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DIFFERENTIATION: THEORY AND APPLICATIONS

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MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National Aerospace University
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Tutorial

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P75

Наведено детальний теоретичний матеріал, зокрема методи диференціювання, застосування похідних при дослідженні функцій і побудові їх графіків, а також додано зразки розв'язання типових задач.

Для студентів першого курсу з англійською мовою навчання.

Reviewers:
Dr. of Sciences in Physics and Mathematics, Prof. S. S. Zub,
Candidate of Technical Sciences, Prof. O. B. Akhiezer

Poliakov, O. G.

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Detailed theoretical material is given, including methods of differentiation, application of derivatives to the investigation of functions, and plotting their graphs, and samples of solving typical problems are added.

For the first year students with English language tuition.

Figs 39. Tables 6. Bibliogr.: 29 titles

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CONTENTS

DERIVATIVES AND DIFFERENTIALS OF THE FIRST ORDER

The concept of derivative

We consider a point moving in a straight line. The path s traversed by the point, measured from some definite point of the line, is evidently a function of time t:

$$s = f(t).$$

A corresponding value of s is defined for every definite value of t. If t receives an increment Δt, the path $s+\Delta s$ will then correspond to the new instant $t+\Delta t$, where Δs is the path traversed in the interval Δt. In the case of uniform motion, the increment of path is proportional to the increment of time, and the ratio $\Delta s/\Delta t$ represents the constant velocity of the motion. This ratio is in general dependent both on the choice of the instant t and on the increment Δt, and represents the average velocity of the motion during the interval from t to $t+\Delta t$. This average velocity is the velocity of an imaginary point which moves uniformly and traverses path Δs in time Δt. For example, we have in the case of uniformly accelerated motion:

$$s = \frac{1}{2}gt^2 + v_0 t$$

and

$$\frac{\Delta s}{\Delta t} = \frac{\frac{1}{2}g\left(t+\Delta t\right)^2 + v_0\left(t+\Delta t\right) - \frac{1}{2}gt^2 - v_0 t}{\Delta t} = gt + v_0 + \frac{1}{2}g\Delta t.$$

The smaller the interval of time t, the more we are justified in taking the motion of the point in question as uniform in this interval, and the limit of the ratio $\frac{\Delta s}{\Delta t}$, with Δt tending to zero, defines the velocity v at the given instant t :

$$v = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}.$$

Thus, in the case of uniformly accelerated motion:

$$v = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}\left(gt + v_0 + \frac{1}{2}g\Delta t\right) = gt + v_0.$$

The velocity v, like the path s, is a function of t; this function is called the derivative of function $f(t)$ with respect to t; thus, the velocity is the derivative of the path with respect to time.

Suppose that a substance takes part in a chemical reaction. The quantity x of this substance, taking part in the reaction at the instant t, is a function of t. There is a corresponding increment Δx of magnitude x for an increment of time Δt, and the ratio $\Delta x/\Delta t$ gives the average speed of the reaction in the interval Δt, whilst the limit of this ratio as Δt tends to zero gives the speed of the chemical reaction at the given instant t.

We considered above the quantity of heat Q, absorbed by a body, as a function of its temperature t^0. Let Δt^0 and ΔQ be the corresponding increments of temperature and quantity of heat.

Accurate measurements indicate that ΔQ is not proportional to Δt^0, and the ratio $\Delta Q/\Delta t$ gives the so-called average specific heat of the body in the temperature interval from t^0 to $t^0+\Delta t^0$, whilst the limit of this ratio as Δt^0 tends to zero gives the specific heat of the body at t^0, this being the derivative of the quantity of heat with respect to temperature. The above examples lead us to the following concept of the derivative of a function:

The derivative of a given function $y = f(x)$ is defined as the limit of the ratio of the increment Δy of the function to the corresponding increment Δx of the independent variable, when the latter tends to zero.

The symbols y' or $f'(x)$ are used to denote the derivative:

$$y' = f'(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The operation of finding the derivative is called differentiation.

It is possible for the above limit not to exist, in which case the derivative does not exist. Assuming that the derivative exists, we can write:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)+\alpha,$$

where $\alpha\to 0$, as $\Delta x\to 0$.

We now have:

$$f(x+\Delta x)-f(x) = \left[f'(x)+\alpha\right]\Delta x,$$

whence it is immediately clear that $\left[f\left(x+\Delta x\right)-f\left(x\right)\right]\to 0$ if $\Delta x\to 0$, i.e. if the derivative exists for some value of x, the function is continuous for this value of x. The converse statement does not hold, i.e. nothing can be said about the existence of a derivative from the continuity of a function. We note that in finding the derivative we take the fraction $\Delta y/\Delta x$, with the numerator and denominator both tending to zero; but we suppose that Δx never in fact becomes zero.

Geometrical significance of the derivative

We turn to the graph of the function $y = f(x)$ to see the geometrical meaning of the derivative. We take a point M of the graph with coordinates (x, y), and an adjacent point N of the graph with coordinates $(x + \Delta x, y + \Delta y)$. We draw the ordinates $\overline{M_1 M}$ and $\overline{N_1 N}$ of these points, and take a line through M parallel to axis OX. We evidently have (Fig. 1):

$$\overline{MP} = \overline{M_1 N_1} = \Delta x\,,\ \overline{M_1 M} = y\,,\ \overline{N_1 N} = y + \Delta y\,,\ \overline{PN} = \Delta y\,. \qquad (1)$$

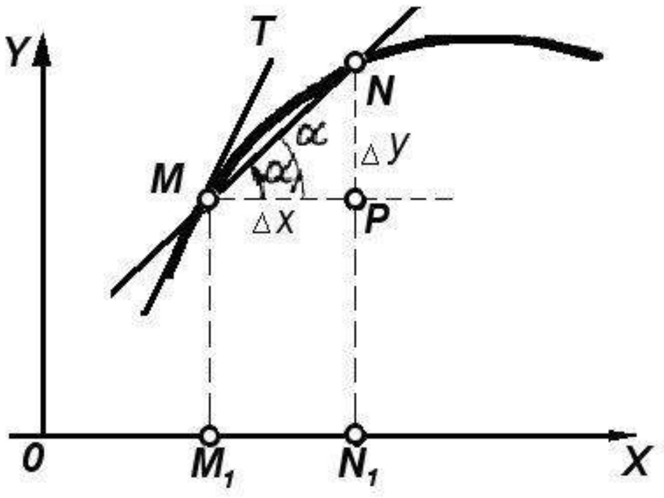

Fig. 1

The ratio $\Delta y/\Delta x$ is clearly equal to the tangent of the angle α_1 that MN forms with the positive direction of OX. As Δx tends to zero, point N will tend to point M, whilst remaining on the curve; MN becomes, in the limiting position, the tangent MT to the curve at the point M; hence, the derivative $f'(x)$ is equal to the tangent of the angle α formed by the tangent to the curve at the point $M(x, y)$ with the positive direction of axis OX, i.e. is equal to the slope of this tangent.

Attention must be paid to the rule of signs when working out the segments in accordance with formula (1), remembering that increments Δx and Δy can be either negative or positive.

We see that the existence of a derivative $f'(x)$ is bound up with the existence of a tangent to the curve corresponding to $y = f(x)$. A continuous curve may have no tangent at all at certain points, or it may have a tangent parallel to axis OY, with infinitely large slope (Fig. 2); the function $f(x)$ then has no derivative for the corresponding value of x.

A curve can have any number of such singular points, and it is even possible to construct a continuous function, as may be shown, such that it has no derivative for any value of x. The curve corresponding to this function cannot be represented geometrically.

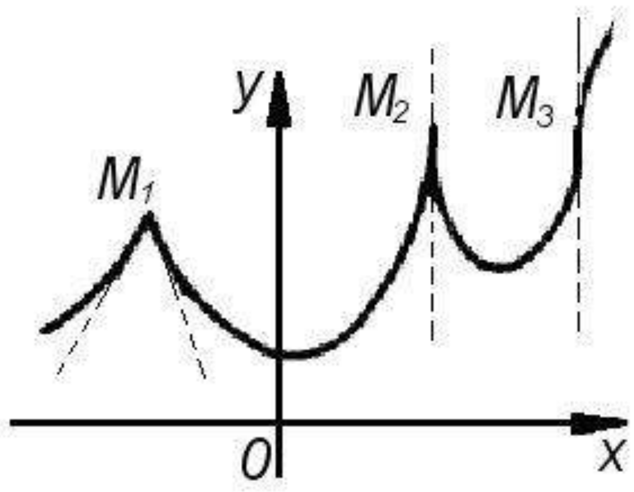

Fig. 2

Denoting for simplicity the increment of the independent variable by h, we have the ratio:

$$\frac{f(x+h)-f(x)}{h}. \qquad (2)$$

If the number x is fixed in the interval in which f(x) is defined, the ratio (2) is a function of h, defined for all h sufficiently close to zero, except $h=0$. The limit of this ratio as $h \to 0$ has to be determined in accordance with what was said in the theory of limits. If the limit exists, it gives us the derivative $f'(x)$. The existence of the limit is equivalent to the following. For any given positive ε there exists a positive η such that

$$\left| f'(x) - \frac{f(x+h)-f(x)}{h} \right| < \varepsilon \text{ for } |h| < \eta \text{ and } h \neq 0.$$

It can happen that ratio B) has a limit for h tending to zero on the side of positive values (on the right), and on the side of negative values (on the left). These limits are usually denoted by $f'(x+0)$ and $f'(x+0)$, being called respectively the derivative on the right and the derivative on the left. If these limits differ, they give the slopes of the tangents to the curve at its bend point (if the tangents exist).

Fig. 2 shows these tangents at the point M_1. The existence of the derivative is equivalent to the existence of derivatives $f^{/}(x+0)$ and $f^{/}(x-0)$ and to the fact of these being equal, so that we have $f^{/}(x)=f^{/}(x+0)=f^{/}(x-0)$.
It is possible for a continuous function to have points for which there is neither derivative $f^{/}(x+0)$ nor $f^{/}(x+0)$. Such a curve is shown in Fig. 3. It has neither derivative for $x=c$.

If a continuous function is given solely in an interval (a,b) , we can only form the derivative on the right, $f^{/}(x+0)$, at $x=a$, and only the derivative on the left, $f^{/}(x-0)$, at $x=b$. When $f(x)$ is said to have a derivative $f^{/}(x)$ in the (closed) interval (a,b), this must be taken to mean the derivative in the ordinary sense for interior points of the interval, and in the special sense indicated at the ends of the interval.

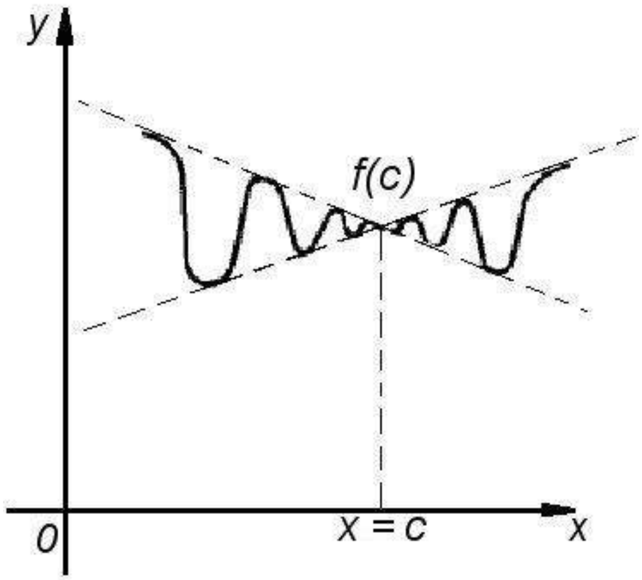

Fig. 3

If $f(x)$ is defined in the interval (A,B), wider than (a,b), i.e. $A<a$ and $B>b$, and has the ordinary derivative $f^{/}(x)$ inside (A,B) it will certainly have a derivative in the sense indicated over (a,b).

Derivatives of some simple functions

It follows from the concept of derivative that, to find the derivative, the increment given to the function must be divided by the corresponding increment of the independent variable, the limit of their ratio then being found as the increment of the independent variable tends to zero. We use this rule for some elementary functions.

I. $y = b$ (constant).

$$y' = \lim_{h\to 0}\frac{b-b}{h} = \lim_{h\to 0} 0 = 0,$$

i.e. the derivative of a constant is zero.

II. $y = x^n$ (*n* a positive integer).

$$y' = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} =$$

$$= \lim_{h\to 0}\frac{x^n + nhx^{n-1} + \frac{n(n-1)}{2!}h^2x^{n-2} + \ldots + h^n - x^n}{h} =$$

$$= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2!}hx^{n-2} + \ldots + h^{n-1}\right) = nx^{n-1}.$$

In particular, if $y = x$, $y' = 1$. We later generalize this rule for differentiation of a power function for any value of the exponent n.

III. $y = \sin x$.

$$y' = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} =$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$$

since $\frac{\sin\frac{h}{2}}{\frac{h}{2}} \to 1$ for $\frac{h}{2} \to 0$.

IV. $y = \cos x$.

$$y' = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\left(-\frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}\right) =$$

$$= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin x.$$

V. $y = \ln(x)$ $(x > 0)$.

$$y' = \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} =$$

$$= \lim_{h\to 0} \frac{1}{x} \frac{\ln\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}.$$

Since for $h \to 0$, $a = \frac{h}{x}$ also tends to 0, and $\ln(1+a)/a \to 1$.

VI. $y = cu(x)$, where c is a constant, and $u(x)$ a function of x.

$$y' = \lim_{h\to 0} \frac{cu(x+h) - cu(x)}{h} = c \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = cu'(x),$$

i.e. the derivative of the product of a constant and a variable is equal to the product of the constant and the derivative of the variable, or, in other words, the constant can be taken outside the sign of the derivative.

VII. $y = \log_a x$.

As we know, $\log_a x = \ln x / \ln a$. Using Rule VI, we obtain:

$$y' = \frac{1}{x \ln a}.$$

VIII. We consider the derivative of the sum of several variables; we confine ourselves to two terms for clarity:

$$y = u(x) + v(x)$$

$$y' = \lim_{h\to 0} \frac{\left[u(x+h) + v(x+h)\right] - \left[u(x) + v(x)\right]}{h} =$$

$$= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] = u'(x) + v'(x),$$

i.e. the derivative of the sum of any given number of functions is equal to the sum of the derivatives of these functions.

IX. We now consider the derivative of the product of two functions:

$$y' = \lim_{h\to 0} \frac{u(x+h) \times v(x+h) - u(x) \times v(x)}{h}.$$

Adding and subtracting $u(x+h) \times v(x)$ in the numerator, then rearranging, we get:

$$y' = \lim_{h\to 0} \frac{u(x+h) \times v(x+h) - u(x+h) \times v(x) + u(x+h) \times v(x) - u(x) \times v(x)}{h} =$$

$$= \lim_{h\to 0} u(x+h) \frac{v(x+h) - v(x)}{h} + \lim_{h\to 0} v(x) \frac{u(x+h) - u(x)}{h} =$$

$$= u(x) \times v'(x) + v(x) \times u'(x),$$

i.e. we have shown that for two factors, the derivative of the product is equal to the sum of the products of each factor with the derivative of the other.

We prove the applicability of this rule to three factors by combining two factors in one group and using the rule for two:

$$y = u(x) \times v(x) \times w(x),$$

$$y' = \left(u(x) \times v(x) \times w(x)\right)' = \left[u(x) \times v(x)\right] \times w'(x) +$$

$$+ w(x) \times \left[u(x) \times v(x)\right]' =$$

$$= u(x) \times v(x) \times w'(x) + u(x) \times v'(x) \times w(x) + u'(x) \times v(x) \times w(x).$$

Using the well-known method of mathematical induction, the rule for two can easily be extended to the case of any finite number of factors.

X. Now let y be a quotient: $y = \frac{u(x)}{v(x)}$,

$$y' = \lim_{h \to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} =$$

$$= \lim_{h \to 0} \frac{1}{v(x)v(x+h)} \cdot \frac{u(x+h)v(x) - v(x+h)u(x)}{h} .$$

Adding and subtracting $u(x)v(x)$ in the numerator of the second fraction, and taking the continuity of $v(x)$ into account, we get:

$$y' = \lim_{h \to 0} \frac{1}{v(x)v(x+h)} \cdot \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{h} =$$

$$= \lim_{h \to 0} \frac{1}{v(x)v(x+h)} \left[v(x) \frac{u(x+h) - u(x)}{h} - u(x) \frac{v(x+h) - v(x)}{h} \right] =$$

$$= \frac{u'(x)v(x) - v'(x)u(x)}{\left(v(x)\right)^2},$$

i.e. the derivative of a fraction (quotient) is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

XI. $y = \tan x$.

$$y' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

XII. $y = \cot x$.

$$y' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x} =$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

In deducing Rules VI, VIII, IX and X, we assumed that the functions $u(x), v(x), w(x)$ have derivatives, and proved the existence of a derivative of the function y.

Derivatives of functions of a function, and of inverse functions

We recall the concept of function of a function. Let $y = f(x)$ be a function, continuous in some segment $a \le x \le b$, with its value lying in the segment $c \le y \le d$. Further, let $z = F(y)$ be a function continuous in the segment $c \le y \le d$. Taking the above function of x as y, we obtain a function of a function (composite function) of x:

$$z = F(y) = F(f(x)).$$

This function is said to depend on x expressed by means of y. It is easily seen that the function is continuous in the segment $a \le x \le b$. In fact, given an infinitesimal increment of x, there is a corresponding infinitesimal increment of y, due to the continuity of $f(x)$, and given the infinitesimal increment of y, there is a corresponding infinitesimal increment of z, due to the continuity of $F(y)$.

We make one remark, before deducing the rule for differentiation of a function of a function. If $z = F(y)$ has a derivative for $y = y_0$, it follows from the above, that we can write:

$$\Delta z = F(y_0 + \Delta y) - F(y_0) = \left[F'(y_0) + \alpha\right]\Delta y, \qquad (3)$$

where α is a function of Δy, defined for all positive values of Δy approaching zero, and where $\alpha \to 0$ as $\Delta y \to 0$ {$\Delta y \ne 0$).Equation (3) remains valid for $\Delta y = 0$ with any choice of α, since for $\Delta y = 0$, Δz also = 0. It is natural from the above to take $\alpha = 0$ for $\Delta y = 0$.

Having agreed on this, we can take $\alpha \to 0$ in formula (3) for $\Delta y \to 0$ in any manner, even whilst taking values equal to zero. We now formulate the theorem about the derivative of a function of a function.

Theorem. *If $y=f(x)$ has a derivative $f'(x_0)$ at $x=x_0$ and $z=F(y)$ has a derivative $F'(y_0)$ at $y_0=f(x_0)$, the function of a function $F(f(x))$ has a derivative at $x=x_0$ equal to the product $F'(y_0)f'(x_0)$.*

Let Δx be the increment (not zero) that we give to the value x_0 of the independent variable x, and $\Delta y=f(x_0+\Delta x)-f(x_0)$ be the corresponding increment of variable y (its value can be zero). Further, let $\Delta z=F(y_0+\Delta y)-F(y_0)$. The derivative of the function of a function $z=F(f(x))$ with respect to x, at $x=x_0$, is evidently equal to the limit of the ratio $\Delta z/\Delta x$ as $\Delta x\to 0$, if this limit exists.

We divide both sides of (3) by Δx:

$$\frac{\Delta z}{\Delta x}=\left[F'(y_0)+\alpha\right]\frac{\Delta y}{\Delta x}.$$

As $\Delta x\to 0$, Δy also $\to 0$, due to the continuity of function $y=f(x)$ at the point $x=x_0$; and hence, as we have shown above, $\alpha\to 0$. The ratio $\Delta y/\Delta x$ now tends to the derivative $f'(x_0)$, and on passing to the limit in the equation above, we obtain:

$$\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}=F'(y_0)f'(x_0),$$

which proves the theorem. We remark that the continuity of $f(x)$ at $x=x_0$ follows from the assumption of the existence of a derivative $f'(x_0)$.

This theorem can be put in the following form, as a rule for differentiation of functions of a function:

the derivative of a function of a function is equal to the product of the derivative with respect to the intermediate variable and the derivative of the intermediate variable with respect to the independent variable:

$$z'_x=F'(y)f'(x).$$

We pass to the rule for differentiation of inverse functions. If $y=f(x)$ is continuous and increasing in the interval (a,b) (i.e. the greatest value of y corresponds to the greatest value of x), with $A=f(a)$ and $B=f(b)$, we know, that a single-valued, continuous, and likewise increasing, inverse function $x=\varphi(y)$ exists in the interval (A,B). Since it is increasing, if $\Delta x=0$, $\Delta y=0$, and conversely; and due to continuity, $\Delta x\to 0$ implies $\Delta y\to 0$, and conversely. (The case of a decreasing function is exactly similar.)

Theorem. *If $f(x)$ has a non-zero derivative $f'(x_0)$ at the point x_0, the inverse function $\varphi(y)$ has a derivative at the point $y_0 = f(x_0)$:*

$$\varphi'(y_0) = \frac{1}{f'(x_0)}. \tag{4}$$

Denoting corresponding increments of x and y by Δx and Δy, i.e.

$$\Delta x = \varphi(y_0 + \Delta y) - \varphi(y_0),$$
$$\Delta y = f(x_0 + \Delta x) - f(x_0),$$

and noting that both these differ from zero, we can write:

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}.$$

As we have seen above, Δx and Δy tend simultaneously to zero, and the last equation leads to (4) in the limit. The present theorem can be put in the form of the following rule for differentiation of inverse functions:
the derivative of an inverse function is equal to unity divided by the derivative of the direct function at the corresponding point.

The rule for differentiation of inverse functions has a simple geometrical interpretation. The functions $x = \varphi(y)$ and $y = f(x)$ have the same graph in the XOY plane, the only difference being that the axis of the independent variable is OY, and not OX, for the function $x = \varphi(y)$. On drawing the tangent MT, and recalling the geometrical significance of the derivative, we get:

$$f'(x) = \tan(OX, MT) = \tan\alpha;$$
$$\varphi'(y) = \tan(OY, MT) = \tan\beta,$$

angle β as well as a being reckoned positive, as in Fig. 4.

But evidently, $\beta = \frac{1}{2}\pi - \alpha$, and hence:

$$\tan\beta = \frac{1}{\tan\alpha}, \text{ i.e. } \varphi'(y) = \frac{1}{f'(x)}.$$

If $x = \varphi(y)$ is the inverse of $y = f(x)$, the converse is evidently true, i.e. $y = f(x)$ can be considered the inverse of $x = \varphi(y)$.

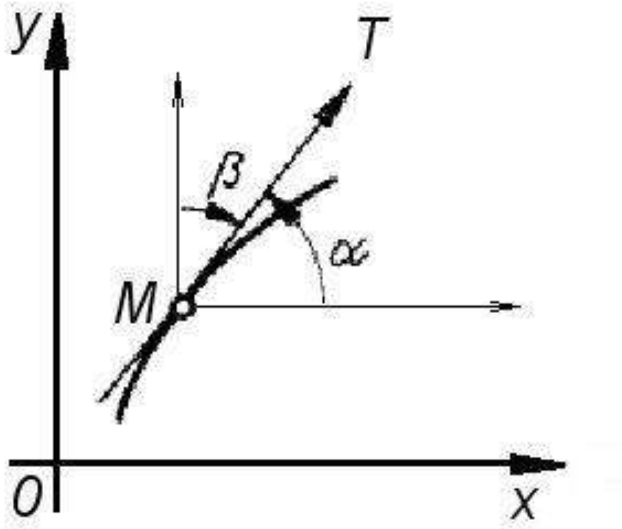

Fig. 4

We use the rule for differentiation of inverse functions for the exponential function.

XIII. $y = a^x, \quad (a > 0)$.

The inverse function in this case will be:

$$x = \varphi(y) = \log_a y,$$

and by VII:

$$\varphi'(y) = \frac{1}{y} \cdot \frac{1}{\log a},$$

whence by the rule for differentiation of inverse functions:

$$y' = \frac{1}{\varphi'(y)} = y \log a \text{ , or } (a^x)' = a^x \log a .$$

In the particular case of $a = e$, we have:

$$(e^x)' = e^x .$$

The formula obtained, together with the rule for differentiation of a function of a function, enables us to calculate the derivative of a power function.

XIV. $y = x^n \quad (x > 0;\ n \in \mathbb{R})$.

This function is defined, and is positive, for all $x > 0$.

Using the definition of logarithm, we can express our function as a function of a function:

$$y = x^n = e^{n \log x}.$$

Using the rule for differentiation of a function of a function, we get:

$$y' = e^{n \log x} \cdot \frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1}.$$

This result can be easily generalized for the case of negative x, provided the function exists for such values, as for instance $y = x^{\frac{1}{3}} = \sqrt[3]{x}$.

We use the rule for differentiation of inverse functions for obtaining the derivatives of the inverse circular functions.

XV. $y = \arcsin x$.

We consider the principal value of this function, i.e. the arc lying in the interval $\left(-\frac{\pi}{2},\ +\frac{\pi}{2}\right)$. We can consider this as the inverse of function $x = \sin y$, and in accordance with the rule for differentiation of inverse functions, we have:

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}},$$

where the plus sign must be taken with the radical, since $\cos y$ has a positive sign in the interval $\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$. We can similarly obtain:

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}},$$

the principal value of $\arccos x$ being taken, i.e. the arc contained in the interval $(0,\ \pi)$.

XVI. $y = \arctan x$.

The principal value of $\arctan x$; lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and we can consider this function as the inverse of $x = \tan y$; hence:

$$y'_x = \frac{1}{x'_y} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

We obtain similarly:

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}.$$

XVII. We also consider the differentiation of a function of the form:

$$y = u^v,$$

where u and v are functions of x (exponential function). We can write:

$$y = e^{v \ln u},$$

and on using the rule for differentiation of a function of a function, we obtain:

$$y' = e^{v \log u} \left(v \ln u \right)'.$$

Using the rule for differentiation of a product, and differentiating $\ln u$ as a function of a function of x, we finally have:

$$y' = e^{v \log u} \left(v' \ln u + \frac{v}{u} u' \right)$$

or

$$y' = u^v \left(v' \ln u + \frac{v}{u} u' \right).$$

Table of derivatives, and examples

A list follows of all the rules that we have deduced for differentiation.

1. $(c)' = 0$.
2. $(cu)' = cu'$.
3. $(u_1 + u_2 + ... + u_n)' = u_1' + u_2' + ... + u_n'$.
4. $(u_1 u_2 ... u_n)' = u_1' u_2 ... u_n + u_1 u_2' ... u_n + ... + u_1 u_2 ... u_n'$.
5. $\left(\frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}$.
6. $(x^n)' = nx^{n-1}$ and $(x)' = 1$.
7. $(\log x)' = \frac{1}{x}$ and $(\log_a x)' = \frac{1}{x} \cdot \frac{1}{\log a}$.
8. $(e^x)' = e^x$ and $(a^x)' = a^x \log a$.
9. $(\sin x)' = \cos x$.
10. $(\cos x)' = -\sin x$.
11. $(\tan x)' = \frac{1}{\cos^2 x}$.

12. $\left(\cot x\right)' = -\frac{1}{\sin^2 x}$.

13. $\left(\arcsin x\right)' = \frac{1}{\sqrt{1-x^2}}$.

14. $\left(\arccos x\right)' = -\frac{1}{\sqrt{1-x^2}}$.

15. $\left(\arctan x\right)' = \frac{1}{1+x^2}$.

16. $\left(\operatorname{arccot} x\right)' = -\frac{1}{1+x^2}$.

17. $\left(u^v\right)' = vu^{v-1}u' + u^v v' \left(\log u\right)$.

18. $y'_x = y'_u \cdot u'_x$ (y depends on x through the medium of u).

19. $x'_y = \frac{1}{y'_x}$.

We use the above rules to solve a few examples.

1. $y - x^3 - 3x^2 + 7x - 10$.

Using Rules 3, 6, and 2, we obtain $y' = 3x^2 - 6x + 7$.

2. $y = \frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$.

Using Rule 6, we obtain: $y' = -\frac{2}{3}x^{-\frac{5}{3}} = -\frac{2}{3x\sqrt[3]{x^2}}$.

3. $y = \sin^2 x$.
We put $u = \sin x$ and use Rules 18, 6, and 9:

$$y' = 2u \cdot u' = 2\sin x \cos x = \sin 2x .$$

4. $y = \sin\left(x^2\right)$.

We put $u = x^2$ and use the same rules:

$$y' = \cos u \cdot u' = 2x\cos\left(x^2\right).$$

5. $y = \log\left(x + \sqrt{1+x^2}\right)$,

We first put $u = x + \sqrt{1+x^2}$, then $v = x^2 + 1$, and use Rules 18 (twice), 7, 3, and 6:

$$y' = \frac{u'}{u} = \frac{\left(1+\sqrt{v}\right)'}{u} = \frac{1}{u}\left(1+\frac{1}{2\sqrt{v}}v'\right) = \frac{1+\frac{x}{\sqrt{v}}}{u} =$$
$$= \frac{1}{x+\sqrt{1+x^2}}\,\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

6. $y = \left(\frac{x}{2x+1}\right)^n$.

We put $u = \frac{x}{2x+1}$ and use Rules 18, 6 and 5:

$$y' = nu^{n-1}u' = nu^{n-1}\frac{2x+1-2x}{\left(2x+1\right)^2} = \frac{nx^{n-1}}{\left(2x+1\right)^{n+1}}.$$

7. $y = x^x$.
Using Rule 17, we obtain:

$$y' = x^{x-1}\cdot x + x^x \log x = x^x\left(1+\log x\right).$$

8. The function y is given as an implicit function of x by the equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0. \qquad (5)$$

The problem is to find the derivative of y.

If we solved the given equation for y, obtaining $y = f(x)$, the left-hand side of the equation would evidently be identical with zero after substituting $y = f(x)$. But the derivative of zero is the same as the derivative of a constant equal to zero, and hence we must obtain zero on differentiating the left-hand side of the given equation with respect to x, y being taken as the function of x given by this equation:

$$\frac{2x}{a^2}+\frac{2y}{b^2}y' = 0, \text{ whence } y' = -\frac{b^2x}{a^2y}.$$

We see that in this case y' is expressed in terms of y as well as x; but we did not need to solve equation (5) for y, i.e. obtain an explicit expression for the function, in order to obtain the derivative.

Equation (5) represents an ellipse, as is known from analytic geometry, and the expression obtained for y' gives the slope of the tangent to the ellipse at the point with coordinates (x, y).

The concept of differential

Let Δx be the arbitrary increment of the independent variable, which we still take as not depending on x. We call it the differential of the independent variable, and denote it by the symbol Δx or dx ;. The latter symbol in no circumstances stands for the product of d and x, being used only as a symbol for denoting an arbitrary quantity, independent of x, which we take as the increment of the independent variable.

The product of the derivative of a function and the differential of the independent variable is called the differential of the function.

The differential of a function is denoted by the symbol dy or $df(x)$:

$$dy \text{ or } df(x) = f'(x)dx. \qquad (6)$$

This formula gives us an expression for the derivative in the form of the quotient of the differentials:

$$f'(x) = \frac{dy}{dx} = \frac{df(x)}{dx}.$$

The differential of a function does not coincide with its increment. To explain the difference between these concepts, we turn to the graph of a function.

We take a certain point $M(x, y)$ on the graph, with a second point N. We draw the tangent MT, the ordinates corresponding to M and N, and the line MP parallel to axis OX (Fig. 5). We have: $\overline{MP} = \overline{M_1N_1} = \Delta x$ (or dx), $PN = \Delta y$ (increment of y), $\tan\angle PMQ = f'(x)$, whence

$$dy = f'(x)dx = \overline{MP}\tan\angle PMQ = \overline{PQ}.$$

The differential of the function, consisting of $\overline{PQ}$, does not coincide with $\overline{PN}$, which gives the increment of the function. The segment $\overline{PQ}$ gives the increment that would be obtained if we replaced segment $\overline{MN}$ of the curve in the interval $(x, x+dx)$ by the segment $\overline{MQ}$ of the tangent, i.e. if we took the increment of the function as proportional to the increment of the independent variable in this interval, the coefficient of proportionality being taken as equal to the slope of the tangent MT, or, what amounts to the same thing, to the derivative $f'(x)$.

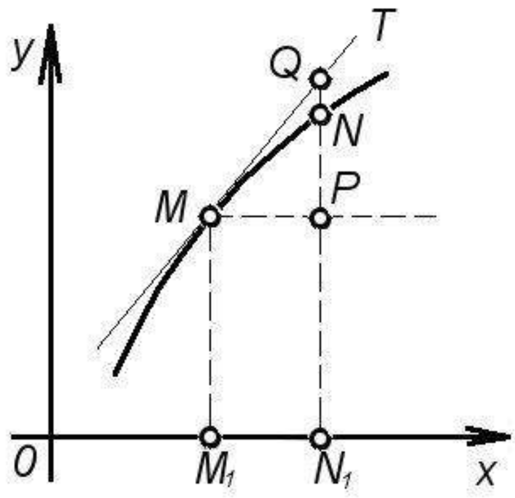

Fig. 5

The segment $\overline{NQ}$ gives the difference between the differential and the increment. We show that, if Δx tends to zero, this difference is an infinitesimal of higher order with respect to Δx .

The ratio $\Delta y/\Delta x$ gives the derivative in the limit, hence:

$$\frac{\Delta y}{\Delta x} = f^{/}(x) + \varepsilon \,,$$

where ε is an infinitesimal along with Δx. We get from this equation:

$$\Delta y = f^{/}(x)\Delta x + \varepsilon\Delta x$$

or

$$\Delta y = dy + \varepsilon\Delta x \,,$$

whence it is clear that the difference between dy and Δy is ($-\varepsilon\Delta x$).

But the ratio of ($-\varepsilon\Delta x$) to Δx, equal to ($-\varepsilon$), tends to zero along with Δx, i.e. the difference between dy and Δy is an infinitesimal of higher order with respect to Δx. We note that this difference can take either sign. Both Δx and the difference take the positive sign in our figure.

Formula (6) gives the rule for finding the differential of a function. We use it in some particular cases.

I. If c is a constant,

$$dc = (c)^{/}\, dx = 0 \cdot dx = 0 \,,$$

i.e. the differential of a constant is zero.

II.

$$d\left[cu(x)\right]=\left[cu(x)\right]' dx=cu'(x)dx=c\cdot du(x),$$

i.e. a constant factor can be taken outside the differentiation sign.

III.

$$d\left[u(x)+v(x)+w(x)\right]=\left[u(x)+v(x)+w(x)\right]' dx=$$
$$=\left[u'(x)+v'(x)+w'(x)\right]dx=u'(x)dx+v'(x)dx+w'(x)dx=du(x)+dv(x)+dw(x),$$

i.e. the differential of a sum is equal to the sum of the differentials of the terms.

IV.

$$d\left[u(x)v(x)w(x)\right]=\left[u(x)v(x)w(x)\right]' dx=$$
$$=u'(x)v(x)w(x)dx+u(x)v'(x)w(x)dx+u(x)v(x)w'(x)dx=$$
$$=du(x)v(x)w(x)+u(x)dv(x)w(x)+u(x)v(x)dw(x),$$

i.e. *the differential of a product is equal to the sum of the products of the differential of each with the remaining factors*.

We confine ourselves to the case of three factors. The same result is obtained for any finite number of factors.

V.

$$d\frac{u(x)}{v(x)}=\left[\frac{u(x)}{v(x)}\right]' dx=\frac{u'(x)v(x)dx-u(x)v'(x)dx}{\left[v(x)\right]^2},$$

i.e. *the differential of a quotient (fraction) is equal to the denominator times the differential of the numerator, minus the numerator times the differential of the denominator, all divided by the square of the denominator*.

VI. We consider a function of a function, $y=f(u)$, where u is a function of x. We find dy, assuming y dependent on x:

$$dy=y'_x dx=f'(u)\cdot u'_x dx=f'(u)du,$$

i.e. *the differential of a function of a function has the same form as would be obtained by treating the auxiliary function as independent variable*.

We consider a numerical example, so as to compare the magnitudes of the increment of a function and its differential.

We take the function:

$$y=f(x)=x^3+2x^2+4x+10$$

and consider its increment:

$$f(2.01)-f(2)=2.01^3+\ 2\cdot 2.01^2+\ 4\cdot 2.01+\ 10-\left(2^3+\ 2\cdot 2^2+\ 4\cdot 2+\ 10\right).$$

On performing all the operations, we obtain the magnitude of the increment:

$$\Delta y= f(2.01)-f(2)=0.240801.$$

Calculation of the differential is much easier. Here, $dx=2.10-2=0.01$, and the differential of the function is:

$$dy=3\left(x^2+4x+4\right)dx=\left(2\cdot 2^2+\ 4.2\ +\ 4\right)0.01=0.24.$$

On comparing Δy and dy, we see that they coincide to three decimal places.

Estimation of errors

When a magnitude x is found in practice or is roughly calculated, an error Δx is obtained, called the absolute error of the observation or calculation. It does not characterize the accuracy of tho observation. For instance, an error of about 1 cm in giving the length of a room would be permissible in practice, whereas the same error in determining the distance between two nearby objects (say the source and screen of a photometer) would point to great inaccuracy in the measurement. Hence follows the further concept of relative error, this being equal to the absolute value of the ratio $|\Delta x/x|$ of the absolute error to the measured magnitude.

We now suppose that a certain magnitude y is defined by the equation $y=f(x)$. An error Δx in defining x leads to a corresponding error Δy. For small values of Δx, Δy can be approximately replaced by the differential dy, so that the relative error in defining magnitude y is given by

$$\left|\frac{dy}{y}\right|.$$

Examples.

1. The current i is given with a tangent galvanometer by the well-known formula:

$$i=c\tan\varphi.$$

Let $d\varphi$ be the error in reading angle φ:

$$di=\frac{c}{\cos^2\varphi}d\varphi,\ \frac{di}{i}=\frac{c}{\cos^2\varphi(c\tan\varphi)}d\varphi=\frac{2}{\sin 2\varphi}d\varphi,$$

whence it is clear that the relative error $|di/i|$ in determining i will be smaller, the nearer φ is to 45°.

2. We take the product uv:

$$d(uv)=vdu+udv,\ \frac{d(uv)}{uv}=\frac{du}{u}+\frac{dv}{v},$$

whence it follows:

$$\left|\frac{d(uv)}{uv}\right|\le\left|\frac{du}{u}\right|+\left|\frac{dv}{v}\right|,$$

i.e. *the relative error of a product is not greater than the sum of the relative errors in the terms*.

The same rule is obtained for a quotient, since:

$$d\frac{u}{v}=\frac{vdu-udv}{v^2},\ \frac{d\frac{u}{v}}{\frac{u}{v}}=\frac{du}{u}-\frac{dv}{v};$$

$$\left|\frac{du\cdot v}{uv}\right|\le\left|\frac{du}{u}\right|+\left|\frac{dv}{v}\right|.$$

3. We consider the formula for the area of a circle:

$$Q=\pi r^2;\ dQ=2\pi rdr;\ \frac{dQ}{Q}=\frac{2\pi rdr}{\pi r^2}=2\frac{dr}{r},$$

i.e. the relative error in determining the area of a circle is by the above formula equal to twice the relative error in determining the radius.

4. We suppose that angle φ is defined by the logarithms of its sine and tangent. We have by the rules of differentiation:

$$d(\log_{10}\sin\varphi)=\frac{\cos\varphi d\varphi}{\ln 10\cdot\sin\varphi},\ d(\log_{10}\tan\varphi)=\frac{d\varphi}{\ln 10\cdot\tan\varphi\cdot\cos^2\varphi}$$

whence

$$d\varphi=\frac{\ln 10\cdot\sin\varphi}{\cos\varphi}d(\log_{10}\sin\varphi),\ \mathrm{d}\varphi=\ln 10\cdot\sin\varphi\cdot\cos\varphi\cdot d(\log_{10}\tan\varphi). \qquad (7)$$

We suppose that we make the same error in finding $\log_{10}\sin\varphi$ and $\log_{10}\tan\varphi$ (this error depends on the number of decimal places in the logarithmic tables that we use). The first of formulae (7) gives a value of $d\varphi$ of greater absolute value than that obtained with the second of formulae (7), since the product $\ln 10\sin\varphi$ is divided in the first case, and multiplied in the second case, by $\cos\varphi$, and $|\cos\varphi|<1$. It is thus better to calculate the angles using the table for $\log_{10}\tan\varphi$.

DERIVATIVES AND DIFFERENTIALS OF HIGHER ORDERS

Derivatives of higher orders

The derivative $f^{/}(x)$ of the function $y=f(x)$ is also a function of x, as we know. On differentiating it, we obtain a new function, called the second derivative or the derivative of the second order of the original function $f(x)$, and denoted by:

$$y^{//} \text{ or } f^{//}(x).$$

On differentiating the second derivative, we obtain the derivative of the third order, or simply, the third derivative:

$$y^{//} \text{ or } f^{//}(x).$$

Using the operation of differentiation in this way, we obtain the derivative of any order n, $y^{(n)}$ or $f^{(n)}(x)$.

We consider some examples.

1. $y=e^{ax}$, $y^{/}=ae^{ax}$, $y=a^2e^{ax}$, ..., $y=a^ne^{ax}$.
2. $y=(ax+b)^k$, $y^{/}=ak(ax+b)^{k-1}$, $y^{//}=a^2k(k-1)(ax+b)^{k-2}$, ...,

$$y^{(n)}=a^nk(k-1)(k-2)...(k-n+1)(ax+b)^{k-n}.$$

3. We know that:

$$(\sin x)^{/}=\cos x=\sin\left(x+\frac{\pi}{2}\right);\ (\cos x)^{/}=-\sin x=\cos\left(x+\frac{\pi}{2}\right),$$

i.e. differentiation of $\sin x$ and $\cos x$ amounts to increasing the argument by $\frac{\pi}{2}$, and hence:

$$(\sin x)^{//}=\left[\sin\left(x+\frac{\pi}{2}\right)\right]^{/}=\sin\left(x+2\frac{\pi}{2}\right)\left(x+\frac{\pi}{2}\right)^{/}=\sin\left(x+2\frac{\pi}{2}\right),$$

and in general:

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right) \text{ and } (\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right).$$

4. $y = \log(1+x),\ y' = \frac{1}{1+x},\ y'' = -\frac{1}{(1+x)^2},\ y''' = \frac{1\cdot 2}{(1+x)^3},\ \ldots,\ y^{(n)} = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$.

5. We consider the sum of functions:

$$y = u + v + w.$$

Using the rule for differentiation of a sum, and assuming that the corresponding derivatives of functions u, v and w exist, we have:

$$y = u + v + w,\ y' = u' + v' + w',\ y'' = u'' + v'' + w'',\ \ldots,\ y^{(n)} = u^{(n)} + v^{(n)} + w^{(n)},$$

i.e. the derivative of any order of a sum is equal to the sum of the derivatives of the same order. For example:

$$y = x^3 - 4x^2 + 7x + 10;\ y' = 3x^2 - 8x + 7;\ y'' = 6x - 8;\ y''' = 6;\ y^{(4)} = 0$$

and generally, $y^{(n)} = 0$ for $n > 3$.

It can be shown in the same way that, in general, *the n-th derivative of a polynomial of degree m is zero if $n > m$*.

We now consider the derivatives of the product of two functions $y = uv$. We use the rules for differentiation of a product and a sum:

$$y' = u'v + uv';\ y'' = u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv'';$$
$$y''' = u'''v + 3u''v' + 3u'v'' + uv'''.$$

We note the following rule: *in order to obtain the n-th derivative of the product uv, $(u+v)^n$ should be determined by Newton's binomial formula, and the exponents of powers of u and v in the result should be taken to indicate the orders of derivatives, the zero powers ($u^0 = v^0 = 1$) appearing in the extreme terms of the result being taken as the functions themselves*.

This rule is known as Leibniz's theorem, and is written symbolically as:

$$y^{(n)} = (u + v)^{(n)}.$$

We prove this theorem by induction. We suppose that the rule is true for the n-th derivative, i.e:

$$y^{(n)}=(u+v)^{(n)}=u^{(n)}v\ +\ \frac{n}{1}u^{(n-1)}v^{/}\ +\ \frac{n(n-1)}{2!}u^{(n-2)}v^{//}\ +\ ...$$
$$...\ +\ \frac{n(n-1)...(n-k+1)}{k!}u^{(n-k)}v^{(k)}+\ ...\ +\ uv^{(n)}. \qquad (8)$$

To obtain $y^{(n+1)}$, we differentiate the above sum with respect to x. The general term $u^{(n-k)}v^{(k)}$ of the sum gives, by the rule for differentiation of a product, the derivative $u^{(n-k+1)}v^{(k)}+u^{(n-k)}v^{(k+1)}$. But this latter sum can be written symbolically as:

$$u^{n-k}v^{k}(u+v).$$

In fact, we have regarded the orders of the derivatives as indices and we have factorized the expression

$$u^{(n-k+1)}v^{(k)}+u^{(n-k)}v^{(k+1)}.$$

Hence we see that $y^{(n+1)}$ must be obtained by multiplying each term of the sum (1), and hence all the sum, symbolically by $(u+v)$, so that:

$$y^{(n+1)}=(u+v)^{(n)}\cdot(u+v)=(u+v)^{(n+1)}.$$

We have shown that if Leibniz's theorem is true for any given n, it is true for $(n+1)$. But we have shown directly that it is true for $n=1,\ 2,\ 3$; and hence it is true for all n.

We take as an example:

$$y=e^{x}\left(3x^{2}-1\right)$$

and find $y^{(100)}$:

$$y^{(100)}=\left(e^{x}\right)^{(100)}\left(3x^{2}-1\right)\ +\ \frac{100}{1}\left(e^{x}\right)^{(99)}\left(3x^{2}-1\right)^{/}\ +\ \frac{100\cdot 99}{1\cdot 2}\left(e^{x}\right)^{(98)}\left(3x^{2}-1\right)^{//}\ +\ ...$$
$$...\ +\ \left(e^{x}\right)\left(3x^{2}-1\right)^{(100)}.$$

All the derivatives of a polynomial of the second degree are zero as from the third order, and $\left(e^{x}\right)^{(n)}=e^{x}$, whence:

$$y^{(100)} = e^x\left(3x^2-1\right)+100\,e^x\cdot 6x+4950\,e^x\cdot 6 = e^x\left(3x^2+600x+29,699\right).$$

Mechanical significance of the second derivative

We consider the motion of a point on a straight line:

$$s = f(t),$$

where, as usual, t denotes time, and s is the path, measured from some fixed point of the line. We obtain the velocity of the motion by differentiating once with respect to t:

$$v = f'(t).$$

We obtain the second derivative as the limit of the ratio $\Delta v/\Delta t$ as Δt tends to zero. This ratio characterizes the rate of change of velocity in the interval Δt, and gives the average acceleration in this interval, whilst the limit of the ratio as $\Delta t \to 0$ gives the acceleration w of the observed motion at time t:

$$w = f''(t).$$

We take $f(t)$ as a polynomial of the second degree:

$$s = at^2+bt+c,\ v = 2at+b,\ w = 2a\ ,$$

i.e. the acceleration w is constant, and the coefficient $a = \frac{w}{2}$. setting $t = 0$, we get $b = v_0$, i.e. the coefficient b is equal to the initial velocity, and $c = s_0$, i.e. c is equal to the distance of the point at $t = 0$ from the origin on the line. On putting these values for $a, b,$ and c in the expression for s, we get the formula for the path with uniformly accelerated $(w > 0)$ or uniformly decelerated $(w < 0)$ motion:

$$s = \frac{1}{2}wt^2+v_0t+s_0.$$

In general, on knowing the law for the change of path, we can find the acceleration w by differentiating twice with respect to t, and hence find the force f producing the motion, since, by Newton's second law, $f = mw$, where m is the mass of the moving point.

All the above applies only to linear motion. In the case of curvilinear motion, as is shown in mechanics, $f^{//}(t)$ gives only the projection of the acceleration vector on the tangent to the trajectory.

We take the example of a point M oscillating harmonically on a line, so that its distance s from a fixed point O of the line is defined by the formula:

$$s = a\sin\left(\frac{2\pi}{\tau}t + \omega\right),$$

where the amplitude a, the period of oscillation τ, and the phase ω are constants. Differentiations give us the velocity v and the force f:

$$v = \frac{2\pi a}{\tau}\cos\left(\frac{2\pi}{\tau}t + \omega\right),\ f = mw = -\frac{4\pi^2 m}{\tau^2}a\sin\left(\frac{2\pi}{\tau}t + \omega\right) = -\frac{4\pi^2 m}{\tau^2}s,$$

i.e. the force is proportional in magnitude to the length of the interval $\overline{OM}$ and acts in the opposite direction. In other words, the force is always directed from the point M to the point O, being proportional to their distance apart.

Differentials of higher orders

We now introduce the concept of higher order differentials of a function $y = f(x)$. Its differentail

$$dy = f^{/}(x)dx$$

is clearly a function of x, though it must be remembered that the differential dx; of the independent variable is reckoned as independent of x, so that it must be taken outside the differentiation sign as a constant factor on further differentiation. The differential of dy can be obtained by treating it as a function of x; this is called the second order differential of the original function $f(x)$ and is denoted by d^2y or $d^2 f(x)$:

$$d^2y = d(dy) = \left[f^{/}(x)dx\right]^{/}dx = f^{//}(x)dx^2.$$

On obtaining the differential of this further function of x, we arrive at the third order differential:

$$d^3y = d\left(d^2y\right) = \left[f^{//}(x)dx^2\right]^{/}dx = f^{///}(x)dx^3,$$

and in general, we arrive by successive differentiation at the concept of the n-th order differential of function $f(x)$, which is expressed as

$$d^n f(x) \text{ or } d^n y = f^{(n)}(x) dx^n . \tag{9}$$

This formula allows of the expression of the n-th derivative as a fraction:

$$f^{(n)}(x) = \frac{d^n y}{dx^n} . \tag{10}$$

We now consider a function of a function, $y = f(u)$, where u is a function of some independent variable. We know that the first differential of this function has the same form as when u is the independent variable:

$$dy = f'(u) du .$$

Formula (9) is no longer valid for obtaining higher order differentials, since we are not justified in treating du as a constant when u is not the independent variable. For example, we obtain the second differential by using the rule for finding the differential of a product, with the result:

$$d^2 y = d\left[f'(u) du \right] = du \cdot d\left[f'(u) \right] + f'(u) \cdot d(du) = f''(u) du^2 + f'(u) d^2 u ,$$

which has the extra term $f'(u) d^2 u$, compared with formula (9).

If u is the independent variable, du must be treated as constant and $d^2 u = 0$. We now take u as a linear function of the independent variable t, i.e.,

$$u = at + b .$$

Now, $du = adt$, i.e., du is again a constant, so that the higher order differentials of the function of a function are given by (9):

$$d^n f(u) = f^{(n)}(u) du^n ,$$

i.e. *formula (9) for the higher order differentials is valid when x is either the independent variable, or a linear function of the independent variable*.

Finite differences of functions

We denote the increment of the independent variable by h. The corresponding increment of $y = f(x)$ will be:

$$\Delta y = f(x+h) - f(x). \tag{11}$$

An alternative name is *the first-order difference* of function $f(x)$. This difference is, for its part, also a function of x, and we can find its difference by subtracting from its value at $(x+h)$ its value at x. This new difference is called the *second-order difference* of the original function $f(x)$ and is denoted by $\Delta^2 y$. We can easily express $\Delta^2 y$ in terms of values of $f(x)$:

$$\Delta^2 y = \left[f(x+2h) - f(x+h) \right] - \left[f(x+h) - f(x) \right] = f(x+2h) - 2f(x+h) + f(x). \tag{12}$$

This second order difference is also a function of x, and the difference of this function can be defined, giving the *third-order difference* of the original function $f(x)$, denoted by $\Delta^3 y$. Replacing x by $(x+h)$ on the right of (12), and subtracting the right of (12) from the result, we obtain the expression for $\Delta^3 y$:

$$\Delta^3 y = \left[f(x+3h) - 2f(x+2h) + f(x+h) \right] - \left[f(x+2h) - 2f(x+h) + f(x) \right] =$$
$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).$$

We can thus go on to define the difference of any order, the n-th order difference $\Delta^n y$ being expressed as follows in terms of values of $f(x)$:

$$\Delta^n y = f(x+nh) - \frac{n}{1!} f\left(x+(n-1)h\right) + \frac{n(n-1)}{2!} f\left(x+(n-2)h\right) - \ldots$$
$$\ldots + (-1)^k \frac{n(n-1)\ \ldots\ (n-k+1)}{k!} f\left(x+(n-k)h\right) + \ \ldots\ + (-1)^n f(x). \tag{13}$$

We have seen above that this formula is true for n= 1, 2, and 3. A rigorous proof requires us to pass from n to $(n+1)$ in the usual way. We note that $(n+1)$ values of $f(x)$, for the values of the argument: $x,\ x+h,\ x+2h,\ \ldots,\ x+nh$, are needed for calculating $\Delta^n y$.

These values of the argument form an arithmetical progression with difference h, or as we say, represent equidistant values.

For small h, Δy differs little from the differential dy. Similarly, higher-order differences give approximate values of the differentials of corresponding order, and conversely. Whilst we lack an analytic expression for a function, we may be given a table for it for equidistant values of the argument; we cannot then calculate the various derivatives of the function accurately, but we can obtain approximate values by calculating the ratio $\Delta^n y / \Delta x^n$, instead of using the accurate formula (10). As an example, we give a table of differences and differentials of the function $y = x^3$ in the interval $(2,3)$, taking:

$$\Delta x = h = 0.1.$$

In setting up this table, the successive values of $y = x^3$ were calculated, then the values of Δy obtained from these, by subtraction in accordance with formula (11), then the values of $\Delta^2 y$ obtained from the values of Δy by further subtractions, and so on. This method of calculating the differences successively is naturally simpler than using formula (13). The differentials were calculated in the ordinary way, the formulae being given at the top of the table, where we must take $dx = h = 0.1$.

We compare the accurate and approximate values of the second derivative $y^{//}$ for $x = 2$. Here, $y^{//} = 6x$ and $y^{//} = 12$ with $x = 2$. The approximate value is given by the ratio $\Delta^2 y / h^2$, and we have for $x = 2$:

$$\frac{0.126}{(0.1)^2} = 12.6 .$$

Table 1

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	dy	$d^2 y$	$d^3 y$	$d^4 y = 0$
2	8	1.261	0.126	0.006	0	1.2	0.12	0.006	0
2.1	9.261	1.387	0.132	0.006	0	1.323	0.126	0.006	0
2.2	10.648	1.519	0.138	0.006	0	1.452	0.132	0.006	0
2.3	12.167	1.657	0.144	0.006	0	1.587	0.138	0.006	0
2.4	13.824	1.801	0.15	0.006	0	1.728	0.144	0.006	0
2.5	15.625	1.951	0.156	0.006	0	1.875	0.15	0.006	0
2.6	17.576	2.107	0.162	0.006	0	2.028	0.156	0.006	0
2.7	19.683	2.269	0.168	0.006	–	2.187	0.162	0.006	–
2.8	21.952	2.437	0.174	–	–	2.352	0.168	–	–
2.9	24.389	2.611	–	–	–	2.523		–	–
3	27	–	–	–	–			–	–

If $f(x)$ is a polynomial of x:

$$y = f(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ \dots \ + a_{m-1} x + a_m ,$$

it is easily seen that, on calculating Δy by (11), we get a polynomial of degree $(m-1)$ for Δy, with highest term $m a_0 h x^{m-1}$. Thus, with $y = x^3, \Delta y$ is a polynomial of second degree in x, $\Delta^2 y$ is a polynomial of first degree, $\Delta^3 y$ is constant, and $\Delta^4 y$ is zero (see table). It is suggested as an exercise to the reader to show that the values of $d^2 y$ must be one step behind those of $\Delta^2 y$ in our example, as is obvious from the table.

APPLICATION OF DERIVATIVES TO THE STUDY OF FUNCTIONS

Tests for increasing and decreasing functions

Knowledge of the derivative enables us to study the various properties of a function. We begin with the simplest and most basic question, of whether a function is increasing or decreasing.

The function $f(x)$ *is said to be increasing in an interval if it increases correspondingly with the variable thraughout that interval*, i.e. *if*

$$f(x+h)-f(x)>0 \text{ for } h>0.$$

On the other hand, if

$$f(x+h)-f(x)<0 \text{ for } h>0,$$

the function is said to be decreasing.

Turning to the graph of the function, the interval in which the function is increasing corresponds to the section of the graph for which greater values of x imply greater values of y. If OX is directed to the right, and OY upwards, as in Fig. 6, the interval of increase of the function will correspond to the part of the graph where movement along it to the right in the direction of increasing abscissa implies movement upwards. An interval of decrease, on the other hand, corresponds to a part of the curve where we move downwards on moving along the curve to the right.

In Fig. 6, the interval of increase corresponds to section AB of the graph, and the interval of decrease to BC. It is immediately clear from the figure that in the first section, the tangent forms with the direction of OX an angle α measured from OX to the tangent,
the tangent of which is positive but the tangent of this angle is also the first derivative $f'(x)$. In section BC, on the contrary, the direction of the tangent forms with the direction of OX an angle α (in the fourth quadrant), the tangent of which is negative, i.e. $f'(x)$ is negative in this case. Putting these results together, we arrive at the following rule:
a function is increasing in the intervals for which $f'(x)$ *is positive, and is decreasing in intervals where* $f'(x)<0$.

We have arrived at this rule by using the figure. A rigorous analytic proof is given later. We now make use of the rule in some examples.

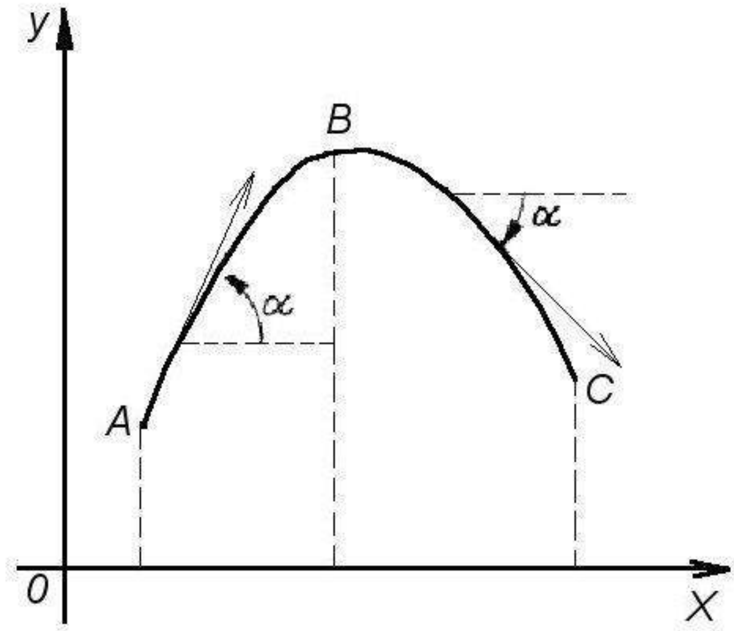

Fig. 6

1. We prove the inequality:

$$\sin x > x - \frac{x^3}{6} \text{ for } x > 0 .$$

For this, we form the difference:

$$f(x) = \sin x - \left(x - \frac{x^3}{6} \right).$$

We find the derivative $f'(x)$:

$$f'(x) = \cos x - 1 + \frac{x^2}{2} = \frac{x^2}{2} - (1 - \cos) = \frac{x^2}{2} - 2\sin^2 \frac{x}{2} =$$
$$= 2\left[\left(\frac{x}{2} \right)^2 - \left(\sin \frac{x}{2} \right)^2 \right].$$

Since the absolute value of the arc is greater than its sine, we can say that $f'(x) > 0$ in the interval $(0,\infty)$, i.e. $f(x)$ is increasing in this interval; and $f(0) = 0$, so that:

$$f(x) = \sin x - \left(x - \frac{x^3}{6} \right) > 0 \text{ for } x > 0,$$

i.e.

$$\sin x > x - \frac{x^3}{6} \quad \text{for} \quad x > 0 \text{.}$$

2. It can be shown in exactly the same way that:

$$x > \log(1+x) \quad \text{for} \quad x > 0 \text{.}$$

We form the difference:

$$f(x) = x - \log(1+x) \text{,}$$

whence

$$f'(x) = 1 - \frac{1}{(1+x)} \text{.}$$

It is evident from this expression that $f'(x) > 0$ for $x > 0$, i.e. $f(x)$ is increasing in the interval $(0, +\infty)$; but $f(0) = 0$, so that:

$$f(x) = x - \log(1+x) > 0 \quad \text{for} \quad x > 0 \text{,}$$

i.e.

$$x > \log(1+x) \quad \text{for} \quad x > 0 \text{.}$$

3. We take the equation:

$$f(x) = 3x^5 - 25x^3 + 60x + 15 = 0 \text{.}$$

We find the derivative $f'(x)$ and set it equal to zero:

$$f(x) = 15x^4 - 75x^2 + 60 = 15\left(x^4 - 5x^2 + 4\right) = 0 \text{.}$$

We solve this biquadratic equation and find that $f'(x)$ is zero for

$$x = -2, \ -1, \ +1, \ +2 \text{.}$$

We can now divide the total interval $(-\infty, +\infty)$ into five intervals:

$$(-\infty, \ +2), \ (-2, \ -1), \ (-1, \ +1), \ (+1, \ +2), \ (+2, \ +\infty) \text{,}$$

so that $f'(x)$ has the same sign in each, $f(x)$ being therefore monotonic, i.e. either increasing or decreasing, and hence having not more than One root in each interval. If $f(x)$ has different signs at the ends of a given interval, $f(x)=0$ has one root in the interval; whereas if it has the same sign, there is no root in the interval. Thus, by finding the signs of $f(x)$ at the ends of the five intervals above, we can find the number of roots of the equation.

To find the signs of $f(x)$ at $x=\pm\infty$, we write $f(x)$ in the form:

$$f(x)=x^5\left(3-\frac{25}{x^2}+\frac{60}{x^4}+\frac{15}{x^5}\right).$$

For $x\to-\infty$, $x^5\to-\infty$, and the expression in brackets tends to 3, so $f(x)\to-\infty$. Similarly it can be seen that $f(x)\to+\infty$ for $x\to+\infty$. On substituting the values $x=-2,-1,+1,+2$, we get the following table 2.

Table 2

x	$-\infty$	-2	-1	1	2	$+\infty$
$f(x)$	$-$	$-$	$-$	$+$	$+$	$+$

We see that $f(x)$ has different signs only at the ends of the interval $(-1,1)$, so that the equation concernedhas only one realroot,which lies in this interval.

We defined above a function increasing or decreasing in an interval. A function is sometimes said to be increasing or decreasing at a point $x=x_0$. The exact meaning is: $f(x)$ is increasing at $x=x_0$, if $f(x)<f(x_0)$ for $x<x_0$, and $f(x)>f(x_0)$ for $x>x_0$, x being taken sufficiently close to x_0. Similarly for a function decreasing at a point. The concept of derivative leads directly to a sufficient condition for a function to be increasing or decreasing at a point, if $f'(x_0)>0$, $f(x)$ is increasing at x_0, and if $f'(x_0)<0$, $f(x)$ is decreasing at x_0.
If, e. g., $f'(x_0)>0$, the ratio:

$$\frac{f(x_0+h)-f(x_0)}{h}.$$

with limit $f'(x_0)$, will also be positive for all h with sufficiently small absolute values, i.e. numerator and denominator will have the same sign. In other words, we shall have $f(x_0+h)-f(x_0)>0$ for $h>0$, and $f(x_0+h)-f(x_0)<0$ for $h<0$, so that $f(x)$ is increasing at x_0.

Maxima and minima of functions

We again turn to the graph of some function $f(x)$ (Fig. 7).

We have a successive alternation of intervals of increase and decrease in this case. Segment AM_1 corresponds to an interval of increase, the next segment M_1M_2 to one of decrease, the next, M_2M_3 again to one of increase, and so on. Peaks of the curve separate intervals of increase from those of decrease. Take the peak M_1 for example; its ordinate is greater than all the ordinates of the curve that are sufficiently close to the peak, whether to the left or right. Such a peak is said to correspond to a maximum of $f(x)$.

This leads us to the following general analytic definition: *the function* $f(x)$ *attains a maximum at the point* $x=x_1$ *if its value* $f(x_1)$ *at this point is greater than its values at all neighbouring points, i.e. if the increment of the function*

$$f(x_1+h)-f(x_1)<0$$

for all positive or negative h*, sufficiently small in absolute value.*

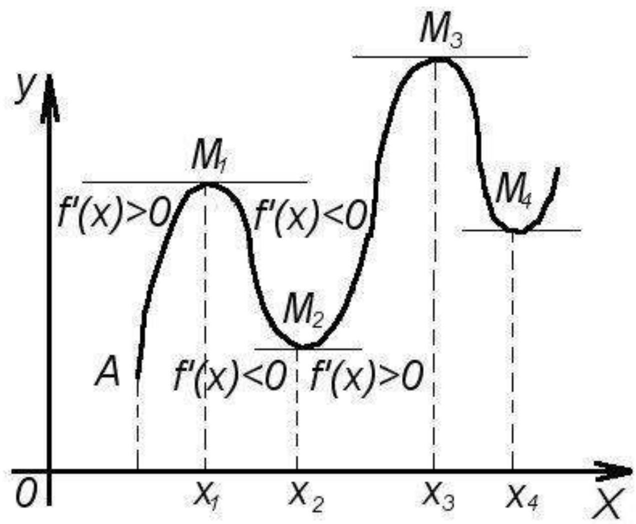

Fig. 7

We now consider the peak M_2. The ordinate in this case is less than all neighbouring ordinates, whether on the left or right; the peak is said to correspond to a minimum of the function, which is defined analytically as follows: *the function* $f(x)$ *attains a minimum at the point* $x=x_2$*, if we have*

$$f(x_2 + h) - f(x_2) > 0$$

for all positive or negative h, sufficiently small in absolute value.

We see from the figure that the tangents at peaks corresponding to either maxima or minima of the function $f(x)$ lie parallel to the axis OX, i.e. their slope $f'(x)$ is zero. The tangent to a curve can be parallel to OX, however, elsewhere than at a peak.

In Fig. 8, for example, point M of the curve is not a peak, yet the tangent at M is parallel to OX.

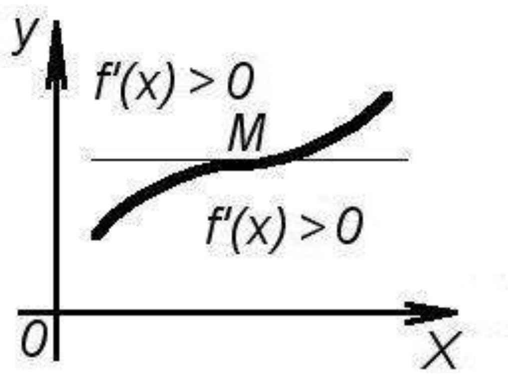

Fig. 8

Suppose that $f'(x)$ becomes zero for some value $x = x_0$, i.e. the tangent at the corresponding point of the graph is parallel to OX.
We consider the sign of $f'(x)$ for x near x_0.

We take the following three cases:

I. For x less than, and sufficiently near x_0, $f'(x)$ is positive, whilst $f'(x)$ is negative for x greater than, and sufficiently near x_0, i.e. in other words, as x passes through x_0, $f'(x)$ passes through zero from positive to negative values.

In this case, we have an interval of increase to the left of $x = x_0$, and an interval of decrease to the right, i.e. $x = x_0$ corresponds to a peak of the curve, where $f(x)$ is a maximum (see Fig. 7).

II. $f'(x)$ is negative for x less than x_0, and positive for x greater than x_0, i.e. $f'(x)$ passes through zero from negative to positive values.

In this case, we have an interval of decrease to the left of $x = x_0$, and one of increase to the right, i.e. $x = x_0$ corresponds to a peak of the curve that gives a minimum of the function (see Fig. 7).

III. $f'(x)$ has the same sign for x either less than or greater than x_0. Suppose, for instance, that the sign is (+).

The corresponding point of the graph in this case lies in an interval of increase, and is clearly not a peak (see Fig. 8).

The above remarks lead us to the following rules for finding the values of x for which $f(x)$ has a maximum or minimum:

1) find $f'(x)$;

2) find the x for which $f'(x)$ is zero, i.e. solve the equation $f'(x)=0$;

3) find how the sign of $f'(x)$ varies on passing through these values, using the following arrangement (Table 3).

Table 3

x	x_0-h	x_0	x_0+h	$f(x)$
$f'(x)$	+	0	–	max
	–		+	min
	+		–	increasing
	–		+	decreasing

In the above table x_0-h and x_0+h indicate that the sign of $f'(x)$ is to be determined for x less than, and greater than, x_0, and always sufficiently close to x_0 so that h is taken as a sufficiently small positive number.

It is assumed in this investigation that $f'(x)=0$, whilst $f'(x)$ differs from zero for all x sufficiently near, but not coinciding with, x_0.

We return to the fact that the tangent at M in Fig. 8 (with abscissa x_0) lies on different sides of the curve in the neighbourhood of M. Here, $f'(x_0)=0$ and $f'(x)>0$ for all x near to, but not at, x_0, all sections of the curve with x_0 as an interior point giving intervals of increase, notwithstanding the fact that $f'(x_0)=0$.

Sometimes a rather different definition from the above is given of a maximum, the function $f(x)$ has a maximum at $x=x_0$ if $f(x_1)$ is not less than the values of $f(x)$ at neighbouring points, i.e. if we have for the increment of the function, $f(x_1+h)-f(x_1)<0$, for all h sufficiently small in absolute value, and either positive or negative. A minimum at the point $x=x_2$ can be similarly defined by the inequality $f(x_2+h)-f(x_2)\geq 0$. If the function in this definition has a derivative at its maximum or minimum, this derivative must be zero, as above.

To take an example, let it be required to find the maxima and minima of the function

$$f(x)=(x-1)^2(x-2)^3.$$

We obtain the first derivative:

$$f(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2 =$$
$$= (x-1)(x-2)^2(5x-7) = 5(x-1)(x-2)^2\left(x-\frac{7}{5}\right).$$

It is clear from the last expression that $f'(x)$ is zero for the following values of the independent variable: $x_1 = 1,\ x_2 = 7/5,\ x_3 = 2$.

We now investigate these. For $x=1$, the factor $(x-2)^2$ has a plus sign, and $(x-7/5)$ a minus sign. For all x sufficiently close to unity, and either greater than or less than unity, each of these factors keeps the same sign, so that their product is unconditionally minus for all x sufficiently near unity. We finally consider the factor $(x-1)$, which is in fact zero at $x=1$. It is negative for $x<1$, and positive for $x>1$. Thus the complete derivative, i.e. $f'(x)$, has a plus sign for $x<1$ and a minus sign for $x>1$. Hence it follows that $x=1$ represents a maximum of the function $f(x)$. We set $x=1$ in the expression for $f(x)$ itself, thus obtaining the value of the maximum, i.e. the ordinate of the corresponding peak of the graph of $f(x)$:

$$f(1) = 0^2 \cdot (-1)^3 = 0.$$

By using the same sort of argument for the other values $x_2 = 7/5$ and $x_3 = 2$, we obtain the following table 4.

Table 4

x	$1-h$	1	$1+h$	$\frac{7}{5}-h$	$\frac{7}{5}$	$\frac{7}{5}+h$	$2-h$	2	$2+h$
$f'(x)$	+	0	–	–	0	+	+	0	+
$f(x)$	increase	0, maximum	decrease		$-\frac{108}{3125}$, minimum	increase			

The method we have outlined for studying the maxima and minima of a function may have the drawback, especially in more complicated examples, that it is not too easy to find the sign of $f'(x)$ for x both greater than, and less than, the value in question. The trouble may be avoided in many cases by taking the second derivative $f''(x)$ into account. Suppose we consider $x = x_0$, for which $f'(x_0) = 0$. We shall assume that, on substituting $x = x_0$ in the expression for the second derivative, we obtain a positive value, i.e. $f''(x_0) > 0$. If $f'(x)$ is taken as the basic function, the fact that its derivative $f''(x)$ is positive at $x = x_0$

means that the basic function itself, $f'(x)$, is increasing at this point, i.e. $f'(x)$ passes from negative to positive values at its zero-point x_0. Thus, *for* $f''(x_0)>0$ *at the point* $x=x_0$, $f(x)$ *will have a minimum*. *It can similarly be shown that, for* $f''(x_0)<0$ *at* $x=x_0$, $f(x)$ *has a maximum*. Finally, if we get zero on substituting $x=x_0$ in the expression for $f''(x)$, i.e. $f''(x_0)=0$, this prevents us from using the second derivative to investigate $x=x_0$, and we must turn back to direct consideration of the sign of $f'(x)$. The arrangement shown in the table 5 is thus obtained.

Table 5

x	$f'(x)$	$f''(x)$	$f(x)$
x_0	0	–	maximum
		+	minimum
		0	doubtful case

It follows directly from the above discussion that, given the existence of the second derivative, a necessary condition for a maximum is $f''(x)\le 0$, and a necessary condition for a minimum is $f''(x)\ge 0$. We can determine the maximum here by the condition $f(x_1+h)-f(x_1)\le 0$, and the minimum by $f(x_2+h)-f(x_2)\ge 0$, as described above.

Example. It is required to find the maxima and minima of the function

$$f(x)=\sin x+\cos x.$$

This function has period 2π, i.e. remains unchanged on substituting $x+2\pi$ for x.

It is sufficient to consider x varying in the interval $(0,2\pi)$.

We find the first and second derivatives:

$$f'(x)=\cos x-\sin x;\ f''(x)=-\sin x-\cos x.$$

We put the first derivative equal to zero, obtaining the equation:

$$\cos x-\sin x=0 \text{ or } \tan x=1.$$

The roots of this equation in $(0,2\pi)$ are:

$$x_1 = \frac{\pi}{4} \text{ and } x_2 = \frac{5\pi}{4}.$$

We investigate these values by finding the sign of $f''(x)$:

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2} < 0\text{; maximum } f\left(\frac{\pi}{4}\right) = \sqrt{2};$$

$$f''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = \sqrt{2} > 0\text{; minimum } f\left(\frac{5\pi}{4}\right) = -\sqrt{2}.$$

We note in conclusion a point that sometimes arises in finding maxima and minima. Points may occur on the graph of the function where there is either no tangent at all, or the tangent is parallel to OY (Fig. 9). The derivative $f'(x)$ will not exist at points of the first kind, whilst it is infinite at points of the second kind, since the slope of a line parallel to OY is infinity. As is evident at once from the figure, however, these points can represent maxima or minima of the function. The above rule for finding maxima and minima should thus be amplified as follows, strictly speaking: *maxima and minima of a function $f(x)$ can occur not only at points where $f'(x)$ is zero, but also at points where $f'(x)$ either does not exist or becomes infinite*. Investigation of these latter points must be carried out by the first of the arrangements given above, i.e. by finding the sign of $f'(x)$ for x less than, and greater than, the value in question.

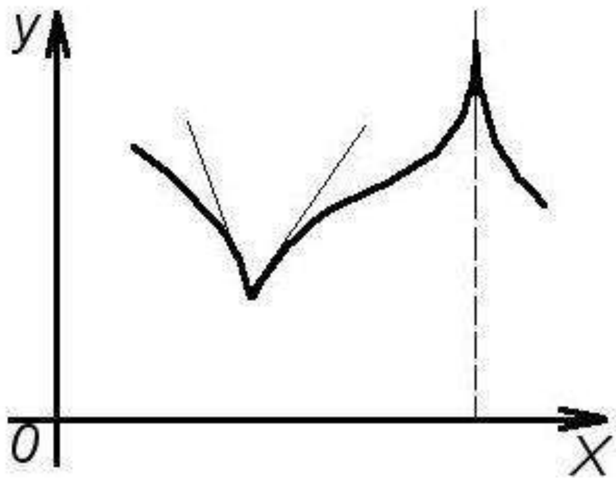

Fig. 9

Example. It is required to find the maxima and minima of $f(x) = (x-1)\sqrt[3]{x^2}$
We find the first derivative:

$$f'(x) = \sqrt[3]{x^2} + \frac{2(x-1)}{3\sqrt[3]{x}} = \frac{5}{3}\frac{x - \frac{2}{5}}{\sqrt[3]{x}}.$$

This is zero for $x = 2/5$ and infinity for $x = 0$.
We consider the latter value: the numerator of the above fraction is minus for zero x, and also minus for all positive or negative x that are near zero. The denominator of the fraction is negative for $x < 0$, and positive for $x > 0$. The fraction as a whole is therefore positive for x less than, and near, zero, and negative for $x > 0$, i.e. we have a maximum at $x = 0$, $f(0) = 0$. We have a minimum at $x = 2/5$:

$$f\left(\frac{2}{5}\right) = -\frac{3}{5}\sqrt[3]{\frac{4}{25}} = -\frac{3}{25}\sqrt[3]{20}.$$

Curve tracing

Finding the maxima and minima of a function $f(x)$ considerably simplifies the problem of tracing its curve. This is explained by means of a few examples.

1. Let it be required to trace the graph of:

$$y = (x-1)^2 (x-2)^3,$$

which we considered in a previous paragraph. We then obtained two peaks of the function: a maximum $(1,0)$ and a minimum $(7/5, -108/3125)$. We fill in these points in the figure. It is also useful to mark the intercepts of the curve on the axes. We have for $x = 0$, $y = -8$, i.e. the point on axis OY is $y = -8$.
We put y equal to zero, i.e.,

$$(x-1)^2 (x-2)^3 = 0,$$

and obtain the points on axis OX. One of these, $x = 1$, is a peak, as already explained, whilst the second, $x = 2$, is not a peak, as was also explained earlier, but corresponds to a point of the graph where the tangent is parallel to OX. The required curve is shown in Fig. 10.

2. We trace the curve:

$$y = e^{-x^2}.$$

We find the first derivative:

$$y' = -2xe^{-x^2}.$$

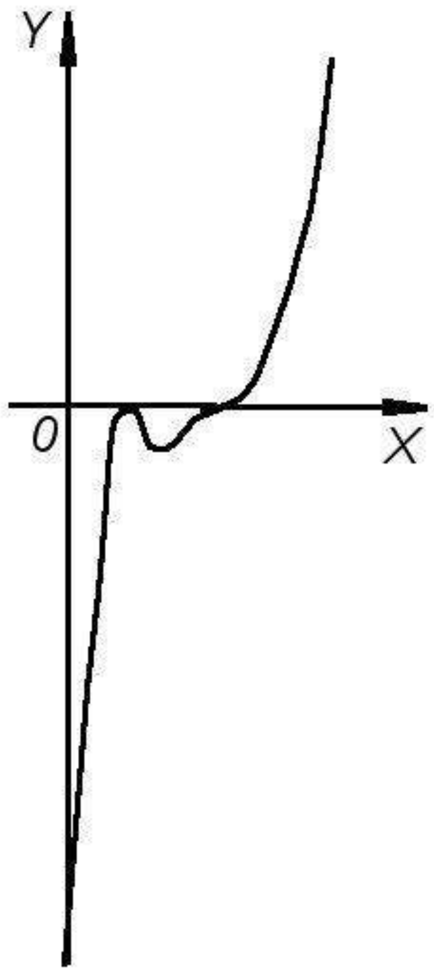

Fig. 10

Putting $y^{/}=0$, we get the value $x=0$, corresponding to a peak (maximum) of the curve, as is easily seen, with ordinate $y=1$. This also gives us the point of the curve on OY. Putting y zero, we get the equation $e^{-x^2}=0$, which has no solution, i.e. the curve has no points on OX. But we note that as x tends to $(+\infty)$ or $(-\infty)$, the power in e^{-x^2} tends to $(-\infty)$ and the expression as a whole tends to zero, i.e. on infinite displacement to left or right, the curve indefinitely approaches the axis OX.

The curve is shown in Fig. 11, in accordance with the data obtained.

3. We draw the curve

$$y=e^{-ax}\sin bx \quad (a>0),$$

representing a damped oscillation. The factor $\sin bx$ does not exceed unity in absolute value, and the graph as a whole is situated between the two curves:

$$y=e^{-ax} \text{ and } y=-e^{-ax}.$$

As x tends to $(+\infty)$, factor e^{-ax}, and hence the product $e^{-ax}\sin bx$ as a whole, tends to zero, i.e. the curve approaches OX asymptotically on infinite displacement to the right. The points of the curve on OX are given by the equation

$$\sin bx = 0\,,$$

so that they are

$$x = \frac{k\pi}{b}\,,\ (k \text{ is an integer}).$$

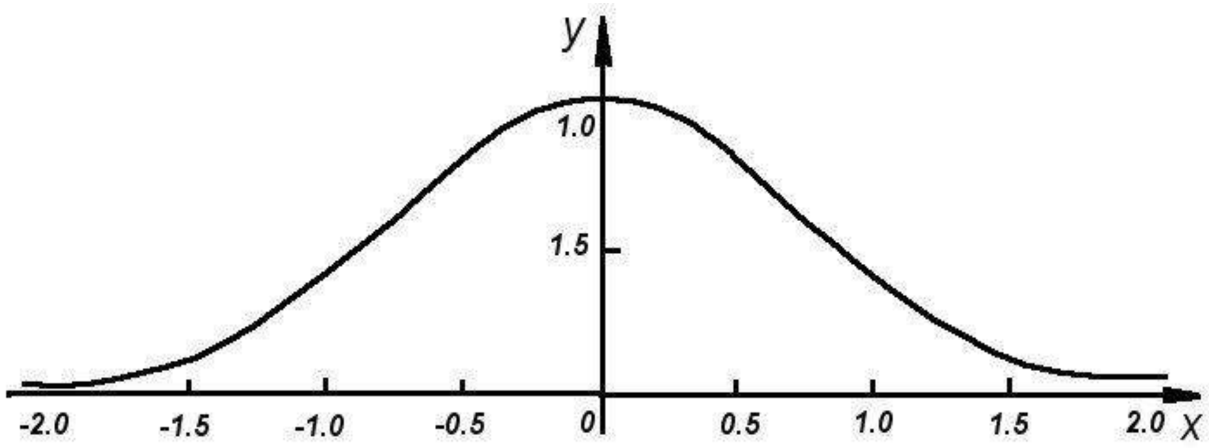

Fig. 11

We find the first derivative:

$$y' = -ae^{-ax}\sin bx + be^{-ax}\cos bx = e^{-ax}\left(b\cos bx - a\sin bx\right).$$

The expression in brackets can be put in the well-known form:

$$b\cos bx - a\sin bx = K\sin\left(bx + \varphi_0\right),$$

where K and φ_0 are constants. On putting the first derivative equal to zero we get the equation:

$$\sin\left(bx + \varphi_0\right) = 0\,,$$

which gives:

$$bx + \varphi_0 = k\pi\,, \text{ i.e. } x = \frac{k\pi - \varphi_0}{b}\ (k \text{ is an integer}). \qquad (14)$$

When x passes through these values, $\sin(bx+\varphi_0)$ changes its sign each time. The same can evidently be said of the derivative y', since,

$$y' = Ke^{-ax}\sin(bx+\varphi_0),$$

and the sign of the factor e^{-ax} is constant. There are thus alternate maxima and minima corresponding to those roots. We should have a sine wave without the exponential factor:

$$y = \sin bx,$$

and the abscissae of its peaks would be given by:

$$\cos bx = 0,$$

i.e.

$$x = \frac{(2k-1)\pi}{2b} \quad (k \text{ an integer}). \tag{15}$$

We thus see that the exponential factor not only reduces the amplitude of the oscillation, but also displaces the abscissae of the peaks of the curve. It is evident, on comparing equations (14) and (15), that this displacement is equal to the constant $(-\pi/2b-\varphi_0/b)$. The graph of the damped oscillation is shown in Fig. 12 for $a=1$ and $b=2\pi$. The peaks of the curve do not lie on the dotted lines corresponding to $y=\pm e^{-ax}$. This is due to the above mentioned displacement of the peaks.

4. We draw the curve:

$$y = \frac{x^3-3x}{6}.$$

We find the first and second derivatives:

$$y' = \frac{x^2-1}{2}; \; y'' = x.$$

We put the first derivative equal to zero, and find the two values $x_1 = 1$ and $x_2 = -1$. Substituting these values in the second derivative, we see that the first value represents a minimum, and the second a maximum. We substitute these values in the expression for y, finding the corresponding peaks of the curve:

$$\left(-1, \frac{1}{3}\right), \quad \left(1, -\frac{1}{3}\right).$$

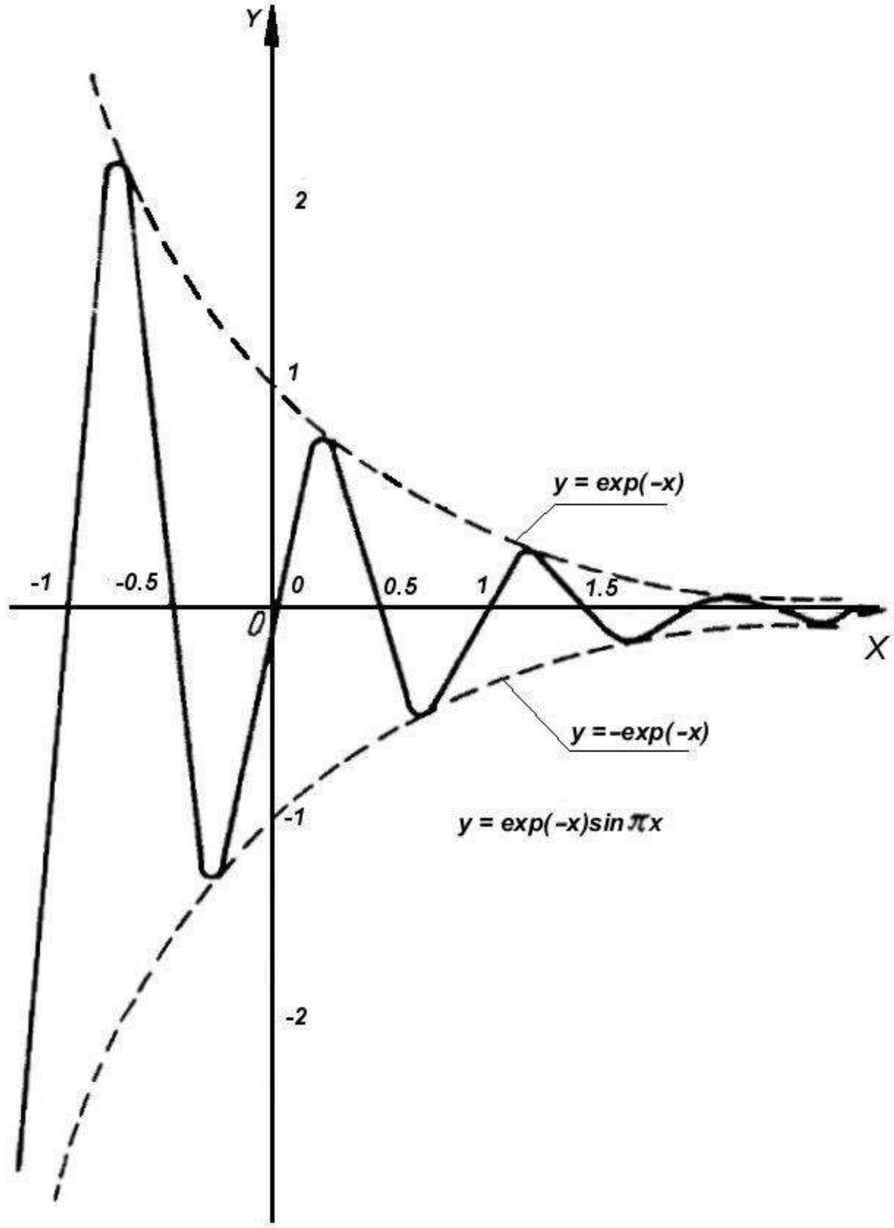

Fig. 12

Putting $x=0$, we get $y=0$, i.e. the origin $(0,0)$ lies on the curve. Finally, we get for zero y, apart from $x=0$, two further points $x=\pm\sqrt{3}$, so that the curve cuts the axes in three points, $(0,0)$, $\left(\sqrt{3},0\right)$ and $\left(-\sqrt{3},0\right)$. We note further that, on simultaneously replacing x and y by $-x$ and $-y$, both sides of the equation of the curve only change sign, i.e. the origin is a centre of symmetry of the curve (Fig. 13).

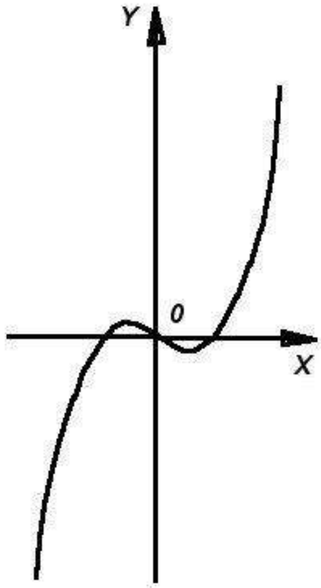

Fig. 13

The greatest and least values of a function

We assume that the values of the function $f(x)$ are considered for values of the independent variable x in the interval (a, b); it is required to find the greatest and the least of these values. If $f(x)$ is continuous, it will certainly attain a greatest and least value, so that the corresponding graph will have a greatest and a least ordinate in the interval in question. We can find all the maxima and minima of the function inside the interval (a, b) in accordance with the rule given above. If the function $\varphi(x)$ has its a greatest ordinate inside this interval, this will evidently coincide with the greatest maximum of the function inside the interval. It can happen, however, that the greatest ordinate is not inside the interval, but at one of the ends $x = a$ or $x = b$. Hence it is not sufficient, in finding say the greatest value of a function, to compare all its maxima inside the interval and take the greatest, since its values at the ends of the interval must also be taken into account. Similarly, the least value of a function must be found by taking all its minima inside the interval, together with the boundary values of the function or $x = a$ and $x = b$. We remark here that maxima and minima can be completely absent, yet a greatest and least value must exist for a continuous function in a bounded interval (a, b).

We note some particular cases, when the greatest and least values can be found very simply. If, for example, $f(x)$ is increasing in (a, b), it will obviously take its least value at $x = a$, and its greatest value at $x = b$. The opposite will be the case for $f(x)$ decreasing.

If the function has a single maximum inside the interval, with no minima, this single maximum gives the greatest value of the function (Fig. 14), so that it is quite unnecessary to find the values at the ends of the interval in order to find the greatest value of the function in this case. Similarly, if the function has a single minimum inside the interval, with no maxima, the minimum gives the least value of the function. The cases just mentioned are met with in the first of the four examples given below.

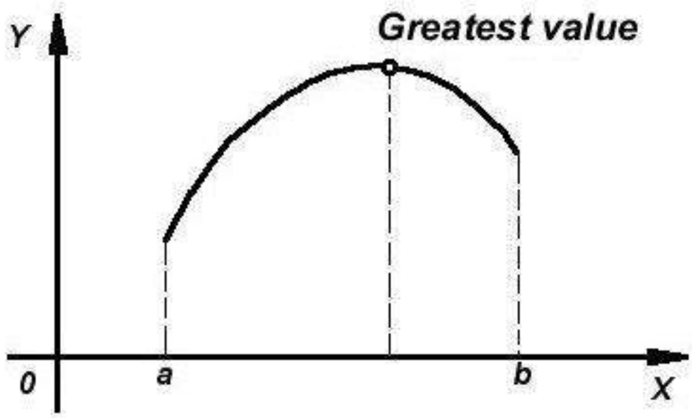

Fig. 14

1. It is required to cut a given line, of length l, into two parts, such that the rectangle formed from these parts has a maximum area.

Let x be the length of one part of the line, so that the length of the other part is $(l-x)$. Since the area of a rectangle is the product of its neighboring sides, we see that the problem reduces to finding the value of x in the interval $(0,\ l)$ for which the function:

$$f(x)=x(l-x)$$

attains its greatest value.

We find the first and second derivatives:

$$f'(x)=(l-x)-x=l-2x;\ f''(x)=-2<0.$$

Equating the first derivative to zero, we get a single value $x=l/2$, which corresponds to a maximum, since $f''(x)$ is a negative constant. The greatest area is thus obtained with a square of side $l/2$.

2. *A sector is cut out from a circle of radius R, and the remainder is glued together into a cone. It is required to find for what angle of the sector cut out the volume of the cone has its greatest value (Fig. 15).*

Instead of taking the angle of the sector cut out as the independent variable, we take the difference between this angle and 2π as x, i.e. x is the angle of the remaining sector. For x near 0 or 2π, the volume of the cone approaches zero, so that there is evidently a value of x in the interval $(0,2\pi)$ for which the volume is greatest.

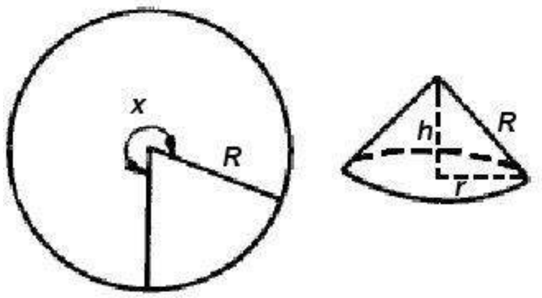

Fig.15

The cone obtained by glueing together the remainder of the circle must have a generator of length R, a length of circular base Rx, a radius of base $r = Rx/2\pi$, and height:

$$h = \sqrt{R^2 - \frac{R^2x^2}{4\pi^2}} = \frac{R}{2\pi}\sqrt{4\pi^2 - x^2}.$$

The volume of this cone will be:

$$v(x) = \frac{1}{3}\pi\frac{R^2x^2}{4\pi^2}\frac{R}{2\pi}\sqrt{4\pi^2 - x^2} = \frac{R^3}{24\pi^2}x^2\sqrt{4\pi^2 - x^2}.$$

The constant factor $R^3/24\pi^2$ can be neglected when finding the greatest value of this function. The remaining product $x^2\sqrt{4\pi^2 - x^2}$ is positive, and therefore attains its greatest value for the same x as that for which its square attains its greatest value. We can thus consider the function:

$$f(x) = 4\pi^2x^4 - x^6$$

inside the interval $(0, 2\pi)$.

We find the first derivative:

$$f'(x) = 16\pi^2x^3 - 6x^5.$$

It exists for all x. Equating it to zero, we obtain three values:

$$x_1 = 0,\ \ x_2 = -2\pi\sqrt{\frac{2}{3}},\ \ x_3 = 2\pi\sqrt{\frac{2}{3}}.$$

The first two values do not lie inside the interval $(0,\ 2\pi)$. The single value $x_3 = 2\pi\sqrt{2/3}$ remains, inside the interval; and since we saw above that the x giving the greatest volume of cone must lie inside the interval, we can say without investigating x_3 that this is the required x.

3. A straight line L divides a plane into two parts (media) I and II. A point moves in medium I with speed v_1, and in medium II with speed v_2. In what path must a point move in order to pass as quickly as possible from point A in medium I to point B in medium II?

Let AA_1 and BB_1 be the perpendiculars from A and B onto L. We introduce the following notation:

$$\overline{AA_1} = a, \quad \overline{BB_1} = b, \quad \overline{A_1B_1} = c,$$

and the abscissae will be measured on L in the direction $\overline{A_1B_1}$ (Fig. 16).

It is obvious that the point must have a straight-line path in both media, I and II, though generally speaking, line AB will not represent the "quickest path". The "quickest path" will thus consist of two straight sections AM

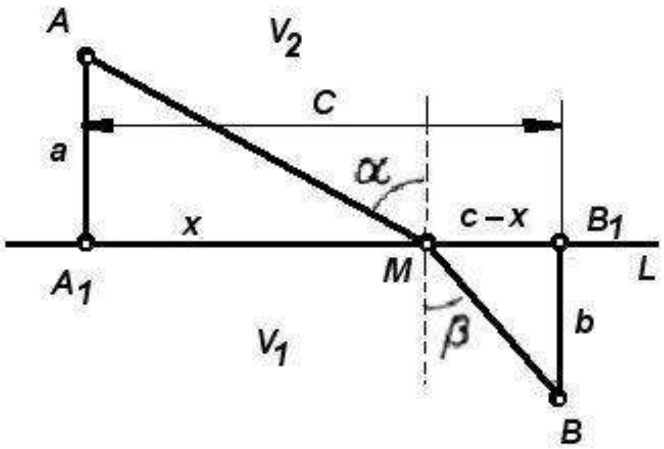

Fig. 16

and $\overline{MB}$, with point M lying on L. We take the abscissa of M as independent variable x $\left(x = \overline{A_1M}\right)$. We want the least value for time t, given by:

$$t = f(x) = \frac{AM}{v_1} + \frac{MB}{v_2} = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c-x)^2}}{v_2}$$

in the interval $(-\infty, +\infty)$.

We find the first and second derivatives:

$$f'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{c-x}{v_2\sqrt{b^2 + (c-x)^2}};$$

$$f''(x) = \frac{a^2}{v_1\left(a^2 + x^2\right)^{3/2}} + \frac{b^2}{v_2\left(b^2 + (c-x)^2\right)^{3/2}}.$$

Both derivatives exist for all x, and $f^{//}(x)$ is always positive. Hence, $f^{/}(x)$ is increasing in the interval $(-\infty,+\infty)$, and cannot become zero more than once. But

$$f^{/}(0)=-\frac{c}{v_2\sqrt{b^2+(c)^2}}<0$$

and

$$f^{/}(c)=\frac{c}{v_1\sqrt{a^2+c^2}}>0,$$

and hence the equation

$$f^{/}(x)=0$$

has a single root x_0 between 0 and c, giving the single minimum of $f(x)$, since $f^{//}(x)>0$. The abscissae 0 and c represent points A_1 and B_1 so that the required point M will lie between A_1 and B_1 as could also be shown by elementary geometry.

We give the geometrical interpretation of the solution obtained. Let α and β denote the angles formed by segment $\overline{AM}$ and $\overline{BM}$ respectively with the perpendicular to L at M. The abscissa x of the required point M mustmake $f^{/}(x)$ zero, i.e. must satisfy the equation:

$$\frac{x}{v_1\sqrt{a^2+x^2}}=\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}$$

which can be rewritten as:

$$\frac{A_1M}{v_1AM}=\frac{MB_1}{v_2BM}$$

or

$$\frac{\sin\alpha}{v_1}=\frac{\sin\beta}{v_2}\text{ , i.e. }\frac{\sin\alpha}{\sin\beta}=\frac{v_1}{v_2};$$

"the quickest path" will be that for which the ratio of the sines of two angles α and β is equal to the ratio of the speeds in media I and II. This result gives us the well-known law for the refraction of light, so that refraction takes place with the ray of light choosing the "quickest path" from a point in one medium to a point in another.

4. We suppose that the experimental determination of a magnitude x involves making n individual careful observations, giving n values:

$$a_1,\ a_2,\ a_3,\ ...,\ a_n\ ,$$

which are not identical due to instrument inaccuracies. The value of x giving the least sum of squares of errors is taken as the "best" value. Finding this best value thus means finding the x giving the least value of the function

$$f(x) = (x - a_1)^2 + (x - a_2)^2 + \ ...\ + (x - a_n)^2$$

in the interval $(-\infty, +\infty)$.

We find the first and second derivatives:

$$f'(x) = 2(x - a_1) + 2(x - a_2) + ... + 2(x - a_n);$$
$$f''(x) = 2 + 2 + ... + 2 = 2n > 0\ .$$

Making the first derivative zero, we get a single value:

$$x = \frac{a_1 + a_2 + ... + a_n}{n},$$

which represents a minimum, since the second derivative is positive. Thus*, the "best" value of x is the arithmetic mean of the observational results*.

5. To find the shortest distance from a point M to a circle.
We take the centre of the circle as origin O, and the line OM as axis OX.
Let $OM = a$, and let R be the radius of the circle. The equation of the circle is:

$$x^2 + y^2 = R^2,$$

and the distance of M, with coordinates $(a,0)$, to any point of the circle will be:

$$\sqrt{(x-a)^2 + y^2}\ .$$

We shall find the least value of the square of this distance. On substituting for y^2 from the equation of the circle, we get the function:

$$f(x) = (x-a)^2 + (R^2 - x^2) = -2ax + a^2 + R^2,$$

where the independent variable x lies in the interval $-R \le x \le R$.

Since the first derivative $f'(x) = -2a$ is negative for all x, function $f(x)$ is decreasing, and hence attains its least value for $x = R$ at the right-hand end of the interval. The shortest distance is thus given by $\overline{PM}$ (Fig. 17).

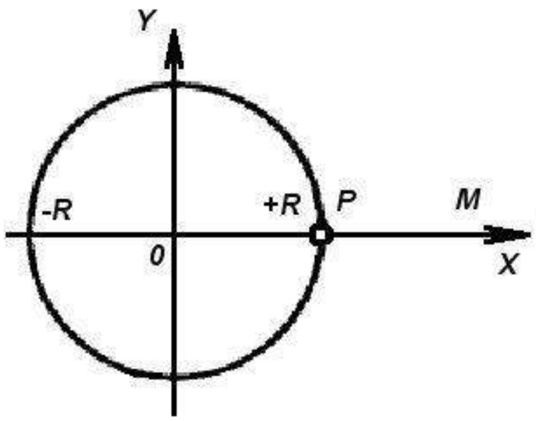

Fig. 17

6. *Given a right circular cone, it is required to inscribe in it a cylinder with the greatest total surface area.*

Let the radius of base of the cone be R and its height H, and let the radius of base and height of the cylinder be r and h. The function whose greatest value is required is here:

$$S = 2\pi r^2 + 2\pi rh.$$

Variables r and h are connected, due to the fact of inscribing the cylinder in the given cone. We have from the similar triangles ABD and AMN (Fig. 18):

$$\frac{MN}{AN} = \frac{BD}{AD},$$

or

$$\frac{h}{R-r} = \frac{H}{R},$$

whence

$$h = \frac{R-r}{R} H.$$

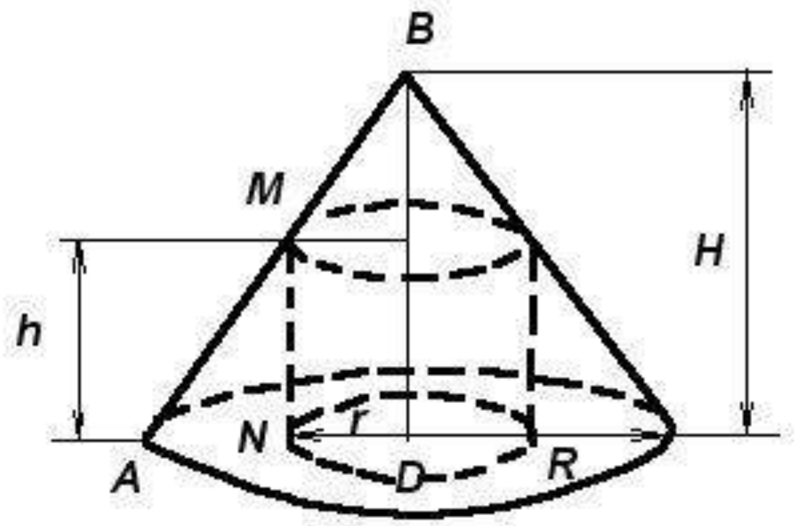

Fig. 18

We obtain on substituting for h in the expression for S:

$$S = 2\pi\left(r^2 + rH\left(1-\frac{r}{R}\right)\right).$$

Hence, S is a function of the single independent variable r, which must lie in the interval $0 \le r \le R$. We find the first and second derivatives:

$$\frac{dS}{dr} = 2\pi\left(2r + H - \frac{2r}{R}H\right), \quad \frac{d^2S}{dr^2} = 4\pi\left(1-\frac{H}{R}\right).$$

We find a single value of r, on making dS/dr zero:

$$r = \frac{HR}{2(H-R)}. \qquad (16)$$

For this value to lie inside $(0, R)$, the inequalities must be satisfied:

$$0 < \frac{HR}{2(H-R)} \text{ and } \frac{HR}{2(H-R)} < R. \qquad (17)$$

The first inequality is equivalent to the condition that $H > R$. We multiply both sides of the second inequality by the positive term $2(H-R)$ and get:

$$R < \frac{H}{2}.$$

With fulfilment of this condition, d^2S/dr^2 is negative; the value (16) gives the single maximum of function S, and the greatest surface-area of the cylinder. The size of the latter can easily be found by substituting for r from (16) in the expression for S.

We now take the case of value (16) not lying inside $(0,R)$ i.e. one of the inequalities (17) is not satisfied. Two possibilities can now arise: either $H<R$ or $H>R$, but $R\geq H/2$. Both these can be characterized by the single inequality:

$$H\leq 2R. \tag{18}$$

We rewrite dS/dr as:

$$\frac{dS}{dr}=2\pi\left(2r+H-\frac{2r}{R}H\right)=\frac{2\pi}{R}\left((2R-H)r+H(R-r)\right).$$

It is clear from the new expression that $dS/dr>0$ for $0<r<R$ on satisfying (18), i.e. S is increasing in $(0,R)$, and thus attains its greatest value for $r=R$. Evidently, $h=0$ with this value of r, and we can look on the result obtained as a flattened cylinder, the base of which coincides with the base of the cone, and the total surface-area of which gives $2\pi R^2$.

Fermat's theorem

We have used elementary geometry above, to give methods for studying the increase and decrease of a function, as well as for finding its maxima and minima and its greatest and least values. We now turn to the rigorous analytic statement of some theorems and formulae, which give us an analytic proof of the rules found above, and also enable the study of functions to be carried somewhat further. When stating the next theorems and formulae, we shall include a detailed and precise account of all the conditions for which they are valid.

Fermat's Theorem. *If $f(x)$ is continuous in (a,b), has a derivative at every interior point of the interval, and attains its greatest (or least) value at some interior point $x=c$, the first derivative must be zero at $x=c$, i.e. $f'(c)=0$.*

We suppose for clarity that $f(c)$ is the greatest value of the function. The proof will be exactly similar, in the case of its being the least value. In accordance with the condition that $x=c$ lies inside the interval, the difference $f(c+h)-f(c)$ will be negative, or at any rate not positive, for any positive or negative h:

$$f(c+h)-f(c)\leq 0.$$

We take the ratio:

$$\frac{f(c+h)-f(c)}{h}.$$

The numerator of this fraction is less than or equal to zero, as remarked, so that:

$$\frac{f(c+h)-f(c)}{h}\leq 0 \quad for \quad h>0; \qquad (19)$$

$$\frac{f(c+h)-f(c)}{h}\geq 0 \quad for \quad h<0.$$

Point $x=c$ lies inside the interval, and a derivative exists at the point by hypothesis, i.e. the above fraction tends to a definite limit $f'(c)$, when h tends to zero in any manner. We first suppose that h tends to zero through positive values. Passing to the limit in the first of inequalities (19), we get:

$$f'(c)\leq 0.$$

Similarly, passing to the limit with $h\to 0$ in the second inequality of (19) gives:

$$f'(c)\geq 0.$$

Taking these two inequalities together, we obtain the required result:

$$f'(c)=0.$$

Rolle's theorem

If $f(x)$ is continuous in (a,b), has a derivative at every interior point of the interval, and has equal values at the ends of the interval, i.e. $f(a)=f(b)$, then there exists at least one interior point, $x=c$, such that the derivative is zero, i.e. $f'(c)=0$.

A continuous function $f(x)$ must have a last value m and a greatest value M in the interval considered. If the least and greatest values happened to coincide, i.e. $m=M$, it would follow that the function kept a constant value m (or M) throughout the interval. We know that the derivative of a constant iszero, so that in this simple case, the derivative would be zero at every interior point of the interval. Turning to the general case, we can thus suppose that $m<M$. Since the function has the same value at the limits, i.e. $f(a)=f(b)$, by hypothesis, at least one of the numbers m or M must differ from this common value. Suppose, for example, that it is M, i.e. the greatest value of the function is reached inside, and not at the limits, of the interval. Let $x=c$ be the point at which this value is reached. By Fermat's theorem, we shall have $f'(c)=0$ at this point, which proves Rolle's theorem.

In the particular case of $f(a)=f(b)=0$, Rolle's theorem can be briefly formulated as: *the first derivative of* $f(x)$ *will vanish at at least one point in* (a,b).

There is a simple geometrical interpretation of Rolle's theorem. By hypothesis, $f(a)=f(b)$, i.e. the ordinates of the curve $y=f(x)$ are equal at the ends of the interval, and a derivative exists inside the interval, i.e. the curve has a definite tangent. Rolle's theorem says that there exists at least one interior point in this case, where the derivative is zero, i.e. where the tangent is parallel to axis OX (Fig. 19).

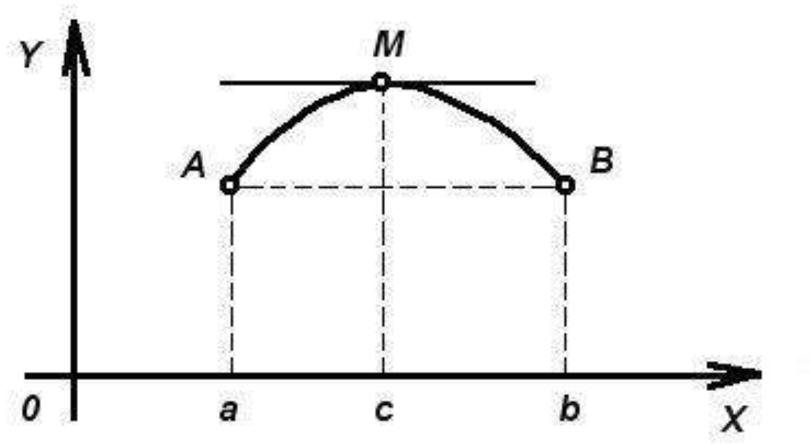

Fig. 19

Remark. If the condition regarding the existence of the derivative $f'(x)$ is not fulfilled for every interior point of the interval in Rolle's theorem, the theorem can be proved false.

Take, for example,

$$f(x)=1-\sqrt[3]{x^2};$$

this is continuous in the interval $(-1,1)$, and $f(-1)=f(1)=0$, but the derivative,

$$f'(x)=-\frac{2}{3\sqrt[3]{x}}$$

does not vanish inside the interval. This follows from the fact that $f'(x)$ does not exist (tends to infinity) for $x=0$ (Fig. 20). Another example is the curve shown in Fig. 21. Here we have the curve $y=f(x)$, with $f(a)=f(b)=0$. But it is evident from the figure that the tangent cannot be parallel to OX inside the interval (a,b), i.e. $f'(x)$ does not vanish. This happens because the curve has two different tangents at $x=c$, to the left and right of this point, and hence a unique derivative does not exist at this point, so that the condition of Rolle's theorem regarding the existence of the derivative at every interior point is not fulfilled.

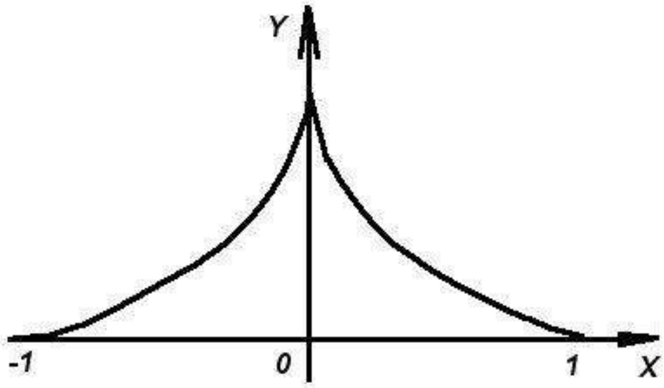

Fig. 20

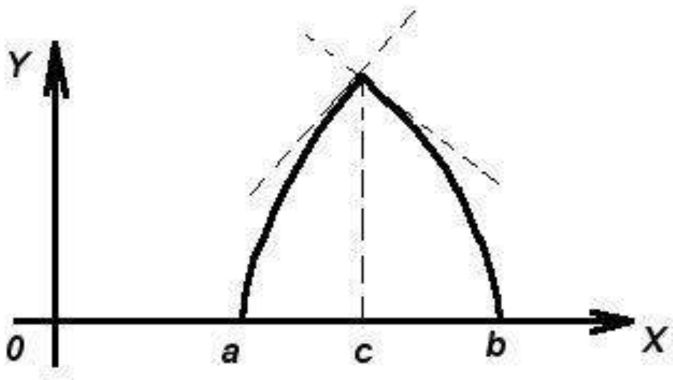

Fig. 21

Lagrange's formula

We assume that $f(x)$ is continuous in (a,b), and has a derivative inside this interval, but that the condition $f(a)=f(b)$ of Rolle's theorem cannot be satisfied. We form a new function:

$$F(x)=f(x)+\lambda x,$$

where λ is a constant, so defined that the new function satisfies the condition mentioned for Rolle's theorem, i.e. we make:

$$F(a)=F(b),$$

so that

$$f(a)+\lambda a = f(b)+\lambda b,$$

whence

$$\lambda = -\frac{f(b)-f(a)}{b-a}.$$

Applying Rolle's theorem now to $F(x)$, we can say that there will be a point $x=c$ between a and b, where

$$F'(c) = f'(c)+\lambda = 0 \quad (a<c<b),$$

whence, on substituting the above expression for λ:

$$f'(c) = -\lambda \text{ or } f'(c) = \frac{f(b)-f(a)}{b-a}.$$

The last equation can be written:

$$f(b)-f(a) = f'(c)(b-a).$$

This equation is called *Lagrange's formula*. The value c lies between a and b, so that the ratio $(c-a)/(b-a)=\theta$ lies between zero and unity, and we can write:

$$c = a+\theta(b-a) \quad (0<\theta<1),$$

and Lagrange's formula now takes the form:

$$f(b)-f(a) = f'\left(a+\theta(b-a)\right)(b-a) \quad (0<\theta<1).$$

Putting $b=a+h$, the formula becomes:

$$f(a+h)-f(a) = hf'(a+\theta h).$$

Lagrange's formula gives an accurate expression for the increment $f(b)-f(a)$ of the function $f(x)$, so that it is also called the *formula of finite increments.*

We know that the derivative of a constant is zero. Lagrange's formula allows us to state the converse: *if the derivative* $f'(x)$ *is zero for every point of* (a,b), $f(x)$ *is constant in the interval*.

In fact, taking an arbitrary x of (a,b), and applying Lagrange's formula to the interval (a,x), we get:

$$f(x)-f(a)=f'(\xi)(x-a) \quad (a<\xi<x);$$

but $f'(\xi)=0$ by hypothesis, hence:

$$f(x)-f(a)=0, \text{ i.e. } f(x)=f(a)=\text{constant}.$$

All we know as regards the magnitude c that appears in Lagrange's formula is that it lies between a and b, so that the formula does not enable us accurately to calculate the increment of a function from its derivative; the formula can be used, however, to estimate the error involved in replacing the increment of a function by its differential.

Example. Let

$$f(x)=\log_{10} x.$$

The derivative is

$$f'(x)=\frac{1}{x}\cdot\frac{1}{\log 10}=\frac{M}{x} \quad (M=0.43429...),$$

and Lagrange's formula gives us:

$$\log_{10}(a+h)-\log_{10} a=h\frac{M}{a+\theta h} \quad (0<\theta<1)$$

or

$$\log_{10}(a+h)=\log_{10} a+h\frac{M}{a+\theta h}.$$

Replacing the increment by the differential gives us the approximate formula:

$$\log_{10}(a+h)-\log_{10} a=h\frac{M}{a}; \quad \log_{10}(a+h)=\log_{10} a+h\frac{M}{a}.$$

On comparing this approximate equation with the accurate one, obtained with Lagrange's formula, we see that the error is:

$$h\frac{M}{a}-h\frac{M}{a+\theta h}=\frac{\theta h^2 M}{a\left(a+\theta h\right)}.$$

Putting $a=100$ and $h=1$, we get the approximate equation:

$$\log_{10}101=\log_{10}100+\frac{M}{100}=2.00434...$$

with the error

$$\frac{\theta\cdot M}{100\left(100+\theta\right)}\quad\left(0<\theta<1\right).$$

Replacing θ by unity in the numerator, and by zero in the denominator, of this fraction, we can say on evaluating the fraction that the error in cal-calculating the value of $\log_{10}101$ is less than

$$\frac{M}{100^2}=0.00004...$$

We can rewrite Lagrange's formula as:

$$\frac{f\left(b\right)-f\left(a\right)}{b-a}=f'\left(c\right)\quad\left(a<c<b\right).$$

We see from the graph of $y=f\left(x\right)$ (Fig. 22) that the ratio

$$\frac{f\left(b\right)-f\left(a\right)}{b-a}=\frac{\overline{CB}}{\overline{\overline{AC}}}=\tan\angle CAB$$

gives slope of the chord AB, while $f'\left(c\right)$ gives the slope of the tangent at some point M of the segment AB of the curve. Lagrange's formula thus amounts to the assertion: there is a point in a segment of a curve where the tangent is parallel to the chord. Rolle's theorem is a special case of this assertion, when the chord is parallel to OX, i.e. $f\left(a\right)=f\left(b\right)$.

Remark. The tests for increase and decrease, which we established above from the graph, follow immediately from Lagrange's formula.
We suppose that the first derivative is positive inside a certain interval, and we let x and $x+h$ be two points of this interval.

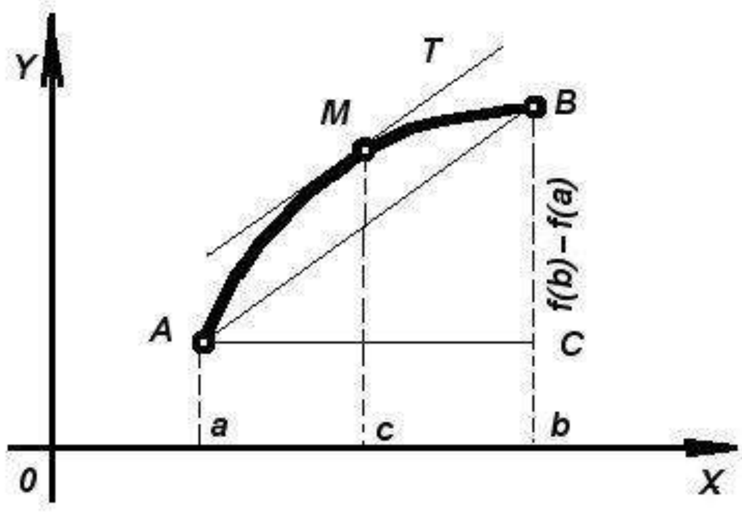

Fig. 22

Evidently, from Lagrange's formula:

$$f(x+h)-f(x)=hf'(x+\theta h) \quad (0<\theta<1),$$

the difference on the left will be positive for positive h, since both factors of the product on the right are positive in this case. The assumption that the derivative is positive in some interval thus leads us to:

$$f(x+h)-f(x)>0,$$

i.e. the function is increasing in this interval. The test for a decreasing function similarly follows directly from the formula written above.

We also note here that the argument used in proving Fermat's theorem is fully applicable to the case when the function reaches a maximum or minimum, and not necessarily its greatest or least value, at the point in question. This argument shows us that the first derivative must be zero at such points, if it exists.

Cauchy's formula

We take $f(x)$ and $\varphi(x)$ continuous in (a,b), with derivatives at every interior point of the interval, and with $\varphi'(x)$ not zero at any interior point. We apply Lagrange's formula to $\varphi(x)$, giving

$$\varphi(b)-\varphi(a)=(b-a)\varphi'(c_1) \quad (a<c_1<b),$$

with $\varphi(c_1) \neq 0$ by hypothesis, whence

$$\varphi(b) - \varphi(a) \neq 0.$$

We form the function:

$$F(x) = f(x) + \lambda \varphi(x),$$

where λ is a constant, so defined that $F(a) = F(b)$, i.e.

$$f(a) + \lambda \varphi(a) = f(b) + \lambda \varphi(b),$$

whence

$$\lambda = -\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)}.$$

Rolle's theorem is applicable to $F(x)$ with this choice of λ, giving us the existence of $x = c$ such that

$$F'(c) = f'(c) + \lambda \varphi'(c) = 0 \quad (a < c < b).$$

This equation gives:

$$\frac{f'(c)}{\varphi'(c)} = -\lambda \quad (\varphi'(c) \neq 0),$$

whence, on substituting the value found for λ, we get:

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(c)}{\varphi'(c)} \quad (a < c < b)$$

or

$$\frac{f(b) - f(a)}{\varphi(b) - \varphi(a)} = \frac{f'(a + \theta(b - a))}{\varphi'(a + \theta(b - a))} \quad (0 < \theta < 1) \tag{20}$$

or

$$\frac{f(a+h)-f(a)}{\varphi(a+h)-\varphi(a)}=\frac{f'(a+\theta h)}{\varphi'(a+\theta h)}.$$

This is Cauchy's formula. On putting $\varphi(x)=x$ in this formula, we have $\varphi'(x)=1$, and the formula becomes:

$$\frac{f(b)-f(a)}{b-a}=\frac{f'(c)}{1}$$

or

$$f(b)-f(a)=(b-a)f'(c).$$

i.e. we get Lagrange's formula as a special case of Cauchy's formula.

Evaluating indeterminate forms

If two functions $\varphi(x)$ and $\psi(x)$ vanish at $x=a$, the fraction $\varphi(x)/\psi(x)$ is an indeterminate form of the type $0/0$ at $x=a$. We indicate a method of evaluating such indeterminate forms. We suppose that $\varphi(x)$ and $\psi(x)$ are continuous and have a first derivative near $x=a$, whilst $\psi'(x)$ does not vanish for x near, but not equal to, a.

We prove the following theorem: *if, with the above assumptions,* $\varphi'(x)/\psi'(x)$ *tends to a limit* b *as* x *tends to* a*, then* $\varphi(x)/\psi(x)$ *tends to the same limit.*

Noting that:

$$\psi(a)=\varphi(a)=0,$$

and using Cauchy's formula, we get:

$$\frac{\varphi(x)}{\psi(x)}=\frac{\varphi(x)-\varphi(a)}{\psi(x)-\varphi(a)}=\frac{\varphi'(\xi)}{\psi'(\xi)} \quad (\xi \text{ between } a \text{ and } x). \tag{21}$$

We remark that we are justified in applying Cauchy's formula, given the assumptions made regarding $\varphi(x)$ and $\psi(x)$.

If $x\to a$, ξ, lying between x and a, will tend to the same limit. The right-hand side of (7) now tends to b, by hypothesis, and hence $\varphi(x)/\psi(x)$ on the left-hand side of (21) tends to the same limit.

The theorem just proved gives us the following rule for evaluating an indeterminacy of the form $0/0$:

To find the limit of $\varphi(x)/\psi(x)$, *in the case of an indeterminacy of the form* $0/0$, *the ratio of the functions can be replaced by the ratio of the derivatives, and the limit found of this new ratio*.

This rule was given by the French mathematician l'Hopital, and is usually named after him.

If the ratio of the derivatives also leads to an indeterminacy of the form $0/0$, the rule can be applied to this ratio, and so on.

We give some examples of using the rule:

1. $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}=\lim\limits_{x\to 0}\dfrac{n(1+x)^{n-1}}{1}=n$.
2. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}=\lim\limits_{x\to 0}\dfrac{\sin x}{6x}=\lim\limits_{x\to 0}\dfrac{\cos x}{6}=\dfrac{1}{6}$,

i.e. the difference $x-\sin x$ is an infinitesimal of the third order with respect to x.

3. $\lim\limits_{x\to 0}\dfrac{x-x\cos x}{x-\sin x}=\lim\limits_{x\to 0}\dfrac{1-\cos x+x\sin x}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x+\sin x+x\cos x}{\sin x}=$

$$=\lim_{x\to 0}\frac{2\cos x+\cos x-x\sin x}{\cos x}=3.$$

The result of this example leads to a practical method for rectifying the arc of a circle.

We take a circle of radius unity. A diameter of the circle is taken as OX, and the tangent at one end of the diameter as OY (Fig. 23).

We take a certain arc OM, and let ON be an interval along OY equal to arc OM. We draw NM and let P be its point of intersection with OX.

Let u denote the length of arc OM (radius taken as unity). The equation of NM can be written in terms of its intersections with the axes:

$$\frac{x}{OP}+\frac{y}{u}=1.$$

We find the length of $\overline{OP}$ by noting that M lies on NM and has the coordinates:

$$x=OQ=1-\cos u,\quad y=QM=\sin u.$$

These coordinates must satisfy the following equation:

$$\frac{1-\cos u}{OP}+\frac{\sin u}{u}=1,$$

whence

$$OP=\frac{u-u\cos u}{u-\sin u}.$$

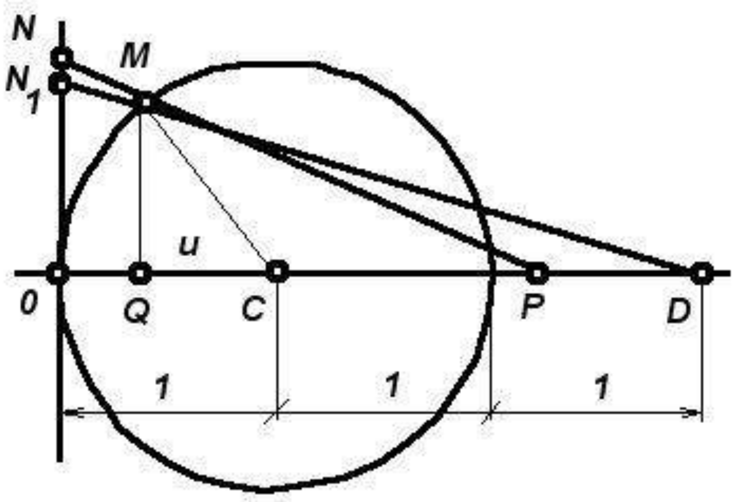

Fig. 23

The result of Example 3 shows that $OP \to 3$ as $u \to 0$, i.e. P tends to D on OX, the distance of D from the origin being three times the radius. This gives a simple method of approximate rectification of the arc of a circle. To rectify arc OM, section $\overline{OD}$ must be taken from O, three times the radius of the circle, then the line DM must be drawn. The intercept of DM on OY, $\overline{ON_1}$ gives the approximate length of arc OM. This method gives a very satisfactory result, especially for small arcs; even for an arc of $\pi/2$, the relative error is about 5%.

Other indeterminate forms

The theorem above can be justified for an indeterminacy of the form ∞/∞. Let

$$\lim_{x\to a}\varphi(x)=\lim_{x\to a}\psi(x)=\infty \tag{22}$$

and

$$\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)}=b. \tag{23}$$

We show that $\varphi(x)/\psi(x)$ tends to the same limit b, assuming that $\psi'(x)$ does not vanish for x near a.

We consider two values x and x_0 of the independent variable, near a, and such that x lies between x_0 and a. We have by Cauchy's formula:

$$\frac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)}=\frac{\varphi'(\xi)}{\psi'(\xi)} \quad (\xi \text{ between } x \text{ and } x_0),$$

whilst, on the other hand,

$$\frac{\varphi(x)-\varphi(x_0)}{\psi(x)-\psi(x_0)}=\frac{\varphi(x)}{\psi(x)}\cdot\frac{1-\dfrac{\varphi(x_0)}{\varphi(x)}}{1-\dfrac{\psi(x_0)}{\psi(x)}}.$$

We remark that it follows directly from (22) that $\varphi(x)$ and $\psi(x)$ differ from zero for x near a.

Comparison of these two equations gives us, after re-arrangement:

$$\frac{\varphi(x)}{\psi(x)}=\frac{\varphi'(\xi)}{\psi'(\xi)}\cdot\frac{1-\dfrac{\psi(x_0)}{\psi(x)}}{1-\dfrac{\varphi(x_0)}{\varphi(x)}}, \tag{24}$$

where ξ lies between x and x_0, and hence between a and x_0. We take x_0 sufficiently near a, so that, by (23), the first factor on the right of (24) differs from b by an arbitrarily small amount, for any choice of x between x_0 and a. Having thus fixed x_0, we let x approach a. The second factor on the right of (24) now tends to unity, by (22), and hence we can say that the left-hand side of (24) differs from b by an arbitrarily small amount for values of x near a, i.e.

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=b.$$

It follows from the theorem just proved that l'Hopital's rule can be used for evaluating an indeterminacy of the form ∞/∞.

We note some further indeterminate forms. We take the product $\varphi(x)\psi(x)$, and let

$$\lim_{x\to a}\varphi(x)=0 \quad \text{and} \quad \lim_{x\to a}\psi(x)=\infty .$$

This gives the indeterminate form $0\cdot\infty$. It is easily transformed to the form $0/0$ or ∞/∞:

$$\varphi(x)\psi(x)=\frac{\varphi(x)}{\frac{1}{\psi(x)}}=\frac{\psi(x)}{\frac{1}{\varphi(x)}} .$$

We consider finally the expression $\varphi(x)^{\psi(x)}$ and let

$$\lim_{x\to a}\varphi(x)=1 \quad \text{and} \quad \lim_{x\to a}\psi(x)=\infty .$$

This gives the indeterminate form 1^{∞}. We take the logarithm of the above expression:

$$\log\left[\varphi(x)^{\psi(x)}\right]=\psi(x)\log\varphi(x),$$

giving the indeterminate form $0\cdot\infty$. By evaluating this indeterminacy, i.e. finding the limit of the logarithm of the given expression, we can discover the limit of the expression itself. The indeterminate forms ∞^0 and 0^0 are similarly evaluated.

We now take some examples:

1. $\lim\limits_{x\to+\infty}\frac{e^x}{x}=\lim\limits_{x\to+\infty}\frac{e^x}{1}=+\infty$,

$\lim\limits_{x\to+\infty}\frac{e^x}{x^2}=\lim\limits_{x\to+\infty}\frac{e^x}{2x}=\lim\limits_{x\to+\infty}\frac{e^x}{2}=+\infty$.

It can similarly be shown that e^x/x^n tends to infinity as x tends to $+\infty$, for any positive n, i.e. the exponential function e^x increases more rapidly than any positive power of x, on indefinite increase of x.

2. $\lim\limits_{x\to+\infty}\frac{\log x}{x^n}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}=\lim\limits_{x\to+\infty}\frac{1}{nx^n}=0 \quad (n>0)$,

i.e. $\log x$ increases more slowly than any positive power of x.

3. $\lim\limits_{x\to+0}x^n\log x=\lim\limits_{x\to+0}\frac{\log x}{\frac{1}{x^n}}=\lim\limits_{x\to+0}\frac{\frac{1}{x}}{\frac{-n}{x^{n+1}}}=-\lim\limits_{x\to+0}\frac{x^n}{n}=0 \quad (n>0)$.

4. We find the limit of x^x for $x \to 0$. The logarithm of the expression gives the indeterminate form $0 \cdot \infty$. This indeterminate form has limit 0, by Example 3, and hence:

$$\lim_{x \to +0} x^x = 1.$$

5. We find the limit of the ratio:

$$\lim_{x \to \infty} \frac{x + \sin x}{x}.$$

Numerator and denominator both tend to infinity. Using the rule for replacing the ratio of the functions by the ratio of their derivatives, we get:

$$\lim_{x \to \infty} \frac{1 + \cos x}{1}.$$

But $1 + \cos x$ tends to no definite limit on indefinite increase of x, since $\cos x$ always oscillates between 1 and — 1; yet it is easy to see that the given ratio itself tends to a limit:

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \left(1 + \frac{\sin x}{x}\right) = 1.$$

The indeterminate form has a limit here, yet the rule for finding it fails. This result does not contradict the theorem, since the theorem states only that, if the ratio of the derivatives tends to a limit, the ratio of the functions tends to the same limit: the theorem does not state the converse.

6. We also note the indeterminate form $(\infty \pm \infty)$. It usually leads to the form $0/0$. For example:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x + x^2} \right) = \lim_{x \to 0} \frac{x + x^2 - \sin x}{\left(x + x^2\right) \sin x}.$$

The last expression consists of the indeterminate form $0/0$. We evaluate this by the method given above:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x + x^2} \right) = 1.$$

SOME GEOMETRICAL APPLICATIONS OF THE DIFFERENTIAL CALCULUS

The differential of arc

It will be shown in the integral calculus how to find the length of an arc of a curve; the expression for the differential of the length of arc will then be investigated, and it will be shown that the ratio of the length of the chord to the length of the arc subtending it tends to unity, when the arc contracts indefinitely to a point.

Let $y = f(x)$ be a given curve, and let the length of arc be measured on it from some fixed point A in a specified direction (Fig. 24). Let s be the length of arc from A to a variable point M.

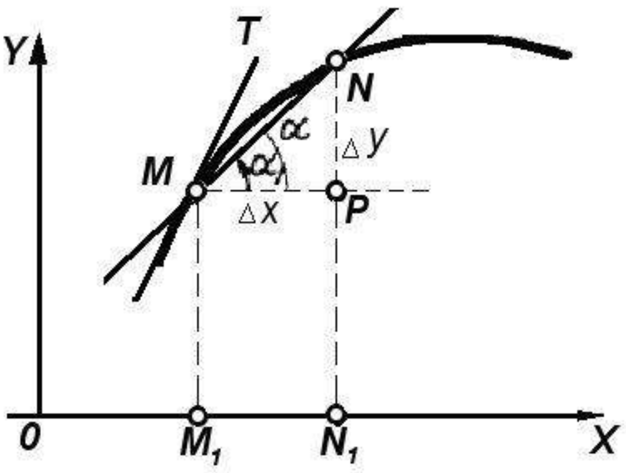

Fig. 24

The magnitude s, like the ordinate y, will be a function of the abscissa x of M. If the direction of AM coincides with the specified direction of the curve, $s > 0$, whilst in the opposite case, $s < 0$. Let $M(x, y)$ and $N(x + \Delta x, y + \Delta y)$ be two points of the curve, and Δs the difference of the lengths of arc AN and AM, i.e. the increment in length of arc in passing from M to N. The absolute value of Δs is the length of arc MN taken with the plus sign. We have from the right-angled triangle:

$$\left(\overline{MN}\right)^2 = \Delta x^2 + \Delta y^2 ,$$

whence

$$\frac{\left(\overline{MN}\right)^2}{\Delta x^2}=1+\left(\frac{\Delta y}{\Delta x}\right)^2,$$

or

$$\left(\frac{\left(\overline{MN}\right)}{\Delta s}\right)^2\left(\frac{\Delta s}{\Delta x}\right)^2=1+\left(\frac{\Delta y}{\Delta x}\right)^2.$$

Passing to the limit, and noting that $\left(\overline{MN}/\Delta s\right)^2\to 1$ from what was remarked above, we get

$$\left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$$

or

$$\frac{ds}{dx}=\pm\sqrt{1+\left(y'\right)^2}. \qquad (25)$$

We must take the positive sign if s increases as x increases, and the negative sign if s decreases with increasing x. We suppose the former case for clarity (illustrated in Fig. 24). It follows from (25) that

$$ds=\sqrt{1+\left(y'\right)^2}\,dx$$

or

$$ds=\sqrt{\left(dx\right)^2+\left(dy\right)^2}. \qquad (26)$$

The length s of arc AM is a natural parameter for defining the position of a point M on a curve. We can take s as independent variable, the coordinates (x,y) of a point M then being functions of s:

$$x=\varphi(s);\ y=\psi(s).$$

A more detailed discussion of the "parameters of a curve" is given later. We now explain the geometrical meaning of the derivatives of x and y with respect to s.

We take the point N so that the direction of arc MN coincides with the specified direction of the curve, i.e. $\Delta s > 0$. The direction of the chord $\overline{MN}$ gives in the limit, as N tends to M, a specified direction of the tangent at M. We take this as the positive direction of the tangent. It is associated with the direction specified for the curve itself.

Let a_1 be the angle between the direction of $\overline{MN}$ and the positive direction of axis OX. The increment Δx of the abscissa x is the projection of $\overline{MN}$ on OX, and hence:

$$\Delta x = \overline{MN}\cos\alpha_1;$$
$$\left(\overline{MN} = \sqrt{\Delta x^2 + \Delta y^2}\right),$$

$\overline{MN}$ being reckoned positive in this equation. On dividing both sides of the equation by the length of arc MN, equal to Δs, we get:

$$\frac{\Delta x}{\Delta s} = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta s}\cos\alpha_1.$$

By hypothesis, $\Delta s > 0$, and hence the ratio $\sqrt{\Delta x^2 + \Delta y^2}/\Delta s$ tends to (+ 1) as N tends to M, whilst a_1 tends to a, the angle between the positive direction of the tangent $\overline{MT}$ and the positive direction of OX. The equation just given becomes in the limit:

$$\cos\alpha = \frac{dx}{ds}. \tag{27}$$

Similarly, by projecting $\overline{MN}$ on OY, we get:

$$\sin\alpha = \frac{dy}{ds}. \tag{28}$$

Concavity, convexity and curvature

Curves, convex and concave towards positive ordinates, are shown in Figs. 25 and 26.

The same curve $y = f(x)$ can, of course, have convex and concave portions (Fig. 27). *Points separating convex from concave portions of a curve are called points of inflexion.*

If we move along the curve in the direction of increasing x, and observe the variation of a, the angle formed by the tangent with the positive direction of OX, we see that (Fig. 27) a *is decreasing in convex, and increasing in concave,* *portions*. It follows that $\tan\alpha$, i.e. the derivative $f^{/}(x)$, will undergo the same variation, since $\tan\alpha$ increases (decreases) with increasing (decreasing) α. But the interval in which $f^{/}(x)$ is decreasing is the interval where its derivative is negative, i.e. $f^{//}(x)<0$; and similarly, the interval of increase of $f^{/}(x)$ is that where $f^{//}(x)>0$. We thus have the theorem:

The portions of a curve are convex towards positive ordinates where $f^{//}(x)<0$*, being concave where* $f^{//}(x)>0$*. Points of inflexion are the points where* $f^{//}(x)$ *changes sign.*

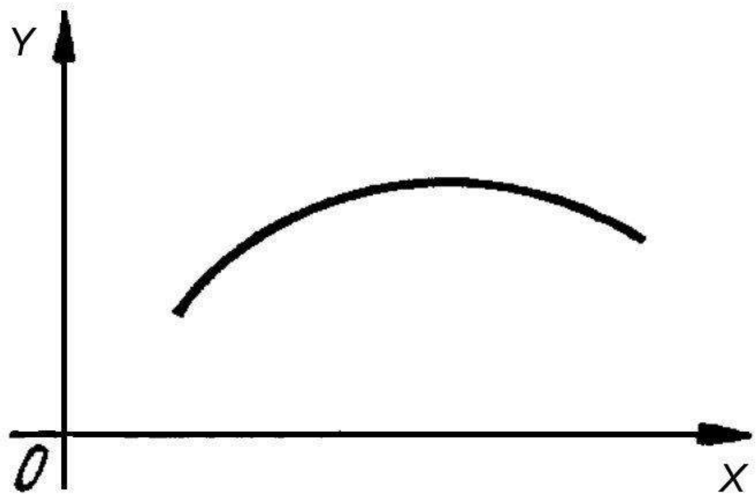

Fig. 25

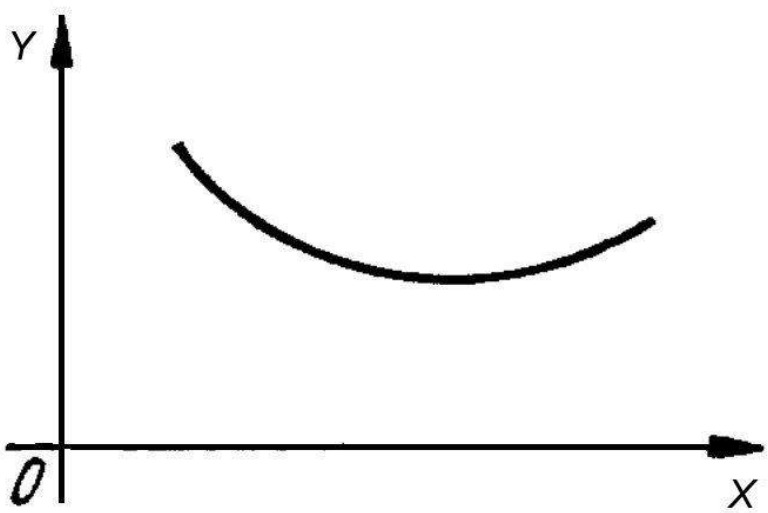

Fig. 26

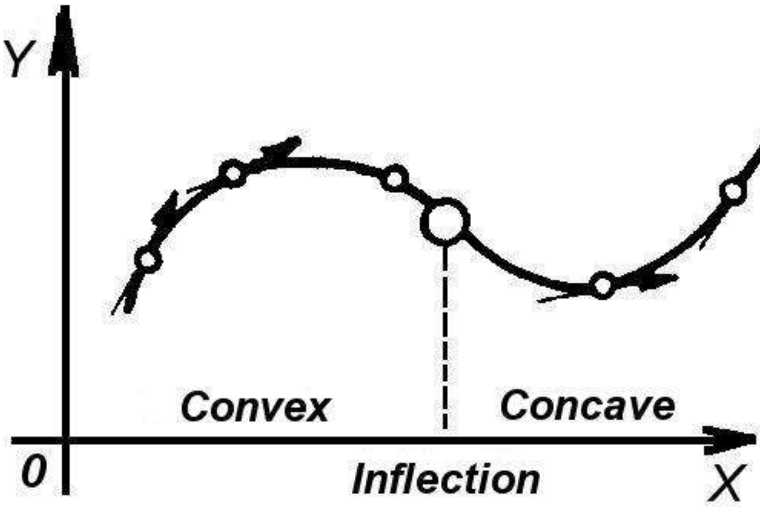

Fig. 27

By using arguments analogous to the earlier ones, we obtain from this theorem a rule for finding the points of inflexion of a curve: *to find the points of inflexion of a curve, the values of x must be found for which $f^{//}(x)$ vanishes or does not exist, and the variation in sign of $f^{//}(x)$ must be investigated on passage through these values, using the following table 6.*

Table 6

	Point of inflexion		Not a point of inlexion	
$f^{//}(x)$	+–	–+	––	++
	concave convex	convex concave	convex	concave

The most natural way of representing the bending of a curve is by following the variation of a, the angle made by the tangent with OX, as we move along the curve. Given two arcs of the same length Δs, the more curved arc will be that for which the tangent moves through a greater angle, i.e. for which the increment $\Delta\alpha$ is greater.

This remark leads us to the concepts of the mean curvature of Δs and of the curvature at a given point: *the mean curvature of an arc Δs is defined as the absolute value of the ratio of $\Delta\alpha$, the angle between the tangents at the ends of the arc, to the length of arc Δs. The limit of this ratio as Δs tends to zero is called the curvature of the curve at the given point (Fig. 28).*

We thus have for the curvature C:

$$C = \left|\frac{d\alpha}{ds}\right|.$$

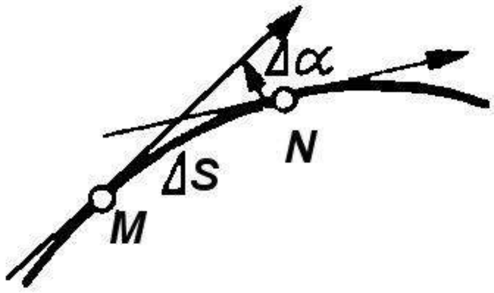

Fig. 28

But $\tan\alpha$ is the first derivative y', i.e.

$$\alpha = \arctan y',$$

whence, differentiating the function of a function $\arctan y'$ with respect to x:

$$d\alpha = \frac{y''}{1+\left(y'\right)^2}dx.$$

As shown above,

$$ds = \pm\sqrt{1+\left(y'\right)^2}dx.$$

Dividing da by ds, we get a final expression for the curvature:

$$C = \pm\frac{y''}{\left(1+\left(y'\right)^2\right)^{3/2}}. \qquad (29)$$

The minus sign is taken in convex parts, and the plus sign in concave parts, so as to give C a positive value.

No curvature exists at points of a curve where the derivatives y' or y'' do not exist. A curve resembles a straight line in the neighborhood of points where y'', and hence the curvature, vanishes; this will happen, for instance, near points of inflexion.

Suppose the coordinates x, y of a point of a curve are given in terms of the length of arc s. Here, as we have seen:

$$\cos\alpha = \frac{dx}{ds}, \quad \sin\alpha = \frac{dy}{ds}.$$

Angle α is also a function of s, and on differentiating the above equations with respect to s, we get:

$$-\sin\alpha \frac{d\alpha}{ds} = \frac{d^2x}{ds^2}, \quad \cos\alpha \frac{d\alpha}{ds} = \frac{d^2y}{ds^2}.$$

Squaring both sides of these equations and adding, we have:

$$\left(\frac{d\alpha}{ds}\right)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \text{ or } (C)^2 = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$$

whence:

$$C = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}.$$

The reciprocal of the curvature, $1/C$, is called the *radius of curvature.* By (29), we have the following expression for the radius of curvature R:

$$R = \left|\frac{ds}{d\alpha}\right| = \pm\frac{\left(1+\left(y^{/}\right)^2\right)^{3/2}}{y^{//}} \tag{30}$$

or

$$R = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}},$$

taking the positive value of the square root.

In the case of a straight line, y is a polynomial of the first degree in x, and hence $y^{//}$ is identically zero, i.e. the curvature is zero everywhere on the line, the radius of curvature being infinity.

We evidently have for a circle of radius r (Fig. 29):

$$\Delta s = r\Delta\alpha \quad \text{and} \quad R = \lim\frac{\Delta s}{\Delta\alpha} = r,$$

i.e. the radius of curvature is constant for the entire circle. We shall see later that the circle alone has this property.

We remark that the variation of the radius of curvature is by no means as easily seen as that of the tangent. We take the curve made up of an arc BC of a circle and a section AB of the tangent to the circle at B (Fig. 30). The radius of curvature is infinity for the portion AB, whilst for the portion BC it is equal to the radius of the circle r; it thus suffers a break in continuity at B, whereas the direction of the tangent varies continuously here.

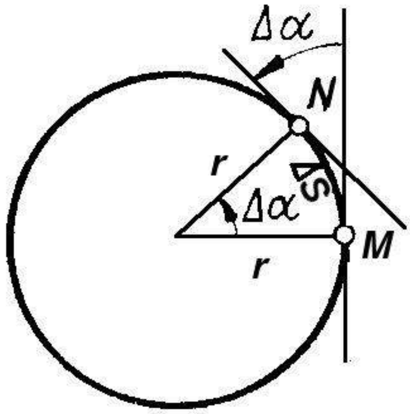

Fig. 29

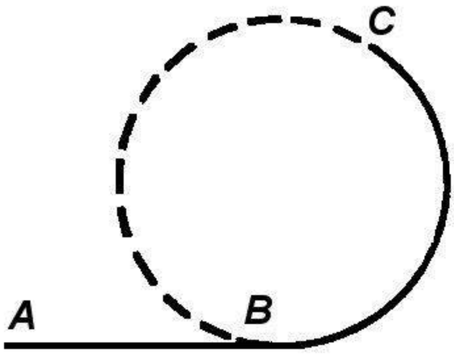

Fig. 30

This accounts for the jolting of railway carriages at bends. Assuming a carriage travelling with fixed speed v, we know from mechanics that a force is exerted along the normal to the trajectory, equal to mv^2/R, where m is the mass of the moving body, and R is the radius of curvature of the trajectory. Hence the force suffers a break in continuity at points where the radius of curvature suffers a break in continuity, which explains the jolts.

Asymptotes

We now turn to considering curves with infinite branches, where one or both of the coordinates x and y increase indefinitely. The hyperbola and parabola are examples of such curves.

A straight line is referred to as an asymptote of a curve with an infinite branch when the distance of points of the curve from the line tends to zero on indefinite displacement along the infinite branch.

We first show how to find the asymptotes of a curve parallel to axis OY. The equation of an asymptote of this sort must have the form:

$$x = c,$$

where c is a constant, and x must tend to c, whilst y tends to infinity, on moving along the corresponding infinite branch (Fig. 31). We thus get the following rule:

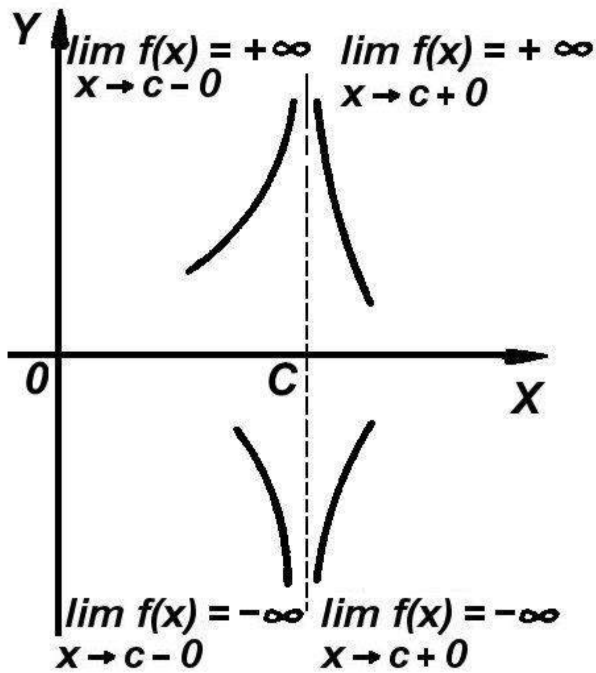

Fig. 31

All the asymptotes parallel to OY of the curve

$$y = f(x)$$

can be found by finding all the values $x = c$, on approach to which $f(x)$ tends to infinity.

To find the position of the curve relative to the asymptote, the sign of $f(x)$ must be determined as x tends to c on the left and right.

We pass to finding the asymptotes, not parallel to OY. The equation of the asymptote must now have the form:

$$\eta = a\xi + b,$$

where $\xi,\ \eta$ are the current coordinates of the asymptote, as distinct from x, y, the current coordinates of the curve.

Let ω be the angle that the asymptote forms with the positive direction of OX, let $\overline{MK}$ be the distance of a point of the curve from the asymptote, and $\overline{MK_1}$ be the difference between the ordinates of the curve and asymptote for the same abscissa x (Fig. 32). We have from the right-angled triangle:

$$\overline{MK_1} = \frac{\overline{MK}}{\cos\omega} \qquad \left(\omega \neq \frac{\pi}{2}\right),$$

and hence we can replace the condition:

$$\lim_{x\to\infty} \overline{MK} = 0$$

by the condition

$$\lim_{x\to\infty} \overline{MK_1} = 0 . \tag{31}$$

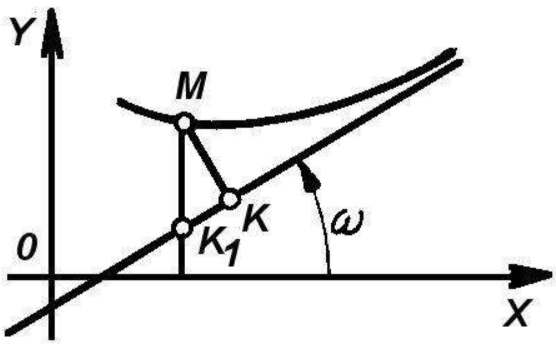

Fig. 32

For an asymptote not parallel to OY, x tends to infinity on moving along the infinite branch. Recalling that $\overline{MK_1}$ is the difference between the ordinates of the curve and asymptote for the same abscissa, we can rewrite condition (31) as:

$$\lim_{x\to\infty}\left[f(x) - ax - b\right] = 0, \tag{32}$$

where the values of a and b have to be found.

We can rewrite (32) in the form:

$$\lim_{x\to\infty} x\left[\frac{f(x)}{x} - a - \frac{b}{x}\right] = 0;$$

the first factor x tends to infinity, so that the expression in square brackets must tend to zero:

$$\lim_{x\to\infty}\left[\frac{f(x)}{x}-a-\frac{b}{x}\right]=\lim_{x\to\infty}\frac{f(x)}{x}-a=0,$$

i.e.

$$a=\lim_{x\to\infty}\frac{f(x)}{x}.$$

Having found a, we obtain b from the basic condition (32), which can be written as:

$$b=\lim_{x\to\infty}\left[f(x)-ax\right].$$

Thus, a necessary and sufficient condition that the curve:

$$y=f(x)$$

has an asymptote not parallel to OY *is that, with* x *increasing indefinitely on movement along the infinite branch, the limits exist:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}, \qquad b=\lim_{x\to\infty}\left[f(x)-ax\right]$$

the equation of the asymptote then being :

$$\eta=a\xi+b.$$

To find the position of the curve relative to its asymptote, the cases of x tending to $(+\infty)$ and to $(-\infty)$ have to be worked out separately, and the sign of the difference,

$$f(x)-(ax+b),$$

determined in each case. If the sign is positive, the curve is situated above the asymptote, and if negative, below the asymptote. If the difference does not keep the same sign on indefinite increase of x, the curve will oscillate about the asymptote (Fig. 33).

Curve-tracing

We now give a fuller indication than above of the series of operations to be carried out in tracing the curve

$$y = f(x).$$

We must:

(a) define the interval of variation of the independent variable x;
(b) find the points of intersection of the curve with the coordinate axes;
(c) find the peaks of the curve;
(d) find the convexities, concavities, and points of inflexion of the curve;
(e) find the asymptotes of the curve;
(f) examine the symmetry of the curve relative to the coordinate axes, if such symmetry exists.

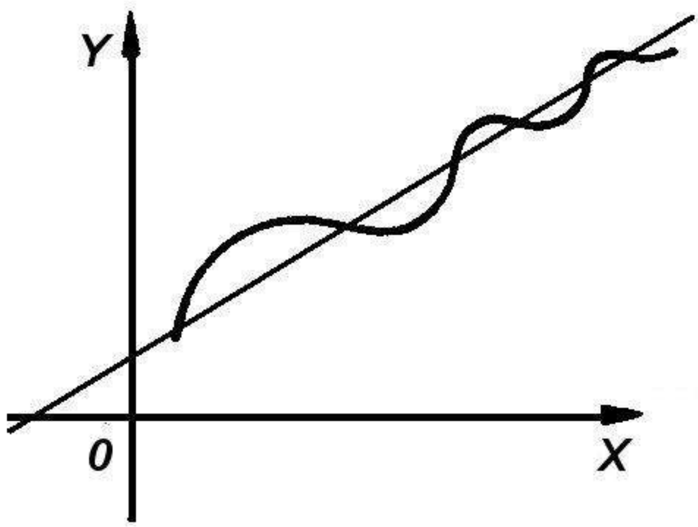

Fig. 33

The curve can be traced more accurately if an extra series of points on it are plotted. The coordinates of these points can be found from the equation of the curve.

1. We trace the curve

$$y = \frac{(x-3)^2}{4(x-1)}:$$

(a) x can vary in the interval $(-\infty, +\infty)$;

(b) Putting $x=0$, we get $y=-9/4$; putting $y=0$, we get $x=3$, i.e. the curve intercepts the axes at the points $(0,-9/4)$ and $(3,0)$;

(c) We find the first and second derivatives:

$$f^{/}(x)=\frac{(x-3)(x+1)}{4(x-1)^2}, \quad f^{//}(x)=\frac{2}{(x-1)^3}.$$

We apply the usual rule to find the peaks: $(3,0)$ is a minimum, $(-1,-2)$is a maximum;

(d) It is clear from the expression for the second derivative that it is positive for $x>1$, and negative for $x<1$, i.e. the curve is concave in the interval $(1,\infty)$, and convex in $(-\infty,1)$. There is no point of inflexion, since $f^{//}(x)$ changes sign only at $x=1$, where the curve has an asymptote parallel to OY, as we shall see next;

(e) y becomes infinite at $x=1$, and the curve has an asymptote:

$$x=1.$$

We now look for asymptotes, not parallel to OY:

$$a=\lim_{x\to\infty}\frac{(x-3)^2}{4(x-1)x}=\lim_{x\to\infty}\frac{\left(1-\frac{3}{x}\right)^2}{4\left(1-\frac{1}{x}\right)}=\frac{1}{4};$$

$$b=\lim_{x\to\infty}\left[\frac{(x-3)^2}{4(x-1)}-\frac{x}{4}\right]=\lim_{x\to\infty}\frac{-5x+9}{4(x-1)}=\lim_{x\to\infty}\frac{-5+\frac{9}{x}}{4\left(1-\frac{1}{x}\right)}=-\frac{5}{4}.$$

The asymptote is thus:

$$y=\frac{1}{4}x-\frac{5}{4}.$$

We propose to the reader the investigation of the position of the curve relative to the asymptotes;

(f) Symmetry does not exist.
Transferring all the data obtained to the figure, we get the curve of Fig. 34.

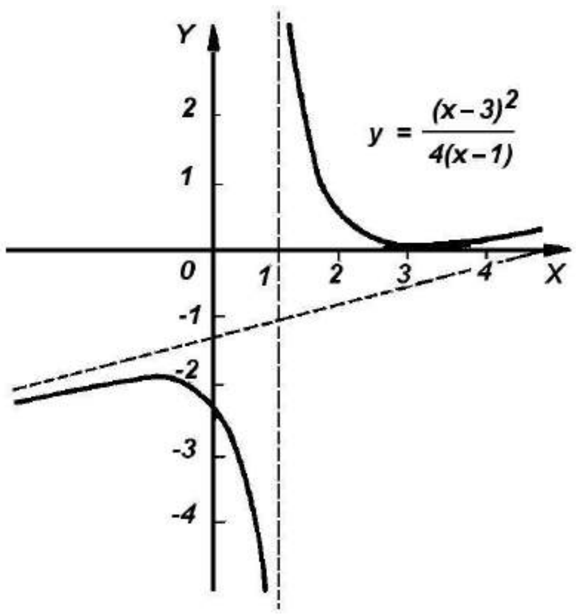

Fig. 34

2. We investigate the curves:

$$y = c\left(a^2 - x^2\right)\left(5a^2 - x^2\right) \quad (c < 0);$$
$$y_1 = c\left(a^2 - x^2\right)^2,$$

which give the shape of a heavy beam, bending under its own weight, the first curve referring to the case when the ends are freely supported, and the second to the case when the ends are constrained. The total length of the beam is $2a$, the origin is at the centre of the beam, and axis OY is directed vertically upwards:

(a) The variation of x evidently only interests us in the interval $(-a, +a)$;

(b) Putting $x = 0$, we get $y = 5ca^4$ and $y_1 = ca^4$, i.e. the bending at the centre of the beam is five times greater in the first case than in the second. For $x = \pm a$, $y = y_1 = 0$, corresponding to the ends of the beam;

(c) We find the derivatives:

$$y' = -4cx\left(3a^2 - x^2\right); \qquad y'' = -12c\left(a^2 - x^2\right);$$
$$y_1' = -4cx\left(a^2 - x^2\right); \qquad y_1'' = -4c\left(a^2 - 3x^2\right).$$

There will be a minimum at $x = 0$ in the interval $(-a, +a)$ in both cases, corresponding to the bending of the centre of the beam, mentioned above;

(d) In the first case, $y'' > 0$ in the interval $(-a, +a)$, i.e. the entire beam is concave upwards. In the second case, y_1'' vanishes at $x = -a/\sqrt{3}$; its sign changes here, and the corresponding points are points of inflexion of the beam;

(e) There are no infinite branches;

(f) Both equations remain unchanged on replacing x by $(-x)$, i.e. both curves are symmetrical about OY.

The two curves are shown in Fig. 35. We have taken the cases $a=1, c=-1$, for simplicity; the length of the beam is considerably greater than its bending, in practice, i.e. a is considerably greater than c, so that the curve, of bending looks rather different.

We suggest that the points of inflexion of the curve:

$$y = e^{-x^2}$$

might be found by the reader, and a comparison made with the graph of Fig. 11.

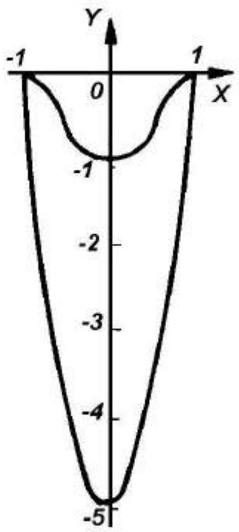

Fig. 35

The parameters of a curve

Given the properties of a geometrical locus, it is not always convenient or possible to find its equation by expressing its properties directly in terms of an equation connecting the current coordinates x and y. It is useful in this case to introduce a third, auxiliary variable, in terms of which the abscissa x and ordinate y of any given point of the locus can be separately expressed.

The combination of two equations obtained in this way:

$$x = \varphi(t), \qquad y = \psi(t) \tag{33}$$

can also be used for plotting and investigating a curve, since each value of t defines the position of a corresponding point of the curve. This method is referred to as *parametric representation of a curve*, the auxiliary variable t being a parameter. To obtain the equation of the curve in the usual (explicit or implicit) form as a relationship between x and y, the parameter t must be eliminated from equations (33), as might possibly be done by solving one of the equations with respect to t, and substituting the result in the other. Curves given by parameters are especially met with in mechanics, as when finding the trajectory of motion of a point, the position of which depends on time t, so that its coordinates are functions of t. The trajectory is given by a parameter, when these functions are known.

For instance, the equation of a circle with centre at (x_0, y_0) and radius r is given in terms of a parameter as:

$$x = x_0 + r\cos t; \qquad y = y_0 + r\sin t. \tag{34}$$

We rewrite these equations as:

$$x - x_0 = r\cos t; \qquad y - y_0 = r\sin t.$$

We eliminate the parameter t by squaring both sides and adding, which gives the ordinary equation of a circle:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Similarly, it is immediately obvious that

$$x = a\cos t; \qquad y = b\sin t \tag{35}$$

are the parametric equations of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let y be defined parametrically by formula (33) as a function of x.

An increment Δt of the parameter produces corresponding increments Δx and Δy, and by dividing numerator and denominator of Δx by Δt, we find the following expression for the derivative of y with respect to x:

$$y'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}} = \frac{\psi'(t)}{\varphi'(t)}$$

or

$$\frac{dy}{dx} = \frac{\psi'(t)}{\varphi'(t)}. \qquad (36)$$

We find the second derivative of y with respect to x:

$$y'' = \frac{d\left(\frac{dy}{dx}\right)}{dx}.$$

Using the rule for finding the differential of a fraction, we get:

$$y'' = \frac{d^2y \cdot dx - d^2x \cdot dy}{(dx)^3}. \qquad (37)$$

But by (33):

$$dx = \varphi'(t)dt; \qquad d^2x = \varphi''(t)dt^2;$$
$$dy = \psi'(t)dt; \qquad d^2y = \psi''(t)dt^2.$$

Substituting in (37) and cancelling dt^3, we finally have:

$$y'' = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{\left[\varphi'(t)\right]^2}. \qquad (38)$$

We remark that the expression (37) for y'' differs from the expression obtained early,

$$y'' = \frac{d^2y}{dx^2}, \qquad (39)$$

this latter formula being obtained when x is the independent variable, whereas t is the independent variable in the parametric form of (33). When x is the independent variable, dx is treated as constant, i.e. independent of x, so that $d^2x = d(dx) = 0$, being the differential of a constant. Formula (37) now reduces to (39). Now that we can determine y' and y'', we can supply information regarding the direction of the tangent to the curve and its convexity and concavity, etc.

Elements of a curve

We give basic formulae connected with the concepts of tangent and curvature of a curve, and introduce some new concepts associated with the concept of tangent.

If the equation of the curve has the form:

$$y = f(x), \tag{40}$$

the derivative $f^{/}(x)$ of y with respect to x is the slope of the tangent, the equation of which can be written in the form:

$$Y - y = y^{/}(X - x); \qquad \left(y^{/} = f^{/}(x)\right), \tag{41}$$

where (x, y) are the coordinates of the point of contact, and (X, Y) are the current coordinates of the tangent. *The normal to a curve at any point of it* (x, y) *is the perpendicular drawn through the point to the tangent at the point*. We know from analytic geometry that the perpendicular to a line has a slope equal to the reciprocal with changed sign, i.e., the slope of the normal is $\left(-1/y^{/}\right)$, and the equation of the normal can be written as:

$$Y - y = -\frac{1}{y^{/}}(X - x)$$

or

$$(X - x) + y^{/}(Y - y) = 0. \tag{42}$$

Let M be any point of the curve, and let T and N be the points of intersection of the tangent and normal to the curve at M with axis OX; also, let Q be the base of the perpendicular dropped from M to OX (Fig. 36).

The segments QT *and* QN *on* OX *are called respectively the subtangent and subnormal of the curve at* M; there axe definite numbers corresponding to these segments, positive or negative, depending on their direction along OX. The lengths of segments MT and MN are referred to respectively as the lengths of the tangent and normal to the curve at M, these lengths always being reckoned positive. The abscissa of Q on OX is evidently equal to the abscissa x of M. Since T and N are the points of intersection of the tangent and normal with OX, their abscissas must be found by setting $Y=0$ in the equations of the tangent and normal, then solving the equations obtained with respect to X. We thus get $\left(x-y/y'\right)$ for the abscissa of T, and $\left(x+yy'\right)$ for the abscissa of N. The magnitudes of the subtangent and subnormal are now easily found:

$$\left.\begin{aligned} \overline{QT}&=\overline{OT}-\overline{OQ}=x-\frac{y}{y'}-x=-\frac{y}{y'},\\ \overline{QN}&=\overline{ON}-\overline{OQ}=x+yy'-x=yy'. \end{aligned}\right\} \tag{43}$$

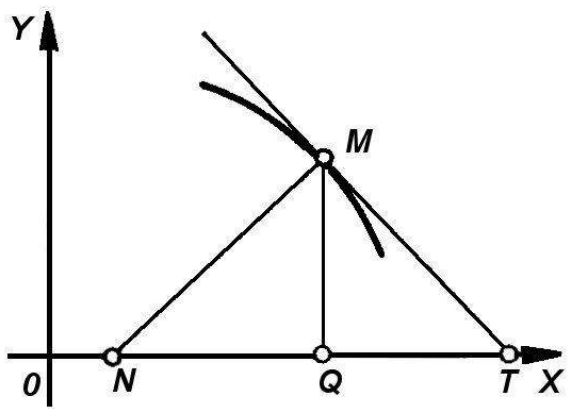

Fig. 36

The lengths of the tangent and normal can now be found from the right-angled triangles MQT and MQN:

$$\begin{aligned} \left|\overline{MT}\right|&=\sqrt{\overline{MQ}^2+\overline{QT}^2}\sqrt{y^2+\frac{y^2}{y'^2}}=\pm\frac{y}{y'}\sqrt{1+y'^2},\\ \left|\overline{MN}\right|&=\sqrt{\overline{MQ}^2+\overline{QN}^2}\sqrt{y^2+y^2y'^2}=\pm y\sqrt{1+y'^2}, \end{aligned} \tag{44}$$

where the sign (±) must be chosen so that the expression on the right-hand side is positive.

We recall the formula for the radius of curvature of a curve:

$$R=\pm\frac{\left(1+y'^2\right)^{3/2}}{y''}. \tag{45}$$

Denoting the length of the normal by n, we get from the second of formulae (44):

$$\sqrt{1+y'^2}=\pm\frac{n}{y},$$

and, on substituting this expression for $\sqrt{1+y'^2}$ in (45), we get the following further expression for the radius of curvature:

$$R=\pm\frac{n^3}{y^3 y''}. \tag{46}$$

If the curve is given parametrically:

$$x=\varphi(t);\quad y=\psi(t),$$

the first and second derivatives of y with respect to x are given by:

$$y'=\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)};$$

$$y''=\frac{d^2y dx-d^2x dy}{dy^3}=\frac{\psi''(t)\varphi'(t)-\varphi''(t)\psi'(t)}{\left[\varphi'(t)\right]^3}. \tag{47}$$

In particular, substituting this expression in (45), we get the following expression for the radius of curvature in this case:

$$R=\pm\frac{\left(dx^2+dy^2\right)^{3/2}}{d^2y dx-d^2x dy}=\pm\frac{\left\{\left[\varphi'(t)\right]^2+\left[\psi'(t)\right]^2\right\}^{3/2}}{\psi''(t)\varphi'(t)-\psi'(t)\varphi''(t)}=\pm\frac{ds}{d\alpha}, \tag{48}$$

where a is the angle formed by the tangent with OX.

Curves in polar coordinates

The position of a point M on the plane (Fig. 37) is defined in polar coordinates: (1) by its distance r from a given point o (the pole), and (2) by the angle θ between the direction of OM and some given direction L (the polar axis). It is usual to refer to r as the radius vector and to θ as the polar angle. If the polar axis is taken as axis OX, and the pole O as origin, we obviously have (Fig. 38):

$$x=r\cos\theta,\quad y=r\sin\theta. \tag{49}$$

To a given position of the point M there corresponds a single determinate positive value of r, but an infinite number of values of θ, differing by multiples of 2π. If M coincides with O, $r=0$, and θ is completely indeterminate.
Any functional relationship of the form

$$r = f(\theta) \tag{50}$$

has a corresponding graph in the polar system of coordinates.

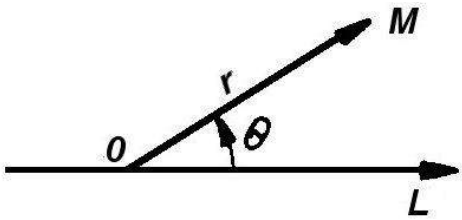

Fig. 37

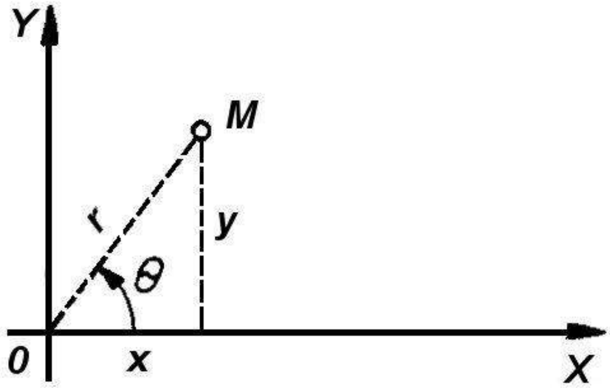

Fig. 38

We shall later consider negative, as well as positive, values of r, it being agreed to take r in the opposite direction to that corresponding to θ, in the case when the value of r corresponding to θ is negative.

Assuming that r is a function of θ for a given curve, equations (49) are seen to represent the parametric equations of this curve, where x and y depend on the parameter θ both directly and through the medium of r. We can thus apply formulae (47) and (48) in this case. On letting a denote the angle formed by the tangent with axis OX, we have by using the first of formulae (47):

$$\tan\alpha = y' = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta},$$

where r' denotes the derivative of r with respect to θ.

We now further introduce μ, the angle between the positive direction of the radius vector and the tangent to the curve (Fig. 39).
We have:

$$\mu = \alpha - \theta,$$

and hence:

$$\cos\mu = \cos\alpha\cos\theta + \sin\alpha\sin\theta ;$$
$$\sin\mu = \sin\alpha\cos\theta - \cos\alpha\sin\theta .$$

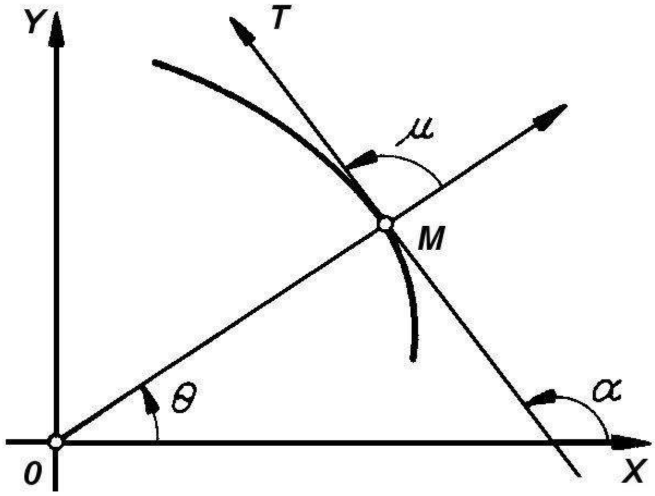

Fig. 39

Differentiating equations (39) with respect to s, and noting that dx/ds and dy/ds are respectively equal to $\cos\alpha$ and $\sin\alpha$, we get:

$$\cos\alpha = \cos\theta\frac{dr}{ds} - r\sin\theta\frac{d\theta}{ds}; \quad \sin\alpha = \sin\theta\frac{dr}{ds} + r\cos\theta\frac{d\theta}{ds} .$$

Substituting these expressions for $\cos\alpha$ and $\sin\alpha$ in the above expressions for $\cos\mu$, and $\sin\mu$, we have:

$$\cos\mu = \frac{dr}{ds}; \quad \sin\mu = \frac{rd\theta}{ds} \tag{51}$$

and hence:

$$\tan\mu = \frac{rd\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r}{r'} . \tag{52}$$

It follows from (39):

$$dx = \cos\theta dr - r\sin\theta d\theta ;$$
$$dy = \sin\theta dr + r\cos\theta d\theta ,$$

and thus:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dr)^2 + r^2(d\theta)^2}\,; \qquad (53)$$

using $a = \mu + \theta$, then dividing numerator and denominator by $d\theta$, we also have:

$$R = \pm\frac{ds}{d\alpha} = \frac{\left[(dr)^2 + r^2(d\theta)^2\right]^{1/2}}{d\mu + d\theta} = \pm\frac{\left(r^2 + r'^2\right)^{1/2}}{1 + \dfrac{d\mu}{d\theta}}.$$

We get by using (52)

$$\mu = \arctan\frac{r}{r'}, \frac{d\mu}{d\theta} = \frac{1}{1 + \left(\dfrac{r}{r'}\right)^2} \cdot \frac{r'^2 - rr''}{r'^2} = \frac{r'^2 - rr''}{r^2 + r'^2},$$

where r' and r'' are the first and second derivatives of r with respect to θ. Substituting this expression for $d\mu/d\theta$ in the above expression for R, we have:

$$R = \pm\frac{\left(r^2 + r'^2\right)^{3/2}}{r^2 + 2r'^2 - rr''}. \qquad (54)$$

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