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THE METHOD OF DETERMINING OPTIMAL CONTROL OF THE THERMOELASTIC STATE OF PIECE-HOMOGENEOUS BODY USING A STATIONARY TEMPERATURE FIELD

This paper proposes a new highly effective method for determining the optimal control of the stress-strain state of spatially multi-connected composite bodies using a stationary temperature field. The proposed method is considered based on the example of a stationary axisymmetric thermoelastic problem for a space with a spherical inclusion and cavity. The proposed method is based on the generalized Fourier method and reduces the original problem to an equivalent problem of optimal control, in which the state of the object is determined by an infinite system of linear algebraic equations, the right-hand side of which parametrically depends on the control. At the same time, the functional of the cost of the initial problem is transformed into a quadratic functional, which depends on the state of the equivalent system and parametrically on the control. The limitation on the temperature distribution is replaced by the value of the control norm in the space of square summable sequences. In fact, this paper considers for the first time the problem of optimal control of an infinite system of linear algebraic equations and develops a method for its solution. The proposed method is based on presenting the solutions of infinite systems in a parametric form, which makes it possible to reduce equivalent problem to the problem of conditional extremum of a quadratic functional, which explicitly depends on the control. A further solution to this problem A further solution to this problem is found by the Lagrange method using the spectral decomposition of the quadratic functional matrix. found by the Lagrange method using the spectral decomposition of the quadratic functional matrix. The method developed in this paper is strictly justified. For all infinite systems, the Fredholm property of their operators is proved. As an important result necessary for substantiation, for the first time, an estimate from below of the module of the multi-parameter determinant of the resolving system of the boundary value problem of conjugation – space with a spherical inclusion – was obtained when solving it using the Fourier method. The theorem that establishes the conditions for the existence and uniqueness of the solution of equivalent problem or optimal control problem without restrictions in the space of square summable sequences is proved. The numerical algorithm is based on a reduction method for infinite systems of linear algebraic equations. Estimates of the practical accuracy of the numerical algorithm demonstrated the stability of the method and sufficiently high accuracy even with close location of the boundary surfaces. Graphs showing the optimal temperature distribution for various geometric parameters of the problem and their analysis are provided. The proposed method extends to boundary value problems with different geometries.

Keywords: *optimal control; thermoelastic state; stationary temperature field; multi-connected piecewise homogeneous body; generalized Fourier method; infinite system of linear algebraic equations; Fredholm operator; quadratic functional; spectral expansion; reduction method.*

1. Introduction

1.1. Motivation

The theory of optimal control today is the key to solving many practically important engineering problems in all areas of human activity. Let's list only some of them, which are not quite traditional: management and economics [1], construction design [2], biology [3], insurance [4], and finance [5]. Separately, the problems of designing mechanical objects that are planned to be operated under temperature fields should be addressed. Here, the temperature can act as a control in minimizing stresses in the zones of their possible concentration, particularly at the interphase boundaries in composite bodies.

The mathematical theory of optimal control began to take shape in the 1950s. Its basis was the maximum principle, which was formulated by L. S. Pontryagin and proved for linear systems by R. V. Gamkrelidze, and for nonlinear systems by V. G. Boltyanski. These results were first announced at the Edinburgh Mathematical Congress in 1958 and were printed in the West in the article [6] and monograph [7]. Around the same time, R. Bellman made an important contribution to theory [8, 9] by using the dynamic programming method he created to solve some optimal processes problems. The next qualitative step in this area was connected to the development of the optimal control theory for systems with distributed parameters. At the first stage, when control was considered as the right-hand parts or coefficients of the minor derivatives in differential equations that described the state of a certain class of physical and technical systems, the necessary optimality conditions were obtained by direct generalization of the classical principle of maxima. Many printed works from this period can be found in monograph bibliographies [10]. At the same time, studies have described optimal systems by linear differential operators of elliptic, parabolic, and hyperbolic types. In the monograph [11], similar problems are considered when the cost functional is quadratic. The necessary extremity condition was obtained in the form of variational inequalities, which were further reduced to so-called one-sided boundary value problems. The author relied on the results of the article [12], which proved the theorem of existence and uniqueness of the solution of variational inequalities for bilinear functionals in an abstract equipped Hilbert space. Note that for the first time, the technique of variational inequalities and one-sided boundary value problems was proposed by G. Fichera [13] when solving the Signorini problem in the theory of elasticity. In further studies, optimal problems were considered in which control was included in the main parts of linear differential operators, which specified the state equations of the system. The monograph [14] showed that the necessary optimality conditions for such problems depend on the shape of the small region in which control variation occurs. In addition, the author showed that writing the differential equation of the problem in a certain form (analogous to the normal Cauchy form for systems of ordinary differential equations) makes it possible to obtain the same derivation of optimality conditions for different types of differential equations. It must be said that L. S. Pontryagin's formulation of an optimization problem with control constraints, for which it was impossible to directly apply the classical calculus of variations, and the appearance of the maximum principle led to a large number of theoretical and applied studies in the field of optimal control. For more than 60 years of research in this field, thousands of scientific articles have been published. It is impossible to give them a superficial review. However, even today, many problems have not yet been solved and require the creation of new methods for research. The following is an analysis of some publications in recent years on key areas of the development of the theory of optimal control of systems with distributed parameters.

1.2. State of the art

All research was aimed at generalizing the results obtained from the classics of theory in the first decades of its formation. First, this concerns the expansion of the classes of differential equations that control optimal systems due to the consideration of equations with a certain type of nonlinearity: semi-linear or quasi-linear, with the presence of degeneracy in the higher terms or singularities in the coefficients or in the control, with different types of inclusion control in the differential equation: on the right-hand side or in the coefficients of the derivatives and even in the senior terms. This led to the development of the differential equation theory. In practical terms, all studies have proved theorems of the existence and uniqueness of solutions to boundary value problems, which act as constraints on the state of the system in optimization problems. The existence of optimal controls and the necessary conditions for them are also established. Usually, a mathematical apparatus based on Sobolev spaces, embedding theorems, a priori estimates, and weak or soft solutions of differential equations is used. Optimization problems with control included in the equations and boundary conditions are considered, with different types of cost functionals and pure or mixed constraints on the state and control of the system in the presence of concentrated or distributed delay. A separate research direction is devoted to the development of numerical methods for solving optimal control problems. A characteristic feature of such works is the establishment of various types of convergence of the approximate solutions to exact ones.

We now provide an overview of individual works that characterize the indicated areas of research. One of the pioneering works devoted to the extension of Pontryagin's maximum principle to differential equations in partial derivatives is [15]. It considers the optimal control of a semi-linear elliptic equation in the case of distributed control in the domain and at its boundary with additional restrictions on the state of the system. This study constructs a conjugate boundary value problem in which penalty functions are included as additional terms on the right-hand side of the differential equation and boundary condition. The Hamiltonian function is separately formed for the control domain and its boundary. The necessary condition for the existence of optimal control is obtained, where both Hamiltonian functions reach a minimum. In [16], a class of optimal control problems for quasi-linear elliptic equations, in which the coefficients of the elliptic differential operator depend on the state function, is considered. The conditions of the first and second order of optimality were found. For this purpose, the solutions of the system state equation and its linearization were studied in detail. An analogue of Pontryagin's maximum principle and sufficient optimality conditions are derived. In [17], a strong convergence of numerical discretization of the problem of optimal control for a quasi-linear elliptic equation was proved using the method of finite elements for the state and different types of discretization for control. For this purpose, the authors estimated the errors in the discretization of the equation of state and the associated conjugate equation. The regular solutions of these equations, which are required for this analysis, are found from the necessary optimality conditions of the first order obtained in the article [16]. The work [18] considers the problem of optimal control for one class of nonlinear elliptic equations of the Reynolds equation type. The existence and uniqueness theorem of the weak solution of the equation of state of the system is proved. The existence of optimal control is also proved. The proposed method is based on Sobolev embedding theorems. Numerical results were obtained by discretization using the characteristic functions of partition intervals. The application is described to determine the optimal distance of the recording head from the magnetic disk. The article [19] considers the problem of optimal control for the stationary Stokes equation. The speed limit is given by the L_2-norm. The optimality conditions of the first order for continuous and discretized systems are derived from the Karush-Kuhn-Tucker conditions for the Lagrange functional after replacing them with weak differential conditions. When discretizing the system, Galerkin spectral approximations are used to obtain a priori error estimates. Numerical solutions to the problem were obtained with the help of Yuzawa's iterative algorithm and projection scheme. New problems related to the optimal control of distributed systems are considered in [20], and they are described by boundary value problems for an elliptic equation in the union of two bounded strictly Lipschitz domains $\Omega = \Omega_1 \cup \Omega_2$. The domains Ω_i have a common boundary section on which the conjugation conditions are set in the form of the heat conduction equation. Control is included on the right-hand side of the differential equation. The existence and uniqueness of the solution to the boundary value problem is demonstrated using the Lax-Milgram theorem. A similar theorem for optimal control, which belongs to the convex closed set of the equipped Hilbert space, is proved using the generalized solution of the conjugate equation and the quadratic cost functional. For all described cases, theorems of existence and uniqueness of optimal controls are proved. The article [21] considers the problem of optimal control in two- and three-dimensional Lipschitz polytope domains using a semi-linear elliptic equation with boundary conditions of the first kind and additional control restrictions. For the computational scheme proposed by the authors, which discretized the state equations and conjugate equations with piecewise linear functions and the control variable with piecewise constant functions, an estimate of the accuracy of the approximate solution was obtained. The error estimate is decomposed into the sum of three components related to the discretization of the state and conjugate state equations, as well as the control variable. Such estimation is important for error control in adaptive finite element method. A new approach to error control and adaptation of the finite element method for the discretization of optimization problems governed by partial differential equations was developed in [22]. The Lagrange formalism is used to calculate the stationary points of the necessary first-order optimality conditions. Grid adaptation is driven by a posteriori error estimates based on grid cell residuals. A feature of the considered problem is the natural selection of the error control functional, which coincides with the cost functional of the optimization problem. The Lagrange multiplier is used to weight the cell residuals in the error estimation. In [23] considers an optimization problem for a linear parabolic equation with control, which is included on the right-hand side of the equation and in the boundary condition. The primary result of this work is the application of the gradient projection method to find optimal control. The formula for the gradient of the cost functional was obtained by solving the conjugate boundary value problem. The theorem on different types of convergence of the control sequence that minimizes the cost functional is also proved. The paper [24] considers the optimization problem for a quasilinear parabolic equation, in which the control is included as a vector parameter in the coefficients (including the senior one) and in the boundary conditions. The peculiarity of the formulation of the problem is that control is not considered a process but a point on the sphere in a finite-dimensional Euclidean space. This study implements a classical approach according to which the necessary optimality condition is found from the maximum principle, which is formulated for the Hamiltonian function constructed using the solutions of the original and conjugate boundary value problems. The numerical algorithm for solving the problem uses an iterative scheme and conjugate gradient method. Optimization problems governed by semi-linear parabolic equations with control under boundary conditions were studied in [25]. A feature of these problems is the pointwise mixed constraints on the control and state of the system. The necessary conditions for the existence of optimal control in certain functional spaces are studied, as well as the conditions for the regularity of Lagrange multipliers, which are used in the construction of conjugate equations. The paper [26] considers a linear parabolic equation in the domain $(0,T)\times(-1,1)$ with bilinear control, which acts on a subset of the interval $(-1,1)$. The diffusion coefficient degenerates at the endpoints of the interval. The authors proved the existence and uniqueness of a weak solution to the boundary value problem for the original equation, the existence of an optimal control for a quadratic functional with a regularizing term. The necessary condition for the first order in the optimal control problem is obtained in the form of a variational inequality with respect to the Frechet derivative of the cost functional. A sufficient condition for the existence of optimal control is also derived. In work [27] mixed integer optimization with constraints in the form of a partial differential equation of evolutionary type was considered. The operator of the equation is an infinitesimal generator of a uniformly continuous semigroup of linear bounded operators. Discretization of the problem is carried out in time after rotation of the evolutionary operator using a convolution operation. Discretization by spatial variables is based on the grid method. The proposed method solves the problem of optimal placement of heat sources in the domain while limiting their number. In [28], the problem of optimal control of the heat conduction equation with non-convex constraints was considered. The problem is related to the practical process of additive layer production, which is used to manufacture three-dimensional details from metal powders by layer-by-layer melting of the material. The laser beam scans the surface of a detail covered with powder, heating it to the required level. A quadratic functional with components that depend on the deviation of the temperature of the detail from the nominal value, the gradient of the temperature field, and the control: the trajectory of the laser beam along the scanning area is minimized. Trajectory conditions lead to non-convex control constraints. The result of this work is the necessary optimality condition obtained in the form of a variational inequality. In [29] considers the optimization problem for a semi-linear parabolic equation, in the right-hand part of which the control and state are included in the form of a bilinear form. The control constraints are given by two boundary functions from space $L^\infty((0,T)\times\Omega)$. Under certain conditions of monotonicity of the integral functional, it is proved in this work that any solution of the optimization problem is given by a bang-bang function, which is constructed in the form of a linear combination of the characteristic functions of a measurable set $E\subset((0,T)\times\Omega)$ and its complement. The coefficients of characteristic functions are boundary functions that specify the admissible control set. In this problem, optimization is actually carried out in domains that are carriers of characteristic functions. Similar problems refer to form optimization problems. In the article [30], an abstract evolutionary equation of the parabolic type with delay was studied, in which the operator was a generator of an exponentially stable semigroup. The delay is considered to be distributed on the segment of the real axis and is included in the state of the system, on which the additional term of the equation depends. The cost functional is formed in a similar manner. The necessary optimality condition is expressed in the form of the maximum principle of the Pontryagin type. Similar problems arise, for example, when controlling heat flow in bodies made of materials with memory, where the generalized (not local) Fourier law is already fulfilled. In [31], the problem of optimal control for a semilinear vector parabolic equation in partial derivatives was considered. The control variable is included in the matrix of coefficients for higher derivatives. The necessary condition of optimality leads to the maximum principle, which is expressed analogously to the classic case. The proof is based on the use of the method of needle variations, which is chosen in such a way as to obtain the necessary differentiability of the state of the system with respect to control. The article [32] investigated a system described by an implicit differential operator equation of the parabolic type, which is insoluble with respect to the higher derivative. Control is included on the right-hand side of the equation in both conventional and pulsed forms. Optimization is carried out according to two types of control, where pulses are considered at fixed moments in time and are controlled using their intensities. The problem of optimizing the time points of impulses is separately considered. The research method is based on the solution of the differential equation constructed using the operator semigroup, which is represented by the integral of the pseudo-resolvent of the operator bundle. The article [33] examines the issues of optimal control of the non-stationary temperature state of homogeneous and layered plates with simultaneous control of the temperature and power of the heat flow in separate areas of the plate. A quadratic functional given in a certain Hilbert space is selected as the cost functional. The existence and uniqueness of optimal control in a convex closed domain of the space of functions that are square-integrable on the segment of the time axis are proved. For a folded plate, in addition to the boundary conditions, conjugation conditions for non-ideal thermal contact between layers are added. An approximate solution to the problem is obtained using the gradient method after obtaining an explicit form of the functional residuals.

Separately, we consider some problems related to the optimal control of the thermoelastic states of bodies. A study [34] analyzed a coupled stationary optimization thermoelastic problem for an arbitrary finite body with mixed boundary conditions for the temperature and displacement fields and boundary control. The necessary conditions for optimality are first derived by classical variational calculus methods with the help of Lagrange multipliers. These conditions include a system of differential equations with respect for temperature, displacements, and conjugate functions. For the numerical solution of the problem, a method is proposed in which the spatial discretization of differential equations is achieved using the finite element method, and the conjugate gradient method is used to minimize the cost function. For the numerical solution of the problem, a method is proposed in which the spatial discretization of differential equations is achieved using the finite element method, and the conjugate gradient method is used to minimize the cost

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function. In [35], the setting and construction of a numerical solution to the problem of optimal control (in the sense of speed) of heating the plate by internal heat sources in the presence of control restrictions and the maximum absolute thermal stress is investigated. The method for solving the inverse thermal conductivity problem was combined with the finite difference method for the analysis of the direct problem. The purpose of the work [36] was to control the deformation and temperature of a thermoelastic body by influencing it via an external force acting on its part. The formulation of the problem differs from the classical one because it lacks information about the initial data - movement and temperature distribution in the body. The incompleteness of the data led to the need to use the developed J.-L. Lions concept of win-win control (Pareto control). For its implementation, the authors introduced a sequence of cost functional that depend on an additional parameter. At each value of parameter, the cost functional corresponds to the control with minimal loss. It is shown that lossless control can be obtained from the control with minimal loss via the passage to limit, provided that the specified parameter tends to zero. In work [37], on the basis of the inverse problem of thermomechanics, a mathematical statement of the problem of optimal (in terms of speed) control of heating of thermosensitive canonical bodies (infinite layer, hollow cylinder or sphere) with restrictions on control and maximum tangential thermal stress, taking into account plastic deformation of the material, was developed; an algorithm for numerical construction of the solution was also developed. In [38], a method of the quasi-static inverse problem of thermoelasticity was developed for solving the problem of optimal (in terms of speed) control of a two-dimensional non-axisymmetric non-stationary thermal regime in a long hollow cylinder with restrictions on thermoelastic stresses. With the help of this method, the problem was reduced to a Fredholm integral equation of the first kind, and a method for its stable regularized solution was developed. The paper [39] solves the problem of optimal control of a stationary two-dimensional thermoelastic state in a given section of a plane-deformed half-space. The power of the internal heat sources is determined by the control function. The quality functional is determined by the uniform deviation of the components of the displacement vector or stress tensor on a certain half-space plane. Assuming the existence of optimal control in the space of continuous functions, the initial problems are reduced to integral Fredholm equations of the first kind, which are solved by using the integral Fourier transform and applying the method of the inverse problem of thermoelasticity. The paper [40] considered the problem of determining the optimal stress regime for heating a piece-homogeneous cylindrical glass shell with a constant thickness, provided there is no external load. The shell is heated convectively by continuously distributed external heat sources. The inner surface of the shell is thermally insulating. The goal of this problem is to find the heating mode of the outer surface of the shell from its initial temperature to a given one at a fixed moment of time under certain restrictions on the parameters of thermal stress state and heating rate. The optimality criterion is the minimum condition of meridional and circular normal stresses. A method based on the principle of stepwise parametric optimization with varying values of the control function and discretization step refinement is proposed. After averaging the shell thickness, the problem becomes a one-dimensional one in the spatial variable. A study [41] considered the problem of optimal control of the axisymmetric thermal stress state of a solid cylindrical body by changing the distribution of volumetric heat sources. To solve this problem, an approach based on the variational method of homogeneous solutions, which was developed earlier for solving axisymmetric problems in the theory of elasticity, is used. The article [42] investigates the two-dimensional stationary problem of optimal control of the temperature stresses of a plane-strained half-space. The temperature of the environment at which convective heat exchange occurs through the boundary surface of the half-space is selected as the control function. The quality functional is given by the norm of the deviation of the individual components of the stress tensor from the specified value. The optimal control in the class of continuous functions was found by the method of the inverse problem of thermoelasticity and the Fourier cosine transformation technique. A similar problem was considered for vertical displacements [43]. We also note that with a broader interpretation of optimality, the theory of optimal control intersects with the theories of automatic and adaptive control [44].

1.3. Objective and Approach

Based on the above bibliographic review, among the published scientific works, there are actually no studies on the optimal control of distributed systems in multi-connected spatial domains.

This work presents a new effective method for solving the problem of optimal control of mathematical physics equations for spatially multi-connected canonical domains. The proposed method is considered on the example of the problem of optimal control of the temperature field of the thermo-stressed state of space with a spherical inclusion and cavity. The method is based on the generalized Fourier method (GFM) [45], which was developed by one of the authors of the article and its development [46]. It makes it possible to reduce the original problem to an equivalent problem of optimal control, in which the state of the object is determined by an infinite system of linear algebraic equations. The existence and uniqueness theorem of the optimal solution of the equivalent problem in the class l_2, as well as an effective and stable algorithm for its numerical solution, were obtained.

2. Formulation of the problem

The problem of optimal control of the stress-strain state of a piecewise homogeneous elastic space using a stationary temperature field is considered. We consider that the space Ω contains a spherical inclusion Ω_1 and a spherical cavity Ω_2, the centers of which are located at points O_1 and O_2 ($|\overrightarrow{O_1O_2}| = z_{12}$). The radii of the inclusion and the cavity are equal to R_1 and R_2, respectively ($R_1 + R_2 < z_{12}$), and we denote their boundaries Γ_1 and Γ_2. The two-phase system (Ω_0, Ω_1) $(\Omega_0 = \Omega \setminus \overline{\Omega_1 \cup \Omega_2})$ has thermomechanical characteristics $(G_j, \nu_j, \alpha_j, k_j)$ $(j = 0;1)$, where G is the shear modulus, ν is Poisson's ratio, α is the coefficient of linear temperature expansion, k is the thermal conductivity coefficient.

It is necessary to determine the temperature field in the domains Ω_j $(j = 0;1)$ (actually the temperature distribution T_2 on the surface Γ_2), which satisfies the following conditions:

$$\vec{\nabla}^2 \vec{U}_j + \frac{1}{1-2\nu_j} \vec{\nabla}(\vec{\nabla}\vec{U}_j) = \alpha_j \frac{2+2\nu_j}{1-2\nu_j} \vec{\nabla} T_j, \quad \vec{x} \in \Omega_j; \quad (1)$$

$$\vec{\nabla}^2 T_j = 0, \quad \vec{x} \in \Omega_j; \quad (2)$$

$$T_{0|\Gamma_1} = T_{1|\Gamma_1}, \quad \left(k_0 \frac{\partial T_0}{\partial n_1}\right)_{|\Gamma_1} = \left(k_1 \frac{\partial T_1}{\partial n_1}\right)_{|\Gamma_1}; \quad (3)$$

$$\left(\vec{U}_0\right)_{|\Gamma_1} = \left(\vec{U}_1\right)_{|\Gamma_1}, \quad \left(F\vec{U}_0\right)_{|\Gamma_1} = \left(F\vec{U}_1\right)_{|\Gamma_1}; \quad (4)$$

$$\left(F\vec{U}_0\right)_{|\Gamma_2} = \vec{f}_{|\Gamma_2}, \quad (5)$$

$$\frac{1}{|\Gamma_1|} \int_{\Gamma_1} |F\vec{U}_0|^2 \, ds \to \min, \quad (6)$$

$$\frac{1}{|\Gamma_2|} \int_{\Gamma_2} T_2^2 ds = T^2. \quad (7)$$

Here T_j, $\vec{U}_j$ $(j = 0;1)$ denote the temperature field and the field of displacements in the domain Ω_j, $F\vec{U}_j$ – vector of stresses on the surfaces of the inclusion or cavity, corresponding to the vector of displacements $\vec{U}_j$, $\vec{f}$ – given vector function, $\vec{\nabla}$ – nabla operator, $\vec{n}_1$ – unit vector normal to the surface Γ_1, $|\Gamma_j|$ – surface area Γ_j, T – given positive constant.

Let's introduce two equally directed spherical coordinate systems $(r_j, \theta_j, \varphi_j)$ $(j = 1;2)$, the beginnings of which coincide with the points O_j, and the axis Oz has the direction of the vector $\overrightarrow{O_1O_2}$. Their coordinates are connected by the following formulas

$$r_1 \sin\theta_1 = r_2 \sin\theta_2; \quad r_1 \cos\theta_1 = r_2 \cos\theta_2 + z_{12}.$$

In the entered coordinates, the surface Γ_j has an equation $r_j = R_j$. We assume that the vector function $\vec{f}$ has axial symmetry and is represented by an absolutely and uniformly convergent series of the form

$$\vec{f}(\theta_2) = \\ = 2G_0 \sum_{n=0}^{\infty} [f_n^{(1)} P_n(\cos\theta_2) \vec{e}_{r_2} + f_n^{(2)} P_n^1(\cos\theta_2) \vec{e}_{\theta_2}],$$

where $P_n^m(x)$ are the Legendre functions of the first kind, $\{\vec{e}_{r_j}, \vec{e}_{\theta_j}\}$ are unit vectors of the spherical coordinate system with origin O_j.

Due to the axial symmetry of the problem, we will find the temperature on the surface in the form

$$T_{2|\Gamma_2} = \sum_{n=0}^{\infty} g_n^{(2)} P_n(\cos\theta_2). \quad (8)$$

Therefore, the solution of problem (1) – (7) is a set of coefficients $\{g_n^{(2)}\}_{n=0}^{\infty}$. Condition (7) shows that the function $T_2(\theta_2)$ must belong to the class $L_2(\Gamma_2)$.

First, we assume that temperature (8) is given, and it is necessary to find the temperature field and thermoelastic displacements in the domains (Ω_0, Ω_1) in the problem (1) – (5).

3. Solving the heat conduction problem

Let's solve problem (2), (3), (8) for the thermal conductivity equation. We seek a solution to this problem in the form

$$T_0(\vec{x}) = \sum_{j=1}^{2} \sum_{n=0}^{\infty} t_n^{(j)} R_j^{n+1} w_n^+(r_j, \theta_j), \quad \vec{x} \in \Omega_0; \quad (9)$$

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$$T_1(\vec{x}) = \sum_{n=0}^{\infty} g_n^{(1)} R_1^{-n} w_n^-(r_1,\theta_1),\ \ \vec{x} \in \Omega_1 \qquad (10)$$

with unknown coefficients $\{t_n^{(j)}\}_{n=0,j=1}^{\infty,2}$, $\{g_n^{(1)}\}_{n=0}^{\infty}$. Here, the basic axisymmetric solutions of the Laplace equation for the exterior and interior of the sphere $\Omega^{\pm} = \{(r,\theta,\varphi): r^{>R}_{<R}\}$ are denoted

$$w_n^{\pm}(r,\theta) = r^{\mp(n+1/2)-1/2} P_n(\cos\theta), \qquad (11)$$

where $P_n(x)$ are Legendre polynomials, and the sign + (–) corresponds to the outer (inner) solution.

Let's use the addition theorems of functions (11) [45]

$$w_n^+(r_j,\theta_j) = \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}} w_k^-(r_{3-j},\theta_{3-j}), \qquad (12)$$

$$n = 0 \div \infty,\ \ r_{3-j} < z_{12},\ \ j = 1,2,$$

where

$$h_{n,k}^{(j)} = (-1)^{jk+(j-1)n} \frac{(n+k)!}{n!k!},$$

to write the solution (9) in the coordinate system with the origin at the point O_j ($j = 1,2$)

$$T_0(\vec{x}) = \sum_{n=0}^{\infty} t_n^{(j)} \left(\frac{R_j}{r_j}\right)^{n+1} P_n(\cos\theta_j) +$$

$$+ \sum_{n=0}^{\infty} \left(\frac{r_j}{R_j}\right)^{n} P_n(\cos\theta_j) \sum_{k=0}^{\infty} u_{n,k}^{(j)} t_k^{(3-j)}, \qquad (13)$$

where

$$u_{n,k}^{(j)} = h_{n,k}^{(j)} \omega_{n,k+1}^{(j)},\ \ \omega_{n,k}^{(j)} = \left(\frac{R_j}{z_{12}}\right)^n \left(\frac{R_{3-j}}{z_{12}}\right)^k.$$

Satisfying the conjugation conditions of the thermal fields $T_0(\vec{x})$ and $T_1(\vec{x})$, and the boundary condition (8), we obtain

$$t_n^{(1)} + \sum_{k=0}^{\infty} u_{n,k}^{(1)} t_k^{(2)} = g_n^{(1)},\ \ n = 0 \div \infty; \qquad (14)$$

$$-(n+1) t_n^{(1)} + n \sum_{k=0}^{\infty} u_{n,k}^{(1)} t_k^{(2)} = \frac{k_1}{k_0} n g_n^{(1)}, \qquad (15)$$

$$n = 0 \div \infty;$$

$$t_n^{(2)} + \sum_{k=0}^{\infty} u_{n,k}^{(2)} t_k^{(1)} = g_n^{(2)},\ \ n = 0 \div \infty. \qquad (16)$$

After removing the unknowns $\{g_n^{(1)}\}_{n=0}^{\infty}$ from the system (14), (15) and relative to the coefficients $\{t_n^{(j)}\}_{n=0,j=1}^{\infty,2}$, the resolving system follows

$$t_n^{(1)} - \gamma_n \sum_{s=0}^{\infty} t_s^{(1)} \sum_{k=0}^{\infty} u_{n,k}^{(1)} u_{k,s}^{(2)} = -\gamma_n \sum_{k=0}^{\infty} u_{n,k}^{(1)} g_k^{(2)}, \qquad (17)$$

$$n = 0 \div \infty;$$

$$t_n^{(2)} - \sum_{s=0}^{\infty} t_s^{(2)} \sum_{k=0}^{\infty} \gamma_k u_{n,k}^{(2)} u_{k,s}^{(1)} = g_n^{(2)}, \qquad (18)$$

$$n = 0 \div \infty,$$

where $\gamma_n = \dfrac{(k_1 - k_0)n}{k_1 n + k_0(n+1)}$.

Theorem 1. Under the condition $z_{12} > R_1 + R_2$ the operators of the systems (17) and (18) are Fredholm operators in the space l_2.

Proof. To prove the theorem, it is enough to show the convergence of the double series

$$\sum_{n,s=0}^{\infty} \left(\gamma_n \sum_{k=0}^{\infty} u_{n,k}^{(1)} u_{k,s}^{(2)}\right)^2,\ \ \sum_{n,s=0}^{\infty} \left(\sum_{k=0}^{\infty} \gamma_k u_{n,k}^{(2)} u_{k,s}^{(1)}\right)^2.$$

Since $|\gamma_n| < 1$, then applying Hölder's inequality, we obtain

$$\sum_{n,s=0}^{\infty} \left(\gamma_n \sum_{k=0}^{\infty} u_{n,k}^{(1)} u_{k,s}^{(2)}\right)^2 <$$

$$< \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |u_{n,k}^{(1)}|^2 \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} |u_{k,s}^{(2)}|^2.$$

When the condition of the theorem is satisfied, the convergence of the series on the right-hand side of the inequality follows from the existence of the exact sum of the series

$$\sum_{n,k=0}^{\infty} |u_{n,k}^{(j)}| = \sum_{k,n=0}^{\infty} \frac{(n+k)!}{n!k!} \left(\frac{R_{3-j}}{z_{12}}\right)^k \left(\frac{R_j}{z_{12}}\right)^n =$$

$$= \frac{z_{12}}{z_{12} - R_1 - R_2}.$$

Similarly, the convergence of the second dual series is proved.

It also follows from the condition $\{g_n^{(2)}\}_{n=0}^{\infty} \in l_2$ that

$$\left(\gamma_n \sum_{k=0}^{\infty} u_{n,k}^{(1)} g_k^{(2)}\right)_{n=0}^{\infty} \in l_2 ,$$

and the previous theorem together with the equivalence of problem (2), (3), (8) to solvable systems (17), (18) (can be proved) justify the existence of a correct solution of systems (17), (18) in space l_2. Then formulas (9), (10), (14) restore the temperature field $(T_0(\vec{x}), T_1(\vec{x}))$ in a two-phase body (Ω_0, Ω_1).

4 Solving the direct thermoelasticity problem

Now consider the thermoelastic problem (1), (4), (5), which corresponds to the temperature field constructed above in the domains (Ω_0, Ω_1). For this purpose, we use the results presented in [49]. Consider in domains $\Omega^{\pm}$ the system of basic axisymmetric solutions of the Lamé equation (homogeneous equation (1)) $\{\vec{W}_{1,n}^{\pm}(r,\theta)\}_{n=1}^{\infty}, \vec{W}_{2,n}^{\pm}(r,\theta)\}_{n=0}^{\infty}$, (the definition of basicity is given in [45]), where

$$\vec{W}_{1,n}^{\pm}(r,\theta) = \vec{\nabla} w_n^{\pm}(r,\theta) , \tag{19}$$

$$\vec{W}_{2,n}^{\pm(j)}(r,\theta) = \chi_n^{\pm(j)} \vec{V}_n^{\pm}(r,\theta) - \zeta_n^{\pm(j)} r^2 \vec{W}_{1,n}^{\pm}(r,\theta) , \tag{20}$$

$$\vec{V}_n^{\pm}(r,\theta) = \vec{\nabla}[r^2 w_n^{\pm}(r,\theta)] , \tag{21}$$

$$\chi_n^{+(j)} = n(4\nu_j - 3) + 2\nu_j - 2 ,\ \zeta_n^{+(j)} = (2n-1)(2\nu_j - 2) ,$$

$$\chi_n^{-(j)} = n(4\nu_j - 3) + 2\nu_j - 1 ,\ \zeta_n^{-(j)} = (2n+3)(2\nu_j - 2) .$$

For these vector functions in coordinate systems with origins O_j, the following addition theorems were proved in [46] ($r_{3-j} < z_{12}$, $j = 1, 2$):

$$\vec{W}_{1,n}^{+}(r_j,\theta_j) = \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}} \vec{W}_{1,k}^{-}(r_{3-j},\theta_{3-j}) ,\ n = 0 \div \infty ; \tag{22}$$

$$\vec{W}_{2,n}^{+}(r_j,\theta_j) = \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k-1}} \gamma_{n,k}^{(1)} \vec{W}_{1,k}^{-}(r_{3-j},\theta_{3-j}) - \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}} \gamma_{n,k}^{(2)} \vec{W}_{2,k}^{-}(r_{3-j},\theta_{3-j}) ,\ n = 0 \div \infty ; \tag{23}$$

$$\vec{V}_n^{+}(r_j,\theta_j) = \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k-1}} \lambda_{n,k}^{(1)} \vec{W}_{1,k}^{-}(r_{3-j},\theta_{3-j}) - \sum_{k=0}^{\infty} \frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}} \lambda_{n,k}^{(2)} \vec{V}_k^{-}(r_{3-j},\theta_{3-j}) ,\ n = 0 \div \infty ; \tag{24}$$

where

$$\gamma_{n,k}^{(1)} = \frac{n(2nk - n - k - 4\nu_0 + 4)}{(n+k)(2k-1)} ,$$

$$\gamma_{n,k}^{(2)} = \frac{n(2n-1)}{(2k+3)(k+1)} ,$$

$$\lambda_{n,k}^{(1)} = \frac{2nk - n - k}{(n+k)(2k-1)} ,\ \lambda_{n,k}^{(2)} = \frac{2n-1}{2k+3} .$$

Remark. When $n = 0$ the coefficient $\gamma_{n,k}^{(1)} = \frac{n(2nk - n - k - 4\nu_0 + 4)}{(n+k)(2k-1)}$ is considered equal to zero, and $\lambda_{0,k}^{(1)} = -\frac{1}{(2k-1)}$ for all k.

As is known, the general solution of the inhomogeneous equation (1) in the domain Ω_j ($j = 0;1$) can be written as follows:

$$\vec{U}_j(\vec{x}) = \vec{U}_j^G(\vec{x}) + \vec{U}_j^T(\vec{x}) , \tag{25}$$

where $\vec{U}_j^G(\vec{x})$ is the general solution of the corresponding homogeneous equation, and $\vec{U}_j^T(\vec{x})$ is the partial solution of the inhomogeneous equation (hereinafter referred to as the vector of thermal displacements). Due to the basicity of solutions (19) and (20) in the domains $\Omega^{\pm}$, the general solution of the homogeneous equation in the domains (Ω_0, Ω_1) can be written as follows:

$$\vec{U}_0^G(\vec{x}) = \sum_{j=1}^{2} \sum_{n=0}^{\infty} a_{1,n}^{(j)} R_j^{n+3} \vec{W}_{1,n}^{+}(r_j,\theta_j) + \sum_{j=1}^{2} \sum_{n=0}^{\infty} a_{2,n}^{(j)} R_j^{n+1} \vec{W}_{2,n}^{+}(r_j,\theta_j) ,\ \vec{x} \in \Omega_0 ; \tag{26}$$

$$\vec{U}_1^G(\vec{x}) = \sum_{n=0}^{\infty} b_{1,n}^{(1)} R_1^{-n+2} \vec{W}_{1,n}^{-}(r_1,\theta_1) + \sum_{n=0}^{\infty} b_{2,n}^{(1)} R_1^{-n} \vec{W}_{2,n}^{-}(r_1,\theta_1) ,\ \vec{x} \in \Omega_1 . \tag{27}$$

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Here $\{a_{i,n}^{(j)}\}_{n=0,i,j=1}^{\infty,2}$, $\{b_{i,n}^{(1)}\}_{n=0,i=1}^{\infty,2}$ – unknown coefficients.

It was shown [46] that the vector of thermal displacements

$$\vec{U}_0^T(\vec{x}) = -\alpha_0^{\nu_0}\sum_{j=1}^{2}\sum_{n=0}^{\infty}\frac{t_n^{(j)}}{2n-1}R_j^{n+1}\vec{V}_n^{+}(r_j,\theta_j),\ \vec{x}\in\Omega_0, \quad (28)$$

where $\alpha_0^{\nu_0} = \dfrac{\alpha_0}{2}\dfrac{1+\nu_0}{1-\nu_0}$ corresponds to the temperature field $T_0(\vec{x})$. It is also possible to prove the following vector of thermal displacements

$$\vec{U}_1^T(\vec{x}) = \alpha_1^{\nu_1}\sum_{n=0}^{\infty}\frac{g_n^{(1)}}{2n+3}R_1^{-n}\vec{V}_n^{-}(r_1,\theta_1),\ \vec{x}\in\Omega_1, \quad (29)$$

where $\alpha_1^{\nu_1} = \dfrac{\alpha_1}{2}\dfrac{1+\nu_1}{1-\nu_1}$ corresponds to the temperature field $T_1(\vec{x})$.

Formulas (22) – (24) make it possible to write a vector function $\vec{U}_0(\vec{x})$ in a spherical coordinate system with the origin at a point O_j

$$\vec{U}_0(r_j,\theta_j) = \sum_{n=0}^{\infty}a_{1,n}^{(j)}R_j^{n+3}\vec{W}_{1,n}^{+}(r_j,\theta_j) +$$
$$+\sum_{n=0}^{\infty}\vec{W}_{1,n}^{-}(r_j,\theta_j)\sum_{k=0}^{\infty}\frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}}R_{3-j}^{k+3}a_{1,k}^{(3-j)} +$$
$$+\sum_{n=0}^{\infty}a_{2,n}^{(j)}R_j^{n+1}\vec{W}_{2,n}^{+}(r_j,\theta_j) +$$
$$+\sum_{n=0}^{\infty}\vec{W}_{1,n}^{-}(r_j,\theta_j)\sum_{k=0}^{\infty}\frac{h_{n,k}^{(j)}}{z_{12}^{n+k-1}}\gamma_{k,n}^{(1)}R_{3-j}^{k+1}a_{2,k}^{(3-j)} -$$
$$-\sum_{n=0}^{\infty}\vec{W}_{2,n}^{-}(r_j,\theta_j)\sum_{k=0}^{\infty}\frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}}\gamma_{k,n}^{(2)}R_{3-j}^{k+1}a_{2,k}^{(3-j)} -$$
$$-\alpha_0^{\nu_0}\sum_{n=0}^{\infty}\frac{t_n^{(j)}}{2n-1}R_j^{n+1}\vec{V}_n^{+}(r_j,\theta_j) -$$
$$-\alpha_0^{\nu_0}\sum_{n=0}^{\infty}\vec{W}_{1,n}^{-}(r_j,\theta_j)\sum_{k=0}^{\infty}\frac{h_{n,k}^{(j)}}{z_{12}^{n+k-1}}\frac{\lambda_{n,k}^{(1)}}{2n-1}R_{3-j}^{k+1}t_k^{(3-j)} +$$
$$+\alpha_0^{\nu_0}\sum_{n=0}^{\infty}\frac{\vec{V}_n^{-}(r_j,\theta_j)}{2n+3}\sum_{k=0}^{\infty}\frac{h_{n,k}^{(j)}}{z_{12}^{n+k+1}}R_{3-j}^{k+1}t_k^{(3-j)}. \quad (30)$$

By passing displacements (27), (29), (30) to the stress vector on the surface Γ_j and fulfilling the conjugation conditions (4) and the boundary condition (5), we arrive at an infinite system of linear algebraic equations with respect to the unknowns $\{a_{i,n}^{(j)}\}_{n=0,i,j=1}^{\infty,2}$, $\{b_{i,n}^{(1)}\}_{n=0,i=1}^{\infty,2}$

$$nb_{1,n}^{(1)} + \beta_{1,n}^{-(1)}b_{2,n}^{(1)} = -(n+1)a_{1,n}^{(1)} + \beta_{1,n}^{+(0)}a_{2,n}^{(1)} +$$
$$+n\sum_{k=0}^{\infty}u_{n,k}^{(1,1)}a_{1,k}^{(2)} + \sum_{k=0}^{\infty}[nu_{n,k}^{(2,1)} - \beta_{1,n}^{-(0)}u_{n,k}^{(3,1)}]a_{2,k}^{(2)} +$$
$$+\alpha_0^{\nu_0}\frac{n-1}{2n-1}t_n^{(1)} - \alpha_1^{\nu_1}\frac{n+2}{2n+3}g_n^{(1)} +$$
$$+\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left[\frac{n+2}{2n+3}u_{n,k}^{(5,1)} - nu_{n,k}^{(4,1)}\right]t_k^{(2)}, \quad (31)$$

$$b_{1,n}^{(1)} + \beta_{2,n}^{-(1)}b_{2,n}^{(1)} = a_{1,n}^{(1)} + \beta_{2,n}^{+(0)}a_{2,n}^{(1)} + \sum_{k=0}^{\infty}u_{n,k}^{(1,1)}a_{1,k}^{(2)} +$$
$$+\sum_{k=0}^{\infty}[u_{n,k}^{(2,1)} - \beta_{2,n}^{-(0)}u_{n,k}^{(3,1)}]a_{2,k}^{(2)} - \alpha_0^{\nu_0}\frac{t_n^{(1)}}{2n-1} +$$
$$+\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left[\frac{u_{n,k}^{(5,1)}}{2n+3} - u_{n,k}^{(4,1)}\right]t_k^{(2)} - \alpha_1^{\nu_1}\frac{g_n^{(1)}}{2n+3}, \quad (32)$$

$$(n+2)(n+1)a_{1,n}^{(1)} + \rho_{1,n}^{+(0)}a_{2,n}^{(1)} + n(n-1)\sum_{k=0}^{\infty}u_{n,k}^{(1,1)}a_{1,k}^{(2)} +$$
$$+\sum_{k=0}^{\infty}[n(n-1)u_{n,k}^{(2,1)} - \rho_{1,n}^{-(0)}u_{n,k}^{(3,1)}]a_{2,k}^{(2)} -$$
$$-\alpha_0^{\nu_0}\left[2+\frac{n(n-1)}{2n-1}\right]t_n^{(1)} +$$
$$+\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left\{\left[\frac{(n+1)(n+2)}{2n+3}-2\right]u_{n,k}^{(5,1)} - n(n-1)u_{n,k}^{(4,1)}\right\}t_k^{(2)} =$$
$$=\frac{G_1}{G_0}n(n-1)b_{1,n}^{(1)} + \frac{G_1}{G_0}\rho_{1,n}^{-(1)}b_{2,n}^{(1)} +$$
$$+\frac{G_1}{G_0}\alpha_1^{\nu_1}\left[\frac{(n+1)(n+2)}{2n+3}-2\right]g_n^{(1)}, \quad (33)$$

$$-(n+2)a_{1,n}^{(1)} + \rho_{2,n}^{+(0)}a_{2,n}^{(1)} + (n-1)\sum_{k=0}^{\infty}u_{n,k}^{(1,1)}a_{1,k}^{(2)} +$$
$$+\sum_{k=0}^{\infty}[(n-1)u_{n,k}^{(2,1)} - \rho_{2,n}^{-(0)}u_{n,k}^{(3,1)}]a_{2,k}^{(2)} +$$
$$+\alpha_0^{\nu_0}\frac{n}{2n-1}t_n^{(1)} +$$
$$+\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left[\frac{n+1}{2n+3}u_{n,k}^{(5,1)} - (n-1)u_{n,k}^{(4,1)}\right]t_k^{(2)} =$$

$$= \frac{G_1}{G_0}(n-1)b_{1,n}^{(1)} + \frac{G_1}{G_0}\rho_{2,n}^{-(1)} b_{2,n}^{(1)} + + \frac{G_1}{G_0}\alpha_1^{\nu_1}\frac{n+1}{2n+3} g_n^{(1)}, \qquad (34)$$

$$(n+2)(n+1)a_{1,n}^{(2)} + \rho_{1,n}^{+(0)} a_{2,n}^{(2)} + n(n-1)\sum_{k=0}^{\infty} u_{n,k}^{(1,2)} a_{1,k}^{(1)} + \sum_{k=0}^{\infty}[n(n-1)u_{n,k}^{(2,2)} - \rho_{1,n}^{-(0)} u_{n,k}^{(3,2)}]a_{2,k}^{(1)} - -\alpha_0^{\nu_0}\left[2+\frac{n(n-1)}{2n-1}\right]t_n^{(2)} + +\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left\{\left[\frac{(n+1)(n+2)}{2n+3}-2\right]u_{n,k}^{(5,2)} - n(n-1)u_{n,k}^{(4,2)}\right\}t_k^{(1)} = = f_n^{(1)}, \qquad (35)$$

$$-(n+2)a_{1,n}^{(2)} + \rho_{2,n}^{+(0)} a_{2,n}^{(2)} + (n-1)\sum_{k=0}^{\infty} u_{n,k}^{(1,2)} a_{1,k}^{(1)} + +\sum_{k=0}^{\infty}[(n-1)u_{n,k}^{(2,2)} - \rho_{2,n}^{-(0)} u_{n,k}^{(3,2)}]a_{2,k}^{(1)} + +\alpha_0^{\nu_0}\frac{n}{2n-1}t_n^{(2)} + +\alpha_0^{\nu_0}\sum_{k=0}^{\infty}\left[\frac{n+1}{2n+3}u_{n,k}^{(5,2)} - (n-1)u_{n,k}^{(4,2)}\right]t_k^{(1)} = f_n^{(2)}, \qquad (36)$$

where

$$\beta_{1,n}^{+(j)} = -n(n+3-4\nu_j), \quad \beta_{1,n}^{-(j)} = (n+1)(n+4\nu_j-2),$$

$$\beta_{2,n}^{+(j)} = n+4\nu_j-4, \quad \beta_{2,n}^{-(j)} = n+5-4\nu_j,$$

$$\rho_{1,n}^{+(j)} = n(n^2+3n-2\nu_j), \quad \rho_{1,n}^{-(j)} = (n+1)(n^2-n-2\nu_j-2),$$

$$\rho_{2,n}^{+(j)} = -(n^2+2\nu_j-2), \quad \rho_{2,n}^{-(j)} = n^2+2n+2\nu_j-1,$$

$$u_{n,k}^{(1,j)} = h_{n,k}^{(j)}\omega_{n-2,k+3}^{(j)},$$

$$u_{n,k}^{(2,j)} = h_{n,k}^{(j)}\gamma_{k,n}^{(1)}\omega_{n-2,k+1}^{(j)},$$

$$u_{n,k}^{(3,j)} = h_{n,k}^{(j)}\gamma_{k,n}^{(2)}\omega_{n,k+1}^{(j)},$$

$$u_{n,k}^{(4,j)} = h_{n,k}^{(j)}\lambda_{n,k}\omega_{n-2,k+1}^{(j)},$$

$$u_{n,k}^{(5,j)} = h_{n,k}^{(j)}\omega_{n,k+1}^{(j)}, \quad \lambda_{n,k} = \frac{2nk-n-k}{(n+k)(2k-1)(2n-1)}.$$

After excluding the unknowns $\{b_{i,n}^{(1)}\}_{n=0,i=1}^{\infty,2}$ from equations (31) – (34), we obtain the system from $\{a_{i,n}^{(j)}\}_{n=0,i,j=1}^{\infty,2}$

$$d_n^{(j,1)}a_{1,n}^{(1)} + d_n^{(j,2)}a_{2,n}^{(1)} + \sum_{k=0}^{\infty} m_{n,k}^{(j,1,1)}a_{1,k}^{(2)} + \sum_{k=0}^{\infty} m_{n,k}^{(j,2,1)}a_{2,k}^{(2)} = = -m_{t,n}^{(j,1)}\bar{t}_n^{(1)} - \sum_{k=0}^{\infty} m_{t,n,k}^{(j,2)}\bar{t}_k^{(2)} + \xi_{g,n}^{(j)}\bar{g}_n^{(1)}, \qquad (37)$$

$$\gamma_n^{(j,1)}a_{1,n}^{(2)} + \gamma_n^{(j,2)}a_{2,n}^{(2)} + \sum_{k=0}^{\infty}\gamma_{n,k}^{(j,1,2)}a_{1,k}^{(1)} + \sum_{k=0}^{\infty}\gamma_{n,k}^{(j,2,2)}a_{2,k}^{(1)} = = f_n^{(j)} - \gamma_{t,n}^{(j,2)}\bar{t}_n^{(2)} - \sum_{k=0}^{\infty}\gamma_{t,n,k}^{(j,1)}\bar{t}_k^{(1)}, \qquad (38)$$

$$b_{1,n}^{(1)} = \vartheta_n^{(1,1)}a_{1,n}^{(1)} + \vartheta_n^{(1,2)}a_{2,n}^{(1)} + \sum_{k=0}^{\infty}u_{n,k}^{(1,1)}a_{1,k}^{(2)} + \sum_{k=0}^{\infty}\psi_{n,k}^{(1)}a_{2,k}^{(2)} + +\sigma_{t,n}^{(1)}\bar{t}_n^{(1)} + \sum_{k=0}^{\infty}\psi_{t,n,k}^{(1)}\bar{t}_k^{(2)} + \sigma_{g,n}^{(1)}\bar{g}_n^{(1)}, \qquad (39)$$

$$b_{2,n}^{(1)} = \vartheta_n^{(2,1)}a_{1,n}^{(1)} + \vartheta_n^{(2,2)}a_{2,n}^{(1)} + \sum_{k=0}^{\infty}\psi_{n,k}^{(2)}a_{2,k}^{(2)} + +\sigma_{t,n}^{(2)}\bar{t}_n^{(1)} + \sum_{k=0}^{\infty}\psi_{t,n,k}^{(2)}\bar{t}_k^{(2)} + \sigma_{g,n}^{(2)}\bar{g}_n^{(1)}, \qquad (40)$$

$$j=1,2; \quad n=0\div\infty,$$

where

$$d_n^{(i,k)} = \gamma_n^{(i,k)} - \xi_n^{(i,k)}, \quad m_{n,k}^{(i,j,1)} = \gamma_{n,k}^{(i,j,1)} - \xi_{n,k}^{(i,j,1)},$$

$$m_{t,n}^{(i,k)} = \gamma_{t,n}^{(i,k)} - \xi_{t,n}^{(i,k)}, \quad m_{t,n,k}^{(j,2)} = \gamma_{t,n,k}^{(j,2)} - \xi_{t,n,k}^{(j,2)},$$

$$\gamma_n^{(1,1)} = (n+2)(n+1), \quad \gamma_{n,k}^{(1,1,j)} = n(n-1)u_{n,k}^{(1,j)},$$

$$\gamma_n^{(1,2)} = \rho_{1,n}^{+(0)}, \quad \gamma_{n,k}^{(1,2,j)} = n(n-1)u_{n,k}^{(2,j)} - \rho_{1,n}^{-(0)}u_{n,k}^{(3,j)},$$

$$\gamma_{t,n}^{(1,j)} = -\left[2+\frac{n(n-1)}{2n-1}\right],$$

$$\gamma_{t,n,k}^{(1,j)} = \left[\frac{(n+1)(n+2)}{2n+3}-2\right]u_{n,k}^{(5,j)} - n(n-1)u_{n,k}^{(4,j)},$$

$$\gamma_n^{(2,1)} = -(n+2), \; \gamma_n^{(2,2)} = \rho_{2,n}^{+(0)}, \; \gamma_{n,k}^{(2,1,j)} = (n-1)u_{n,k}^{(1,j)},$$

$$\gamma_{n,k}^{(2,2,j)} = (n-1)u_{n,k}^{(2,j)} - \rho_{2,n}^{-(0)}u_{n,k}^{(3,j)},$$

$$\gamma_{t,n}^{(2,j)} = \frac{n}{2n-1}, \; \gamma_{t,n,k}^{(2,j)} = \frac{n+1}{2n+3}u_{n,k}^{(5,j)} - (n-1)u_{n,k}^{(4,j)},$$

$$\xi_n^{(1,1)} = \frac{G_1}{G_0}\left[n(n-1) + (2n+1)\mu_{1,n}\right],$$

$$\xi_n^{(1,2)} = \frac{G_1}{G_0}n\left[(n-1)\beta_{2,n}^{+(0)} + (2n-1)\mu_{1,n}\right],$$

$$\xi_n^{(2,1)} = \frac{G_1}{G_0}\left[n-1+(2n+1)\mu_{2,n}\right],$$

$$\xi_n^{(2,2)} = \frac{G_1}{G_0}\left[(n-1)\beta_{2,n}^{+(0)} + n(2n-1)\mu_{2,n}\right],$$

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$$\xi^{(1,1,1)}_{n,k}=\frac{G_1}{G_0}n(n-1)u^{(1,1)}_{n,k},\ \xi^{(1,2,1)}_{n,k}=\frac{G_1}{G_0}\Big\{n(n-1)u^{(2,1)}_{n,k}-$$
$$-\Big[n(n-1)\beta^{-(0)}_{2,n}+\mu_{1,n}\Delta^{-(0)}_n\Big]u^{(3,1)}_{n,k}\Big\},$$
$$\xi^{(1,1)}_{t,n}=-\frac{G_1}{G_0}\left[\frac{n(n-1)}{2n-1}+\mu_{1,n}\right],$$
$$\xi^{(1,2)}_{t,n,k}=-\frac{G_1}{G_0}\Big\{n(n-1)u^{(4,1)}_{n,k}-$$
$$-\frac{1}{2n+3}\Big[n(n-1)-2\mu_{1,n}\Big]u^{(5,1)}_{n,k}\Big\},$$
$$\xi^{(1)}_{g,n}=\frac{G_1}{G_0}\frac{2}{2n+3}\Big[\mu_{1,n}-2\Big],$$
$$\xi^{(2,1,1)}_{n,k}=\frac{G_1}{G_0}(n-1)u^{(1,1)}_{n,k},\ \xi^{(2,2,1)}_{n,k}=$$
$$=\frac{G_1}{G_0}\Big\{(n-1)u^{(2,1)}_{n,k}-\Big[(n-1)\beta^{-(0)}_{2,n}+\mu_{2,n}\Delta^{-(0)}_n\Big]u^{(3,1)}_{n,k}\Big\},$$
$$\xi^{(2,1)}_{t,n}=-\frac{G_1}{G_0}\left[\frac{n-1}{2n-1}+\mu_{2,n}\right],$$
$$\xi^{(2,2)}_{t,n,k}=\frac{G_1}{G_0}\left\{\left[\frac{n-1-2\mu_{2,n}}{2n+3}\right]u^{(5,1)}_{n,k}-(n-1)u^{(4,1)}_{n,k}\right\},$$
$$\xi^{(2)}_{g,n}=\frac{G_1}{G_0}\frac{2}{2n+3}\Big[1+\mu_{2,n}\Big],$$
$$\mu_{2,n}=\frac{\rho^{-(1)}_{2,n}-(n-1)\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n},\ \bar{t}^{(j)}_n=\alpha_0^{\nu_0}t^{(j)}_n,$$
$$\mu_{1,n}=\frac{\rho^{-(1)}_{1,n}-n(n-1)\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n},\ \bar{g}^{(1)}_n=\alpha_1^{\nu_1}g^{(1)}_n,$$
$$\Delta^{-(j)}_n=2[(3-4\nu_j)n+1-2\nu_j)],$$
$$\vartheta^{(1,1)}_n=1-\frac{(2n+1)\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n},\ \vartheta^{(1,2)}_n=\beta^{+(0)}_{2,n}-\frac{n(2n-1)\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n},$$
$$\psi^{(1)}_{n,k}=u^{(2,1)}_{n,k}-\left[\beta^{-(0)}_{2,n}-\beta^{-(1)}_{2,n}\frac{\Delta^{-(0)}_n}{\Delta^{-(1)}_n}\right]u^{(3,1)}_{n,k},$$
$$\sigma^{(1)}_{t,n}=\left[\frac{\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n}-\frac{1}{2n-1}\right],\ \sigma^{(1)}_{g,n}=\frac{-1}{2n+3}\left[1+\frac{2\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n}\right],$$
$$\psi^{(1)}_{t,n,k}=\left[1+\frac{2\beta^{-(1)}_{2,n}}{\Delta^{-(1)}_n}\right]\frac{u^{(5,1)}_{n,k}}{(2n+3)}-u^{(4,1)}_{n,k},$$
$$\vartheta^{(2,1)}_n=\frac{2n+1}{\Delta^{-(1)}_n},\ \vartheta^{(2,2)}_n=\frac{n(2n-1)}{\Delta^{-(1)}_n},\ \psi^{(2)}_{n,k}=-\frac{\Delta^{-(0)}_n}{\Delta^{-(1)}_n}u^{(3,1)}_{n,k},$$
$$\sigma^{(2)}_{t,n}=-\frac{1}{\Delta^{-(1)}_n},\ \sigma^{(2)}_{g,n}=\frac{2}{(2n+3)\Delta^{-(1)}_n},$$
$$\bar{g}^{(2)}_n=\alpha_0^{\nu_0}g^{(2)}_n,\ \psi^{(2)}_{t,n,k}=-\frac{2u^{(5,1)}_{n,k}}{(2n+3)\Delta^{-(1)}_n}.$$

5. Analysis of the resolving system

In the Hilbert space l_2, we consider linear operators defined by infinite matrices

$$D^{(1)}_{ik}=\mathrm{diag}(d^{(i,k)}_n)^{\infty}_{n=0},\ D^{(2)}_{ik}=\mathrm{diag}(\gamma^{(i,k)}_n)^{\infty}_{n=0},\ i,k=1,2;,$$
$$M^{(1)}_{sr}=(m^{(s,r,1)}_{n,k})^{\infty}_{n,k=0},\ M^{(2)}_{sr}=(\gamma^{(s,r,2)}_{n,k})^{\infty}_{n,k=0},\ s,r=1,2;$$
$$M^{(1,1)}_{t,j}=\mathrm{diag}(m^{(j,1)}_{t,n})^{\infty}_{n=0},\ M^{(2,2)}_{t,j}=\mathrm{diag}(\gamma^{(j,2)}_{t,n})^{\infty}_{n=0},$$
$$M^{(1,2)}_{t,j}=(m^{(j,1)}_{t,n,k})^{\infty}_{n,k=0},\ M^{(2,1)}_{t,j}=(\gamma^{(j,2)}_{t,n,k})^{\infty}_{n,k=0},\ j=1,2,$$
$$\Xi^{(j)}_g=\mathrm{diag}(\xi^{(j)}_{g,n})^{\infty}_{n=0}.$$

Let us denote $l_2^4=l_2\times l_2\times l_2\times l_2$ the Cartesian product of four exemplars of space l_2. The space l_2^4 is a Hilbert space with elements $x=(x_1,x_2,x_3,x_4)^T$, $x_i\in l_2$, (here the symbol T stands for transposition operation, that is, the vector x is a symbolic column composed of four infinite sequences) and a scalar product

$$(x,y)=\sum_{i=1}^{4}(x_i,y_i)\ \ \forall x=(x_i)^{4\ T}_{i=1},\ y=(y_i)^{4\ T}_{i=1}.$$

Let

$$a=(a^{(1)}_1,a^{(1)}_2,a^{(2)}_1,a^{(2)}_2)^T,$$
$$f_t=(f^{(1,1)}_t,f^{(2,1)}_t,f^{(1,2)}_t,f^{(2,2)}_t)^T$$

where

$$a^{(j)}_i=(a^{(j)}_{i,n})^{\infty}_{n=0},\ i,j=1,2;$$
$$f^{(j,1)}_t=-M^{(1,1)}_{t,j}\bar{t}^{(1)}-M^{(1,2)}_{t,j}\bar{t}^{(2)}+\Xi^{(j)}_g\bar{g}^{(1)},\ j=1,2;$$
$$f^{(j,2)}_t=f^{(j)}-M^{(2,1)}_{t,j}\bar{t}^{(1)}-M^{(2,2)}_{t,j}\bar{t}^{(2)},\ j=1,2;$$
$$\bar{t}^{(j)}=(\bar{t}^{(j)}_n)^{\infty}_{n=0},\ f^{(j)}=(f^{(j)}_n)^{\infty}_{n=0},\ \bar{g}^{(j)}=(\bar{g}^{(j)}_n)^{\infty}_{n=0}.$$

Consider the symbolic matrix

$$\Phi=D+M,$$

where

$$D=\mathrm{diag}(D_1,D_2),\ M=\begin{pmatrix}0 & M_1\\ M_2 & 0\end{pmatrix},$$
$$D_j=\begin{pmatrix}D^{(j)}_{11} & D^{(j)}_{12}\\ D^{(j)}_{21} & D^{(j)}_{22}\end{pmatrix},\ M_j=\begin{pmatrix}M^{(j)}_{11} & M^{(j)}_{12}\\ M^{(j)}_{21} & M^{(j)}_{22}\end{pmatrix}.$$

Now, systems (37), (38) in the previous notations can be written in the following form

$$\Phi a = f_t . \quad (41)$$

Theorem 2. If the conditions $R_1 + R_2 < z_{12}$, $\nu_j \in (0, 0.5)$, $G_j > 0$ are satisfied, the operator Φ of system(41) in space l_2^4 is a Fredholm operator.

An important and fundamental fact for the proof of Theorem 2 is the following new result.

Theorem 3. Determinant

$$\Delta_n^c = \begin{vmatrix} d_n^{(1,1)} & d_n^{(1,2)} \\ d_n^{(2,1)} & d_n^{(2,2)} \end{vmatrix} \neq 0$$

at any $n = 0 \div \infty$, $\nu_j \in (0, 0.5)$, $G_j > 0$. Moreover, an evaluation is carried out

$$| \Delta_n^c | > (n+2)(2n^2+1) . \quad (42)$$

Proof of Theorem 2. Let us show that the operator D is continuously invertible in the space l_2^4. The formal inverse of the operator D is the operator

$$D^{-1} = \mathrm{diag}(D_1^{-1}, D_2^{-1}) ,$$

Where

$$D_j^{-1} = \begin{pmatrix} V_{11}^{(j)} & V_{12}^{(j)} \\ V_{21}^{(j)} & V_{22}^{(j)} \end{pmatrix},$$

$$V_{ii}^{(j)} = \left(D_{ii}^{(j)} - D_{i,3-i}^{(j)} \left[D_{3-i,3-i}^{(j)} \right]^{-1} D_{3-i,i}^{(j)} \right), \; i = 1, 2 ,$$

$$V_{i,3-i}^{(j)} = -\left[D_{ii}^{(j)} \right]^{-1} D_{i,3-i}^{(j)} V_{3-i,3-i}^{(j)} , \; i = 1, 2 .$$

Due to the diagonality of matrices $D_{ik}^{(j)}$, matrices $V_{ik}^{(j)}$ can be written explicitly

$$V_{ii}^{(1)} = \mathrm{diag}\left(\frac{d_n^{(3-i,3-i)}}{\Delta_n^c} \right), \; V_{i,3-i}^{(1)} = -\mathrm{diag}\left(\frac{d_n^{(i,3-i)}}{\Delta_n^c} \right), \; i = 1, 2 ,$$

$$V_{ii}^{(2)} = \mathrm{diag}\left(\frac{\gamma_n^{(3-i,3-i)}}{\Delta_n^{+(0)}} \right), \; V_{i,3-i}^{(2)} = -\mathrm{diag}\left(\frac{\gamma_n^{(i,3-i)}}{\Delta_n^{+(0)}} \right), \; i = 1, 2 ,$$

$$\Delta_n^{+(0)} = \begin{vmatrix} \gamma_n^{(1,1)} & \gamma_n^{(1,2)} \\ \gamma_n^{(2,1)} & \gamma_n^{(2,2)} \end{vmatrix} =$$

$$= 2(n+2)[n^2 + (1-2\nu_0)n + 1 - \nu_0] . \quad (43)$$

Note that the determinant $\Delta_n^{+(0)} \neq 0$ for any $n = 0 \div \infty$, $\nu_0 \in (0, 0.5)$, and estimate (42) is performed for it.

Therefore, based on the previous result and Theorem 3, the inverse of the matrix D exists. Let us prove that it defines a bounded operator acting in the space l_2^4.

For any vector $x = (x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)})^T \in l_2^4$, the estimate

$$\| D^{-1} x \|^2 = \sum_{i,j=1}^{2} \| V_{ii}^{(j)} x_i^{(j)} + V_{i,3-i}^{(j)} x_{3-i}^{(j)} \|^2 \leq$$

$$\leq 2 \sum_{i,j=1}^{2} \left(\| V_{ii}^{(j)} x_i^{(j)} \|^2 + \| V_{i,3-i}^{(j)} x_{3-i}^{(j)} \|^2 \right),$$

is correct and each term in the sum can be evaluated from above by one of the sums

$$\sum_{n=0}^{\infty} \left| \frac{d_n^{(i,k)}}{\Delta_n^c} x_{k,n}^{(1)} \right|^2 , \; \sum_{n=0}^{\infty} \left| \frac{\gamma_n^{(i,k)}}{\Delta_n^{+(0)}} x_{k,n}^{(2)} \right|^2 , \; i, k = 1, 2 .$$

Here $x_k^{(j)} = (x_{k,n}^{(j)})_{n=0}^{\infty}$. Since the sequences $(d_n^{(i,k)} / \Delta_n^c)_{n=0}^{\infty}$, $(\gamma_n^{(i,k)} / \Delta_n^{+(0)})_{n=0}^{\infty}$, are bounded, there exists a positive constant C, such that

$$\| D^{-1} x \|^2 \leq C \sum_{i,j=1}^{2} \left(\| x_i^{(j)} \|^2 + \| x_{3-i}^{(j)} \|^2 \right) = C \| x \|^2 .$$

The last inequality proves the continuous invertibility of the operator D.

The linear operator defined by the symbolic matrix M is a compact operator in the space l_2^4 which follows from the convergence of the series

$$\sum_{n,k=0}^{\infty} n^s k^r | u_{n,k}^{(i,j)} | < \infty , \; i = 1 \div 5, \; j = 1, 2 ,$$

where s, r – fixed non-negative integers. This result is proved in the same way as in Theorem 1.

The final result of Theorem 2 now follows from the well-known theorem of S. M. Nikolskyi.

We note that under the condition $f^{(j)}, \bar{g}^{(2)} \in l_2$ it is not difficult to verify the belongingness of the vector of the right parts of the system (41) to the space l_2^4. Then the correct solvability of system (41) is a consequence of Theorem 2 and the equivalence of system (41) and the original problem (1), (4), (5).

Remark. The result of Theorem 3 is more significant than just a tool to prove Theorem 2. In fact, for the

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first time, it has been rigorously proven that the axisymmetric conjugation problem for a two-phase system - an elastic space with a spherical inclusion of another material - has a correct solution by the usual Fourier method.

6. Reducing the problem of the optimal control to an equivalent problem

Let's proceed to solving the problem of optimal control (1) – (8). First, let's transform the functional (6). To do this, let's write down the stress vector on the surface Γ_1

$$F\vec{U}_1 = 2G_1 \sum_{n=0}^{\infty}\left[b_{1,n}^{(1)}(n-1)n + b_{2,n}^{(1)}\rho_{1,n}^{-(1)} + \left(-2+\frac{(n+1)(n+2)}{2n+3}\right)\bar{g}_n^{(1)}\right]P_n(\cos\theta_1)\vec{e}_{r_1} +$$

$$+2G_1 \sum_{n=0}^{\infty}\left[b_{1,n}^{(1)}(n-1) + b_{2,n}^{(1)}\rho_{2,n}^{-(1)} + \frac{n+1}{2n+3}\bar{g}_n^{(1)}\right]P_n^1(\cos\theta_1)\vec{e}_{\theta_1}$$

.

For such a vector, the functional (6) has the quadratic form in the infinite-dimensional space of numerical sequences $\{b_{1,n}^{(1)}, b_{2,n}^{(1)}, \bar{g}_n^{(1)}\}_{n=0}^{\infty}$

$$J[\bar{g}^{(2)}] = \frac{1}{|\Gamma_1|}\int_{\Gamma_1} |F\vec{U}_0|^2\, ds =$$

$$= 2G_1^2 \sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{i=1}^{2}\eta_n^{(j,i)} b_{i,n}^{(1)} + \tau_n^{(j)}\bar{g}_n^{(1)}\right]^2, \quad (44)$$

where

$$\eta_n^{(1,1)} = \frac{n(n-1)}{\sqrt{n+1/2}},\quad \eta_n^{(1,2)} = \frac{\rho_{1,n}^{-(1)}}{\sqrt{n+1/2}},$$

$$\eta_n^{(2,1)} = (n-1)\sqrt{\frac{2n(n+1)}{2n+1}},\quad \eta_n^{(2,2)} = \rho_{2,n}^{-(1)}\sqrt{\frac{2n(n+1)}{2n+1}},$$

$$\tau_n^{(1)} = \left(\frac{(n+1)(n+2)}{2n+3} - 2\right)\frac{1}{\sqrt{n+1/2}},$$

$$\tau_n^{(2)} = \frac{n+1}{2n+3}\sqrt{\frac{2n(n+1)}{2n+1}},\quad \bar{g}^{(2)} = (\bar{g}_k^{(2)})_{k=0}^{\infty}.$$

The control constraint (7) can be written as follows:

$$\frac{1}{2}\sum_{n=0}^{\infty}(\bar{g}_n^{(2)})^2 (n+1/2)^{-1} = (\alpha_0^{\nu_0})^2 T^2. \quad (45)$$

Thus, the original problem of optimal control is reduced to an equivalent problem in which the state of the object is determined by an infinite system of linear algebraic equations (14), (17), (18), (37) – (40), and optimal control $(\bar{g}_n^{(2)})_{n=0}^{\infty}$ sets the coefficients of the temperature field (8) on the surface Γ_2, gives the minimum of the quality functional (44) and satisfies the constraint (45).

7. The method for solving the equivalent problem

Bearing in mind that the sequences $\{b_{1,n}^{(1)}, b_{2,n}^{(1)}, \bar{g}_n^{(1)}\}_{n=0}^{\infty}$ satisfy the linear relationship (14), (17), (18), (37) – (40), we can assert their linear dependence on the parameters $\{\bar{g}_n^{(2)}\}_{n=0}^{\infty}$, $\{f_n^{(j)}\}_{j=1,n=0}^{2,\infty}$ as follows

$$b_{i,n}^{(1)} = \sum_{k=0}^{\infty}\left[c_{n,k}^{(i)}\bar{g}_k^{(2)} + \sum_{j=1}^{2}s_{n,k}^{(i,j)}f_k^{(j)}\right],$$

$$\bar{g}_n^{(1)} = \sum_{k=0}^{\infty} q_{n,k}\bar{g}_k^{(2)}, \quad (46)$$

and the unknown coefficients $\{c_{n,k}^{(i)}\}_{n,k=0}^{\infty}$, $\{s_{n,k}^{(i,j)}\}_{n,k=0}^{\infty}$, $\{q_{n,k}\}_{n,k=0}^{\infty}$ of the previous series can be found using these formulas:

$$c_{n,k}^{(i)} = \frac{\partial b_{i,n}^{(1)}}{\partial \bar{g}_k^{(2)}},\quad s_{n,k}^{(i,j)} = \frac{\partial b_{i,n}^{(1)}}{\partial f_k^{(j)}},\quad q_{n,k} = \frac{\partial \bar{g}_n^{(1)}}{\partial \bar{g}_k^{(2)}}. \quad (47)$$

Substituting formulas (46) into the functional (44)

$$J[\bar{g}^{(2)}] = 2G_1^2\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\bar{g}_k^{(2)} + F_n^{(j)}\right]^2, \quad (48)$$

where

$$B_{n,k}^{(j)} = \sum_{i=1}^{2}\eta_n^{(j,i)}c_{n,k}^{(i)} + \tau_n^{(j)}q_{n,k}, \quad (49)$$

$$F_n^{(j)} = \sum_{i=1}^{2}\eta_n^{(j,i)}\sum_{r=1}^{2}\sum_{k=0}^{\infty}s_{n,k}^{(i,r)}f_k^{(r)}. \quad (50)$$

After reducing the functional to a physically dimensionless form, we have:

$$\frac{1}{2}\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\bar{g}_k^{(2)} + F_n^{(j)}\right]^2 \to \min, \quad (51)$$

$$\frac{1}{2}\sum_{n=0}^{\infty}(\bar{g}_n^{(2)})^2(n+1/2)^{-1}=(\alpha_0^{v_0})^2T^2 . \qquad (52)$$

We solve problem (51), (52) for the conditional extremum by the Lagrange method and reduce it to the problem for the unconditional minimum of the functional

$$L[\bar{g}^{(2)}]=\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}B_{n,k}^{(j)}\bar{g}_k^{(2)}+F_n^{(j)}\right]^2+$$
$$+\varsigma\sum_{n=0}^{\infty}(\bar{g}_n^{(2)})^2(n+1/2)^{-1} . \qquad (53)$$

Here ς is the Lagrange multiplier. The existence and uniqueness of the solution to problem (53), (52) are further proved in Theorem 5. The necessary minimum condition of the functional (53) leads to the following system:

$$(\varsigma\tilde{D}+B)\bar{g}^{(2)}=-F , \qquad (54)$$

where

$$B=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}B_{n,m}^{(j)}B_{n,k}^{(j)}\right)_{n,k=0}^{\infty} , \qquad (55)$$

$$\tilde{D}=\operatorname{diag}\left(\frac{2}{2n+1}\right)_{n=0}^{\infty},\ F=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}B_{n,m}^{(j)}F_n^{(j)}\right)_{m=0}^{\infty} . \qquad (56)$$

The Lagrange multiplier can be found from the additional condition (52).

Before examining the operator of system (54), we present a practical method for calculating the elements of the matrices B and F. Based on formulas (14), (17), (18), (39) – (41), (47), we obtain

$$(I-U_{22})\frac{\partial\bar{t}^{(2)}}{\partial\bar{g}^{(2)}}=I , \qquad (57)$$

$$\frac{\partial\bar{g}^{(1)}}{\partial\bar{g}^{(2)}}=\frac{\alpha_1^{v_1}}{\alpha_0^{v_0}}U_g\frac{\partial\bar{t}^{(2)}}{\partial\bar{g}^{(2)}} , \qquad (58)$$

$$\frac{\partial\bar{t}^{(1)}}{\partial\bar{g}^{(2)}}=-U_{12}\frac{\partial\bar{t}^{(2)}}{\partial\bar{g}^{(2)}} , \qquad (59)$$

$$\Phi\frac{\partial a}{\partial\bar{g}^{(2)}}=\frac{\partial f_t}{\partial\bar{g}^{(2)}},\ \Phi\frac{\partial a}{\partial f^{(j)}}=\frac{\partial f_t}{\partial f^{(j)}} , \qquad (60)$$
$$j=1,2;\ m=0\div\infty ;$$

$$\frac{\partial b_{i,n}^{(1)}}{\partial\bar{g}_m^{(2)}}=\vartheta_n^{(i,1)}\frac{\partial a_{1,n}^{(1)}}{\partial\bar{g}_m^{(2)}}+\vartheta_n^{(i,2)}\frac{\partial a_{2,n}^{(1)}}{\partial\bar{g}_m^{(2)}}+\delta_{i,1}\sum_{k=0}^{\infty}u_{n,k}^{(1,1)}\frac{\partial a_{1,k}^{(2)}}{\partial\bar{g}_m^{(2)}}+$$
$$+\sum_{k=0}^{\infty}\psi_{n,k}^{(i)}\frac{\partial a_{2,k}^{(2)}}{\partial\bar{g}_m^{(2)}}+\sigma_{t,n}^{(i)}\frac{\partial\bar{t}_n^{(1)}}{\partial\bar{g}_m^{(2)}}+\sum_{k=0}^{\infty}\psi_{t,n,k}^{(i)}\frac{\partial\bar{t}_k^{(2)}}{\partial\bar{g}_m^{(2)}}+\sigma_{g,n}^{(i)}\frac{\partial\bar{g}_n^{(1)}}{\partial\bar{g}_m^{(2)}} , \qquad (61)$$

$$\frac{\partial b_{i,n}^{(1)}}{\partial f_m^{(j)}}=\vartheta_n^{(i,1)}\frac{\partial a_{1,n}^{(1)}}{\partial f_m^{(j)}}+\vartheta_n^{(i,2)}\frac{\partial a_{2,n}^{(1)}}{\partial f_m^{(j)}}+\delta_{i,1}\sum_{k=0}^{\infty}u_{n,k}^{(1,1)}\frac{\partial a_{1,k}^{(2)}}{\partial f_m^{(j)}}+$$
$$+\sum_{k=0}^{\infty}\psi_{n,k}^{(i)}\frac{\partial a_{2,k}^{(2)}}{\partial f_m^{(j)}} , \qquad (62)$$
$$i=1,2;\ n=0\div\infty ,$$

where

$U_{12}=\left(\gamma_n u_{n,k}^{(1)}\right)_{n,k=0}^{\infty}$, $U_g=\left((1-\gamma_n)u_{n,k}^{(1)}\right)_{n,k=0}^{\infty}$

$U_{22}=\left(\sum_{k=0}^{\infty}\gamma_k u_{n,k}^{(2)}u_{k,s}^{(1)}\right)_{k,s=0}^{\infty}$,

$I=(\delta_{n,m})_{n=0}^{\infty}$, $\delta_{i,k}$ – Kronecker's delta symbol, $\frac{\partial\bar{t}^{(2)}}{\partial\bar{g}^{(2)}},\frac{\partial\bar{t}^{(2)}}{\partial\bar{g}^{(2)}},\frac{\partial\bar{g}^{(1)}}{\partial\bar{g}^{(2)}}$ – matrices of first derivatives, $\frac{\partial a}{\partial\bar{g}^{(2)}},\frac{\partial a}{\partial f^{(j)}},\frac{\partial f_t}{\partial\bar{g}^{(2)}},\frac{\partial f_t}{\partial f^{(j)}}$ are block matrices of first derivatives.

Remark. To find the derivatives $\frac{\partial\bar{t}^{(j)}}{\partial\bar{g}_m^{(2)}}$, $\frac{\partial a}{\partial\bar{g}_m^{(2)}}$, $\frac{\partial a}{\partial f_m^{(j)}}$ it is necessary to solve the same systems (17), (18), (41), with other right-hand parts. Thus, the solvability of systems (57), (60) is substantiated in Theorems 1 and 2.

For further, we will enter the notation

$$\left(\frac{\bar{t}_k^{(2)}}{\sqrt{k+1/2}}\right)_{k=0}^{\infty}=\left(\tilde{t}_k^{(2)}\right)_{k=0}^{\infty}=\tilde{t}^{(2)} ,$$

$$\left(\frac{\bar{g}_m^{(2)}}{\sqrt{m+1/2}}\right)_{m=0}^{\infty}=\left(\tilde{g}_m^{(2)}\right)_{m=0}^{\infty}=\tilde{g}^{(2)} ,$$

$$\tilde{B}=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}\tilde{B}_{n,m}^{(j)}\tilde{B}_{n,k}^{(j)}\right)_{m,k=0}^{\infty},\ \tilde{B}_{n,m}^{(j)}=B_{n,m}^{(j)}\sqrt{m+1/2} ,$$

$$\tilde{F}=\left(\sum_{j=1}^{2}\sum_{n=0}^{\infty}\tilde{B}_{n,m}^{(j)}F_n^{(j)}\right)_{m=0}^{\infty} .$$

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In the new notation, system (54) can be written in the following form:

$$(\varsigma I+\tilde{B})\tilde{g}^{(2)}=-\tilde{F}. \qquad (63)$$

with a constraint

$$\sum_{n=0}^{\infty}(\tilde{g}_n^{(2)})^2=2(\alpha_0^{\nu_0})^2T^2. \qquad (64)$$

Theorem 4. When the conditions of Theorem 2 are met, the matrix $\tilde{B}$ of system (63) defines a symmetric, positive, compact operator in space l_2.

Proof. The symmetry of the matrix $\tilde{B}$ is obvious. In addition, it is positive because for any vector $x=(x_k)\in l_2$ quadratic form

$$(\tilde{B}x,x)=\sum_{m,k=0}^{\infty}x_m x_k\sum_{j=1}^{2}\sum_{n=0}^{\infty}\tilde{B}_{n,m}^{(j)}\tilde{B}_{n,k}^{(j)}=$$

$$=\sum_{j=1}^{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\tilde{B}_{n,k}^{(j)}x_k\right)^2\geq 0.$$

Let us prove that the matrix $\tilde{B}$ defines a compact operator in the Hilbert space l_2. First, we prove the compactness of the operator defined by a matrix with elements $\tilde{B}_{n,m}^{(j)}$. Because it is impossible to strictly estimate the matrix coefficients of this matrix, we apply the following artificial technique. For simplicity, we show it by examining one of the constituent elements of this matrix, namely $\left(\tau_n^{(j)}\frac{\partial\overline{g}_n^{(1)}}{\partial\overline{g}_m^{(2)}}\sqrt{m+1/2}\right)_{n,m=0}^{\infty}$. It follows from formulas (57) and (58) that

$$\left(\tau_n^{(j)}\frac{\partial\overline{g}_n^{(1)}}{\partial\overline{g}_m^{(2)}}\sqrt{m+1/2}\right)_{n,m=0}^{\infty}=\tilde{U}_g^{(j)}(I-\tilde{U}_{22})^{-1}, \qquad (65)$$

where

$$\tilde{U}_g^{(j)}=\left((1-\gamma_n)\tau_n^{(j)}u_{n,k}^{(1)}\sqrt{k+1/2}\right)_{n,k=0}^{\infty},$$

$$\tilde{U}_{22}=\left(\sum_{k=0}^{\infty}\frac{\gamma_k}{\sqrt{n+1/2}}u_{n,k}^{(2)}u_{k,s}^{(1)}\sqrt{s+1/2}\right)_{n,s=0}^{\infty}.$$

Matrices $\tilde{U}_g^{(j)},\tilde{U}_{22}$ define the compact operators, and $(I-\tilde{U}_{22})^{-1}$ is a bounded operator in the space l_2. The latter is a consequence of the convergence of any series of the species

$$\sum_{n,k=0}^{\infty}n^s k^r\,|u_{n,k}^{(j,i)}|<\infty,$$

where s,r are fixed non-negative integers. Then, according to the properties of compact operators, the matrix (65) defines a compact operator.

In the same manner, the theorem is proved for other constituent components of matrices $(\tilde{B}_{n,m}^{(j)})_{n,m=0}^{\infty}$. The final result follows from the fact that the matrix $\tilde{B}$ is the sum of products of matrices that define compact operators in l_2.

Remark. 1. Using these ideas, it is possible to prove that if $f^{(j)}\in l_2$ the column $\tilde{F}\in l_2$.

2. In addition to the properties of the matrix $\tilde{B}$ listed in Theorem 4, it can also be proved that it is a nondegenerate matrix, that is, $(Bx,x)=0$ only if $x=0$.

The further solution to the problem is based on the spectral method. It follows from the properties of the operator that its spectrum consists of a counted sequence of positive eigenvalues

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq\cdots,$$

that converges to zero. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete orthonormal system of eigenvectors of the operator $\tilde{B}$ in the space l_2 corresponding to the eigenvalues of $\{\lambda_n\}_{n=1}^{\infty}$. Let us denote by $f_n=(\tilde{F},\varphi_n)$ the Fourier coefficients of the development of the vector $\tilde{F}$ according to the system $\{\varphi_n\}_{n=1}^{\infty}$.

Theorem 5. Let the conditions of Theorem 2 are held. If

$$\sum_{n=1}^{\infty}\frac{f_n^2}{\lambda_n^2}\leq 2(\alpha_0^{\nu_0})^2T^2, \qquad (66)$$

then there is a solution to equation (63) at $\varsigma=0$

$$\tilde{g}^{(2)}=-\sum_{n=1}^{\infty}\frac{f_n}{\lambda_n}\varphi_n\in l_2, \qquad (67)$$

and constraint (64) was not considered. If

$$\sum_{n=1}^{\infty}\frac{f_n^2}{\lambda_n^2} > 2(\alpha_0^{\nu_0})^2 T^2 , \qquad (68)$$

then under $\varsigma > 0$ there is a unique solution to problem (63), (64) in space l_2.

Proof. The first conclusion of the theorem is directly verified. Let's prove the second statement. In the new notation introduced above in this paragraph, problem (53), (52) can be written as follows:

$$L[\tilde{g}^{(2)}] = (S_\varsigma \tilde{g}^{(2)}, \tilde{g}^{(2)}) + 2(\tilde{F}, \tilde{g}^{(2)}) +$$

$$+\sum_{j=1}^{2} \| F^{(j)} \|^2 \to \min , \qquad (69)$$

$$\| \tilde{g}^{(2)} \|^2 = 2(\alpha_0^{\nu_0})^2 T^2 , \qquad (70)$$

where $S_\varsigma = \varsigma I + \tilde{B}$. An operator S_ς is a symmetric positive definite bounded operator given in space l_2. The last statement follows from an obvious inequality for any $\tilde{g}^{(2)} \in l_2$

$$\varsigma \| \tilde{g}^{(2)} \|^2 \le (S_\varsigma \tilde{g}^{(2)}, \tilde{g}^{(2)}) \le (\varsigma + \lambda_1) \| \tilde{g}^{(2)} \|^2 .$$

The last inequality allows us to introduce another norm in space l_2

$$\| \tilde{g}^{(2)} \|_S = \sqrt{(S_\varsigma \tilde{g}^{(2)}, \tilde{g}^{(2)})} ,$$

which is equivalent to the main one and is induced by the scalar product

$$(f, g)_S = (S_\varsigma f, g) \quad f, g \in l_2 .$$

Now, the existence of a unique $\tilde{g}^{(2)} \in l_2$, which sets the minimum of the functional (69) for each fixed value of the parameter $\varsigma > 0$, follows from the fact that the scalar product $(\tilde{F}, \tilde{g}^{(2)})$ is a bounded functional also in the norm $\| \cdot \|_S$, and therefore, according to the Riesz theorem, is equal to

$$(\tilde{F}, \tilde{g}^{(2)}) = (F_0, \tilde{g}^{(2)})_s$$

for some unique element $F_0 \in l_2$. Then

$$L[\tilde{g}^{(2)}] = (\tilde{g}^{(2)}, \tilde{g}^{(2)})_S + 2(F_0, \tilde{g}^{(2)})_S + \sum_{j=1}^{2} \| F^{(j)} \|^2 =$$

$$= \| \tilde{g}^{(2)} + F_0 \|_S^2 - \| F_0 \|_S^2 + \sum_{j=1}^{2} \| F^{(j)} \|^2 .$$

Hence, the minimum of the functional (69) is reached on the element $\tilde{g}^{(2)} = -F_0$.

The necessary condition (63) for the minimum of the functional (69) makes it possible to explicitly construct the optimal control $\tilde{g}^{(2)}$ using the spectral expansion of the operator S_ς

$$\tilde{g}^{(2)} = -\sum_{n=1}^{\infty} \frac{f_n}{\varsigma + \lambda_n} \varphi_n . \qquad (71)$$

It remains to satisfy condition (70). For this, the parameter ς must be selected as the positive root of the equation

$$\sum_{n=1}^{\infty} \frac{f_n^2}{(\varsigma + \lambda_n)^2} = 2(\alpha_0^{\nu_0})^2 T^2 . \qquad (72)$$

The left-hand side of equation (72) on the semiaxis $\varsigma \in (0, \infty)$ is a continuous, monotonically decreasing function with the domain of

$$\left(0, \sum_{n=1}^{\infty} \frac{f_n^2}{\lambda_n^2}\right)$$

(the right limit of the interval can be equal to ∞, if the series diverges). Then, based on condition (68), there is a unique value of the parameter $\varsigma > 0$ for which equality (70) holds. Thus, problem (69), (70) has a unique solution $\tilde{g}^{(2)} \in l_2$.

Remark. When condition (66) is fulfilled, optimal control (67) is a solution of the original problem (1) – (6) without restriction (7).

8. Computer experiment

We divide the numerical solution of the equivalent problem into two stages. In the first stage, we will form the matrix $\tilde{B}$ and the column $\tilde{F}$ of the right-hand parts of the system (63). For this purpose, it is necessary to solve the systems (57), (60) and find the derivatives

$\frac{\partial \bar{g}^{(1)}}{\partial \bar{g}_m^{(2)}}$, $\frac{\partial b_{i,n}^{(1)}}{\partial \bar{g}_m^{(2)}}$, $\frac{\partial b_{i,n}^{(1)}}{\partial f_m^{(j)}}$ according to formulas (59), (61), (62). As demonstrated in points 3, 5, the specified systems have Fredholm operators and are uniquely solvable. It is known that such systems can be correctly solved by the reduction method. At the second stage, system (63) also with a Fredholm operator is solved. Here again, the reduction method is used. A spectral decomposition is applied to the reduced matrix of system (63), as a result of which the rational equation (72) is obtained with respect to the parameter ς from condition (64).

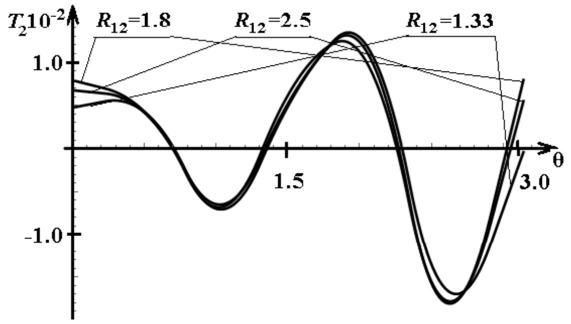

Fig. 1. Graphs of temperature distribution $T_2(\theta)\cdot 10^{-2}$ on the surface Γ_2 with the relative distance between the surfaces $d_{12} = 0.3$ and the ratio of their radii R_{12}

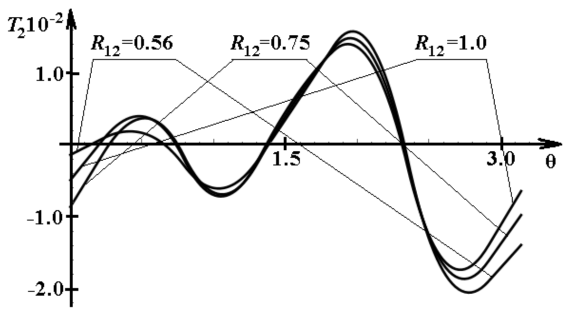

Fig. 2. Graphs of temperature distribution $T_2(\theta)\cdot 10^{-2}$ on the surface Γ_2 with the relative distance $d_{12} = 0.3$ between the surfaces and the ratio of their radii R_{12}

Calculations were carried out for materials in the following areas: Ω_0 – steel, Ω_1 – aluminum, for which $G_1 / G_0 = 0.317$, $\alpha_1 / \alpha_0 = 1.71$, $k_1 / k_0 = 4.61$, $\nu_0 = 0.28$, $\nu_1 = 0.34$. It was assumed that hydrostatic pressure acts on the surface Γ_2 with stresses $\sigma_r / (2G_0) = -10^{-3}$, $\tau_{r\theta} / (2G_0) = 0$. The results of the computer experiment are shown in Figures 1-5 and Table 1.

In Figure 1 shows graphs of temperature $T_2(\theta)\cdot 10^{-2}$ at a given relative distance $d_{12} \equiv 1 - R_1 / z_{12} - R_2 / z_{12} = 0.3$ and different ratios of surface radii $R_{12} \equiv R_1 : R_2 = 5:2,\ 9:5,\ 4:3$. It follows from the graphs that the optimal temperature distribution under $R_{12} \in [1.33; 2.5]$ depends little on the ratio of surface radii in almost the entire segment $\theta \in [0; \pi]$, except for the areas around its ends. A similar behavior of the optimal control curves is observed under $R_{12} \in [0.56; 1.0]$ (Fig. 2) with the difference that the temperature distribution around the poles of the spherical surfaces had the opposite sign.

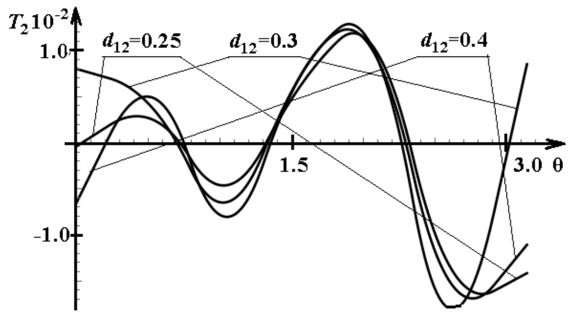

Fig. 3. Graphs of temperature distribution $T_2(\theta)\cdot 10^{-2}$ on the surface Γ_2 with the relative distance d_{12} between the surfaces and the ratio of their radii $R_{12} = 2$

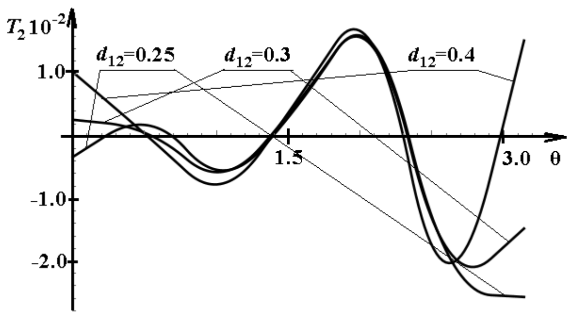

Fig. 4. Graphs of temperature distribution $T_2(\theta)\cdot 10^{-2}$ on the surface Γ_2 with the relative distance d_{12} between the surfaces and the ratio of their radii $R_{12} = 0.5$

In Figures 3-5, the graphs show the dependence of the optimal temperature distribution on the relative distance d_{12} between the surfaces at a fixed ratio of their radii: $R_{12} = 2$ (Fig. 3), $R_{12} = 0.5$ (Fig. 4), $R_{12} = 1$ (Fig. 5). As expected, when the surfaces approach each other, the greatest temperature effects occur in regions located around the axis of the problem. At the same time, the character of the distribution of the optimal temperature on the surface Γ_2 on the segment $\theta \in [\pi/4; 3\pi/4]$ is practically the same for different geometric parameters of the problem.

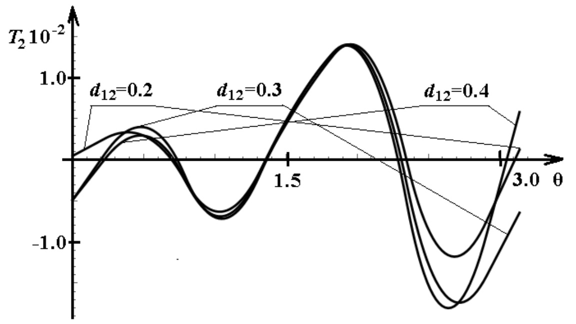

Fig. 5. Graphs of temperature distribution $T_2(\theta)\cdot 10^{-2}$ on the surface Γ_2 with the relative distance d_{12} between the surfaces and the ratio of their radii $R_{12} = 1$

Since the reduction method is used many times when solving the equivalent problem, in addition to its theoretical convergence, proved in Theorems 1, 2, 4, it is necessary to determine its practical convergence. Such a study was carried out in the work. His results are presented in Table 1. In this table, the value ε specifies in percent the relative error of calculating the optimal control values at the points $\theta_k = \dfrac{\pi k}{6}$, $k = 0 \div 6$ of the segment $[0; \pi]$. The error is calculated by the formula

$$\varepsilon = | T_{2,40}(\theta) - T_{2,30}(\theta) | / | T_{2,40}(\theta) | \cdot 100 ,$$

where $T_{2,n}(\theta)$ is approximate solution to the equivalent problem and corresponds to the reduction parameter n. The reduction parameter n specifies n+1 unknowns of each type, which is retained in systems (54), (57), (60) after their reduction to finite systems.

Table 1

Relative error $\varepsilon \cdot 10^3$ (%)in the temperature calculation $T_2(\theta)$ for various geometric parameters

$\theta \backslash R_{12}$	5/2	9/5	4/3	1	3/4	5/9	2/5
0	97	7,2	4.4	0.5	0.8	33	36
$\pi/6$	16	2.3	0.8	0.1	0.4	6.3	53
$\pi/3$	0.1	0.2	0.07	0.02	0.2	1.2	7.3
$\pi/2$	1.8	0.2	0.08	0.01	0.1	0.8	4.5
$2\pi/3$	2.4	0.3	0.01	0.01	0.1	0.2	0.6
$5\pi/6$	1.4	0.2	0.08	0.03	0.2	1.2	6.2
π	21	2.5	16	0.5	2.3	12	64

The convergence speed of the reduction method decreases as the surfaces approach. In the table, the percentage error $\varepsilon \cdot 10^3$ is calculated for the relative distance $d_{12} = 0.3$. Naturally, its value is different at different points of the segment $[0; \pi]$, and the worst results are observed at its ends. However, a computer experiment showed that even when the surfaces approach the distance $d_{12} = 0.2 \div 0.25$ in the most problematic points, the accuracy remains, which is determined by two correct significant figures after the point in the 10-year record of the result.

Discussions

Let's summarize the obtained results. The initial problem is reduced to an equivalent problem by the generalized Fourier method, in which the state of the object is determined by an infinite system of linear algebraic equations, the right-hand side of which parametrically depends on the control $(\tilde{g}_n^{(2)})_{n=0}^{\infty} \in l_2$. In this case, the cost functional becomes a quadratic functional in the space l_2 of numerical sequences linearly dependent on the state of the object and temperature field, and the control satisfies a certain quadratic constraint. To the best of our knowledge, this paper is the first to consider an optimization problem in which an object is controlled by an infinite system of linear algebraic equations. The main problem in solving an equivalent problem is the impossibility of an explicit solution to an infinite system. Therefore, this paper proposes a method of presenting the solutions of infinite systems in parametric form through the components of the derivatives of the state of the object. This method reduces the equivalent problem to the problem of the conditional extremum of the quadratic functional, which already clearly depends on the control. The last problem was solved using the Lagrange method. The necessary extremum condition leads to finding the optimal control from an infinite system of linear algebraic equations with a numerical parameter satisfying the additional quadratic equation. Theorem 5 establishes the final theoretical result of this study, which consists in the conditions for the existence and uniqueness of optimal control $\tilde{g}^{(2)}$ in space l_2. Two possibilities are formally obtained, depending on which of the constraints (66) or (68) are fulfilled for the given problem. In the first case, there is a solution to the unconditional optimization problem (1) – (6), in the second – to the optimal control problem with constraints (1) – (7). In fact, a computer experiment conducted with different data showed that the series (66) diverges, that is, only the second case is realized. Thus far, the authors have failed to prove this fact theoretically due to insurmountable analytical difficulties in finding matrix elements of inverse operators of infinite

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systems. In the future, this problem may become an area of additional research.

Note that the additional condition (7) in the formulation of the optimal control problem plays the role of a regularizing procedure when solving an incorrect problem.

The computer experiment showed good stability and convergence of the proposed method.

Thus, for the first time, a practical highly efficient method for solving the optimal control problem of a linear stationary system of differential equations of thermoelasticity with a quadratic constraint in a multi-connected spatial domain based on the generalized Fourier method is developed. The obtained results provide further possibilities for the application of the proposed method to boundary value problems for various differential equations in multi-connected spatial domains of different geometries.

It should be noted that one of the possible directions of practical application of the proposed technique may be the modeling of optimal control of the temperature field during crystal growth in order to reduce the zones and level of concentration of residual stresses near macroscopic pores and foreign inclusions.

Conclusions

This paper proposes a highly effective method for determining the optimal control of the stress-strain state of spatially multi-connected composite bodies using a stationary temperature field. The proposed method is considered based on the example of a stationary axisymmetric thermoelastic problem for a space with a spherical inclusion and a cavity. The proposed method is based on the generalized Fourier method and makes it possible to reduce the original problem to an equivalent problem of optimal control, in which the state of the object is determined by an infinite system of linear algebraic equations, the right part of which parametrically depends on the control. In this case, the cost functional of the initial problem is transformed into a quadratic functional that depends on the state of the equivalent system and parametrically on the control. The limitation on the temperature distribution is replaced by the value of the control norm in space l_2. The proposed method also solves the main problem of the equivalent problem–the impossibility of obtaining a clear dependence of the system state on control. In this study, it is proposed to present the solutions of non-finite systems in a parametric form through the components of the derivatives of the state of the object, due to which the equivalent problem was reduced to the problem of the conditional extremum of the quadratic functional, which already clearly depends on the control. The above representation is based on the solutions of some infinite systems of linear algebraic equations that differ only in their right-hand sides. A further solution of the problem to the conditional extremum of the cost functional is found by the Lagrange method, which reduces this problem to an infinite system of linear algebraic equations with a parameter and a constraint in the form of an additional quadratic equation. The last problem was investigated using the spectral method.

The method developed in this paper is strictly justified. For all infinite systems, the Fredholm properties of their operators have been proved, and for a system with a parameter, the properties of the system operator have been established, which allow us to use its spectral expansion. As an important result, without which it would be impossible to justify the proposed method, for the first time an estimate from below of the module of the multiparameter determinant of the resolving system of the boundary value problem of the conjugation – a space with a spherical inclusion – was obtained when solving it using the Fourier method. The main result of this study is a theorem that establishes the conditions of existence and uniqueness in the space l_2 the solution of equivalent or optimal control problems without restrictions.

The numerical algorithm is based on a reduction method for solving infinite systems of linear algebraic equations. It is known that this is correct for systems with Fredholm operators, that is, the approximate solution converges to the exact solution as the reduction parameter increases. The practical accuracy of the numerical algorithm was investigated by comparing the optimal control obtained using different reduction parameters. The calculations demonstrated the stability of the method and a fairly high accuracy even when the boundary surfaces were approached by a relative distance of 0.2. Graphs of the optimal temperature distribution for various geometric parameters of the problem and their analysis are presented in this paper.

The proposed method extends to boundary value problems with different geometries.

Contribution of the authors: formulation of the problem – **Oleksii Nikolaev**, the idea of the method – **Oleksii Nikolaev,** method development – **Oleksii Nikolaev, Mariia Skitska**, solving direct problems – **Mariia Skitska**, justification of the method – **Oleksii Nikolaev, Mariia Skitska**, computer program and experiment – **Oleksii Nikolaev, Mariia Skitska**, analysis and discussion of results – **Oleksii Nikolaev, Mariia Skitska**, manuscript of the article – **Oleksii Nikolaev,** abstract – **Oleksii Nikolaev**.

Conflict of Interest

The authors declare that they have no conflicts of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

Financing

This study was conducted without financial support.

Data Availability

The associated data are in the data repository.

Use of Artificial Intelligence

The authors confirm that they did not use artificial intelligence technologies in their work.

All authors have read and approved the version of this manuscript for publication.

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МЕТОД ВИЗНАЧЕННЯ ОПТИМАЛЬНОГО КЕРУВАННЯ ТЕРМОПРУЖНИМ СТАНОМ КУСКОВО-ОДНОРІДНОГО ТІЛА ЗА ДОПОМОГОЮ СТАЦІОНАРНОГО ТЕМПЕРАТУРНОГО ПОЛЯ

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У статті запропоновано новий високоефективний метод визначення оптимального керування напружено-деформованим станом просторового багатозв’язного складеного тіла за допомогою стаціонарного температурного поля. Метод розглянуто на прикладі стаціонарної осесиметричної термопружної задачі для простору зі сферичними включенням і порожниною. Він базується на узагальненому методі Фур’є і зводить вихідну задачу до еквівалентної задачі оптимального керування, в якій стан об’єкту визначається нескінченною системою лінійних алгебраїчних рівнянь, права чистина яких параметрично залежить від керування. При цьому функціонал вартості вихідної задачі перетворюється на квадратичний функціонал, який залежить від стану еквівалентної системи і параметрично від керування, а обмеження на розподіл температури замінюється значенням норми керування в просторі сумовних з квадратом послідовностей. В роботі фактично вперше розглянуто задачу оптимального керування нескінченною системою лінійних алгебраїчних рівнянь і розроблено метод її розв'язання. Він заснований на поданні розв'язків нескінченних систем у параметричній формі, що дозволило звести еквівалентну задачу до задачі на умовний екстремум квадратичного функціонала, який явно залежить від керування. Подальший розв'язок цієї задачі знаходиться методом Лагранжа із застосуванням спектрального розкладу матриці квадратичного функціонала. Розроблений у статті метод строго обґрунтовано. Для всіх нескінченних систем доведено фредгольмовість їх операторів. Як важливий, необхідний для обґрунтування результат, вперше отримано оцінку знизу модуля багатопараметричного визначника розв’язувальної системи крайової задачі спряження – простір зі сферичним включенням – при розв'язанні її методом Фур’є. Доведено теорему, яка встановлює умови існування та єдиності в просторі сумовних з квадратом послідовностей розв'язку еквівалентної задачі або задачі оптимального керування без обмеження. Чисельний алгоритм засновано на методі редукції для нескінченних систем лінійних алгебраїчних рівнянь. Оцінки практичної точності чисельного алгоритму показали стійкість методу і достатньо високу точність навіть при близькому розташуванні граничних поверхонь. Наведено графіки оптимального розподілу температури при різних геометричних параметрах задачі та їх аналіз. Метод припускає розповсюдження на інші крайові задачі з різною геометрією.

Ключові слова: оптимальне керування; термопружений стан; стаціонарне температурне поле; багатозв’язне кусково-однорідне тіло; узагальнений метод Фур’є; нескінченна система лінійних алгебраїчних рівнянь; фредгольмів оператор; квадратичний функціонал; спектральний розклад; метод редукції.

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