frac{x}{a} + \frac{x}{2}\sqrt{x^2 - a^2},$$

and by the substitution $x = a\sinh t$

$$\int \sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\operatorname{arsinh}\frac{x}{a} + \frac{x}{2}\sqrt{x^2+a^2}$$

$$(\operatorname{arsinh} z = \ln\left(z+\sqrt{z^2+1}\right), \quad \operatorname{arcosh} z = \ln\left(z+\sqrt{z^2-1}\right)).$$

The substitution $t = \frac{a}{x}$, $dx = -\frac{a}{t^2}dt$ leads to the formulae

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\arcsin\frac{a}{x} + C; \quad \int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\operatorname{arsinh}\frac{a}{x} + C;$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\operatorname{arcosh}\frac{a}{x} + C.$$

Exrercises

Find primitive functions:

1. $\int \frac{xdx}{x^2-x+1}$.
2. $\int \frac{xdx}{\sqrt{x^2-4x+1}}$.
3. $\int \frac{dx}{\sqrt{5+2x+x^2}}$.
4. $\int \frac{x^4}{1-x}dx$.
5. $\int \frac{dx}{x^2+x+1}$.
6. $\int \frac{dx}{\sqrt{3-2x-x^2}}$.
7. $\int \frac{\cos x}{\sin^2 x}dx$.
8. $\int \frac{x^7}{\left(1-x^4\right)^2}\,dx$.
9. $\int x^3\left(\sqrt{1-x^2}\right)^5 dx$.
10. $\int \cos^n x \sin x\,dx$.
11. $\int x^3 e^{-x^2}dx$.
12. $\int \cos^6 x dx$.
13. $\int x^2\cos x\,dx$.
14. $\int x^3\cos x^2 dx$.
15. $\int x^2\left(\ln x\right)^2 dx$.
16. $\int \frac{x\cos x}{\sin^2 x}\,dx$.
17. $\int xe^{x^2}dx$.
18. $\int x^2\sqrt{1+x^3}dx$.

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19. $\int \frac{3dx}{9x^2-6x+2}$.
20. $\int \frac{dx}{x\left(\ln x\right)^n}$.
21. $\int x^3 e^x dx$.

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22. $\int \frac{\ln x}{x}\,dx$.
23. $\int_0^1 \frac{\arctan x}{1+x^2}\,dx$.
24. $\int \frac{6x}{2+3x}\,dx$.

1.9. Integration of Rational Functions

The most important general class of functions integrable in terms of elementary functions consists of the rational functions

$$R(x)=\frac{f(x)}{g(x)},$$

where $f(x)$ and $g(x)$ are polynomials:

$$f(x)=a_m x^m+a_{m-1}x^{m-1}+...+a_0,$$
$$g(x)=b_n x^n+b_{n-1}x^{n-1}+...+b_0\ .$$

.

We suppose the polynomials be of degree m and n respectively, i.e. $a_m\neq 0,\ b_n\neq 0$. In case $m<n$ the function is called *proper rational function* or just *proper fraction*. Otherwise it is called *improper rational function* or *improper fraction*.

We need only consider those rational functions for which the denominator is not a constant, because every polynomial can be integrated at once and that the integral is itself a polynomial. Moreover, we can always assume that the degree of the numerator is less than that of the denominator. For otherwise we can divide the polynomial $f(x)$ by the polynomial $g(x)$ and obtain a remainder of degree less than n in other words, we can write $f(x)=q(x)g(x)+r(x)$, where $q(x)$ and $r(x)$ are also polynomials and $r(x)$ is of lower degree than n. The integration of $\frac{f(x)}{g(x)}$ is therefore reduced to the integration of the polynomial $q(x)$ and of the proper fraction $\frac{r(x)}{g(x)}$.

1.9.1. Integration of the Fundamental Types

We shall not at once proceed to the consideration of the most general proper fraction, but shall instead study only those functions in which the denominator $g(x)$ is of a particularly simple type, namely,

$$g(x)=x,\ \ g(x)=1+x^2,$$

or, more generally,

$$g(x)=x^n,\ \ g(x)=\left(1+x^2\right)^n,$$

where n is any positive integer.

To this case we can reduce the somewhat more general case in which $g(x)=(\alpha x+\beta)^n$, a power of a linear expression $\alpha x+\beta$ $(\alpha\neq 0)$, or $g(x)=(ax^2+bx+c)^n$, a power of a definite quadratic expression. A quadratic expression $Q(x)=ax^2+bx+c,\quad (a\neq 0)$ is said to be definite if for all real values of x it takes values having one and the same sign, i.e. if the equation $Q(x)=0$ has no real roots. For this it is necessary and sufficient that $D=b^2-4ac$ should be negative.

In the first case we introduce a new variable $\xi=ax+\beta$. Then $d\xi/dx=\alpha$, and $x=(\xi-\beta)/\alpha$ is also a linear function of ξ. Each numerator $f(x)$ becomes a polynomial $\varphi(\xi)$ of the same degree, and consequently

$$\int\frac{f(x)}{(\alpha x+\beta)^n}dx=\frac{1}{\alpha}\int\frac{\varphi(\xi)}{\xi^n}d\xi.$$

In the second case, we write (this rearrangement is called "*taking complete square"*)

$$ax^2+bx+c=\frac{1}{a}\left(ax+\frac{b}{2}\right)^2-\frac{D}{4a}\qquad\left(D=b^2-4ac,\,\mathrm{D}<0\right).$$

By introducing the new variable

$$\xi=\frac{2ax+b}{\sqrt{-D}}$$

we arrive at an integral with the denominator $\left[\frac{-D}{4a}\left(1+\xi^2\right)\right]^n$.

Hence in order to integrate rational functions whose denominators are powers of a linear expression or of a definite quadratic expression it is sufficient to be able to integrate the following types of functions:

$$\frac{1}{x^n},\quad\frac{x^{2\nu}}{\left(x^2+1\right)^n},\quad\frac{x^{2\nu+1}}{\left(x^2+1\right)^n}.$$

We shall, in fact, see that even these types need not be treated in general, for we can reduce the integration of every rational function to the integration of the very special forms of these three functions obtained by taking $\nu=0$. Accordingly we now consider the integration of the three expressions

$$\frac{1}{x^n},\quad\frac{1}{\left(x^2+1\right)^n},\quad\frac{x}{\left(x^2+1\right)^n}.$$

Integration of the first type of function, $\frac{1}{x^n}$, immediately yields the expression $\ln|x|$ if $n=1$, and the expression $-\frac{1}{(n-1)x^{n-1}}$ if $n>1$, so that in both cases the integral is again an elementary function. Functions of the third type can be integrated immediately by introducing the new variable $\xi = x^2+1$, whence we obtain $2xdx = d\xi$ and

$$\int \frac{x}{\left(x^2+1\right)^n}dx = \frac{1}{2}\int \frac{d\xi}{\xi} = \begin{cases} \frac{1}{2}\ln\left(x^2+1\right) & \text{if} \quad n=1, \\ -\frac{1}{2(n-1)\left(x^2+1\right)^{n-1}} & \text{if} \quad n>1. \end{cases}$$

Finally, in order to calculate the integral

$$I_n = \int \frac{1}{\left(x^2+1\right)^n}dx ,$$

where n has any value exceeding 1, we make use of a recurrence method. For it we put

$$\frac{1}{\left(x^2+1\right)^n} = \frac{1}{\left(x^2+1\right)^{n-1}} - \frac{x^2}{\left(x^2+1\right)^n} ,$$

so that $\int \frac{1}{\left(x^2+1\right)^n}dx = \int \frac{1}{\left(x^2+1\right)^{n-1}}dx - \int \frac{x^2}{\left(x^2+1\right)^n}dx$,

then we can transform the right-hand side by integrating by parts:

$$\int \frac{x^2}{\left(x^2+1\right)^n}dx = \left| \begin{matrix} u(x)=x, & u'(x)=1, \\ v'(x)=\frac{x}{\left(x^2+1\right)^n}, & v(x)=-\frac{1}{2(n-1)\left(x^2+1\right)^{n-1}} \end{matrix} \right| =$$

$$= -\frac{x}{2(n-1)\left(x^2+1\right)^{n-1}} + \frac{1}{2(n-1)}\int \frac{1}{\left(x^2+1\right)^{n-1}}dx ,$$

and then we get recurrence formula

$$I_n = \int \frac{1}{\left(x^2+1\right)^n}dx = \frac{x}{2(n-1)\left(x^2+1\right)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{1}{\left(x^2+1\right)^{n-1}}dx$$

or

$$I_n = \frac{x}{2(n-1)\left(x^2+1\right)^{n-1}} + \frac{2n-3}{2(n-1)} I_{n-1}.$$

The calculation of the integral I_n is thus reduced to that of the integral I_{n-1}. If $n-1>1$ we apply the same process to the latter integral, and continue the process until we finally arrive at the expression

$$\int \frac{dx}{1+x^2} = \arctan x .$$

We thus see that the integral I_n can be explicitly expressed in terms of rational functions and the function $\arctan x$.

The integral of the function $\dfrac{1}{\left(x^2-1\right)^n}$ can be calculated in the same way; by the corresponding recurrence method we reduce it to the integral $\int \frac{dx}{1-x^2} = \operatorname{arth} x \ (\text{or} \ \operatorname{arcth} x)$.

Incidentally, we could also have integrated the function $\dfrac{1}{\left(x^2+1\right)^n}$ directly using the substitution $x = \tan t$; we should then have obtained $dx = \sec^2 t dt$ and $1/\left(1+x^2\right) = \cos^2 t$, so that

$$\int \frac{dx}{\left(x^2+1\right)^n} = \int \cos^{2n-2} t \, dt ,$$

and we have already learned how to evaluate this integral.

1.9.2. Partial Fractions

We are now in a position to integrate the most general rational functions, in virtue of the fact that every such function can be represented as the sum of so-called partial fractions, i.e. as the sum of a polynomial and a finite number of rational functions, each one of which has either a power of a linear expression for its denominator and a constant for its numerator, or else a power of a definite quadratic expression for its denominator and a linear function for its numerator. If the degree of the numerator $f(x)$ is less than that of the denominator $g(x)$ the polynomial does not occur. We are now in a position to integrate each partial fraction. The denominator can be reduced to one of the special forms x^n and $\left(x^2+1\right)^n$, and the fraction is then a combination of the fundamental types integrated earlier.

We shall not give the general proof of the possibility of this resolution into partial fractions. On the contrary, we shall confine ourselves to making the statement of the theorem intelligible to the reader and to showing by examples how the resolution into partial fractions can be carried out in typical cases. In actual practice only comparatively simple functions are dealt with, for otherwise the computations become far too complicated.

It is known from elementary algebra, every polynomial $g(x)$ can be written in the form

$$g(x)=a\left(x-\alpha_1\right)^{l_1}\dots\ \left(x-\alpha_k\right)^{l_k}\left(x^2+2b_1x+c_1\right)^{r_1}\dots\ \left(x^2+2b_mx+c_m\right)^{r_m}.$$

Here the numbers $\alpha_1,\dots,\alpha_k$ are the real and distinct roots of the equation $g(x)=0$, and the positive integers $l_1,\dots,l_k$ indicate the numbers of times they are repeated; the factor $x^2+2b_\nu x+c_\nu$ indicate definite quadratic expressions, of which no two are the same, with conjugate complex roots, and the positive integers $r_1,\dots,r_m$ give the numbers of times that these roots are repeated.

We assume that the denominator is either given to us in this form or that we have brought it to this form by calculating the real and imaginary roots. Let us further suppose that the numerator $f(x)$ is of lower degree than the denominator. Then the theorem on decomposition into partial fractions can be stated as follows. For each factor $(x-\alpha)^l$, where α is any one of the real roots and l is the number of times it is repeated, we can determine an expression of the form

$$\frac{A_1}{(x-\alpha)}+\frac{A_2}{(x-\alpha)^2}+\dots+\frac{A_l}{(x-\alpha)^l},$$

and for each quadratic factor $Q(x)=x^2+2bx+c$ in our product which is raised to the power r we can determine an expression of the form

$$\frac{B_1+C_1x}{Q}+\frac{B_2+C_2x}{Q^2}+\dots+\frac{B_r+C_rx}{Q^r},$$

in such a way that the function $\dfrac{f(x)}{g(x)}$ is the sum of all these expressions. In other words, the quotient $\dfrac{f(x)}{g(x)}$ can be represented as a sum of fractions each of which belongs to one or other of the types integrated earlier.

Here we give a brief sketch of the method by which the possibility of this decomposition into partial fractions can be proved. If $g(x)=(x-\alpha)^k h(x)$ and $h(\alpha)\neq 0$, then on the right-hand side of the equation

$$\frac{f(x)}{g(x)}-\frac{f(\alpha)}{h(\alpha)(x-\alpha)^k}=\frac{1}{h(\alpha)}\frac{f(x)h(\alpha)-f(\alpha)h(x)}{(x-\alpha)^k h(x)}$$

the numerator obviously vanishes for $x=\alpha$ it is therefore of the form $h(\alpha)(x-\alpha)^m f_1(x)$ where $f_1(x)$ is also a polynomial, the integer $m\geq 1$, and $f_1(\alpha)\neq 0$. Writing $\dfrac{f(\alpha)}{h(\alpha)}=\beta$, this gives us

$$\frac{f(x)}{g(x)}-\frac{\beta}{(x-\alpha)^k}=\frac{f_1(x)}{(x-\alpha)^{k-m}h(x)}.$$

Continuing the process, we can keep on diminishing the degree of the power of $(x-\alpha)$ occurring in the denominator until finally no such factor is left. On the remaining fraction we repeat the process for some other root of $g(x)$, and do this as many times as $g(x)$ has distinct factors. This being done not only for the real but also for the complex roots, we eventually arrive at the complete analysis into partial fractions. Two terms corresponding to a pair of complex conjugate roots have complex conjugate numerators and can be reduced to a common denominator; for example

$$\frac{K+iL}{x-(\alpha+i\beta)}+\frac{K-iL}{x-(\alpha-i\beta)}=\frac{B+Cx}{x^2+2bx+c},$$

$$b=-\alpha, c=\alpha^2+\beta^2,\ b^2<c,$$

$$B=-2(K\alpha+L\beta),\ C=2K.$$

In particular cases the splitting up into partial fractions can be done easily by inspection.

If, for example, $g(x)=x^2-1$, we see at once that

$$\frac{1}{x^2-1}=\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x+1},$$

so that

$$\int\frac{dx}{x^2-1}=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C.$$

More generally, if $g(x)=(x-\alpha)(x-\beta)$, that is, if $g(x)$ is a non-definite quadratic expression with two real zeros α and β, we have

$$\frac{1}{(x-\alpha)(x-\beta)}=\frac{1}{(\alpha-\beta)}\frac{(x-\beta)-(x-\alpha)}{(x-\alpha)(x-\beta)}=\frac{1}{(\alpha-\beta)}\left[\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right]$$

so that

$$\int\frac{dx}{(x-\alpha)(x-\beta)}=\frac{1}{(\alpha-\beta)}\int\left[\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right]dx=\frac{1}{\alpha-\beta}\ln\left|\frac{x-\alpha}{x-\beta}\right|.$$

1.9.3. Further Examples of Resolution into Partial Fractions. The Method of Undetermined Coefficients

If $g(x)=(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$, where $\alpha_i\neq\alpha_k$ if $i\neq k$, i.e. if the equation $g(x)=0$ has only single real roots, the expression in terms of partial fractions has the simple form

$$\frac{1}{g(x)}=\frac{A_1}{(x-\alpha_1)}+\frac{A_2}{(x-\alpha_2)}+...+\frac{A_n}{(x-\alpha_n)}.$$

We obtain explicit expressions for the coefficients $A_1, A_2,...$ if we multiply both sides of this equation by $(x-\alpha_1)$, cancel the common factor $(x-\alpha_1)$ in the numerator and the denominator on the left and in the first term on the right, and then put $x=\alpha_1$. This gives

$$A_1=\frac{1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)...(\alpha_1-\alpha_n)}.$$

Note that the denominator on the right is $g'(\alpha_1)$, i.e. the derivative of the function $g(x)$ at the point $x=\alpha_1$. Really,

$$g'(x)=(x-\alpha_2)...(x-\alpha_n)+(x-\alpha_1)(x-\alpha_3)...(x-\alpha_n)+$$
$$+\ldots+(x-\alpha_1)\ldots(x-\alpha_{k-1})(x-\alpha_{k+1})...(x-\alpha_n)+\ldots+$$
$$+(x-\alpha_1)...(x-\alpha_{n-1})=(x-\alpha_2)...(x-\alpha_n)+(x-\alpha_1)\,p(x)$$

where $p(x)$ is a polynomial, so

$$g'(\alpha_1)=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)...(\alpha_1-\alpha_n)+\underbrace{(\alpha_1-\alpha_1)\,p(\alpha_1)}_{=0}.$$

As a typical example of a denominator $g(x)$ with multiple roots, we consider the function $\frac{1}{x^2(x-1)}$. The preliminary statement

$$\frac{1}{x^2(x-1)}=\frac{A}{x-1}+\frac{B}{x}+\frac{C}{x^2}$$

leads us to the required result. If we multiply both sides of this equation by $x^2(x-1)$ we obtain the equation

$$1=(A+B)x^2-(B-C)x-C,$$

true for all values of x, from which we have to determine the coefficients A, B, C. This condition cannot hold unless all the coefficients of the polynomial $(A+B)x^2-(B-C)x-C-1$ are zero, i.e. we must have $A+B=B-C=C+1=0$ or $C=-1, B=-1, A=1$. We thus obtain the resolution

$$\frac{1}{x^2(x-1)}=\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2},$$

and consequently

$$\int\frac{dx}{x^2(x-1)}=\ln|x-1|-\ln|x|+\frac{1}{x}\ .$$

We shall now split up the function $\frac{1}{x(x^2+1)}$ (which is an example of the case where the zeros of the denominator are complex) in accordance with the equation

$$\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}.$$

For the coefficients we obtain $A+B=C=A-1=0$, so that

$$\frac{1}{x(x^2+1)}=\frac{1}{x}-\frac{x}{x^2+1},$$

and consequently

$$\int \frac{dx}{x\left(x^2+1\right)} = \ln|x| - \frac{1}{2}\ln\left(x^2+1\right).$$

As a third example we consider the function $\dfrac{1}{x^4+1}$. Even Leibnitz found this a troublesome integration. We can represent the denominator as the product of two quadratic factors:

$$x^4+1 = \left(x^2+1\right)^2 - 2x^2 = \left(x^2+1+\sqrt{2}x\right)\left(x^2+1-\sqrt{2}x\right).$$

We know, therefore, that the resolution into partial fractions will have the form

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}.$$

To determine the coefficients A, B, C, D, we have

$$\left(A+C\right)x^3 + \left(B+D-A\sqrt{2}+C\sqrt{2}\right)x^2 + (A+C-B\sqrt{2}+D\sqrt{2})x + (B+D) = 1,$$

so A, B, C, D satisfy the set of equations

$$\begin{cases} \left(A+C\right)=0, \\ B+D-A\sqrt{2}+C\sqrt{2}=0, \\ A+C-B\sqrt{2}+D\sqrt{2}=0, \\ B+D=1, \end{cases}$$

from which

$$A=\frac{1}{2\sqrt{2}},\ B=\frac{1}{2},\ C=-\frac{1}{2\sqrt{2}},\ D=\frac{1}{2}.$$

We therefore have

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{1}{2\sqrt{2}}\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1},$$

and finally we obtain

$$\int \frac{1}{x^4+1}dx = \frac{1}{4\sqrt{2}}\ln\left|x^2+\sqrt{2}x+1\right| - \frac{1}{4\sqrt{2}}\ln\left|x^2-\sqrt{2}x+1\right| +$$

$$+\frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right) + \frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}x-1\right),$$

which may easily be verified by differentiation.

Exercises

Integrate:

1. $\int \frac{dx}{2x-3x^2}$.
2. $\int \frac{dx}{x^2\left(x^2+1\right)^2}$.
3. $\int \frac{x^2\,dx}{\left(x-1\right)^2\left(x^2+1\right)}$.
4. $\int \frac{dx}{x^2-x}$.
5. $\int \frac{dx}{1-x^3}$.
6. $\int \frac{\left(x-4\right)dx}{\left(x^2+1\right)\left(x-2\right)}$.
7. $\int \frac{3\,dx}{x\left(x+1\right)^3}$.
8. $\int \frac{x^6}{1-x^4}\,dx$.
9. $\int \frac{x^2}{x^4+x^2-2}\,dx$.

1.10. Integration of Trigonometric Functions

The integration of some other general classes of functions can be reduced to the integration of rational functions. We shall be better able to understand this reduction if we begin by stating certain elementary facts about the trigonometric and irrational functions.

Let's consider integral

$$\int R\left(\sin x, \cos x\right)dx,$$

where R is a rational function of its arguments, i.e. the quotient of polynomials in these arguments.

If we put $t=\tan\frac{x}{2}$, elementary trigonometry gives us the simple formulae

$$\sin x=\frac{2t}{1+t^2}, \quad \cos x=\frac{1-t^2}{1+t^2}.$$

These equations show that $\sin x,\ \cos x$ can be expressed rationally in terms of t. On taking differential of t, we get $dx=\frac{2dt}{1+t^2}$. Therefore, by using substitution $t=\tan\frac{x}{2}$ we can reduce the integral

$$\int R\left(\sin x, \cos x\right)dx,$$

to the integral of rational function

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2}$$

that can be taken by the method developed above.

The substitution $t = \tan\frac{x}{2}$ is called *general trigonometric substitution*.

Example,

$$\int \frac{3+\cos x}{5-\sin x}dx = \begin{bmatrix} \tan\frac{x}{2} = t, & \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2}, & dx = \frac{2dt}{1+t^2} \end{bmatrix} =$$

$$= \int \frac{3+\frac{1-t^2}{1+t^2}}{5-\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = 2\int \frac{2t^2+4}{5t^2-2t+5}\frac{dt}{1+t^2},$$

so the initial integral has been reduced to the integral of the rational function. However, the general trigonometric substitution sometimes requires us to use difficult transformations.

In the three cases we can avoid general trigonometric substitution, by using another:

1. If the integrand is odd by $\sin x$, i.e. $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ we have to use substitution: $\cos x = t$.
2. If the integrand is odd by $\cos x$, i.e. $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ we have to use substitution: $\sin x = t$.
3. If the integrand is even by $\sin x$, and $\cos x$ simultaneously, i.e. $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ we have to use substitution: $\tan x = t$.

For example:

1. $\int \sin x \cos^2 x\,dx = \begin{vmatrix} \cos x = t \\ dt = -\sin x\,dx \end{vmatrix} = -\int t^2\,dt = -\frac{t^3}{3} + C = -\frac{\cos^3 x}{3} + C$.

2. $\int \frac{\cos x}{\sin^4 x}dx = \begin{vmatrix} \sin x = t \\ dt = \cos x\,dx \end{vmatrix} = \int \frac{dt}{t^4} = \int t^{-4}\,dt = \frac{t^{-3}}{-3} + C = \frac{-1}{3\sin^3 x} + C$.

3. $\int \frac{\sin^2 x}{\cos^4 x}dx = \begin{vmatrix} \tan x = t \\ dt = \frac{dx}{\cos^2 x} \end{vmatrix} = \int t^2 x\,dt = \frac{t^3}{3} + C = \frac{\tan^3 x}{3} + C$.

If Integrals of form $\int \sin^n x \cos^m x\,dx$ they both powers are even, it is more convenient to integrate using power reducing formulae:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

For example,

$$\int \sin^2 x \cos^4 x\,dx = \int \frac{(\sin 2x)^2}{4}\cos^2 x\,dx = \frac{1}{4}\int \frac{1-\cos 4\text{x}}{2}\cdot\frac{1+\cos 2x}{2}dx =$$

$$= \frac{1}{16}\int (1+\cos 2x - \cos 4x - \cos 2x \cos 4x)dx =$$

$$= \frac{1}{16}\int \left(1+\cos 2x - \cos 4x - \frac{1}{2}(\cos 2x + \cos 6x)\right)dx =$$

$$= \frac{1}{16}\int \left(1+\frac{1}{2}\cos 2x - \cos 4x - \frac{1}{2}\cos 6x\right)dx =$$

$$= \frac{1}{16}\left(x+\frac{1}{4}\sin 2x - \frac{1}{4}\sin 4\text{x} - \frac{1}{12}\sin 6x\right)+C.$$

1.11. Integration of expressions containing radicals

We consider some other types of integral, which reduce to integrals of rational fractions.

1. The integral

$$\int R\left[x, \left(\frac{\alpha x+\beta}{ax+b}\right)^{m_1}, \left(\frac{\alpha x+\beta}{ax+b}\right)^{m_2}, \ldots, \left(\frac{\alpha x+\beta}{ax+b}\right)^{m_n}\right]dx,$$

where R is a rational function of its arguments, whilst $m_1, m_2, \ldots, m_n$ are rational numbers.

Let M be the least common denominator of these fractions. We introduce a new variable t:

$$t^M = \left(\frac{\alpha x+\beta}{ax+b}\right).$$

Evidently, after this x, $\frac{dx}{dt}$ and all the expressions $\left(\frac{\alpha x+\beta}{ax+b}\right)^{m_k}$ $(k=1,\ldots,n)$ will be rational functions of t, and the integral reduces to that taken of a rational fraction.

2. Integration of quadratic irrationality.

An integral of form $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$, where R is a rational function of its arguments, leads to the integral of rational function with the aid of Euler's substitutions:

I. $\sqrt{ax^2+bx+c}=\sqrt{a}x+t$, if $a>0$.

II. $\sqrt{ax^2+bx+c}=\sqrt{c}+xt$, if $c>0$.

III. $\sqrt{ax^2+bx+c}=t(x-\lambda)$, if $D=b^2-4ac>0$, where λ is a real root of equation $ax^2+bx+c=0$.

For example, $\int\frac{dx}{x\sqrt{x^2+8x+1}}$.

We apply Euler's first substitution: $\sqrt{x^2+8x+1}=t+x$. Squaring this we get

$$x^2+8x+1=(t+x)^2=t^2+2tx+x^2,$$

then on canceling x^2, we get $8x+1=t^2+2tx$ and

$$x=\frac{t^2-1}{8-2t},\quad dx=-\frac{\left(t^2-8t+1\right)}{2(4-t)^2}dt.$$

After the substitution we have

$$\int\frac{-\frac{\left(t^2-8t+1\right)}{2(4-t)^2}dt}{\frac{t^2-1}{8-2t}\left(t+\frac{t^2-1}{8-2t}\right)}=2\int\frac{-\left(t^2-8t+1\right)}{\left(t^2-1\right)\left(8t-t^2-1\right)}dt=\ln\left|\frac{t-1}{t+1}\right|+C=$$

$$=\ln\left|\frac{\sqrt{x^2+8x+1}-x-1}{\sqrt{x^2+8x+1}-x+1}\right|+C.$$

Despite Euler's substitutions can rationalize any integral of the form $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$, they require us to do extensive calculations.

Sometimes it is more comfortable to use the following trigonometric substitutions (arbitrary integral $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ can be reduced to one of 3 forms listed below by taking complete square in ax^2+bx+c; see 1.9.1):

1. $x=\alpha\sin t$ or $x=\alpha\cos t$ for $\int R\left(x,\sqrt{\alpha^2-x^2}\right)dx\ (\alpha>0)$.

2. $x=\dfrac{\alpha}{\cos t}$ for $\int R\left(x,\sqrt{x^2-\alpha^2}\right)dx\ (\alpha>0)$.

3. $x=\alpha\tan t$ for $\int R\left(x,\sqrt{x^2+\alpha^2}\right)dx\ (\alpha>0)$,

or the following hyperbolic trigonometric substitutions (see example 23, sec. 1.8):

4. $x=\alpha\tanh t$ for $\int R\left(x,\sqrt{\alpha^2-x^2}\right)dx\ (\alpha>0)$.

5. $x=\alpha\cosh t$ or $x=\alpha\coth t$ for $\int R\left(x,\sqrt{x^2-\alpha^2}\right)dx\ (\alpha>0)$.

6. $x=\alpha\sinh t$ for $\int R\left(x,\sqrt{x^2+\alpha^2}\right)dx\ (\alpha>0)$.

For example,

$$\int x^2\sqrt{9-x^2}\,dx=\left|\begin{matrix}x=3\sin t\\ dx=3\cos t\,dt\end{matrix}\right|=$$

$$=\int 9\sin^2 t\sqrt{9-3\sin^2 t}\,3\cos t\,dt=81\int\sin^2 t\sqrt{1-\sin^2 t}\cos t\,dt=$$

$$=81\int\sin^2 t\cos^2 t\ dt=-\frac{81}{4}\int\sin^2 2t\ dt=$$

$$=-\frac{81}{4}\int\frac{1-\cos 4t}{2}\ dt=-\frac{81}{8}\left(t-\frac{\sin 4t}{4}\right)+C.$$

With the aim to get the answer in terms of x we note that $\mathrm{t}=\operatorname{asin}\left(\dfrac{x}{3}\right)$,

$$\sin 4t=2\sin 2t\cos 2t=4\sin t\sqrt{1-\sin^2 t}\left(1-2\sin^2 t\right)=\frac{4}{81}\left(9x-2x^3\right)\sqrt{9-x^2}\ ,$$

hence

$$\int x^2\sqrt{9-x^2}\,dx=\frac{1}{8}\left(81\cdot\operatorname{asin}\left(\frac{x}{3}\right)+\left(2x^3-9x\right)\sqrt{9-x^2}\right)+C\,.$$

Exercises

Integrate:

1. $\int \frac{x+1}{x^3-1}dx$.

2. $\int \frac{dx}{x^4+16}$.

3. $\int \frac{3x+1}{x^2-7x+12}dx$.

4. $\int \frac{dx}{\cos x}$.

5. $\int \frac{dx}{\sin^3 x}$.

6. $\int \frac{x+1}{(x-1)^2(x+2)}dx$.

7. $\int \tan^3 x\, dx$.

8. $\int \frac{dx}{1+\cos^2 x}$.

9. $\int x\sqrt{x^2+4x}\, dx$.

10. $\int \frac{dx}{\cos^6 x}$.

11. $\int \frac{dx}{2+\sin x}$.

12. $\int \frac{dx}{3+\sin^2 x}$.

13. $\int \frac{\cos x\, dx}{\cos x+\sin x}$.

14. $\int \frac{dx}{1+\cos x}$.

15. $\int \frac{dx}{\sin x+\cos x}$.

16. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

17. $\int_0^{\pi/2} \frac{dx}{3+\cos x}$.

18. $\int \cos^2 x\cdot\sin^3 x\, dx$.

19. $\int \frac{\sqrt{x}+1dx}{\sqrt[3]{x}-1}$.

20. $\int \frac{dx}{1+\sin x}$.

21. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$.

22. $\int \sqrt{x^2-4}\, dx$.

23. $\int x\sqrt{\frac{x-1}{x+1}}\, dx$.

24. $\int \sqrt[3]{\frac{x+1}{x-1}}\, dx$.

2. PROPERTIES OF THE DEFINITE INTEGRAL

2.1. Basic properties of the definite integral

We have seen that the definite integral

$$\int_a^b f(x)\,dx \qquad (2.1)$$

is the limit of a sum of the form:

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \quad (x_{k-1} \le \xi_k \le x_k). \qquad (2.2)$$

We have assumed here that $a < b$ and correspondingly $x_{k-1} < x_k$.

If we have $a > b$, integral (2.1) can be defined as before as the limit of the sum (2.2), except that now we have:

$$a = x_0 > x_1 > x_2 > \ldots > x_{k-1} > x_k > \ldots > x_{n-1} > x_n = b.$$

i.e. all the differences $x_k - x_{k-1}$ are negative. Finally, if we reverse limits a and b, i.e. we take a as upper limit and b as lower limit, the points x_k of the interval must now be taken in the reverse order whilst all $(x_k - x_{k-1})$ in (2.2) change sign, so that the sum itself and its limit change sign, i.e.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \qquad (2.3)$$

Further, on interpreting the definite integral as an area, it is natural to take

$$\int_a^a f(x)\,dx = 0. \qquad (2.4)$$

We also note the obvious equality:

$$\int_a^b dx = b - a. \qquad (2.5)$$

The function under the integral is here equal to unity for all x, so that

$$\int_a^b dx = \lim\left[(x_1 - a) + (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_{n-1} - x_{n-2}) + (b - x_{n-1})\right],$$

but the expression in square brackets is equal to the constant $(b-a)$. Evidently, (2.5) gives the area of a rectangle of base $(b-a)$ and unit height.

We can now start by noting three properties of definite integrals:

I. *The value of a definite integral with identical upper and lower limits is zero.*

II. *A definite integral preserves its absolute value and merely changes sign, on interchanging the upper and lower limits :*

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

For $a < b$, this property can be taken as defining the integral from b to a. It is naturally assumed that the integral on the right exists.

III. *The magnitude of a definite integral is independent of the notation for the variable of integration :*

$$\int_a^b f(x)dx = \int_a^b f(t)dt.$$

The functions we consider will in future be assumed continuous in the interval of integration, unless there is a proviso to the contrary.

IV. *Given a series of numbers* $a, b, c, ..., k, l$*, arranged in any order, we have:*

$$\int_a^l f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + ... + \int_k^l f(x)dx. \qquad (2.6)$$

It suffices to establish this formula for the case of three numbers a, b, c, the proof being then easily extended to cover any required number of terms.

We first take $a < b < c$. We have by definition:

$$\int_a^c f(x)dx = \lim \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

this limit being the same, irrespective of how we divide the interval (a, c), provided only that the greatest of the differences $(x_i - x_{i-1})$ tends to zero, and their number increases indefinitely. We can decide to divide (a, c) so that b, lying between a and c, appears as one of the points of division. The sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

can now be split into two sums of the same form, one being found by dividing the interval (a, b) and the other by dividing (b, c), with the number of divisions increasing indefinitely in both cases, and with the greatest of the $(x_i - x_{i-1})$ tending to zero. These two sums will tend respectively to

$$\int_a^b f(x)dx \text{ and } \int_b^c f(x)dx,$$

and we have finally:

$$\int_a^c f(x)dx = \lim\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \int_a^b f(x)dx + \int_b^c f(x)dx,$$

which it was required to prove.

Now let b lie outside (a,c), say $a<c<b$. We can write in this case:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

whence

$$\int_a^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx,$$

But by property II:

$$-\int_c^b f(x)dx = \int_b^c f(x)dx,$$

i.e. we again have

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

All the other possible arrangements of the points can be considered in a similar way.

V. *A constant factor can be taken outside the definite integration sign*, i.e.

$$\int_a^b Af(x)dx = A\int_a^b f(x)dx \ ,$$

since

$$\int_a^b Af(x)dx = \lim\sum_{i=1}^{n} Af(\xi_i)(x_i - x_{i-1}) =$$

$$= A\lim\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = A\int_a^b f(x)dx \ .$$

VI. *The definite integral of an algebraic sum is equal to the algebraic sum of the definite integrals of each term, since, e.g.*

$$\int_a^b [f(x)-\varphi(x)]dx = \lim \sum_{i=1}^{n} [f(\xi_i)-\varphi(\xi_i)](x_i - x_{i-1}) =$$

$$= \lim \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - \lim \sum_{i=1}^{n} \varphi(\xi_i)(x_i - x_{i-1}) = \int_a^b f(x)dx - \int_a^b \varphi(x)dx .$$

2.2. Mean value theorem

VII. *If functions* $f(x)$ *and* $\varphi(x)$ *satisfy the condition*

$$f(x) \le \varphi(x) \quad (2.7)$$

in the interval (a,b) *, then*

$$\int_a^b f(x)dx \le \int_a^b \varphi(x)dx \quad (b > a) \quad (2.8)$$

or briefly, an inequality can be integrated.

We form the difference

$$\int_a^b \varphi(x)dx - \int_a^b f(x)dx = \int_a^b [\varphi(x) - f(x)]dx =$$

$$= \lim \sum_{i=1}^{n} [\varphi(\xi_i) - f(\xi_i)](x_i - x_{i-1}).$$

By inequality (2.7), the terms under the summation are positive, or at least, not negative. The same can therefore be said of the whole sum and of its limit, i.e. of the difference between the integrals, whence follows inequality (2.8).

We also explain the above in geometrical terms. We first suppose that both curves

$$y = f(x), \; y = \varphi(x)$$

lie above OX (Fig. 2.1).

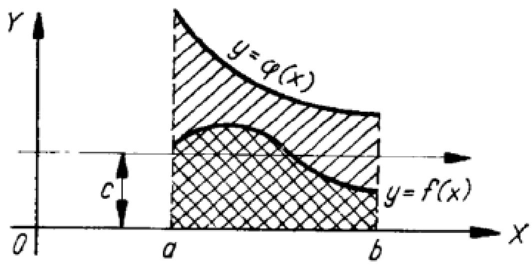

Fig. 2.1

Then the figure bounded by the curve $y = f(x)$, OX and the ordinates $x = a$ and $x = b$, lies entirely within the similar figure bounded by $y = \varphi(x)$, and hence the area of the first figure cannot exceed that of the second figure, i.e.

$$\int_a^b f(x)\,dx \le \int_a^b \varphi(x)\,dx .$$

The general case, with any arrangement of the given curves relative to OX, whilst preserving condition (2.7), follows from the above by giving the figure an upward displacement so that both curves appear above OX; this displacement adds the same term c to both functions $f(x)$ and $\varphi(x)$, and the same rectangular area with base $(b-a)$ and height c to the area of both figures, so that the inequality remains valid.

Corollary. If we have in the interval (a,b):

$$|f(x)| \le \varphi(x) \le M , \tag{2.9}$$

then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \varphi(x)\,dx \le M(b-a) \quad (b > a). \tag{2.10}$$

In fact, (2.9) is equivalent to:

$$-M \le -\varphi(x) \le f(x) \le \varphi(x) \le M .$$

Integrating these inequalities from a to b (property VII) and using (2.5), we get:

$$-M(b-a) \le -\int_a^b \varphi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \varphi(x)\,dx \le M(b-a),$$

which is equivalent to inequality (2.10).

Setting $\varphi(x) = |f(x)|$, (2.10) gives the important inequality:

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx , \tag{2.11}$$

which is a generalization for the case of an integral of the well-known property of a sum: the absolute magnitude of a sum is less than or equal to the sum of the absolute magnitudes of the component terms. It is easily seen that the sign of equality is obtained in the above formula only in the case when $f(x)$ does not change sign in the interval (a,b).

An extremely important theorem also follows from property VII.

Mean value theorem. If the function $\varphi(x)$ preserves its sign in the interval (a,b), then

$$\int_a^b f(x)\varphi(x)dx = f(\xi)\int_a^b \varphi(x)dx, \qquad (2.12)$$

where ξ lies in the interval (a,b).

For clarity, we shall take $\varphi(x) \geq 0$ in the interval (a,b), whilst we denote the least and greatest values of $f(x)$ in (a,b) by m and M respectively. Since clearly

$$m \leq f(x) \leq M$$

(the signs of equality being obtained simultaneously only when $f(x)$ is constant), and $\varphi(x) \geq 0$, then

$$m\varphi(x) \leq f(x)\varphi(x) \leq M\varphi(x),$$

and by property VII, taking $b > a$,

$$m\int_a^b \varphi(x)dx \leq \int_a^b f(x)\varphi(x)dx \leq M\int_a^b \varphi(x)dx.$$

Hence it is clear that there is a number P, satisfying $m \leq P \leq M$, such that

$$\int_a^b f(x)\varphi(x)dx = P\int_a^b \varphi(x)dx. \qquad (2.13)$$

Since $f(x)$ is continuous (see Fig. 2.2), it takes in (a,b) all values included between m and M, one of these being P. Hence there exists ξ in (a,b) such that

$$f(\xi) = P,$$

which proves formula (2.12).

If $\varphi(x) \leq 0$ in (a,b), then $-\varphi(x) \geq 0$ in (a,b). We get by applying the theorem just proved:

$$\int_a^b f(x)\left[-\varphi(x)\right]dx = f(\xi)\int_a^b \left[-\varphi(x)\right]dx;$$

on taking the (—) sign outside the sign of integration and multiplying both sides by (-1), we arrive at formula (2.12).

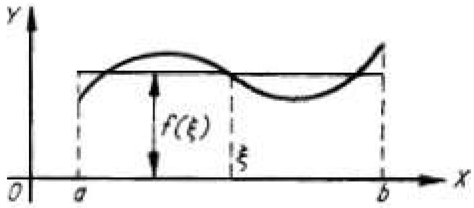

Fig. 2.2

If $b < a$, we have from the above in exactly the same way:

$$\int_b^a f(x)\varphi(x)dx = f(\xi)\int_b^a \varphi(x)dx.$$

Interchanging the limits of integration on both sides and multiplying by (-1), we arrive at (2.12), which is thus proved in general.

On putting $\varphi(x) = 1$, we obtain an important particular case of the mean value theorem :

$$\int_a^b f(x)dx = f(\xi)\int_a^b dx = f(\xi)(b-a). \qquad (2.14)$$

The value of a definite integral is equal to the product of the length of the interval of integration and the value of the integrand for some value of the independent variable lying in the interval.

This length must be taken with the (−) sign if $a > b$. This proposition means geometrically that, given the area bounded by any curve, OX and two ordinates $x = a$ and $x = b$, it is always possible to find a rectangle of the same area with the same base $(b-a)$ and with height equal to some ordinate of the curve in the interval (a,b) (see Fig. 2.2).

It is easily shown that the ξ appearing in (2.13) and (2.14) can always be taken as lying inside (a,b).

2.3. Existence of the primitives

VIII. *If the upper limit of a definite integral is a variable, the derivative of the integral with respect to the upper limit is equal to the value of the integrand at the upper limit.*

We note that the value of

$$\int_a^b f(x)dx$$

depends on the limits of integration a and b, given the integrand $f(x)$. We consider

$$\int_a^x f(t)\,dt,$$

with constant lower limit a and variable upper limit x, the variable of integration being denoted by t to distinguish it from the upper limit x. The value of this integral will be a function of x:

$$F(x)=\int_a^x f(t)\,dt. \tag{2.15}$$

We have to show that

$$\frac{dF(x)}{dx}=f(x).$$

We prove this by finding the derivative of $F(x)$ directly from the definition:

$$\frac{dF(x)}{dx}=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}.$$

We have:

$$F(x+h)=\int_a^{x+h} f(t)\,dt=\int_a^x f(t)\,dt+\int_x^{x+h} f(t)\,dt$$

(by property IV), whence:

$$F(x+h)=F(x)+\int_x^{x+h} f(t)\,dt,$$

and

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Using (2.14), we have:

$$\int_x^{x+h} f(t)\,dt=f(\xi)h,$$

where ξ lies in the interval $(x,x+h)$; hence

$$\frac{F(x+h)-F(x)}{h}=f(\xi).$$

As h tends to zero, ξ, lying between x and $x+h$, tends to x, and by the continuity of $f(x)$, $f(\xi)$ tends to $f(x)$, so that

$$\frac{dF(x)}{dx}=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}f(\xi)=f(x),$$

which it was required to prove.

We note that h can only be given positive values for $x=a$, and only negative values for $x=b$ (with $a<b$), whilst $F(x)$ has the derivative $f(x)$ throughout (a,b) (closed). We have already discussed the definition of the derivative at the ends of a closed interval.

The corollary follows that the definite integral $F(x)$, considered as a function of the upper limit x, is continuous in (a,b), where we must take $F(a)=0$.

We remark that if we apply the mean value theorem to integral (2.15), we get $F(x)=f(\xi)(x-a)$, whence it follows that $F(x)\to 0$ as $x\to a$. It also follows from the above discussion that:

IX. *Every continuous function $f(x)$ has a primitive or indefinite integral.*

Function (2.15) is the primitive of $f(x)$, which vanishes for $x=a$.

If $F(x)$ is one primitive, then:

$$\int_a^b f(x)dx=F(b)-F(a). \tag{2.16}$$

2.4. Discontinuities of the integrand. Improper integral

It has been assumed in all the above discussions that the integrand $f(x)$ is continuous throughout the interval (a,b) of integration. If the dicontinuty is of the 2nd kind, the integral is called *improper integral*.

We now introduce the concept of integral for various discontinuous functions.

If the integrand $f(x)$ has a discontinuity at a point c of the interval (a,b), whilst each of the integrals

$$\int_a^{c-\varepsilon^{/}} f(x)dx,\quad \int_{c+\varepsilon^{//}}^{b} f(x)dx \quad (a<b)$$

tends to a definite limit as the positive numbers $\varepsilon^{/}$ and $\varepsilon^{//}$ tend to zero, these limits are referred to as the definite integrals of $f(x)$ between (a,c) and (c,b) respectively, i.e.

$$\int_a^c f(x)dx = \lim_{\varepsilon' \to +0} \int_a^{c-\varepsilon'} f(x)dx,$$
$$\int_c^b f(x)dx = \lim_{\varepsilon'' \to +0} \int_{c+\varepsilon''}^b f(x)dx,$$

if these limits exist.

We take in this case:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

The function $F(x)$ defined by (2.15) is easily seen to have the following properties:

$F'(x) = f(x)$ for every point of (a,b) except $x = c$, and $F(x)$ is continuous throughout (a,b), excluding $x = c$.

If c coincides with one end of (a,b), only one of the limits has to be considered, either

$$\lim_{\varepsilon \to +0} \int_{a+\varepsilon}^b f(x)dx, \text{ or } \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)dx.$$

Finally, if we have more than one point of discontinuity c in (a,b), the interval must be divided so that there is only one point of discontinuity in each interval.

Having agreed to attach the above meaning to the symbol

$$\int_a^b f(x)dx,$$

property IX and formula (2.16):

$$\int_a^b f(x)dx = F_1(b) - F_1(a)$$

will certainly be valid if $F'(x) = f(x)$ for every point of (a,b) except $x = c$, and $F_1(x)$ is continuous throughout (a,b), excluding $x = c$.

It is sufficient to prove this assertion for one discontinuity c inside (a,b), since the case of several discontinuities and the case of $c = a$ or $c = b$ follow in a similar way.

Since $f(x)$ is continuous in the intervals $(a, c - \varepsilon')$, $(c + \varepsilon'', b)$, we can apply (2.16) to these intervals, which gives:

$$\int_a^{c-\varepsilon'} f(x)dx = F(c-\varepsilon') - F(a),$$

$$\int_{c+\varepsilon''}^{b} f(x)dx = F(b) - F(c+\varepsilon').$$

We can write, by the continuity of $F(x)$:

$$\int_a^c f(x)dx = \lim_{\varepsilon' \to +0} \left[F(c-\varepsilon') - F(a) \right] = F(c) - F(a),$$

$$\int_c^b f(x)dx = \lim_{\varepsilon'' \to +0} \left[F(b) - F(c+\varepsilon'') \right] = F(b) - F(c),$$

i.e.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c) - F(a) + F(b) - F(c) = F(b) - F(a),$$

which it was required to prove.

The case in question is encountered geometrically when a curve $y = f(x)$ has a discontinuity at a point c yet the area under the curve always exists. Take, for instance, the graph of a function defined as follows (Fig. 2.3):

$$f(x) = \frac{1}{2}x + \frac{1}{2} \text{ for } 0 \le x < 2,$$

$$f(x) = x \text{ for } 2 \le x \le 3.$$

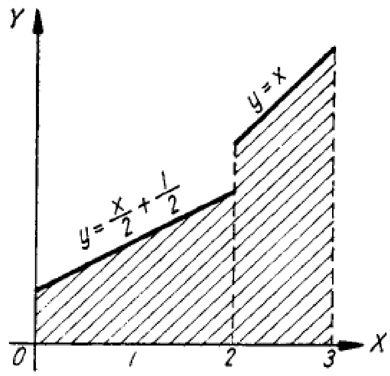

Fig. 2.3

The area bounded by this curve, OX , the ordinate $x = 0$ and the variable ordinate $x = x_1$, is a continuous function of x, in spite of the fact that $f(x)$ has a discontinuity at $x = 2$. On the other hand, it is easy to find a primitive of $f(x)$ that will be continuous throughout the interval $(0;\, 3)$. Take, for instance, the function $F_1(x)$ defined as follows:

$$F(x)=\frac{x^2}{4}+\frac{1}{2}x \text{ for } 0\le x\le 2,$$

$$F(x)=\frac{x^2}{2} \text{ for } 2<x\le 3.$$

We find, in fact, on differentiating:

$$F'(x)=\frac{x}{2}+\frac{1}{2}$$

in $(0,2)$, and $F'(x)=x$; in $(2,3)$. Furthermore, the two expressions for $F(x)$ give the same value 2 for $x=2$, which ensures the continuity of $F(x)$.

The area bounded by our curve, OX and the ordinates $x=0$ and $x=3$, is given by:

$$\int_0^3 f(x)dx=\int_0^2 f(x)dx+\int_2^3 f(x)dx=F(3)-F(0)=\frac{9}{2},$$

which is easily checked directly from the figure.

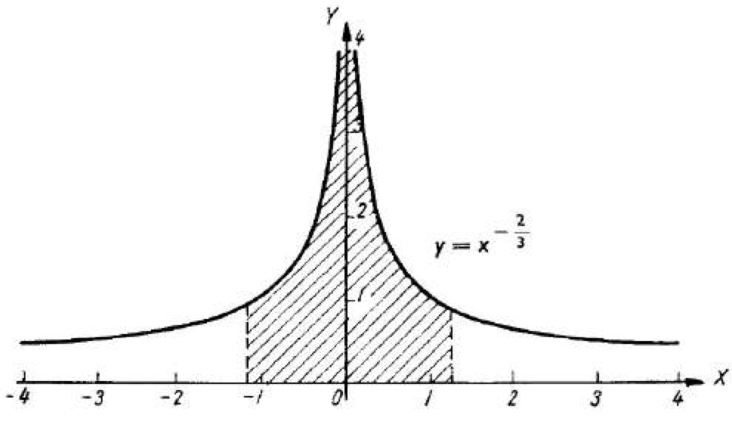

Fig. 2.4

We take further the function $y=x^{-2/3}$(Fig. 2.4). It tends to infinity for $x=0$, but its primitives remain continuous for this value of x, one primitive being $3x^{1/3}$; hence, we can write:

$$\int_{-1}^{+1} x^{-2/3}\,dx=3x^{1/3}\Big|_{-1}^{+1}=6,$$

in other words, although the curve in question rises infinitely as x approaches zero, it still has a perfectly definite area between the ordinates $x=-1$ and $x=1$.

The primitive $(-1/x)$ of the function $1/x^2$ itself tends to infinity for $x=0$, so that (2.16) cannot be applied for this function in the case of zero lying inside the interval (a,b); the curve of $1/x^2$ does not possess a finite area for such an interval.

2.5. Infinite limits

The preceding discussion can be extended to the case of an infinite interval, on taking

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx \ , \tag{2.17}$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \tag{2.18}$$

provided these limits exist. In this case we also say about *improper integral.*

This proviso is certainly fulfilled if the primitive $F_1(x)$ tends to a definite limit as x tends to ($+\infty$) or ($-\infty$). Denoting these limits directly by $F(+\infty)$ and $F(-\infty)$, we have:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\left[F(b)-F(a)\right] = F(+\infty)-F(a),$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\left[F(b)-F(a)\right] = F(b)-F(-\infty), \tag{2.19}$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx = F(+\infty)-F(-\infty), \tag{2.20}$$

this last being a generalization of (2.16) for the case of an infinite interval.

Relationship (2.19) is often written as

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx \ .$$

Geometrically speaking, fulfillment of the above proviso means that the infinite branch of the curve $y=f(x)$, corresponding to $x\to\pm\infty$, has an area.

We have thus extended the concept of definite integral, originally established for a continuous function and a finite interval, to the cases of a discontinuous function and an infinite interval. This extension is characterized by first finding the integral of a continuous function for a shortened interval, then passing to the limit. Integrals obtained in this manner are distinguished from primitives by being referred to as improper integrals.

We note that the integral of a discontinuous function in a finite interval has in some cases a direct significance as the limit of a sum. We shall discuss this later. This is the case, e.g., with the integral expressing the area shown in Fig. 2.3. In essence, therefore, this integral will not be improper If, however, the integrand is unbounded in the interval of integration (tends to infinity), or if this interval is infinite, the integral can then only exist in the improper form.

Example. The curve $y = 1/(1+x^2)$ extends indefinitely for $x = \pm\infty$ yet always bounds a finite area with axis OX (Fig. 2.5), since

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \arctan x \Big|_{-\infty}^{+\infty} = \frac{1}{2}\pi - \left(-\frac{1}{2}\pi\right) = \pi .$$

It should be recalled, when evaluating this integral, that an arbitrary value of the many-valued function $\arctan x$ cannot be taken; the function must be defined so that it becomes single-valued, i.e. its values lie between $-\pi/2$ and $+\pi/2$; if this is not done, the above formula becomes meaningless.

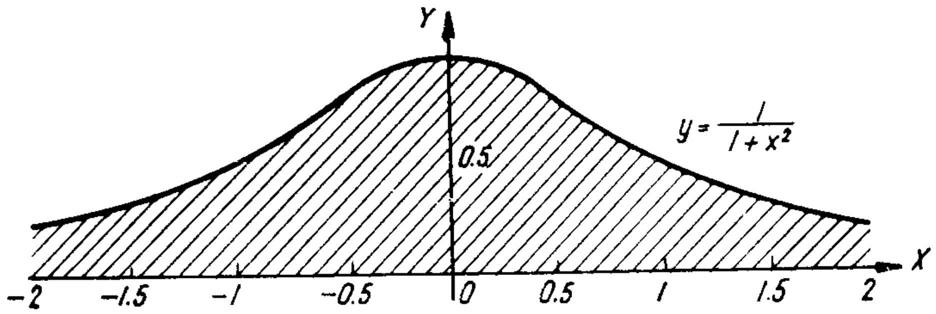

Fig. 2.5

2.6. Change of variable for definite integrals

Let $f(x)$ be continuous in the interval (a,b), or in the wider interval (A,B), discussed below. Further, let $\varphi(t)$ be a single-valued, continuous function with a continuous derivative $\varphi'(t)$ in the interval (α,β), where

$$\varphi(\alpha) = a \text{ and } \varphi(\beta) = b. \qquad (2.21)$$

We further suppose that $\varphi(t)$ does not move outside (a,b), or the wider interval (A,B), in which $f(x)$ is continuous, when t varies in (α,β). The function of a function $f[\varphi(t)]$ is now a continuous function of t in the interval (α,β).

If, with these assumptions, we introduce a new variable of integration t in place of x:

$$x=\varphi(t), \tag{2.22}$$

the formula for transforming the definite integral is:

$$\int_a^b f(x)dx=\int_\alpha^\beta f\left[\varphi(t)\right]\varphi^{/}(t)dt\,. \tag{2.23}$$

We prove this by introducing integrals with variable limits in place of those above:

$$F(x)=\int_a^x f(y)dy\,,\ \Psi(t)=\int_\alpha^t f\left[\varphi(z)\right]\varphi^{/}(z)dz\,.$$

By (2.22), $F(x)$ is a function of a function of t:

$$F(x)=F\left[\varphi(t)\right]=\int_a^{\varphi(t)} f(y)dy.$$

We find its derivative by the rule for differentiation of a function of a function:

$$\frac{dF(x)}{dt}=\frac{dF(x)}{dx}\frac{dx}{dt},$$

whilst by property VIII (sec. 2.3):

$$\frac{dF(x)}{dx}=f(x);$$

it also follows from (2.22) that

$$\frac{dx}{dt}=\varphi(t),$$

whence

$$\frac{dF(x)}{dt}=f(x)\varphi^{/}(t)=f\left[\varphi(t)\right]\varphi^{/}(t).$$

We now find the derivative of $\Psi(t)$. We have by property VIII and the assumptions made:

$$\frac{d\Psi(t)}{dt}=f\left[\varphi(t)\right]\varphi^{/}(t).$$

Functions $\Psi(t)$ and $F(x)$, considered as functions of t, thus have the same derivative in the interval (α,β), and hence can only differ by a constant; whilst we have for $t=\alpha$:

$$x=\varphi(\alpha)=a,\quad F(x)\big|_{t=\alpha}=F(a)=0;\ \ \Psi(\alpha)=0,$$

i.e. these two functions are equal for $t=\alpha$ and hence are equal for all t in (α,β). In particular, we have for $t=\beta$:

$$F(x)\Big|_{t=\beta}=F(b)=\int_a^b f(x)\,dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt,$$

which it was required to prove.

The inverse:

$$t=\psi(x) \tag{2.24}$$

is very often used instead of substitution (2.21):

$$x=\varphi(t).$$

Limits α and β are then immediately defined by:

$$a=\psi(a),\ \ \beta=\psi(b),$$

whilst it must be borne in mind that expression (2.22) for x, obtained by solving (2.24) with respect to x, must satisfy all the conditions mentioned above ; in particular, $\varphi(t)$ must be a single-valued function of t. It can be shown that (2.23) is invalid if $\varphi(t)$ lacks this property.

We replace x in $\int_{-1}^{+1} dx=2$ by the new variable t, where $t=x^2$, and obtain an integral equal to zero from the right-hand side of (2.23), since its limits are the same, +1; but this is impossible; the error arises due to the expression for x in terms of t:

$$x=\pm\sqrt{t}$$

being a many-valued function.

Example. We call $f(x)$ an even function of x if $f(-x)=f(x)$, and an odd function if $f(-x)=-f(x)$.

For instance, $\cos x$ is an even function of x, and $\sin x$ is an odd function.

We show that

$$\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx,$$

if $f(x)$ is even, and that

$$\int_{-a}^{a} f(x)\,dx=0,$$

if $f(x)$ is odd.

We separate the integral into two parts:

$$\int_{-a}^{a} f(x)\,dx=\int_{-a}^{0} f(x)\,dx+\int_0^a f(x)\,dx.$$

We make a change of variable, $x=-t$, in the first integral and use properties II and III (sec. 2.1):

$$\int_{-a}^{0} f(x)\,dx = -\int_{\alpha}^{0} f(-t)\,dt = \int_{0}^{\alpha} f(-t)\,dt = \int_{0}^{a} f(-x)\,dx,$$

whence, substituting in the previous equation:

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} \left[f(-x) + f(x) \right] dx.$$

If $f(x)$ is an even function, $\left[f(-x) + f(x) \right] = 2f(x)$, whilst the sum in the square brackets is zero if $f(x)$ is odd; and this proves our statement.

2.7. Integration by parts

In the case of definite integrals, the formula for integration by parts can be put in the form :

$$\int_{a}^{b} u(x)\,dv(x) = u(x)v(x)\Big|_{a}^{b} - \int_{a}^{b} v(x)\,du(x). \qquad (2.25)$$

Of course it is assumed that $u(x)$ and $v(x)$ have continuous derivatives in the interval (a,b).

Example. To evaluate:

$$\int_{0}^{\frac{\pi}{2}} \sin^n x\,dx, \quad \int_{0}^{\frac{\pi}{2}} \cos^n x\,dx,$$

We put

$$I_n = \int_{0}^{\frac{\pi}{2}} \sin^n x\,dx.$$

Integrating by parts, we have:

$$I_n = \int_{0}^{\frac{\pi}{2}} \sin^{n-1} x \sin x\,dx = -\int_{0}^{\frac{\pi}{2}} \sin^{n-1} x\,d\cos x =$$

$$= -\sin^{n-1} x \cos x \Big|_{0}^{\frac{\pi}{2}} + \int_{0}^{\frac{\pi}{2}} \cos x (n-1) \sin^{n-2} x \cos x\,dx =$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,dx=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\left(1-\sin^2 x\right)dx=$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\,dx=(n-1)I_{n-2}-(n-1)I_n \ ,$$

i.e.

$$I_n=(n-1)I_{n-2}-(n-1)I_n,$$

whence, solving for I_n, we get:

$$I_n=\frac{n-1}{n}I_{n-2}. \qquad (2.26)$$

This is called a reduction formula, since it reduces the evaluation of I_n to the evaluation of a similar integral, but with a lower subscript $(n-2)$,

We take the cases separately, of n even or odd.

1. $n=2k$ (even). We have by (2.26):

$$I_{2k}=\frac{2k-1}{2k}I_{2k-2}=\frac{(2k-1)(2k-3)}{2k(2k-2)}I_{2k-4}=...=\frac{(2k-1)(2k-3)...3\cdot 1}{2k(2k-2)...4\cdot 2}I_0$$

and since

$$I_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$$

we have finally:

$$I_{2k}=\frac{(2k-1)(2k-3)...3\cdot 1}{2k(2k-2)...4\cdot 2}\frac{\pi}{2}.$$

2. $n=2k+1$ (odd). We find, as above:

$$I_{2k+1}=\frac{2k(2k-2)(2k-3)...4\cdot 2}{(2k+1)(3k-1)...5\cdot 3}I_1,\quad I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx=-\cos x\Big|_0^{\frac{\pi}{2}}=1,$$

and hence

$$I_{2k+1}=\frac{2k(2k-2)(2k-3)...4\cdot 2}{(2k+1)(3k-1)...5\cdot 3}.$$

Using these integrals we can get the formula obtained by Wallis. By division this yields

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots\frac{2k\cdot 2k}{(2k-1)\cdot(2k+1)}\frac{\int_0^{\pi/2}\sin^{2k}x\,dx}{\int_0^{\pi/2}\sin^{2k+1}x\,dx}\ .$$

The quotient of the two integrals on the right-hand side converges to 1 as k increases, as we recognize from the following considerations. In the interval $0<x<\pi/2$ we have

$$0<\sin^{2k+1}x\le\sin^{2k}x\le\sin^{2k-1}x,$$

consequently

$$0<\int_0^{\pi/2}\sin^{2k+1}x\,dx\le\int_0^{\pi/2}\sin^{2k}x\,dx\le\int_0^{\pi/2}\sin^{2k-1}x\,dx.$$

If we here divide each term by $\int_0^{\pi/2}\sin^{2k+1}xdx$ and notice that by the first formula proved above

$$\frac{\int_0^{\pi/2}\sin^{2k-1}x\,dx}{\int_0^{\pi/2}\sin^{2k+1}x\,dx}=\frac{2k+1}{2k}=1+\frac{1}{2k},$$

we have

$$1\le\frac{\int_0^{\pi/2}\sin^{2k}x\,dx}{\int_0^{\pi/2}\sin^{2k+1}x\,dx}\le 1+\frac{1}{2k},$$

from which the above statement follows.

The relation

$$\frac{\pi}{2}=\lim_{k\to\infty}\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2k}{2k-1}\frac{2k}{2k+1}$$

consequently holds.

This product formula (due to Wallis), with its simple law of formation, gives a remarkable relation between the number π and the integers. If we observe that $\lim_{k\to\infty}\frac{2k}{2k+1}=1$, we can write

$$\lim_{k\to\infty}\frac{2^2\cdot 4^2\ldots(2k-2)^2}{3^2\cdot 5^2\ldots(2k-1)^2}2k=\frac{\pi}{2}\ ,$$

and if we take the square root and then multiply numerator and denominator by $2\cdot 4\ldots(2k-2)$ we find that

$$\sqrt{\frac{\pi}{2}}=\lim_{k\to\infty}\frac{2\cdot 4\ldots(2k-2)}{3\cdot 5\ldots(2k-1)}\sqrt{2k}=\lim_{k\to\infty}\frac{2^2\cdot 4^2\ldots(2k)^2}{(2k)!}\frac{\sqrt{2k}}{2k}\ .$$

From this we finally obtain

$$\lim_{k\to\infty}\frac{(k!)^2\,2^{2k}}{(2k)!\sqrt{k}}=\sqrt{\pi}\ .$$

The integral

$$I_n=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

can be evaluated in a similar way, though it is easier to reduce it to the former by noting that

$$\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\sin^n\left(\frac{\pi}{2}-x\right)dx\,,$$

whence, putting

$$\frac{\pi}{2}-x=t,\quad x=\frac{\pi}{2}-t\,,$$

we have on the basis of property II (sec. 2.1):

$$\int_0^{\frac{\pi}{2}}\cos^n x\,dx=-\int_{\frac{\pi}{2}}^{0}\sin^n t\,dt=\int_0^{\frac{\pi}{2}}\sin^n t\,dt\ .$$

Taking together the results obtained, we can write:

$$\int_0^{\frac{\pi}{2}}\sin^{2k}x\,dx=\int_0^{\frac{\pi}{2}}\cos^{2k}x\,dx=\frac{(2k-1)(2k-3)\ldots 3\cdot 1}{2k(2k-2)\ldots 4\cdot 2}\frac{\pi}{2}\,, \qquad (2.27)$$

$$\int_0^{\frac{\pi}{2}}\sin^{2k+1}x\,dx=\int_0^{\frac{\pi}{2}}\cos^{2k+1}x\,dx=\frac{2k(2k-2)(2k-3)\ldots 4\cdot 2}{(2k+1)(3k-1)\ldots 5\cdot 3}\ . \qquad (2.28)$$

3. APPLICATIONS OF DEFINITE INTEGRALS

3.1. Calculation of area

We saw that the area bounded by a given curve $y = f(x)$, by the axis OX and two ordinates $x = a$ and $x = b$ is, expressed by the definite integral:

$$\int_a^b f(x)dx.$$

The area thus found, however, does not give us the actual sum of the areas, formed by the given curve with OX, but gives only their algebraic sum, in which every area situated below OX takes the (-) sign. To find the sum of these areas in the ordinary sense, we have to find

$$\int_a^b |f(x)|dx.$$

Thus, the sum of the shaded areas in Fig. 3.1 is equal to

$$\int_a^c f(x)dx - \int_c^g f(x)dx + \int_g^h f(x)dx - \int_h^k f(x)dx + \int_k^b f(x)dx.$$

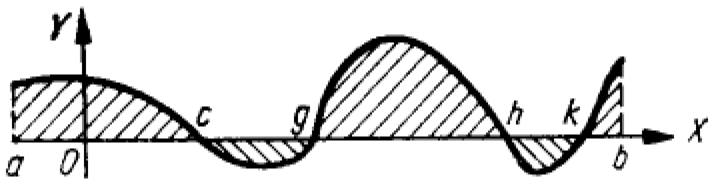

Fig. 3.1

The area contained between two curves

$$y = f(x), \quad y = \varphi(x) \tag{3.1}$$

and two ordinates

$$x = a, \ \ x = b,$$

where one curve lies below the other, i.e.

$$f(x) \ge \varphi(x)$$

in the interval (a,b), is given by the definite integral

$$\int_a^b \left[f(x) - \varphi(x) \right] dx. \tag{3.2}$$

We suppose first that both curves lie above OX. It is at once evident from Fig. 3.2 that the required area S is the difference between the areas bounded by the given curves with OX:

$$S=\int_a^b f(x)dx-\int_a^b \varphi(x)dx=\int_a^b\left[f(x)-\varphi(x)\right]dx,$$

which it was required to prove. The general case, of any disposition of the two curves relative to OX , now follows by displacing OX downwards, so that both curves appear above Ox; this displacement is equivalent to adding the same constant to both functions,with the difference $f(x)-\varphi(x)$ remaining unchanged.

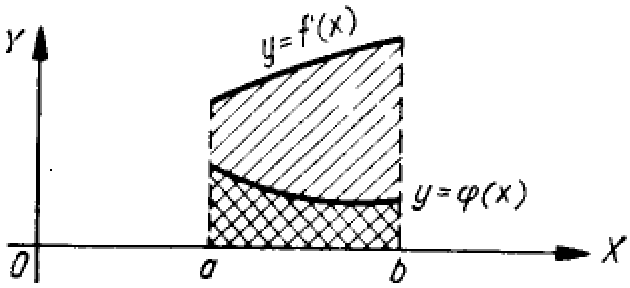

Fig. 3.2

We propose as an exercise the proof that, if two given curves intercept, i.e. part of one curve is below the other, and part above, the sum of the areas contained between the curves and the ordinates $x=a$ and $x=b$ is equal to

$$\int_a^b\left|f(x)-\varphi(x)\right|dx. \tag{3.3}$$

Evaluation of a definite integral is often called quadrature. This is because finding a definite integral often amounts to finding an area, as shown above.

Examples:

1. The area bounded by the second degree parabola

$$y=ax^2+bx+c,$$

the axis OX , and two ordinates, distant h apart, is equal to

$$\frac{h}{6}\left(y_1+y_2+4y_0\right), \tag{3.4}$$

where y_1 and y_2 denote the outer ordinates of the curve, and y_0 lies equidistant from each of these outer ordinates.

We assume here that the curve lies above OX . We can take without loss of generality the left-hand outer ordinate as along OY (Fig. 3.3) in proving (4), since displacement of the entire figure along OX changes neither the magnitude of the area in question, nor the relative disposition of the outer and centre ordinates, nor the size of these ordinates. With this assumption, and with the equation of the parabola in the form $y = ax^2 + bx + c$, we write the required area S as the definite integral:

$$S = \int_0^h \left(ax^2 + bx + c\right)dx = a\frac{x^3}{3} + b\frac{x^2}{2} + cx\Big|_0^h =$$

$$= a\frac{h^3}{3} + b\frac{h^2}{2} + ch = \frac{h}{6}\left(2ah^2 + 3bh + 6c\right).$$

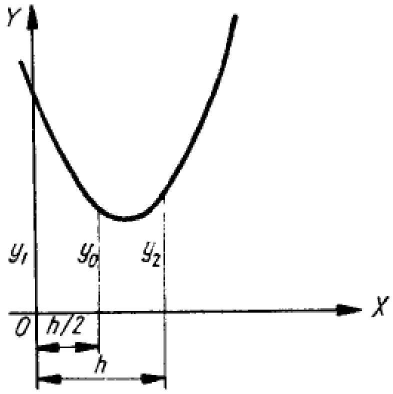

Fig. 3.3

We have from our notation:

$$y_0 = ax^2 + bx + c\Big|_{x=\frac{1}{2}h} = \frac{1}{4}ah^2 + \frac{1}{2}bh + c,$$

$$y_1 = ax^2 + bx + c\Big|_{x=0} = c, y_2 = ax^2 + bx + c\Big|_{x=h} = ah^2 + bh + c,$$

whence it follows:

$$y_1 + y_2 + 4y_0 = 2ah^2 + 3bh + 6c,$$

which proves our example.

2. Area of an ellipse (Fig. 3.4).

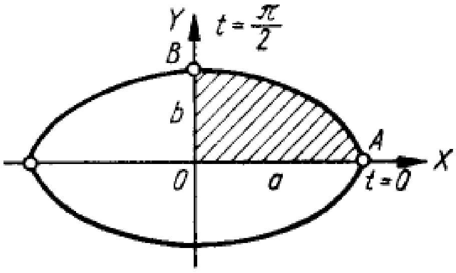

Fig. 3.4

An ellipse having the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is symmetrical relative to the axes, and hence the required area S is equal to four times the area of the part lying in the first quadrant, i.e.

$$S = 4\int_0^a y\,dx.$$

Instead of finding y from the equation of the ellipse and substituting the expression obtained in the integrand, we use the parametric form of the ellipse:

$$x = a\cos t,\ \ y = b\sin t, \tag{3.5}$$

introducing the new variable in place of x; y is now expressed directly by the second of equations (5). When x varies from 0 to a, t varies from $\pi/2$ to 0, and since all the conditions for the rule of change of variables are fulfilled in this case,

$$S = 4\int_{\pi/2}^{0} b\sin t\, d\left(a\cos t\right) = -4ab\int_{\pi/2}^{0}\sin^2 t\,dt = 4ab\int_0^{\pi/2}\sin^2 t\,dt.$$

We have for $k = 1$ from (2.26) (sec. 2.7):

$$\int_0^{\pi/2}\sin^2 t\,dt = \frac{1}{2}\cdot\frac{1}{2}\pi = \frac{\pi}{4},$$

so that we finally get:

$$S = \pi ab. \tag{3.6}$$

With $a = b$, when the ellipse becomes a circle of radius a, we get the well-known expression πa^2 for the area of a circle.

3. To find the area contained between the two curves (Fig. 3.5)

$$y = x^2,\ \ x = y^2.$$

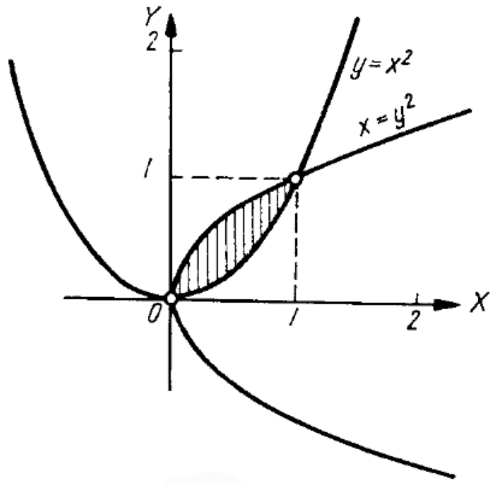

Fig. 3.5

The given curves intercept at two points $(0,0)$, $(1,1)$, their coordinates being obtained by simultaneously solving the equations of the curves. Since we have in $(0,1)$:

$$\sqrt{x} \ge x^2,$$

we find on using (2) that the required area S is given by

$$S=\int_0^1\left(\sqrt{x}-x^2\right)dx=\left(\frac{2}{3}x^{3/2}-\frac{x^3}{3}\right)\Bigg|_0^1=\frac{1}{3}.$$

3.2. Area of a sector

The area of the sector bounded by a curve with polar equation

$$r=f\left(\theta\right), \tag{3.7}$$

and the two radius vectors

$$\theta=\alpha,\quad \theta=\beta \tag{3.8}$$

drawn from the pole at angles α and β to the polar axis, is given by :

$$S=\int_\alpha^\beta\frac{1}{2}r^2\,d\theta=\frac{1}{2}\int_\alpha^\beta\left[f\left(\theta\right)\right]^2d\theta. \tag{3.9}$$

We deduce (3.9) by dividing the area in question into small elements (Fig. 3.6), the angle between the radius vectors (3.8) being divided into n parts. We take the area of one such small sector, bounded by radius vectors θ and $\theta+\Delta\theta$. Let ΔS denote this area, and m and M the least and greatest values respectively of $r=f\left(\theta\right)$ in the interval $\left(\theta,\theta+\Delta\theta\right)$; then we see that ΔS lies between the areas of two circular sectors with the same angle $\Delta\theta$ but with radii m and M, i.e.

$$\frac{1}{2}m^2\Delta\theta\le\Delta S\le\frac{1}{2}M^2\Delta\theta,$$

and hence, denoting some number between m and M by P, we can write:

$$\Delta S=\frac{1}{2}P^2\Delta\theta.$$

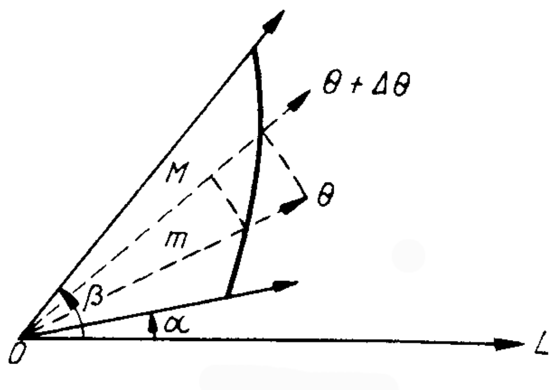

Fig. 3.6

Since the continuous function $f\left(\theta\right)$ takes all values between m and M in the interval $\left(\theta,\theta+\Delta\theta\right)$, there must exist θ' in this interval such that

$$f\left(\theta'\right) = P,$$

so that now

$$\Delta S = \frac{1}{2}\left[f\left(\theta'\right)\right]^2 \Delta\theta. \qquad (3.10)$$

If we now increase the number of elementary sectors ΔS, so that the greatest of the $\Delta\theta$ tends to zero, and if we recall what was said in section 1.2, we get in the limit:

$$S = \lim \sum \Delta S = \lim \sum \frac{1}{2}\left[f\left(\theta'\right)\right]^2 \Delta\theta = \int_\alpha^\beta \frac{1}{2}\left[f\left(\theta\right)\right]^2 d\theta = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta,$$

which it was required to prove.

We remark that the basic idea in the above proof of (3.9) lies in replacing the area ΔS by the area of a circular sector of the same angle $\Delta\theta$ and radius $f\left(\theta'\right)$. If instead of the exact expression (3.10) we use the approximation:

$$\Delta S = \frac{1}{2} r^2 \Delta\theta,$$

where $r = f\left(\theta''\right)$ and θ'' is any value from the interval $\left(\theta, \theta + \Delta\theta\right)$, we get in the limit the same result for the area of a sector:

$$\lim \sum \frac{1}{2}\left[f\left(\theta''\right)\right]^2 \Delta\theta = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta. \qquad (3.11)$$

This leads us to a simple geometrical interpretation of the integral expression in (3.11): $\frac{1}{2} r^2\, d\theta$ is an approximate expression for the area of an elementary sector of angle $d\theta$, and hence is simply referred to as an elementary area in polar coordinates.

Example. To find the area bounded by the closed curve, called a trifolium (Fig. 3.7): $r = a\cos 3\theta \quad \left(a > 0\right)$.

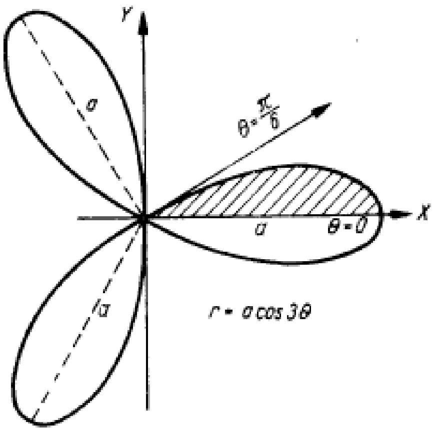

Fig. 3.7

The total area bounded by it is six times the area of the shaded part, corresponding to a variation of θ from 0 to $\pi/6$; so that we have from (3.9):

$$S = 6\int_0^{\pi/6} \frac{1}{2} a^2 \cos^2 3\theta \, d\theta = a^2 \int_0^{\pi/6} \cos^2 3\theta \, d(3\theta) = a^2 \int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi a^2}{4}.$$

3.3. Length of arc

Let AB be the arc of a certain curve. We draw a series of successive chords (Fig. 3.8) and increase their number in such a way that the greatest tends to zero. If the total length of the series of chords now tends to a definite limit, independently of how the chords are drawn, the arc is said to be rectifiable, and the limit in question is called the length of this arc. This definition of length is also suitable for a closed curve.

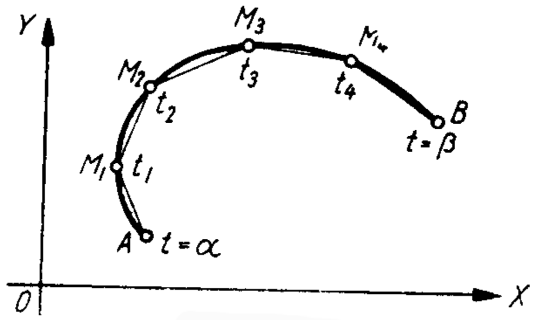

Fig. 3.8

Let the curve be given explicitly as $y = f(x)$, and let A and B correspond to $x = a$ and $x = b$ respectively $(a < b)$; let $f(x)$ also have a continuous derivative in the interval $a \le x \le b$, to which the arc AB corresponds. We show that the arc AB can be rectified in these conditions, and that its length is expressed by a definite integral.

Let the chords be AM_1, M_1M_2, ..., $M_{n-1}B$, and let the coordinates of their ends be

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

and $y_i = f(x_i)$. Using the formula of analytic geometry, we have for the total length of the chords:

$$p = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \left(f(x_i) - f(x_{i-1})\right)^2}.$$

Using the formula for finite increments:

$$f(x_i) - f(x_{i-1}) = f'(\xi_i)(x_i - x_{i-1}) \qquad (x_{i-1} < \xi_i < x_i),$$

we get for the length of a single chord

$$\sqrt{1+f'^2(\xi_1)}(x_i - x_{i-1}),$$

from which we see that the requirement that the greatest chord tends to zero is equivalent to the requirement that the greatest of the $(x_i - x_{i-1})$ tends to zero. We now have

$$p = \sum_{i=1}^{n} \sqrt{1+f'^2(\xi_i)}(x_i - x_{i-1}),$$

which in fact has a limit equal to the integral

$$\int_a^b \sqrt{1+f'^2(x)}dx.$$

Thus, the length l of arc AB is given by:

$$l = \int_a^b \sqrt{1+f'^2(x)}dx. \tag{3.12}$$

Let $x' < x''$ be any two values of x in the interval (a,b), and let M', M'' be the corresponding points of arc AB. Using the mean value theorem, we get the following expression for the length l' of arc M', M'':

$$l' = \int_{x'}^{x''} \sqrt{1+f'^2(x)}dx = \sqrt{1+f'^2(\xi_1)}(x'' - x') \quad (x' < \xi_1 < x'').$$

Using the formula for finite differences, we get for the length of chord M', M'':

$$M'M'' = \sqrt{(x''-x')^2 + (f(x'') - f(x'))^2} =$$

$$= \sqrt{1+f'^2(\xi_2)}(x''-x') \quad (x' < \xi_2 < x'').$$

Hence it follows:

$$\frac{M'M''}{l'} = \frac{\sqrt{1+f'^2(\xi_2)}}{\sqrt{1+f'^2(\xi_1)}}.$$

If M' and M'' tend to M with abscissa x, x' and x'' tend to x, and also ξ_1 and $\xi_2 \to x$, and we get from the above formula:

$$\frac{M'M''}{l'} \to 1.$$

We now suppose that the curve is given parametrically,

$$x = \varphi(t), \quad y = \psi(t),$$

where A and B correspond to $t=\alpha$ and $t=\beta\left(\alpha,\beta\right)$. We assume that points of the curve AB correspond to t in the interval $\alpha\le t\le\beta$, with different points for different t, and with the curve neither closed nor intercepting itself (see Fig. 3.8). We assume further that there exist continuous derivatives $\varphi^{/}\left(t\right)$ and $\psi^{/}\left(t\right)$in the interval $\alpha\le t\le\beta$.

Let chords be drawn as above, $AM_1,\ M_1M_2,\ ...,\ M_{n-1}B$, with their ends corresponding to $t_0=\alpha<t_1<t_2<...<t_{n-1}<t_n=\beta$.

We have for the total length of the chords:

$$p=\sum_{i=1}^{n}\sqrt{\left(\varphi\left(t_i\right)-\varphi\left(t_{i-1}\right)\right)^2+\left(\psi\left(t_i\right)-\psi\left(t_{i-1}\right)\right)^2}$$

or, using the formula for finite increments,

$$p=\sum_{i=1}^{n}\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i^{/}\right)}\left(t_i-t_{i-1}\right),\ \ \left(t_{i-1}<\tau_i<t_i\right),\left(t_{i-1}<\tau_i^{/}<t_i\right). \quad (3.13)$$

It can be shown that the requirement that the greatest of the chords tends to zero is equivalent to the requirement that the greatest of $\left(t_i-t_{i-1}\right)$ tends to zero. This can be proved without assuming the existence of derivatives $\varphi^{/}\left(t\right)$ and $\psi^{/}\left(t\right)$.

Expression (3.13) differs from the sum giving in the limit the integral

$$\int_{\alpha}^{\beta}\sqrt{\varphi^{/2}\left(t\right)+\psi^{/2}\left(t\right)}\,dt, \quad (3.14)$$

since the arguments τ_i and $\tau_i^{/}$ are in general different. We introduce the sum

$$q=\sum_{i=1}^{n}\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i\right)}\left(t_i-t_{i-1}\right),$$

which gives integral (3.14) in the limit. To show that (3.13) tends to the limit (3.14), we have to show that the difference

$$p-q=\sum_{i=1}^{n}\left[\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i^{/}\right)}-\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i\right)}\right]\left(t_i-t_{i-1}\right)$$

tends to zero.

We multiply and divide by the sum of the radicals, and get

$$p-q=\sum_{i=1}^{n}\frac{\left(\psi^{/}\left(\tau_i^{/}\right)+\psi^{/}\left(\tau_i\right)\right)\left(\psi^{/}\left(\tau_i^{/}\right)-\psi^{/}\left(\tau_i\right)\right)}{\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i^{/}\right)}+\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i\right)}}\left(t_i-t_{i-1}\right).$$

Since

$$\psi^{/}\left(\tau_i^{/}\right)+\psi^{/}\left(\tau_i\right)\le\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i^{/}\right)}+\sqrt{\varphi^{/2}\left(\tau_i\right)+\psi^{/2}\left(\tau_i\right)},$$

we have

$$\left|p-q\right|\le\sum_{i=1}^{n}\left|\psi^{/}\left(\tau_i^{/}\right)-\psi^{/}\left(\tau_i\right)\right|\left(t_i-t_{i-1}\right).$$

The numbers τ_i and $\tau_i^{/}$ belong to the interval $\left(t_{i-1},t_i\right)$, and we can say, by the uniform continuity of $\psi^{/}\left(t\right)$ in the interval $\alpha\le t\le\beta$, that the greatest of the $\left|\psi^{/}\left(\tau_i^{/}\right)-\psi^{/}\left(\tau_i\right)\right|$, which we denote by δ, will tend to zero if the greatest of the $\left(t_i-t_{i-1}\right)$ tends to zero. But we have from the above formula:

$$\left|p-q\right|\le\sum_{i=1}^{n}\delta\left(t_i-t_{i-1}\right)=\delta\sum_{i=1}^{n}\left(t_i-t_{i-1}\right)=\delta\left(\beta-\alpha\right),$$

whence it is clear that $p-q\to 0$. Thus the total length of the chords given by (3.13), tends to integral (3.14), i.e.

$$l=\int_{\alpha}^{\beta}\sqrt{\varphi^{/2}\left(t\right)+\psi^{/2}\left(t\right)}\,dt. \qquad (3.15)$$

This formula for the length l remains valid in the case of a closed curve. This can be shown, for instance, simply by dividing the closed curve into two open curves, applying (3.15) to each, and adding the values of l obtained. Similarly, if a curve L consists of a finite number of curves L_k, each with a parametric form that satisfies the conditions given above, the length L_k of each can be found from (3.15), and the total length L obtained by summation.

We take t variable in the interval $\left(\alpha,\beta\right)$, with the corresponding point M varying over the arc AB. The length of arc AM will be a function of t, given by

$$s\left(t\right)=\int_{\alpha}^{t}\sqrt{\varphi^{/2}\left(t\right)+\psi^{/2}\left(t\right)}\,dt. \qquad (3.16)$$

We get by the rule for differentiation of an integral with respect to its upper limit,

$$\frac{ds}{dt}=\sqrt{\varphi^{/2}\left(t\right)+\psi^{/2}\left(t\right)}, \qquad (3.17)$$

i.e.

$$ds=\sqrt{\varphi^{/2}\left(t\right)+\psi^{/2}\left(t\right)}\,dt,$$

whence, noting that

$$\varphi'(t)=\frac{dx}{dt},\quad \psi'(t)=\frac{dy}{dt},$$

we get the formula for the differential of an arc:

$$ds=\sqrt{(dx)^2+(dy)^2},$$

whilst (3.15) can be written without indicating the variable of integration, in the form:

$$l=\int_{(A)}^{(B)} ds=\int_{(A)}^{(B)}\sqrt{(dx)^2+(dy)^2}.$$

Limits (A) and (B) indicate the initial and final points of the arc. If $\varphi'^2(t)+\psi'^2(t)>0$ for all t in (α,β), (3.17) gives us the derivative of the parameter t with respect to s:

$$\frac{dt}{ds}=\frac{1}{\sqrt{\varphi'^2(t)+\psi'^2(t)}}.$$

The existence of continuous derivatives $\varphi'(t)$ and $\psi'(t)$, together with the condition $\varphi'^2(t)+\psi'^2(t)>0$, ensures a tangent varying continuously along AB.

If the curve is given in polar coordinates by the equation

$$r=f(\theta),$$

we can introduce rectangular coordinates x and y, related to r and θ by

$$x=r\cos\theta,\quad y=r\sin\theta, \qquad (3.18)$$

then take these equations as a parametric form of the curve with the parameter θ.

We then have:

$$dx=\cos\theta dr-r\sin\theta d\theta,$$
$$dy=\sin\theta dr+r\cos\theta d\theta,$$
$$dx^2+dy^2=(dr)^2+r^2(d\theta)^2,$$

whence

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{(dr)^2+r^2(d\theta)^2} \qquad (3.19)$$

and if $A,\ B$ correspond to values $\alpha,\ \beta$ respectively of the polar angle θ (Fig. 3.9), (3.15) gives us:

$$s=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta. \qquad (3.20)$$

Expression (3.19) for ds, called the differential of an arc in polar coordinates, can be got directly from the figure (Fig. 3.9), by replacing the infinitely small arc $MM^{/}$ by its chord, then taking the chord as the hypotenuse of the right-angled triangle $MNM^{/}$, the adjacent sides of which, $\overline{MN}$ and $\overline{NM^{/}}$, are respectively approximately equal to $rd\theta$ and dr.

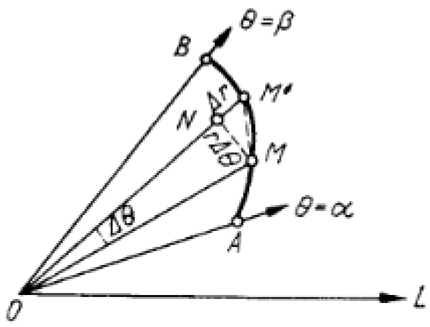

Fig. 3.9

Examples:

1.The length of arc s of the parabola $y = x^2$ measured from the vertex $(0,0)$ to a variable point with abscissa x, is given by (3.12) as the integral:

$$s = \int_0^x \sqrt{1 + y^{/2}}\, dx = \int_0^x \sqrt{1 + 4x^2}\, dx = \frac{1}{2}\int_0^{2x} \sqrt{1 + t^2}\, dt \qquad (3.21)$$

(substituting $t = 2x$).

We have by (example 20, sec. 1.8):

$$\int \sqrt{1 + t^2}\, dt = \frac{1}{2}\left[t\sqrt{1 + t^2} + \ln\left(t + \sqrt{1 + t^2}\right)\right] + C.$$

Substituting this in (3.21), we easily obtain:

$$s = \frac{1}{4}\left[2x\sqrt{1 + 4x^2} + \ln\left(2x + \sqrt{1 + 4x^2}\right)\right].$$

2. The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is equal, by its symmetry with respect to the axes, to four times the length of the part lying in the first quadrant. We use the parametric equations of the ellipse

$$x = a\cos t, \quad y = b\sin t,$$

and note that variation from A to B means variation of the parameter from 0 to $\pi/2$; we then get, by (3.15), the following expression for the required length l:

$$l = 4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt. \tag{3.22}$$

This integral cannot be evaluated in an explicit form; a method of approximate evaluation, given later, has to be used.

3. The length of arc of the logarithmic spiral

$$r = Ce^{a\theta},$$

cut off by the radius-vectors $\theta = \alpha,\ \ \theta = \beta$, is given by (3.20) as the integral:

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = C\sqrt{1+\alpha^2}\int_\alpha^\beta e^{a\theta}\, d\theta = \frac{C\sqrt{1+\alpha^2}}{\alpha}\left[e^{a\beta} - e^{a\alpha}\right].$$

4. Let $M(x,y)$ be a point of the catenary $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)\ \ (a>0)$. We find the length of arc AM (Fig. 3.10).

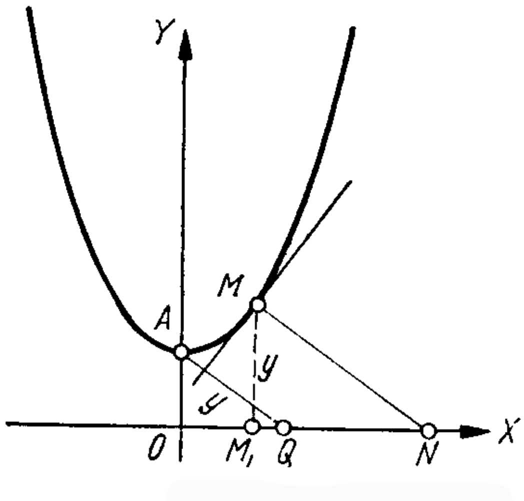

Fig. 3.10

Using the expression for

$$\left(1+y'^2\right) = 1 + \frac{\left(e^{x/a} - e^{-x/a}\right)^2}{4} = \frac{4 + e^{2x/a} - 2 + e^{-2x/a}}{4} = \frac{\left(e^{x/a} + e^{-x/a}\right)^2}{4} = \frac{y^2}{a^2},$$

we get:

$$AM = \int_0^x \sqrt{1+y'^2}\, dx = \int_0^x \frac{y}{a} dx = \frac{1}{2}\int_0^x \left(e^{x/a} + e^{-x/a}\right) dx =$$

$$= \frac{a}{2}\left(e^{x/a} - e^{-x/a}\right) = ay',$$

whence

$$a^2 + \left(\text{arc}\, AM\right)^2 = a^2 + a^2 y'^2 = a^2\left(1+y'^2\right) = y^2,$$

i.e. the length of arc AM is equal to the adjacent side of a right-angled triangle with hypotenuse equal to the ordinate of M, and the other adjacent side equal to a. This gives us the following rule for finding the length of arc AM.

With the vertex A of the catenary as centre, describe a circle with radius equal to the ordinate of M; let the circle cut OX in Q; then OQ is the rectified arc AM (see Fig. 3.10).

We have been guided in the choice of sign in the above formula by the fact that y' is positive for points lying on the right-hand side of the catenary.

5. We take the cycloid (Fig. 3.11)

$$(a = R),$$

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t) \end{cases}$$

and find the length of arc l of the branch OO' (Fig. 3.11), and the area S bounded between this branch and OX:

$$l = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt =$$

$$= \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\, dt = a\int_0^{2\pi} \sqrt{2(1-\cos t)}\, dt =$$

$$= a\int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt = 2a\int_0^{2\pi} \sin\frac{t}{2} dt = 2a\left[-2\cos\frac{t}{2}\right]\Bigg|_0^{2\pi} = 8a,$$

i.e. the length of arc of one branch of a cycloid is four times the diameter of the rolling circle:

$$S = \int_0^{2\pi a} y\,dx = \int_0^{2\pi} y(t)\cdot x'(t)dt = a^2\int_0^{2\pi} (1-\cos t)^2\, dt =$$

$$= a^2\int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right)dt = a^2 t - 2\sin t + \frac{t}{2} + \frac{1}{4}\sin 2t\Bigg|_0^{2\pi} = 3\pi a^2,$$

i.e. the area bounded by one branch of a cycloid and the fixed line along which the circle rolls is equal to three times the area of the rolling circle.

We had to take the (+) sign with $\sqrt{4\sin^2 \frac{t}{2}}$ in finding l, since $\sin\frac{t}{2}$ is positive with t varying from 0 to 2π.

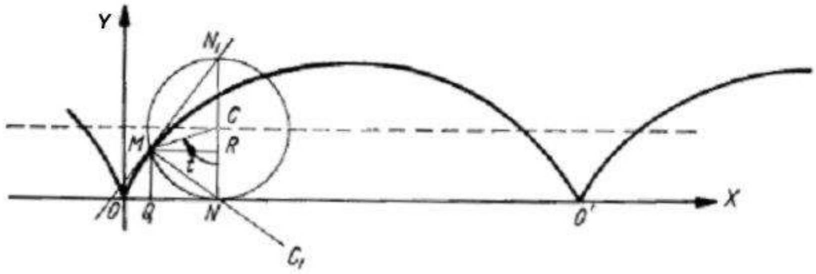

Fig. 3.11

6. The cardioid $r = 2a(1+\cos\theta)$ (where $a = CN$ is a radius of a circle) is symmetrical with respect to the polar axis (Fig. 3.12), and its length l can therefore be found by doubling the length of arc for θ varying from 0 to π:

$$l = 2\int_0^{\pi} \sqrt{r^2 + r'^2}\, d\theta = 2\cdot 2a\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + \sin^2\theta}\, d\theta =$$

$$= 4a\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta = 4a\int_0^{\pi} \sqrt{2\cdot 2\cos^2\frac{\theta}{2}}\, d\theta = 8a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 16a,$$

i.e. the length of arc of a cardioid is eight times the diameter of the rolling (or fixed) circle.

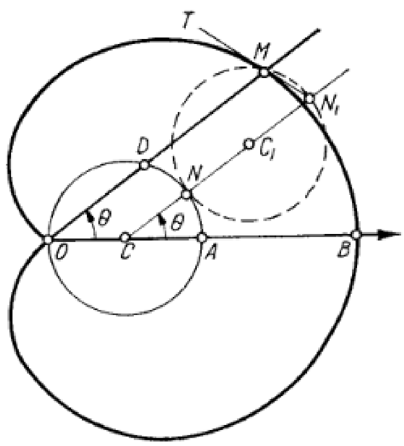

Fig. 3.12

3.4. Calculation of the volumes of solids of given cross-section

The calculation of the volume of a solid also leads to evaluation of a definite integral, when the area of the cross-section, perpendicular to a given direction, is known.

We call the volume of the given solid V (Fig. 3.13) and we assume that the cross-sectional area of the solid is known in planes perpendicular to a given direction, which is taken as OX.

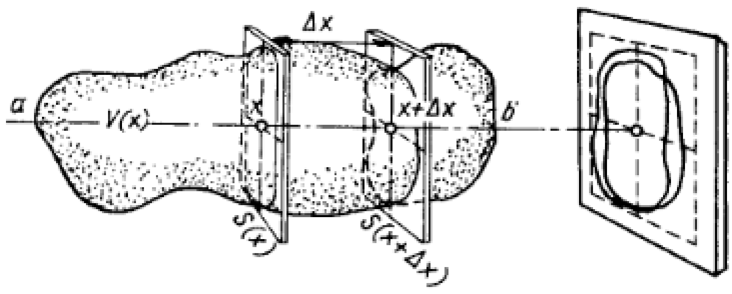

Fig. 3.13

Each cross-section is defined by the abscissa x of its point of intersection with OX, so that the cross-sectional area is a function of x, say $S(x)$, which we assume known.

Further, let a, b denote the abscissae of the extreme sections of the solid. We find V by dividing the solid into elements, by means of a series of cross-sections starting at $x = a$ and ending at $x = b$we take one such element ΔV between the cross-sections with abscissae x and $x + \Delta x$. We replace ΔV by the volume of a right cylinder of height Δx and base equal to the cross-section of the solid at x (Fig. 3.14).

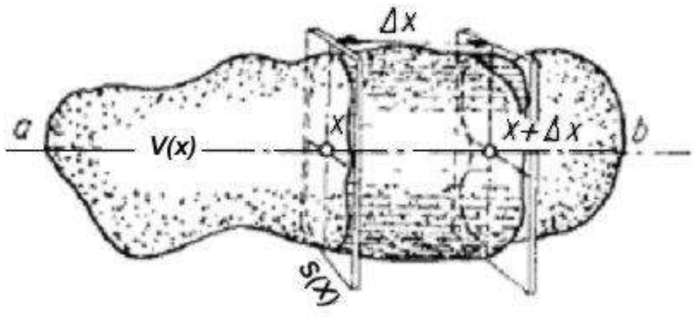

Fig. 3.14

The volume of this cylinder is $S(x)\Delta x$, so that the required volume V will be given approximately by

$$\sum S(x)\Delta x,$$

where the summation is over all the elements into which the solid has been divided. On indefinitely increasing the number of elements, and with the greatest of the Δx tending to zero, the above sum becomes in the limit a definite integral, accurately expressing V; this leads to the following proposition:

If the cross-section of a solid is known for all the planes lying perpendicular to a given direction, which we take as the axis OX, the volume V of the solid is given by :

$$V = \int_a^b S(x)dx, \qquad (3.23)$$

where $S(x)$ denotes the area of the cross-section of abscissa x, and a, b are the abscissae of the extremities of the solid.

Example. To find the volume of a cylindrical ungula, cut from a right circular cylinder by a plane passing through a diameter of its base (Fig. 3.15).

We take the diameter AB as OX, with A as origin; we let r denote the radius of the base of the cylinder, and α the angle between the cutting plane and the base.

A cross-section perpendicular to AB consists of a right-angled triangle PQR, the area of which is:

$$S(x) = \frac{1}{2} PQ \cdot QR = \frac{1}{2} \tan\alpha \cdot PQ^2.$$

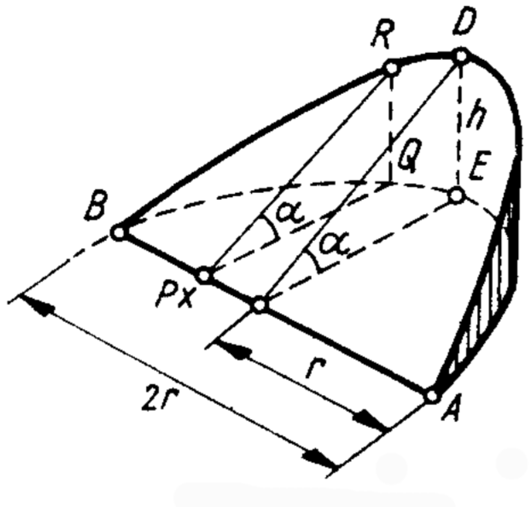

Fig. 3.15

Further, we know from the properties of circles that PQ is the geometric mean of AP, PB, i.e.

$$|PQ|^2 = |AP||PB| = x(2r - x),$$

so that finally,

$$S(x) = \frac{1}{2} x(2r - x)\tan\alpha.$$

Using (3.23), we get for the required volume V:

$$V = \int_0^{2r} S(x)\,dx = \frac{\tan\alpha}{2}\int_0^{2r} x(2r - x)\,dx = \frac{\tan\alpha}{2}\left(rx^2 - \frac{x^3}{3}\right)\Bigg|_0^{2r} = \frac{2r^3\tan\alpha}{3} = \frac{2}{3}r^2 h$$

where the "height" of the ungula $h = r\tan\alpha$.

3.5. Volume of a solid of revolution

When the solid in question is obtained by revolution of a given curve $y = f(x)$ about OX, its cross-section is a circle of radius y (Fig. 3.16), and hence

$$S(x) = \pi y^2,$$

$$V(x) = \int_a^b \pi y^2\, dx,$$

i.e. the volume of the solid, obtained by revolution about OX of the part of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$, is given by :

$$V = \int_a^b \pi y^2\, dx. \tag{3.24}$$

Example. Volume of the ellipsoid of revolution. We get by revolution of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about its major axis the *prolate ellipsoid of revolution* (Fig. 3.17). The extreme abscissae x are $(-a)$and $(+a)$ in this case, so that (3.24) gives:

$$V_{pro} = \pi \int_{-a}^{+a} y^2\, dx = \pi \int_{-a}^{+a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 \left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{+a} = \frac{4}{3}\pi a b^2. \tag{3.25}$$

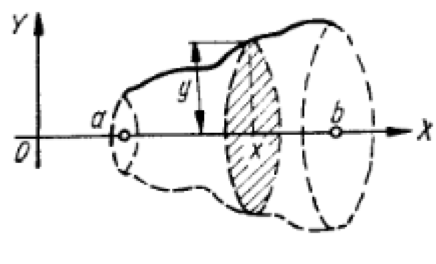

Fig. 3.16

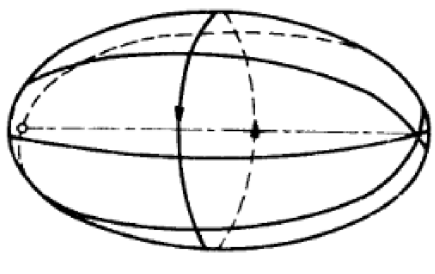
Fig. 3.17

The volume of the *oblate ellipsoid* of revolution, obtained by revolution of the ellipse about its minor axis, is found in the same way. It is only necessary to interchange x, y and a, b, which gives:

$$V_{ob} = \pi \int_{-b}^{+b} x^2\, dy = \pi \int_{-b}^{+b} a^2 \left(1 - \frac{y^2}{b^2}\right) dy = \frac{4}{3}\pi b a^2. \tag{3.26}$$

When $a=b$, both ellipsoids reduce to a sphere of radius a, with volume $\frac{4}{3}\pi a^3$.

3.6. Surface area of a solid of revolution

When a solid is obtained by the revolution of a curve, given in the XOY plane, about the axis OX, its surface area is defined as the limit of the surface area of a second solid, the second solid being obtained by revolution of a successive series of chords, inscribed in the original curve, also about OX, the limit being taken when the number of chords increases indefinitely, and the greatest of them tends to zero (Fig. 3.18).

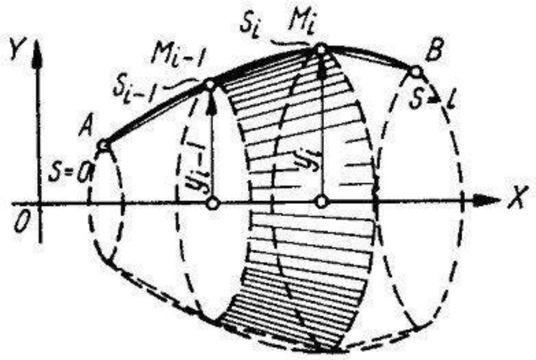

Fig. 3.18

If the part of the rotating curve lies between A and B, the surface area F is given for the solid of revolution by :

$$F=\int_{(A)}^{(B)} 2\pi y\,ds, \tag{3.27}$$

where ds is the differential of the arc of the given curve, i.e.

$$ds=\sqrt{(dx)^2+(dy)^2}.$$

The curve may be given in any convenient form in the above formula, either explicitly or parametrically; symbols (A), (B) indicate that integration is between the limits of the independent variable corresponding to the given points A, B of the curve.

We shall assume that the equation of the curve is given parametrically, with the length of arc s as parameter, measured from the point A; let l denote the total length of curve AB. The curve is of course assumed to be rectifiable. We have: $x=\varphi(s)$, $y=\psi(s)$. We divide the interval $(0,l)$ of variation of s into sub-intervals in the usual way, with

$$0 = s_0 < s_1 < s_2 < \ldots < s_{n-1} < s_n = l .$$

Let $s = s_i$ correspond to the point M_i of the curve, with M_0 evidently coinciding with A, and M_n with B. Let q_i denote the length of the chord $M_{i-1}M_i$ and Δs_i the length of the arc $M_{i-1}M_i$, and let $y_i = \psi(s_i)$. Using the formula for the lateral surface of the frustrum of a cone, we have the following result for the surface area obtained by revolution of the series of chords $AM_1M_2 \ldots M_{n-1}B$:

$$Q = 2\pi \sum_{i=1}^{n} \frac{1}{2}(y_{i-1} + y_i)q_i$$

or

$$Q = 2\pi \sum_{i=1}^{n} y_{i-1}q_i + \pi \sum_{i=1}^{n} (y_i - y_{i-1})q_i .$$

Let δ be the greatest of the $|y_i - y_{i-1}|$. By the uniform continuity of $\psi(s)$ in $0 \le s \le l$, δ tends to zero if the greatest value of $(s_i - s_{i-1})$ tends to zero. But we have:

$$\left| \sum_{i=1}^{n} (y_i - y_{i-1})q_i \right| \le \delta \sum_{i=1}^{n} q_i \le \delta l ,$$

whence it follows that the second term in the expression for Q tends to zero. We consider the first term by rewriting it as:

$$2\pi \sum_{i=1}^{n} y_{i-1}q_i = 2\pi \sum_{i=1}^{n} y_{i-1}\Delta s_i - 2\pi \sum_{i=1}^{n} y_{i-1}(\Delta s_i - q_i).$$

We show that the second term on the right-hand side tends to zero, by remarking that $\psi(s)$, continuous in $(0, l)$, is bounded, so that there exists a positive m, such that $|y_{i-1}| \le m$ for all i. Hence:

$$\left| \sum_{i=1}^{n} y_{i-1}(\Delta s_i - q_i) \right| \le \sum_{i=1}^{n} m(\Delta s_i - q_i) = m\left(l - \sum_{i=1}^{n} q_i \right).$$

But if the greatest of the $(s_i - s_{i-1})$ values tends to zero, the greatest of the chords q_i also tends to zero, and the total length of the chords tends to the length of the arc:

$$\sum_{i=1}^{n} q_i \to l ,$$

whence

$$2\pi \sum_{i=1}^{n} y_{i-1}(\Delta s_i - q_i) \to 0 .$$

The only remaining term to be investigated in the expression for Q is thus:

$$2\pi\sum_{i=1}^{n} y_{i-1}\Delta s_i = 2\pi\sum_{i=1}^{n}\psi\left(s_{i-1}\right)\left(s_i - s_{i-1}\right).$$

But the limit of this sum gives us integral (3.27). Our formula has thus been proved. If the curve is given in terms of a parameter t, we have:

$$F = \int_{\alpha}^{\beta} 2\pi\psi\left(t\right)\sqrt{\varphi'^2\left(t\right)+\psi'^2\left(t\right)}\,dt \qquad (3.28)$$

and in the case of the explicit $y = f\left(x\right)$ equation for AB:

$$F = \int_{a}^{b} 2\pi f\left(x\right)\sqrt{1+f'^2\left(x\right)}\,dx. \qquad (3.29)$$

Example. Surface area of prolate and oblate ellipsoid of revolution.

We take the prolate case first. Using the notation of the example of the previous item, we have by (3.29):

$$F_{pro} = 2\pi\int_{-a}^{a} y\sqrt{1+y'^2}\,dx = 2\pi\int_{-a}^{a}\sqrt{y^2+\left(yy'\right)^2}\,dx.$$

We have from the equation of the ellipse:

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right), \quad yy' = -\frac{b^2 x}{a^2},$$

whence

$$\left(yy'\right)^2 = \frac{b^4 x^2}{a^4},$$

$$F_{pro} = 2\pi\int_{-a}^{a}\sqrt{b^2-\frac{b^2x^2}{a^2}+\frac{b^4x^2}{a^4}}\,dx = 2\pi b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}\left(1-\frac{b^2}{a^2}\right)}\,dx.$$

We introduce here the eccentricity of the ellipse

$$\varepsilon^2 = \frac{a^2-b^2}{a^2},$$

and get:

$$F_{pro} = 2\pi b\int_{-a}^{a}\sqrt{1-\frac{\varepsilon^2x^2}{a^2}}\,dx = 4\pi b\int_{0}^{a}\sqrt{1-\frac{\varepsilon^2x^2}{a^2}}\,dx =$$

$$= \frac{4\pi ba}{\varepsilon}\int_{0}^{a}\sqrt{1-\left(\frac{\varepsilon x}{a}\right)^2}\,d\left(\frac{\varepsilon x}{a}\right) = \frac{4\pi ba}{\varepsilon}\int_{0}^{\varepsilon}\sqrt{1-t^2}\,dt.$$

Integrating by parts, we have:

$$\int\sqrt{1-t^2}\,dt = t\sqrt{1-t^2}+\int\frac{t^2}{\sqrt{1-t^2}}dt = t\sqrt{1-t^2}-\int\sqrt{1-t^2}\,dt+\int\frac{1}{\sqrt{1-t^2}}dt,$$

whence

$$\int\sqrt{1-t^2}\,dt = \frac{1}{2}\left[t\sqrt{1-t^2}+\arcsin t\right],$$

and finally

$$F_{pro} = 2\pi ab\left[\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right]. \tag{3.30}$$

This formula is also valid in the limit for $\varepsilon = 0$, i.e. when $b = a$, and the ellipse reduces to a sphere of radius a. The square brackets contain an indeterminate form in this case, which may be evaluated by l'Hospital rule as:

$$\left.\frac{\arcsin\varepsilon}{\varepsilon}\right|_{\varepsilon=0} = \left.\frac{1/\sqrt{1-\varepsilon^2}}{1}\right|_{\varepsilon=0} = 1.$$

We now turn to the oblate ellipsoid of revolution. We interchange $x,\ y$, and $a,\ b$, and find:

$$F_{ob} = 2\pi\int_{-b}^{b}\sqrt{x^2+\left(xx'\right)^2}\,dy,$$

where x is taken as a function of y.

But we have from the equation of the ellipse:

$$x^2 = a^2\left(1-\frac{y^2}{b^2}\right),\quad xx' = -\frac{a^2y}{b^2},\quad \left(xx'\right)^2 = \frac{a^4y^2}{b^4},$$

whence

$$F_{ob} = 2\pi a\int_{-b}^{b}\sqrt{1+\frac{y^2}{b^2}\left(\frac{a^2}{b^2}-1\right)}\,dy = \left|\frac{ya\varepsilon}{b^2}=t\right| = 4\pi\frac{b^2}{\varepsilon}\int_{0}^{a\varepsilon/b}\sqrt{1+t^2}\,dt =$$

$$= 2\pi\frac{b^2}{\varepsilon}\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]\Bigg|_0^{a\varepsilon/b} =$$

$$= 2\pi\frac{b^2}{\varepsilon}\left[\frac{a\varepsilon}{b}\sqrt{\frac{a^2}{b^2}}+\ln\left(\frac{a\varepsilon}{b}+\sqrt{\frac{a^2}{b^2}}\right)\right] = 2\pi a^2+2\pi\frac{b^2}{\varepsilon}\ln\frac{a\left(1+\varepsilon\right)}{b},$$

and finally:

$$F_{ob} = 2\pi a^2+2\pi\frac{b^2}{\varepsilon}\ln\frac{a\left(1+\varepsilon\right)}{b}. \tag{3.31}$$

3.7. Determination of centre of gravity. Guldin's theorem

Given a system of n point-masses at

$$M_1(x_1, y_1), M_2(x_2, y_2), ..., M_n(x_n, y_n),$$

with respective masses

$$m_1,\ m_2,\ ...,\ m_n,$$

the centre of gravity G of the system is defined as the point whose coordinates x_G, y_G satisfy the conditions:

$$Mx_G = \sum_{i=1}^{n} m_i x_i, \qquad My_G = \sum_{i=1}^{n} m_i y_i\ , \tag{3.32}$$

where M denotes the total mass of the system:

$$M = \sum_{i=1}^{n} m_i\ .$$

The points of a system may be arranged in any desired manner when finding its centre of gravity G; the aim is to group the points in such a way that the points of anyone group may be replaced by a single point at the centre of gravity of the group, the mass at the point being the total mass of the group.

We shall not dwell on the proof of this general principle, since it presents no difficulty and can easily be verified for the simplest particular cases of systems of three, four points etc.

We shall be concerned in future, not with systems of points, but with the continuous distribution of mass over a plane figure (domain) or on a line.

For simplicity, we limit our consideration to homogeneous solids, the density of which we take as unity, so that the mass is equal to the length in the case of linear distributions, and to the area for plane distributions.

Suppose first that it is required to find the centre of gravity of an arc AB of a curve (Fig. 3.19), the length of which is s.

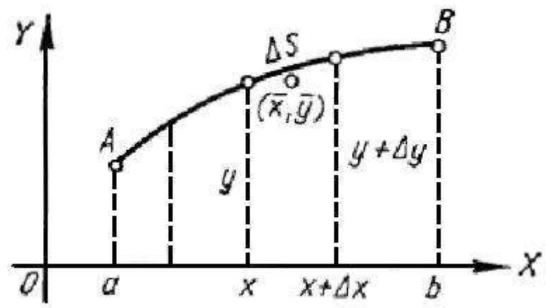

Fig. 3.19

Following the above general principle, we divide arc AB into n small elements Δs. The centre of gravity of the total system can be found by replacing each of these elements with its centre of gravity at which is concentrated the total mass of the element $\Delta m = \Delta s$. (The centre of gravity of an element does not in general lie on the curve, though it approaches closer to the curve, the smaller the element. This is indicated schematically in Fig. 3.19).

We take an element Δs, and let the coordinates of its ends be (x, y), $(x + \Delta x, y + \Delta y)$; the coordinates of its centre of gravity are denoted by $(\overline{x}, \overline{y})$. We can suppose that the point $(\overline{x}, \overline{y})$ differs by as little as may be desired from the point (x, y) on sufficiently diminishing s.

We have by (3.32), as in sec. 3.4:

$$Mx_G = sx_G = \sum \overline{x}\Delta m = \sum \overline{x}\Delta s = \lim \sum x\Delta s = \int_{(A)}^{(B)} x\,ds, \qquad (3.33)$$

$$My_G = sy_G = \sum \overline{y}\Delta m = \sum \overline{y}\Delta s = \lim \sum y\Delta s = \int_{(A)}^{(B)} y\,ds, \qquad (3.34)$$

whence, on obtaining s from the formula:

$$s = \int_{(A)}^{(B)} ds = \int_{(A)}^{(B)} \sqrt{(dx)^2 + (dy)^2},$$

we find the coordinates of the centre of gravity G.

An important theorem follows from (3.33) and (3.34).

Guldin's first theorem*. The surface area of a solid, obtained by revolution of the arc of a given plane curve about some non-intersecting axis in the same plane, is equal to the product of the length of arc and the path described on revolution by the centre of gravity of the arc.*

Taking the axis of revolution as OX, we have for the surface area F of the solid obtained by revolution of arc AB (using (3.27)):

$$F = 2\pi \int_{(A)}^{(B)} y\,ds = 2\pi y_G s$$

[by (3.34)], which it was required to prove.

We now take a plane domain S, its area being also denoted by S. We suppose for simplicity that the domain (Fig. 3.20) is bounded by two curves, the ordinates of which are denoted by

$$y_1 = f_1(x), \quad y_2 = f_2(x).$$

In accordance with the general principle mentioned at the beginning of this article, we divide the figure into n vertical strips ΔS by lines parallel to OY. We can find the coordinates of the centre of gravity G by replacing each strip by its centre of gravity, associated with its mass $\Delta m = \Delta S$. We take the strip bounded by M_1M_2 and $M_1^{/}M_2^{/}$, with abscissae x and $x+\Delta x$, and let its centre of gravity be $\left(\overline{x},\overline{y}\right)$.

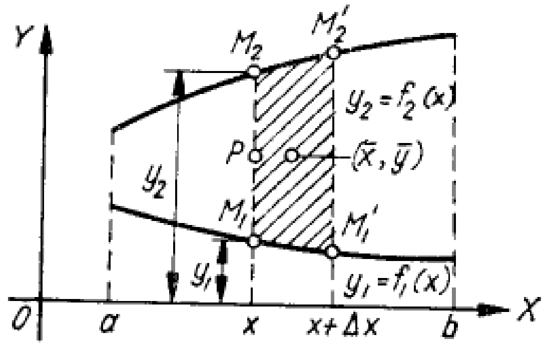

Fig. 3.20

On sufficiently reducing the breadth Δx of the strip, $\left(\overline{x},\overline{y}\right)$ will differ as little as desired from the centre P of M_1M_2, so that we can write the approximate equations:

$$\overline{x}\approx x,\quad \overline{y}\approx\frac{y_1+y_2}{2}.$$

Further, the mass Δm of the strip is equal to its area ΔS and so can be taken as approximately equal to the area of the rectangle of base Δx and height differing by as little as desired from $\overline{M_1M_2}=y_2-y_1$, i.e.

$$\Delta m\approx\left(y_2-y_1\right)\Delta x.$$

Using (3.32), we can write:

$$Mx_G=Sx_G=\sum\overline{x}\Delta m=\lim\sum\left[x\left(y_2-y_1\right)\right]\Delta x=\int_a^b x\left(y_2-y_1\right)dx, \quad (3.35)$$

$$My_G=Sy_G=\sum\overline{y}\Delta m=\lim\sum\left(\frac{y_2+y_1}{2}\right)\left(y_2-y_1\right)\Delta x=$$
$$=\lim\sum\left(\frac{y_2^2-y_1^2}{2}\right)\Delta x=\int_a^b\left(\frac{y_2^2-y_1^2}{2}\right)dx. \quad (3.36)$$

Guldin's second theorem. This follows from (3.36):

When a plane figure revolves about an axis in its plane which does not cut the figure, the volume of the solid obtained is equal to the product of the area of the figure and the length of path resulting from the revolution of the centre of gravity of the figure.

We take the axis of revolution as OX, and notice that the volume V of the solid of revolution in question is equal to the difference between the volumes of the solids obtained by revolution of the curves y_2 and y_1, so that, in accordance with (3.24):

$$V = \pi\int_a^b y_2^2\,dx - \pi\int_a^b y_1^2\,dx = \pi\int_a^b \left(y_2^2 - y_1^2\right)dx = 2\pi y_G S\,,$$

by (3.36), which is what we required to prove.

The above two theorems of Guldin are extremely useful, either for finding the surface area or volume of a solid of revolution, when the centre of gravity of the revolving figure is known, or conversely for finding the centre of gravity of a figure when the volume or surface area of the solid obtained by its revolution is known.

Examples:

1. To find the volume V of the anchor ring (torus) obtained by revolution of a circle of radius r (Fig. 3.21) about an axis in its plane, distant a from the centre (with $r < a$, i.e. the axis of revolution does not cut the circle).

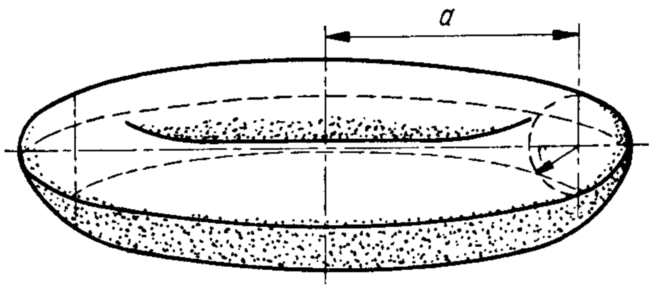

Fig. 3.21

The centre of gravity of the revolving circle evidently lies at its centre, so that the path described by revolution of the centre of gravity is equal to $2\pi a$. The area of the revolving figure is πa^2, and hence we have by Guldin's second theorem:

$$V = \pi r^2 \cdot 2\pi a = 2\pi^2 a r^2\,. \qquad (3.37)$$

2. To find the surface area F of the anchor ring of Example 1.
The length of the revolving circle is $2\pi r$; the centre of gravity coincides with the centre of the circle, as before, so that we have by Guldin's first theorem:

$$F = 2\pi r \cdot 2\pi a = 4\pi^2 a r\,. \qquad (3.38)$$

3. To find the centre of gravity G of a semi-circular disc of radius a. We take the base of the semi-circle as axis OX, with OY upwards through the

centre and perpendicular to OX (Fig. 3.22); it is clear from the symmetry of the figure with respect to OY that G lies on OY.

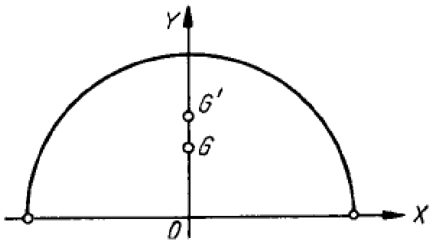

Fig. 3.22

It only remains to find y_G. We apply Guldin's second theorem. The solid obtained by revolution of the semi-circle about OX is a sphere of radius a, and its volume is $\frac{4}{3}\pi a^3$. The area S of the revolving figure is $\frac{1}{2}\pi a^2$, and hence:

$$\frac{4}{3}\pi a^3 = \frac{1}{2}\pi a^2 \cdot 2\pi y_G, \quad y_G = \frac{4}{3}\frac{a}{\pi} .$$

4. To find the centre of gravity $G^/$ of a semi-circular arc of radius a .

We take the same axes as in the previous example, and see that $G^/$ also lies on OY, so that we have to find $y_{G^/}$. We apply Guldin's first theorem, and note that the surface area F of the solid of revolution is $4\pi a^2$ in this case, whilst $s = \pi a$, so that:

$$4\pi a^2 = \pi a \cdot 2\pi y_{G^/}, \quad y_{G^/} = 2\frac{a}{\pi}.$$

As might be expected, the centre of gravity of the semi-circular arc lies closer to it than the centre of gravity of the semi-circular disc bounded by it.

BIBLIOGRAPHY

1. Adams, R. A. Calculus : a Complete Course / R. A. Adams. – Pearson, 2018. – 1174 p.
2. Courant, R. Differential and Integral Calculus / R. Courant. – London, 1961. – 632 p.
3. Herman, E. Calculus / E. Herman, G. Strang. – Houston, 2017. – V. 2. – 829 p.
4. Stewart, J. Calculus / J. Stewart. – Boston, 2016. – 1459 p.
5. Strang, G. Calculus / G. Strang. – Wellesley – Cambridge, 2014. – 670 p.

CONTENT

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