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THE THEORY OF LIMITS. CONTINUOUS FUNCTIONS

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THE THEORY OF LIMITS. CONTINUOUS FUNCTIONS

Tutorial

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Посібник містить дві теми: теорія границь і неперервні функції. Наведено детальний теоретичний матеріал, до якого додаються зразки розв'язування типових задач. Наприкінці посібника подано задачі для самостійного виконання.

Видання відповідає програмі курсу «Вища математика» для технічних навчальних закладів.

Для студентів першого курсу з англійською мовою навчання.

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This manual contains two topics: the theory of limits and continuous functions.
The detailed theoretical material, which is accompanied by solved examples for typical problems, is given. At the end of the manual it is given the tasks for self-execution.

This manual corresponds to the program "Higher Mathematics" for technical schools.

For the first year students with English language tuition.

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Contents

1. ORDERED VARIABLES. SEQUENCES

When referring to the independent variable x, we have only been concerned with the set of the values that x can assume. For example, this can be the set of values satisfying $0 < x < 1$. We shall now consider the variable x taking an infinity of values in sequence, i.e. we are now interested, not only in the set of values of x, but also in the order in which it takes these values. More precisely, we assume the possibility of distinguishing, for every value of x, a value that precedes it and a value that follows one, it being also assumed that no value of the variable is the last, i.e. whatever value we take, there exists an infinity of successive values. A variable of this type is sometimes called *ordered*. If x', x'' are two values of the ordered variable x, a preceding and a succeeding value can be distinguished, whilst if x' precedes x'', and x'' precedes x''', then x' precedes x'''. We shall assume, for example, that the set of values of x is defined by $0 < x < 1$, and that of two distinct values x' and x'' the succeeding value is the greater. We thus obtain an ordered variable, continuously increasing through all real values from zero to unity, without reaching unity. The sequence of values of the variable, for phenomena occurring in time, is established by the temporal sequence, and we shall sometimes make use of this time-scheme below, using terms such as "previous" and "later" in place of "preceding" and "succeeding" values.

An important particular case of an ordered variable is that when the sequence of values of the variable can be enumerated, by arranging them in a series of the form:

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

so that, given two values x_p and x_q, the value succeeds that has the greater subscript. In the case mentioned above, when the variable increases from zero to unity, we can clearly not numerate its successive values. It may also be noted that it is possible to encounter identical values amongst those of an ordered variable. For example, we might have $x_3 = 7$ and $x_{12} = 7$ in the enumerated variable. Abstracting, as we always do, from the concrete nature of the magnitude (length, weight etc.), we must understand by the term "ordered variable", or as we shall say for brevity, "variable", simply the total sequence of its numerical values. We normally introduce one letter, say x, and suppose that it assumes successively the above-mentioned numerical values.

For every value of the variable x, a corresponding point *K* is defined on the axis *OX*. Thus, as x varies in sequence, the point *K* moves along *OX*.

The present book is devoted to the basic theory of limits, which is fundamental to all modern mathematical analysis. This theory considers some extremely simple, and at the same time, extremely important, cases of variation of magnitudes.

2. INFINITESIMALS

We assume that the point *K* constantly remains inside a certain interval of the axis *OX*. This is equivalent to the condition that the length of the interval $\overline{OK}$, where *O* is the origin, remains less than a definite positive number *M*. The magnitude x is said to be *bounded* in this case. Noting that the length of $\overline{OK}$ is $|x|$, we can give the following definition:

Definition. *A variable x is said to be bounded, if there exists a positive number M, such that $|x| < M$ for all values of x.*

We can take $x = \sin a$ as an example of a bounded magnitude, where the angle a varies in any manner. Here, *M* can be taken as any number greater than unity.

We now consider the case when the point *K* is displaced successively, and indefinitely approaches the origin. More precisely, we suppose that successive displacements of point *K* bring it inside any previously assigned small section $\overline{S^{/}S}$ of the axis *OX* with centre *O*, and that it remains inside this section on further displacement. In this case, we say that the *magnitude x tends to zero* or is an *infinitesimal*.

We denote the length of the interval $\overline{S^{/}S}$ by 2*ε*, where *ε* signifies any given positive number. If the point *K* is inside $\overline{S^{/}S}$, then $\overline{OK} < \varepsilon$ and conversely, if $\overline{OK} < \varepsilon$, *K* is inside $\overline{S^{/}S}$. We can thus give the following definition: *The variable x tends to zero or is an infinitesimal, if for any given positive ε there exists a value of x, such that for all subsequent values of x, $|x| < \varepsilon$.*

In view of the importance of the concept of infinitesimal, we give another formulation of the same definition.

Definition. *A magnitude x is said to tend to zero or to be an infinitesimal, if on successive variation $|x|$ becomes, and on further variation remains, less than any previously assigned small positive number ε.*

The term "infinitesimal" denotes the character of the variation of the variable described above, and the underlying concept is not to be confused with that of a *very small magnitude*, which is often employed in practice.

Suppose that, in measuring a certain tract of land, we obtained 1000 m, with some remainder that we considered very small in comparison with the total length, so that we neglected it. The length of this remainder is expressed by a definite positive number, and the term "infinitesimal" is evidently not applicable here. If we were to meet with the same remainder in a second, more accurate measurement, we should cease to consider it as very small, and we should take it into account. It is thus clear that the concept of a small magnitude is a relative concept, bound up with the practical nature of the measurement.

Suppose that the successive values of the variable x are

$$x_1, x_2, x_3, ..., x_n, ...$$

and let ε be any given positive number. To prove that x is an infinitesimal, we must show that, starting with a certain value of n, $|x_n|$ will be less than ε, i.e. we must be able to find a certain integer *N* such that

$$|x_n| < \varepsilon \text{ for } n > N.$$

This *N* depends on ε.

As an example of an infinitesimal, we take the magnitude assuming successively the values:

$$q, q^2, q^3, ..., q^n, ... (0 < q < 1) \quad (2.1)$$

We have to satisfy the inequality:

$$q^n < \varepsilon \text{ or } n \log_{10} q < \log_{10} \varepsilon.$$

Remembering that $\log_{10} q$ is negative, we can rewrite the above inequality as:

$$n > \frac{\log_{10} \varepsilon}{\log_{10} q},$$

since division by a negative number changes the sense of the inequality; thus we can now take *N* as the largest integer in the quotient $\log_{10} \varepsilon / \log_{10} q$. Thus the magnitude in question, or as we usually say, the sequence (1) tends to zero.

If we replace *q* by (—*q*) in the sequence (1), the only difference is the appearance of the minus sign with odd powers; the absolute magnitude of the members of the sequence is as before, and hence we also have an infinitesimal in this case.

The fact that x is infinitesimal is usually denoted by:

$$\lim x = 0 \text{ or } x \to 0.$$

Here, lim is an abbreviation of "limit"

We note two properties of infinitesimals.

1. *The sum of any (definite) number of infinitesimals is also an infinitesimal*.

Take, for example, the sum $w = x + y + z$ of three infinitesimals, and suppose that the variables are enumerated. Let

$$x_1, x_2, ...; y_1, y_2, ...; z_1, z_2, ...;$$

be the successive values of $x, y, z,$ respectively. We obtain successive values for w:

$$w_1 = x_1 + y_1 + z_1, w_2 = x_2 + y_2 + z_2, \ldots.$$

Let ε be any given positive number. Since x, y, z are infinitesimals, we can say that there exists N_1 such that $|x_n| < \varepsilon / 3$ for $n > N_1$; N_2, such that $|y_n| < \varepsilon / 3$ for $n > N_2$; and N_3, such that $|z_n| < \varepsilon / 3$ for $n > N_3$. If *N* denotes the greatest of N_1, N_2 and N_3, we have:

$$|x_n| < \frac{\varepsilon}{3}; \quad |y_n| < \frac{\varepsilon}{3}; \quad |z_n| < \frac{\varepsilon}{3} \text{ for } n > N$$

and hence:

$$|w_n| < |x_n| + |y_n| + |z_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \text{ for } n > N,$$

i.e. $|w_n| < \varepsilon$ for $n > N$, whence $w = x + y + z$ is an infinitesimal. In the general case of non-enumerated variables we can look at x, y, z as functions of some ordered variable $t : x = x(t)$, $y = y(t)$, $z = z(t)$. Variables x, y, z are themselves ordered, so that if $t = t^{/}$ precedes $t = t^{//}$, then $x(t^{/})$ precedes $x(t^{//})$, etc. The sum

$$w(t) = x(t) + y(t) + z(t),$$

obtained by adding the x, y, and z corresponding to the same value of t, is also ordered. The proof is as above, for enumerated variables. In this latter case, t has the role of subscript; or the subscript can be looked at as an increasing, integral t.

2. *The product of a bounded magnitude and an infinitesimal is an infinitesimal*.

We consider the product of the enumerated variables xy, where x is bounded, and y is an infinitesimal. We have the condition that $|x|$ remains less than some positive *M* for any n. If ε is any given positive number, there exists *N*, such that $|y_n| < \varepsilon / M$ for $n > N$.

Thus

$$|x_n y_n| = |x_n| \cdot |y_n| < M \cdot \frac{\varepsilon}{M} \text{ for } n > N.$$

Hence, $|x_n y_n| < \varepsilon$ for $n > N$, so that $xy \to 0$. The proof is analogous for non-enumerated variables.

We note that the second property is all the more readily justified if x is a constant. We can now take *M* as any positive number greater than $|x|$, i.e. the product of a constant and an infinitesimal is an infinitesimal.

In view of the fundamental importance of the concept of infinitesimal for what follows, we shall pause to add some remarks supplementary to the mentioned above.

As we have shown, a variable having the sequence of values (1), tends to zero, only if $0<q<1$ or $-1<q<0$. Setting $q=1/2$, for example, we obtain the sequence:

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},...,\frac{1}{2^n},...$$

Each successive value is less than the previous one in this case, and the variable tends to zero, diminishing all the time. Setting $q=1/2$ we obtain the sequence:

$$-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\frac{1}{16},...$$

Here the variable tends to zero, taking values in turn greater than, and less than, zero.

Suppose that we insert zero in every other place in the above sequence, i. e. we take a variable with the sequence:

$$\frac{1}{2},0,\frac{1}{4},0,\frac{1}{8},0,\frac{1}{16},0,\frac{1}{32},0,\frac{1}{64},0,...$$

Clearly, the variable in this case tends to zero, though in the process it takes exactly the value zero an infinite number of times. This does not contradict the definition of a magnitude tending to zero.

Finally, suppose that all the successive values of a variable are equal to zero. This also comes under the definition of a magnitude tending to zero, all the more since $|x|$ is now zero all the time, i.e. $|x|<\varepsilon$ for any given positive ε, not only from a certain initial point of its variation, but always. In other words, a constant equal to zero comes under the definition of an infinitesimal. No other constant whatever comes under the definition.

There is one further point. We recall the definition of infinitesimal: for any given positive ε, there exists a value of the variable x, such that for all subsequent values, $|x|<\varepsilon$. It follows immediately, that in proving that a given variable x tends to zero, we can confine ourselves to considering only those values of x that succeed a certain definite value of x, where this definite value can be chosen arbitrarily.

Concerning this, it is useful in the theory of limits to add a rider to the definition of a bounded magnitude, viz, there is no need to demand that $|y|<M$ for all values of y; it is sufficient to take the more general definition: *a magnitude y is said to be bounded, if there exists a positive number M and a value of y, such that $|y|<M$ for all subsequent values.*

The proof of the second property of infinitesimals remains unchanged with this definition of a bounded magnitude. For an enumerated variable, the first definition of a bounded magnitude follows from the second, so that the second is not less general. In fact, if $|x_n| < M$ for $n > N$, then denoting by $M^{/}$ the greatest of numbers

$|x_1|, |x_2|, ..., |x_N|$ and *M*,

we can assert that $|x_n| < M^{/} + 1$ for any *n*.

3. THE LIMIT OF A VARIABLE

We have called a variable an infinitesimal, if its corresponding point *K* in the axis *OX* has on displacement the following property: on successive variation the length of the interval $\overline{OK}$ becomes, and on further variation remains, less than any given positive number *ε*. We now suppose that this property is fulfilled, not by the interval $\overline{OK}$, but by $\overline{AK}$, where *A* is a definite point on the axis *OX* with abscissa *a* (Fig. 3.1).

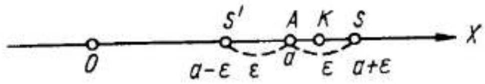

Fig. 3.1

In this case, the interval $\overline{S^{/}S}$ of length 2*ε* will have its centre at the point *A*, abscissa $x = a$, instead of at the origin, and the point *K* must come within this interval on successive displacement, then remain there on further displacement. We say in this case that the constant *a is the limit of variable* x, or that x tends to *a*.

Noting that the length of $\overline{AK}$ is $|a - x|$, we can formulate the following definition:

Definition. *The constant a is called the limit of the variable* x*,when the difference a — x (or x — a) is an infinitesimal.*

Having regard to the definition of an infinitesimal, a limit can be thus defined:

Definition. *The constant a is called the limit of the variable* x*,when we have the following property : for any given positive ε there exists a value of* x *such that, for all subsequent values,* $|a - x| < \varepsilon$.

We note some immediately obvious consequences of this definition, without dwelling on their detailed proof.

No variable can tend to two different limits, and not every variable has a limit. For example, the variable $\sin a$ oscillates between —1 and 1 on successive increase of the angle *a*, and has no limit.

The limit of an infinitesimal is zero.

If x *and* y *vary simultaneously, and each tends to a limit in the course of successive variation, whilst both always satisfy* $x < y$, *their limits* a *and* b *satisfy the condition* $a < b$.

We note here, that if the variables satisfy $x < y$, the sign of equality can be obtained for their limits, i.e. we have $a < b$.
If x, y, z vary simultaneously and always satisfy the condition $x < y < z$ on successive variation, and if x and z tend to the same limit a, y also tends to the limit a.
If a is the limit of x (or x tends to a), we write:

$$\lim x = a \text{ or } x \to a .$$

If x tends to a, the difference $x - a = \alpha$ is an infinitesimal, and we can write:

$$x = a + \alpha \qquad (3.1)$$

i.e. *every variable tending to a limit can be expressed as the sum of two terms: a constant term, equal to the limit, and an infinitesimal. Conversely, if a variable x can be expressed in the form (2), where a is a constant, and α is an infinitesimal, the difference $x - a$ will be an infinitesimal, and hence, a is the limit of x.*

If the sequence $x_1, x_2, ...$ tends to the limit a, every infinite subsequence $x_{n_1}, x_{n_2}, ...$ contained in the first sequence, also tends to a. In this subsequence, the subscript n_k increases with increasing k and runs through some part of the set of positive integers. There is no analogous property, generally speaking, for a non-enumerated variable tending to a limit.

We take as an example the variable x with the sequence of values:

$$x_1 = 0.1, x_2 = 0.11, x_3 = 0.111, ..., x_n = 0.11...11, ...,$$

and we show that its limit is $1/9$. We first form the difference $1/9 - x_n$:

$$\frac{1}{9} - x_1 = \frac{1}{90}, \frac{1}{9} - x_2 = \frac{1}{900}, \frac{1}{9} - x_3 = \frac{1}{9000}, ..., \frac{1}{9} - x_n = \frac{1}{9 \cdot 10^n}.$$

The condition:

$$\frac{1}{9 \cdot 10^n} < \varepsilon$$

is evidently equivalent to the condition:

$$9 \times 10^n > \frac{1}{\varepsilon} \text{ or } n > \log_{10} \frac{1}{\varepsilon} - \log_{10} 9,$$

and we can take *N* as the greatest integer contained in the difference $\log_{10} 1/\varepsilon - \log_{10} 9$. In this example, the difference $1/9 - x_n$ is a positive number for every n, i. e. x tends to the limit $1/9$ whilst always remaining less than it.

We now consider the sum of the first n members of the indefinitely diminishing geometrical progression:

$$s_n = b + bq + bq^2 + bq^3 + ... + bq^{n-1} \quad (0 < |q| < 1).$$

As we know,

$$s_n = \frac{b(1-q^n)}{1-q}.$$

Setting $n = 1, 2, 3, \ldots$, we obtain the sequence:

$$s_1, s_2, s_3, \ldots, s_k, \ldots$$

We have from the expression for s_n:

$$\frac{b}{1-q} - s_n = \frac{bq^n}{1-q}.$$

The right-hand side consists of the product of a constant $b/(1-q)$ and an infinitesimal q^n. Hence, using the second property of infinitesimals, the difference $b/(1-q) - s_n$ is an infinitesimal, and we can say that the constant $b/(1-q)$ is the limit of the sequence

$$s_1, s_2, s_3, \ldots, s_k, \ldots.$$

Suppose that $b > 0$ and $q < 0$. The difference $b/(1-q) - s_n$ is now positive for even n and negative for odd n, so that the variable s_n is alternately greater than, and less than, the limit to which it tends.

The same remarks apply in the case of magnitudes that tend to a given limit as were made in the previous paragraph, *apropos* magnitudes that tend to zero.

Any constant, equal to the number a, comes under the definition of a variable, tending to the limit a. We note, here, that a magnitude, all of those values are equal to a, has in the ordinary way an infinite set of values, though all these values are equal to the same number. This view of a constant as a particular case of a variable comes in useful later on.

Furthermore, there is no need to consider all the values of a variable x when defining its limit; we need only take values subsequent to some arbitrarily given value.

Another point: if a variable x tends to a limit a, it will differ from a by as little as is desired, after a certain initial moment of its variation, and hence it is all the more a bounded variable.

An ordered variable does not always have a limit, as already mentioned. If we take, for example, the enumerated variable

$$x_1 = 0.1, x_2 = 0.11, x_3 = 0.111, \ldots, x_n = 0.11\ldots11, \ldots,$$

those limit is $1/9$, and the variable $y_1 = 1/2, y_2 = 1/2^2, y_3 = 1/2^3, \ldots$ those limit is zero, the enumerated variable

$$z_1 = 0.1, z_2 = 1/2, z_3 = 0.11, z_4 = 1/2^2, z_5 = 0.111, z_6 = 1/2^3; \ldots, ,$$

does not tend to a limit. The sequence of its values $z_1, z_3, z_5, \ldots$ has the limit $1/9$, and the sequence $z_2, z_4, z_6, \ldots$ has the limit zero.

4. BASIC THEOREMS

1. *The limit of the algebraic sum of a finite number of variables is equal to the sum of their limits.*

For the sake of exactness let us take the algebraic sum $x-y+z$ of three simultaneously varying magnitudes. We suppose that x, y and z tend respectively to limits a, b and c. We show that the sum tends to the limit $a-b+c$.

We have by hypothesis :

$$x=a+\alpha, y=b+\beta, z=c+\gamma,$$

where α, β, γ are infinitesimals. We can write for the sum:

$$x-y+z=(a+\alpha)-(b+\beta)+(c+\gamma)=(a-b+c)+(\alpha-\beta+\gamma).$$

The first bracket on the right-hand side of this equation is a con-constant, and the second is an infinitesimal. Hence:

$$\lim(x-y+z)=a-b+c=\lim x-\lim y+\lim z.$$

2. *The limit of the product of a finite number of variables is equal to the product of their limits.*

We confine ourselves to the case of the product xy of two variables. We suppose that x and y vary simultaneously, tending respectively to limits a and b, and we show that xy tends to the limit ab.

We have by hypothesis:

$$x=a+\alpha, y=b+\beta,$$

where a and β are infinitesimals; hence:

$$xy=(a+\alpha)(b+\beta)=ab+(a\beta+b\alpha+\alpha\beta).$$

Using both of the properties of infinitesimals from, we see that the sum in the bracket on the right of this equation is an infinitesimal, and hence we have:

$$\lim(xy)=ab=\lim x\cdot\lim y.$$

3. *The limit of a quotient is equal to the quotient of the limits, provided the limit of the denominator is not zero.*

We take the quotient x/y, and suppose that x and y tend simultaneously to their respective limits a and b, where $b\neq 0$. We show that x/y tends to a/b.

To prove the theorem, it is sufficient to show that the difference $a/b-x/y$ is an infinitesimal. By hypothesis:

$$x=a+\alpha; y=b+\beta \quad (b\neq 0),$$

where α and β are infinitesimals. Hence:

$$\frac{a}{b}-\frac{x}{y}=\frac{1}{b(b+\beta)}\cdot(a\beta-b\alpha).$$

The denominator of the fraction on the right of this equation is the product of two factors, and tends to b^2. Thus, from some initial moment of its variation, it is greater than $b^2/2$, the fraction as a whole being included between zero and $2/b^2$, i.e. the fraction is bounded. The term $(a\beta - b\alpha)$ is an infinitesimal. Hence, the difference $a/b - x/y$ is an infinitesimal, and

$$\lim \frac{x}{y} = \frac{a}{b} = \frac{\lim x}{\lim y}.$$

The theorems proved are of fundamental importance in the theory of limits. The proofs have been given for the general case, and not for the case of enumerated variables, as when proving the properties of infinitesimals. But the remark we made when proving the first property of infinitesimals should be borne in mind. Take the case of a product. We take x and y as functions of some ordered variable t: $x = x(t); y = y(t)$. Then x and y are themselves ordered variables. The same can be said of their product: $w(t) = x(t) \cdot y(t)$. The subscript plays the part of t in enumerated variables, increasing through integral values.

We remark further, that the above theorems establish the existence of the limit of a sum, a product and a fraction. For example, the third theorem can be stated more fully as: if numerator and denominator tend to limits, and the limit of the denominator differs from zero, the quotient then tends to a limit, and this limit is the quotient of the limits of numerator and denominator.

We note some consequences of these theorems. If x tends to the limit a, then bx^k, where b is a constant and k a positive integer, tends to the limit ba^k, in accordance with Theorem 2.

Consider the integral polynomial

$$f(x) = a_0x^m + a_1x^{m-1} + ... + a_kx^{m-k} + ... + a_{m-1}x + a_m,$$

with constant coefficients a_k. Using Theorem 1 and the previous remark, we can say that this polynomial tends to a limit:

$\lim f(x) = f(a)$ as x tends to a where

$$f(a) = a_0a^m + a_1a^{m-1} + ... + a_ka^{m-k} + ... + a_{m-1}a + a_m. \qquad (4.1)$$

.

Similarly, as x tends to a, the rational fraction:

$$\varphi(x) = \frac{a_0x^m + a_1x^{m-1} + ... + a_kx^{m-k} + ... + a_{m-1}x + a_m}{b_0x^p + b_1x^{p-1} + ... + b_{p-1}x + b_p}$$

tends to a limit:

$$\lim \varphi(x) = \varphi(a) = \frac{a_0a^m + a_1a^{m-1} + ... + a_ka^{m-k} + ... + a_{m-1}a + a_m}{b_0a^p + b_1a^{p-1} + ... + b_{p-1}a + b_p} \qquad (4.2)$$

$$\text{if } b_0a^p + b_1a^{p-1} + ... + b_{p-1}a + b_p \neq 0.$$

All these remarks are valid, in whatever way x tends to its limit a.

We can of course take polynomials arranged in powers of several variables, all tending to limits, instead of polynomials arranged in powers of a single variable.

For example, if $\lim x = a$ and $\lim y = b$, then

$$\lim(x^2 + xy + y^2) = a^2 + ab + b^2.$$

5. INFINITELY LARGE MAGNITUDES

If the variable x tends to a limit, it is evidently bounded, as already remarked. We now consider some cases of variation of unbounded magnitudes.

As before, we shall take along with x its corresponding point *K*, displaced on the axis *OX*. Let the point *K* move in such a way that, however large an interval $\overline{T'T}$ we take, with the origin as centre, the point *K* will eventually be displaced outside it, and from then on will remain outside. In this case, x is an infinitely large magnitude, and tends to infinity. Let 2*M* be the length of the interval $\overline{T'T}$. Recalling that the length of the interval $\overline{OK} = |x|$, we can give the following definition:

The magnitude x is said to be infinitely large, or to tend to infinity, if on successive variation of x, $|x|$ becomes, and on further variation remains, greater than any given positive number M. In other words, the magnitude x is called infinitely large if it satisfies the following condition:*given any positive number M, there exists a value of x such that, for all subsequent x, $|x| > M$*.

In particular, if x is infinitely large, and always remains positive during its successive variation as from a certain initial value (point *K* to the right of *O*), we say that x tends to plus infinity ($+\infty$). Similarly, if x remains negative (point *K* to the left of *O*), we say that x tends to minus infinity ($-\infty$).

The following symbols are used for infinitely large magnitudes:

$$\lim x = \infty, \quad \lim x = +\infty, \quad \lim x = -\infty$$

or

$$x \to \infty, \quad x \to +\infty, \quad x \to -\infty.$$

The term "infinitely large" serves merely as a brief designation for the character of variation described above of the variable x, and here, as with the concept of infinitesimal, a distinction must be made between the concepts of "infinitely large" and "very large" magnitudes.

If, for example, x takes the sequence of values 1, 2, 3, ... then evidently, $x \to +\infty$. If its sequence of values is: —1, —2, —3, ...,then

$x \to -\infty$. And finally, if the sequence is: —1, 2, —3, 4, . . ., we can write: $x \to \infty$.

Let us take as a further example the magnitude with the sequence of values:

$$q, q^2, q^3, ..., q^n, ..., \quad (q > 1), \tag{5.1}$$

and let *M* be any given positive number. The condition

$$q^n > M$$

is equivalent to

$$n > \frac{\log_{10} M}{\log_{10} q},$$

and hence, if *N* is the greatest integer contained in the quotient $\log_{10} M \cdot \log_{10} q$, we have:

$$q^n > M \text{ for } n > N,$$

i.e. the variable in question tends to $+\infty$.

If q is replaced by $(-q)$ in the sequence (5), the only change is in the signs of odd powers of q, the absolute values of the members of the sequence remaining as before; thus, for negative q, with absolute value greater than unity, the sequence (5) tends to infinity.

When in future we say that a variable tends to a limit, a finite limit is to be understood. It is occasionally said that a variable "tends to an infinite limit", implying by these words an infinitely large magnitude.

An immediate consequence of the above definitions is: if variable x tends to zero, then m/x, where m is a given constant, differing from zero, tends to infinity; and if x tends to infinity, m/x tends to zero.

6. MONOTONIC VARIABLES

The important thing is often to show that a given variable tends to a limit, without necessarily being able to discover what this limit actually is. We now outline an important test for the existence of a limit.

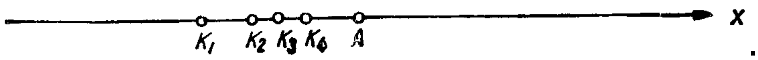

Fig. 6.1

We suppose that the variable x is always increasing (more precisely never decreasing) or else always decreasing (more precisely, never increasing). In the first case, any given value is not less than all preceding values, and not greater than all subsequent values. In the second case, any given value is not greater than all preceding, and not less than all succeeding, values. We speak of *monotonic variation* in these cases.

Point *K* on the axis *OX*, corresponding to x, is now displaced in a single direction, positively, if x increases, and negatively, if x decreases. It is obvious at once that only two possibilities can arise: either *K* moves away indefinitely along the line ($x \to +\infty$ or $-\infty$), or *K* indefinitely approaches some definite point *A* (Fig. 6.1), i.e. x tends to a limit. If x is known to be bounded, as well as varying monotonically, the first possibility drops out, and it can be asserted that the variable tends to a limit.

This argument is based on intuition, and evidently lacks the force of a proof. We shall not give the rigorous proof in the book.

The above test for the existence of a limit is usually formulated as follows: *if a variable is bounded and varies monotonically, it tends to a limit.*

Take the example of the sequence:

$$u_1 = \frac{x}{1},\ \ u_2 = \frac{x^2}{2!},\ \ u_3 = \frac{x^3}{3!}, ...,\ \ u_n = \frac{x^n}{n!}, ..., \qquad (6.1)$$

where x is a given positive number.

We have:

$$u_n = u_{n-1}\frac{x}{n}. \qquad (6.2)$$

For $n > x$, x/n is less than unity, and $u_n < u_{n-1}$,i.e. from some initial value, u_n is always decreasing for n increasing, whilst remaining greater than zero. The variable thus tends to some limit u, in accordance with the test for the existence of a limit. Let the integer n increase indefinitely in equation (7). We obtain in the limit:

$$u = u \cdot 0 \ \text{ or } \ u = 0,$$

i.e.

$$\lim_{n \to +\infty} \frac{x^n}{n!}. \qquad (6.3)$$

If we replace x by $(-x)$ in sequence (6), the only change is in the sign of members with odd n, so that the new sequence also tends to zero, i.e. equation (8) is valid for any given x, positive or negative.

We obtain the limit in this example, after first showing that it exists. If we did not show its existence, our method could lead to a false result. Consider, for instance, the sequence:

$$u_1 = q,\ \ u_2 = q^2,\ \ u_3 = q^3, ...,\ \ u_n = q^n, ..., (q > 1).$$

We have obviously:

$$u_n = u_{n-1}q.$$

We denote the limit of u_n by u, without troubling about its existence. On transition to the limit in the above equation, we obtain:

$$u = uq \text{ ,i.e. } u(1-q) = 0,$$

and hence,

$$u = 0.$$

But this result is false, since we know that for $q > 1$, $\lim q^n = +\infty$.

7. CAUCHY'S TEST FOR THE EXISTENCE OF A LIMIT

The French mathematician Cauchy gave a necessary and sufficient condition for the existence of a limit, which we shall now formulate. If the limit is known, it is characterized by the fact that, starting with a certain value of the variable, the absolute value of the difference between the limit and the variable is less than any given positive *ε*. According to Cauchy's test, a necessary and sufficient condition for a limit to exist is that, starting from a certain value of the variable, the difference between any two successive values of the variable is less than any given positive *ε*. We formulate this rigorously:

Cauchy's test. *A necessary and sufficient condition for a variable* x *to have a limit is that, given any positive number ε, there exists a value of* x *such that, for any two successive values* $x^{/}$ *and* $x^{//}$*, we have* $\left| x^{/} - x^{//} \right| < \varepsilon$.

Suppose that we have the enumerated variable

$$x_1, x_2, ..., x_n, ...$$

According to Cauchy's test, a necessary and sufficient condition for this sequence to have a limit is that, given any positive *ε*, there exists an *N* (depending on *ε*) such that

$$\left| x_m - x_n \right| < \varepsilon \text{, for } m \text{ and } n > N. \quad (7.1)$$

It is easy to show that this condition is necessary. If our sequence has the limit a, we write $x_m - x_n = (x_m - a) + (a - x_n)$, whence it follows:

$$\left| x_m - x_n \right| \le \left| x_m - a \right| + \left| a - x_n \right|.$$

But, by definition of a limit, there exists *N* such that $\left| x_m - a \right| < \varepsilon / 2$ and $\left| a - x_n \right| < \varepsilon / 2$ for m and $n > N$, and therefore $\left| x_m - x_n \right| < \varepsilon$ for m and $n > N$. To put the matter briefly, values of x lying arbitrarily close to a lie arbitrarily close to each other.

We avoid a rigorous proof of the sufficiency of Cauchy's test and give a descriptive explanation instead (Fig. 7.1).

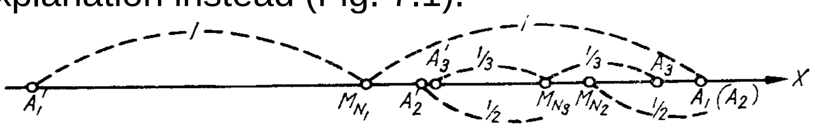

Fig. 7.1

Let M_s be a point of the coordinate axis corresponding to the number x_s. Suppose that condition (9) is fulfilled. In accordance with this condition, there exists a value $N = N_1$ such that

$$\left| x_s - x_{N_1} \right| < 1,$$

for $s > N_1$, i.e. every point M_s, where $s > N_1$, lies inside the interval $\overline{A_1^/ A_1}$, the length of which is equal to two and the mid-point of which corresponds to x_{N_1}.

Similarly, there exists a value $N = N_2 \geq N_1$, such that

$$\left| x_s - x_{N_2} \right| < \frac{1}{2} \text{ for } s > N_2 .$$

We construct an interval, of length unity, with mid-point M_{N_2}; and we let $\overline{A_2^/ A_2}$ be the part of this interval belonging to $\overline{A_1^/ A_1}$.

By virtue of the two conditions above, the point M_s must lie inside interval $\overline{A_2^/ A_2}$ for $s > N_2$.

Similarly there exists $N = N_3 \geq N_2$, such that $\left| x_s - x_{N_3} \right| < \dfrac{1}{3}$ for $s > N_3$. We proceed as before, and construct $\overline{A_3^/ A_3}$, with length not exceeding 2/3 and belonging to $\overline{A_2^/ A_2}$, all values of M_s being interior points of it for $s > N_3$. Setting $\varepsilon = 1/4, 1/5, ..., 1/n, ...$, we obtain in this way a sequence of intervals $\overline{A_n^/ A_n}$, each successive member of which is comprised in the previous member, whilst the length of the members tends to zero. The ends of these intervals obviously tend to the same point *A*, and the number a corresponding to this point is the limit of the variable x, since it follows from the construction described above that, for a sufficiently large value of s, all the points M_s will lie as close as desired to the point *A*.

As an application of Cauchy's test, we take Kepler's equation, which defines the position of a planet in its orbit. This equation has the form:

$x = q \sin x + a$,

where a and q are given numbers, both lying between zero and unity, and x is unknown.

We take an arbitrary x_0 and construct a sequence of numbers:

$$x_1 = q \sin x_0 + a,\ x_2 = q \sin x_1 + a, ...,\ x_n = q \sin x_{n-1} + a,\ x_{n+1} = q \sin x_n + a$$

Subtracting the first equation from the second term by term, we obtain:

$$x_2 - x_1 = q(\sin x_1 - \sin x_0) = 2q \sin \frac{x_1 - x_0}{2} \cos \frac{x_1 + x_0}{2},$$

Noting that $\left| \sin a \right| \leq \left| a \right|$ and $\left| \cos a \right| \leq 1$, we have:

$$|x_2 - x_1| \le 2q\frac{|x_1 - x_0|}{2} = q|x_1 - x_0| \qquad (7.2)$$

We can find in precisely the same way that

$$|x_3 - x_2| \le q|x_2 - x_1|,$$

so that, using (10), we can write:

$$|x_3 - x_2| \le q^2|x_1 - x_0|$$

Proceeding in this manner, we obtain for every n the condition:

$$|x_{n+1} - x_n| \le q^n|x_1 - x_0| \qquad (7.3)$$

We now consider the difference $x_m - x_n$, taking $m > n$ for the sake of clarity:

$$x_m - x_n = x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - x_{m-3} + ... + x_{n+1} - x_n.$$

Using (11), and the formula for the sum of the terms of a geometrical progression, we may write:

$$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + |x_{m-2} - x_{m-3}| + ... + |x_{n+1} - x_n| \le$$

$$\le (q^{m-1} + q^{m-2} + q^{m-3} + ... + q^n)|x_1 - x_0| = q^n\frac{1-q^{m-n}}{1-q}|x_1 - x_0|$$

As n tends to infinity, q^n tends to zero ;$|x_1 - x_0|$ is constant; the fraction $(1-q^{m-n})/(1-q)$ always lies between zero and $1/(1-q)$, i.e. is bounded, since, for $m > n$, q^{m-n} lies between zero and unity. Thus, with indefinite increase of n, and any $m > n$, the difference $x_m - x_n$ tends to zero, and condition (9) is fulfilled. We can say, in accordance with Cauchy's test, that a limit exists:

$$\lim_{n\to+\infty} x_n = \xi.$$

We let n tend to infinity in the equation

$$x_{n+1} = q\sin x_n + a.$$

Using the continuity of the function $\sin x$, we find in the limit:

$$\xi = q\sin\xi + a \qquad (7.4)$$

i.e. the limit ξ of the variable x_n is also the root of Kepler's equation.

We started with an arbitrary x_0 in constructing the sequence x_n. We show, however, that Kepler's equation does not possess two roots, i.e. that $x_n = \xi$ is independent of the choice of x_0, and is equal to the *single* root of Kepler's equation.

We assume there is a root ξ_1 in addition to the root ξ, so that:

$$\xi_1 = q\sin\xi_1 + a.$$

Subtracting equation (12) term by term from this equation, we obtain:

$$\xi_1 - \xi = q(\sin\xi_1 - \sin\xi) = 2q\sin\frac{\xi_1 - \xi}{2}\cos\frac{\xi_1 + \xi}{2},$$

whence, as before,

$$|\xi_1 - \xi| \le q|\xi_1 - \xi|.$$

But q lies between zero and unity, so that the above relationship is only possible for $\xi_1 - \xi = 0$, i.e. $\xi_1 = \xi$, and hence Kepler's equation has only one root ξ.

8. SIMULTANEOUS VARIATION OF TWO VARIABLES, CONNECTED BY A FUNCTIONAL RELATIONSHIP

We consider two variables x and y, connected by the functional relationship:

$$y = f(x)$$

and we let $f(x)$ be defined to the left and right of the point $x = c$. We shall assume that x increases and passes through all real values as it tends to c, without in fact reaching c. In this case, $f(x)$ is an ordered variable. We suppose that it has a limit *A*.

This is usually written as follows:

$$\lim_{x\to c-0} y = \lim_{x\to c-0} f(x) = A, \tag{8.1}$$

where the symbol $x \to c-0$ indicates that x tends to c from the side of lower values.

Similarly, if x tends to c whilst diminishing and passing through all real values, and if $f(x)$ now tends to the limit *B*, we write this as:

$$\lim_{x\to c+0} y = \lim_{x\to c+0} f(x) = B, \tag{8.2}$$

The existence of the limit (8.1) is evidently equivalent to $f(x)$ coming as close as desired to the number *A*, when x comes sufficiently close to the number c, whilst remaining less than c, i.e. (8.1) is equivalent to the following: *for any given positive number ε there exists a positive number η such that*

$$|A - f(x)| < \varepsilon \text{ as soon as } 0 < c - x < \eta.$$

Of course, η depends on ε.

In precisely the same way, (8.2) is equivalent to: *for any given positive number ε there exists a positive number η such that*

$$|B - f(x)| < \varepsilon \text{ as soon as } 0 < x - c < \eta.$$

If limits *A* and *B* are equal, we write this as follows:

$$\lim_{x\to c} y = \lim_{x\to c} f(x) = A. \tag{8.3}$$

It is immaterial here, whether x is on one side of c or the other, and (8.3) implies: for any given positive ε there exists a positive η such that

$$|A - f(x)| < \varepsilon \text{ as soon as } |c - x| < \eta \text{ and } x \neq c. \qquad (8.4)$$

Limit (13) is often denoted by the symbol $f(c-0)$ and limit (8.2) by $f(c+0)$:

$$\lim_{x\to c-0} f(x) = f(c-0); \quad \lim_{x\to c+0} f(x) = f(c+0).$$

Symbols $f(c-0)$ and $f(c+0)$ should be distinguished from $f(c)$, i.e. the value of $f(x)$ for $x = c$. This latter value can differ from $f(c-0)$ and $f(c+0)$, or in fact can be entirely meaningless. The limits $f(c-0)$ and $f(c+0)$ exist in the case of functions having graphs with no discontinuities, when we obviously have: $f(c-0) = f(c+0) = f(c)$, i.e. $\lim_{x\to c} f(x) = f(c)$.

We say in this case that *the function* $f(x)$ *is continuous for* $x = c$ *(at the point* $x = c$*)*. We shall consider the properties of continuous functions in detail later.

We return to the general case. The above definitions are easily generalized for the case when x or $f(x)$ tends to infinity. It is easy to see, for example, on the basis of what has been said, that

$$\lim_{x\to c-0} \frac{1}{x-c} = -\infty; \quad \lim_{x\to c+0} \frac{1}{x-c} = +\infty,$$

$$\lim_{x\to \frac{\pi}{2}-0} \tan x = +\infty; \quad \lim_{x\to \frac{\pi}{2}+0} \tan x = -\infty$$

Taking the principal values of the function $y = \arctan x$, we can write:

$$\lim_{x\to c-0} \arctan\frac{1}{x-c} = -\frac{\pi}{2};$$

$$\lim_{x\to c+0} \arctan\frac{1}{x-c} = \frac{\pi}{2}.$$

If $f(x)$ is defined for all sufficiently large x, the limit can exist:

$$\lim_{x\to +\infty} f(x) = A.$$

If $f(x)$ is defined for all x, either positive or negative, that are sufficiently large in absolute value, the limit can exist:

$$\lim_{x\to \infty} f(x) = A.$$

The latter is equivalent to: for any given positive number ε there exists a positive number *M*, such that

$$|A - f(x)| < \varepsilon \text{ for } |x| > M.$$

The following equations may easily be verified:

$$\lim_{x\to+\infty} x^3 = +\infty;\ \lim_{x\to-\infty} x^3 = -\infty;\ \lim_{x\to\infty}\frac{1}{x} = 0;\ \lim_{x\to\infty} x^2 = +\infty.$$

We also take an example from physics. Suppose that we heat a certain solid, and let t_0 be its initial temperature. The temperature of the body rises on heating, until the melting point is reached. The temperature now remains constant on further heating, till the point when the whole of the substance has passed over to the liquid state; after this, the temperature-rise begins again, in the resultant liquid. The situation is similar on passage from the liquid to the gaseous state. We shall consider the amount of heat *Q* communicated to the substance as a function of the temperature. Figure (8.1) shows the graph of this function, with temperature on the horizontal axis, and the amount of heat absorbed on the vertical axis. Let t_1 be the temperature at which transition to the liquid state begins, and t_2 the temperature at which the transition from the liquid to the gaseous state begins. Evidently:

$$\lim_{t\to t_1-0} Q = ord.\overline{AB} \text{ and } \lim_{t\to t_1+0} Q = ord.\overline{AC}.$$

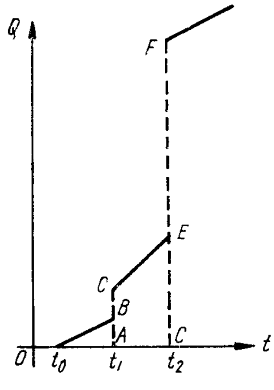

Fig. 8.1

The size of the segment $\overline{BC}$ gives the latent heat of fusion, and that of $\overline{EF}$ the latent heat of vaporization.

If limits $f(c-0)$ and $f(c+0)$ exist and differ, their difference $f(c+0)-f(c-0)$ is called the break, or jump, of function $f(x)$ at x = c (at the point x = c).

The function $y = \arctan 1/(x-c)$ has a jump of π at $x = c$. The function $Q(t)$ just considered has a jump equal to the latent heat of fusion at the melting-point $t = t_1$.

In defining the limit of $f(x)$ as x tends to c, we assumed that x never actually coincides with c. This proviso is made, since the value of $f(x)$ for $x = c$ either sometimes does not exist, or else has nothing in common with the values of $f(x)$ for x close to c. The function $Q(t)$, for example, is not defined for $t = t_1$.

Another explanatory example may be given. We assume that a function is defined as follows in the interval (—1. +1):

$$y = x + 1 \quad \text{for} \quad -1 < x < 0;$$

$$y = x - 1 \quad \text{for} \quad 0 < x < 1; \; y = 0 \quad \text{for} \; x = 0.$$

Fig. 8.2 shows the graph of this function; it consists of two straight sections, with their ends excluded (for $x = 0$), and a single isolated point, the origin. We now have:

$$\lim_{x \to -0} f(x) = 1; \;\; \lim_{x \to +0} f(x) = -1; \;\; f(0) = 0.$$

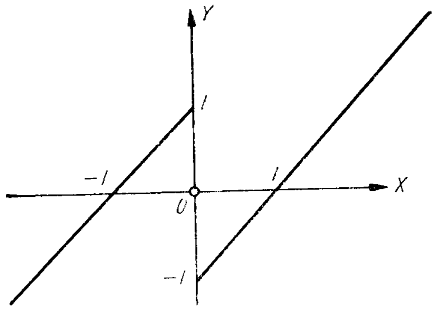

Fig. 8.2

9. IMPORTANT LIMIT $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

We consider an example that is important later on. We take

$$y = \frac{\sin x}{x}.$$

This function is defined for all x, other than $x = 0$, for which both numerator and denominator become zero, so that the fraction loses its meaning. We shall see how y varies as x tends to zero. The magnitude of the fraction does not change when x changes sign, so that it is sufficient to find the limit of the fraction as x tends to zero through positive values, i.e. in the first quadrant. This limit exists, as we shall show. From the above remarks, the same limit is obtained for x tending to zero through negative values. We note that the theorem regarding the limit of a quotient cannot be used, since the denominator tends to zero as $x \to 0$.

We shall take x as the angle subtended at the centre of a circle of unit radius. Measuring angle in radians, we have

$$\sin x = \overline{AC}, \; x = \frac{1}{2} arc\overline{AB}, \; \tan x = \overline{AD},$$

where $\overline{AD}$ is the tangent to the circle at the end of arc x (see Fig 9.1).

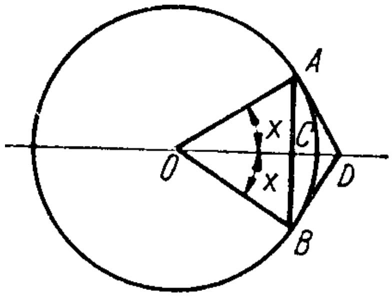

Fig. 9.1

Since the length of the arc is intermediate between the length of the chord and the sum of the tangents, we can write:

$$2\sin x < 2x < 2\tan x,$$

whence, dividing by $2\sin x$, we have:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or

$$1 > \frac{\sin x}{x} > \cos x \qquad (9.1)$$

But as x tends to zero, $\cos x$, given by the distance $\overline{OC}$, evidently tends to unity, i.e. the variable $\sin x / x$ always lies between unity and a magnitude tending to unity, and hence:

$$\lim_{x \to 0} y = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We determine for this case η, encountered in condition (8.4).

Subtracting the three terms of (9.1) from unity, we have:

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x,$$

and this shows that

$$\left| 1 - \frac{\sin x}{x} \right| < \varepsilon \text{ if } \left| 1 - \cos x \right| < \varepsilon .$$

Recalling that the sine of an arc in the first quadrant is less than the arc itself, we obtain:

$$1-\cos x = 2\sin^2\frac{x}{2} < 2\left(\frac{x}{2}\right)^2 = \frac{x^2}{2},$$

and it is sufficient to choose:

$$\frac{x^2}{2} < \varepsilon \text{, i.e. } |x| < \sqrt{2\varepsilon}$$

Thus, $\sqrt{2\varepsilon}$ can act as η in the given case.

10. CONTINUITY OF FUNCTIONS

We have already introduced the definition of the continuity of a function at the point $x = c$, if the function is defined both at the point and in the vicinity to left and right. We give the definition again.

Definition. *The function $f(x)$ is said to be continuous for $x = c$ (at the point $x = c$), if a limit of $f(x)$ exists for $x \to c$ and if this limit is equal to $f(c)$:*

$$\lim_{x\to c} f(x) = f(c) = f(\lim_{x\to c} x) \qquad (10.1)$$

We recall that this is equivalent to the fact that *there exist limits $f(c-0)$ and $f(c+0)$ to left and right*, and to the fact that *these limits are equal to each other and to $f(c)$,* i.e.

$f(c-0) = f(c+0) = f(c)$. Alternatively, the definition given above is equivalent, as we have seen, to: *for any given positive ε, there exists a positive η such that*

$$|f(c) - f(x)| < \varepsilon \text{ for } |c - x| < \eta. \qquad (10.2)$$

It may be remarked that, in view of the arbitrariness of the choice of ε, we can write $|f(c) - f(x)| \le \varepsilon$ in place of $|f(c) - f(x)| < \varepsilon$ in this definition. This remark applies to all previous similar definitions, and in particular, to the definition of an infinitesimal and a limit, as also to the following equivalent definition of continuity.

The difference $x - c$ is the increment of the independent variable, whilst $f(x) - f(c)$ is the corresponding increment of the function, so that the definition of continuity just given is equivalent to the following: *a function is said to be continuous at the point $x = c$, if to an infinitesimal increment of the independent variable (from the initial value $x = c$) there corresponds an infinitesimal increment of the function.*

We note that the property of continuity, as expressed in equation (18), amounts to the possibility of finding the limit of the function by directly replacing the independent variable with its limit.

We saw from formulae (3) and (4), that polynomials in x and the quotients of such polynomials, i.e. rational functions of x, are functions continuous for any x, except those for which the denominator of the rational function becomes zero.

The function $y = b$ is also obviously continuous, its value being the same for all x.

All the elementary functions, discussed in the first chapter (power, exponential, logarithmic, trigonometric and inverse circular), are continuous for all the x for which they exist, except those for which they tend to infinity.

For example, $\log_{10} x$ is a continuous function of x for all positive x; $\tan x$ is a continuous function of x for all x, except

$$x = (2k+1)\frac{\pi}{2},$$

where k is any integer.

Notice further the function u^v, where u and v are continuous functions of x, u being assumed not to take negative values. This is also called an *exponential* function. It likewise has the property of continuity, except for those x for which u and v are simultaneously zero or $u = 0$ and $v < 0$.

We shall accept without proof what has been said about the continuity of the elementary functions, although proof is of course required, and can in fact be given with complete rigour. We shall later examine the question in detail.

It can easily be shown that *the sum or product of any finite number of continuous functions is itself a continuous function; the same is true of the quotient of two continuous functions except for those values of the independent variable for which the denominator tends to zero.*

We only consider the case of a quotient. We assume that functions $\varphi(x)$ and $\psi(x)$ are continuous for $x = a$ and that $\psi(a) \neq 0$. We take the function

$$f(x) = \frac{\varphi(x)}{\psi(x)}.$$

Using the theorem concerning the limit of a quotient, we obtain:

$$\lim_{x \to a} f(x) = \frac{\lim\limits_{x \to a} \varphi(x)}{\lim\limits_{x \to a} \psi(x)} = \frac{\varphi(a)}{\psi(a)} = f(a),$$

which proves the continuity of the quotient $f(x)$ for $x = a$.

We note one simple example. If $y = \sin x$ is a continuous function of x, $y = b\sin x$, where b is a constant, will also be continuous, being the product of the continuous functions $y = b$ (see above) and $y = \sin x$.

We turn again now to the function $y = \sin x / x$. This is not defined for $x = 0$, but we know that $\lim_{x \to 0} y = 1$. Hence, if we put $y = 1$ for $x = 0$, y will be a continuous function at the point $x = 0$.

Such a process of finding the limit of a function for x tending to its point of indeterminacy is called *disclosing the indeterminacy*, and the limit itself, if it exists, is sometimes called a *true value* of the function at this point of indeterminacy. We shall have many examples later on of the disclosure of indeterminacies.

11. THE PROPERTIES OF CONTINUOUS FUNCTIONS

We defined above the continuity of a function for a given value of x. We now suppose that the function is defined in a finite interval $a \le x \le b$. *If it is continuous for any given x in this interval, we say that it is continuous in the interval* (a,b). We note here that continuity of the function at the ends of the interval, $x = a$ and $x = b$, consists in:

$$\lim_{x \to a+0} f(x) = f(a), \lim_{x \to b-0} f(x) = f(b).$$

All continuous functions have the following properties:

1. *If the function $f(x)$ is continuous in the interval (a,b), there exists at least one value of x in this interval at which $f(x)$ takes its maximum value, and at least one value of x for which the function takes its minimum value.*
2. *I f the function $f(x)$ is continuous in the interval (a,b), with $f(a) = m$ and $f(b) = n$, and if k is any number lying between m and n, there exists at least one x in the interval such that $f(x) = k$; and in particular, if $f(a)$ and $f(b)$ have opposite signs, there exists at least one x in the interval such that $f(x)$ is zero.*

These two properties are immediately clear, if we note that the graph corresponding to a continuous function is a continuous curve. This remark cannot serve as a proof, of course. The concept itself of a continuous curve, obvious at first sight, is seen to be unusually complex on closer inspection. The rigorous proof of the two properties mentioned, as also of the third, to follow, is based on the theory of irrational numbers. We accept these properties without proof.

In subsequent paragraphs of the present section, we study the basis of the theory of irrational numbers and the relationship of this theory to the theory of limits and to the properties of continuous functions.

We may remark that the second property of continuous functions can also be formulated thus: on continuous variation of x from a to b, the continuous function $f(x)$ passes at least once through every number lying between $f(a)$ and $f(b)$.

Figures 11.1 and 11.2 show the graphs of functions, continuous in the interval (a,b), for which $f(a)<0$ and $f(b)>0$. In Fig. 11.1 the graph cuts the axis *OX* once, and $f(x)$ is zero for the corresponding x. There are three such values of x, instead of one, in the case of Fig. 11.2.

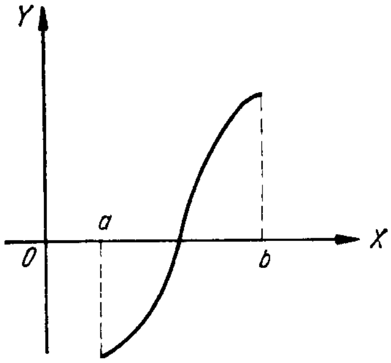

Fig 11.1

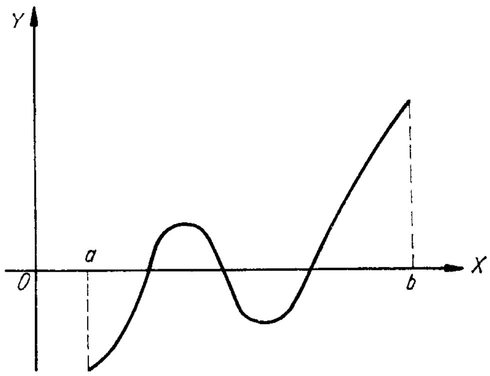

Fig. 11.2

We now pass to the third property of continuous functions, which is less obvious than the two previous ones.

3. If $f(x)$ is continuous in the interval (a,b), and if $x=x_0$ is a certain value of x in this interval, by condition (10.2) (replacing c by x_0), for any given positive ε there exists an η, of course depending on ε, such that

$$|f(x)-f(x_0)|<\varepsilon, \text{ if } |x-x_0|<\eta,$$

it naturally being assumed that x also lies in this interval. (If, for example, $x_0=a$, x must be greater than a, and if $x_0=b$, $x<b$.)

But the number η can depend, not only on ε, but also on just what value of $x=x_0$ we take in the interval. The third property of continuous functions consists in the fact that, for any given ε, there exists the same η for all x_0

in the interval (a,b). In other words, *if* $f(x)$ *is continuous in the interval* (a,b)*, for any given positive* ε *there exists a positive* η *such that*

$$\left| f(x^{//}) - f(x^{/}) \right| < \varepsilon \tag{11.1}$$

for any two values $x^{//}$ *and* $x^{/}$ *in the interval* (a,b) *which satisfy the inequality*

$$\left| x^{//} - x^{/} \right| < \eta . \tag{11.2}$$

This property is referred to as *uniform continuity*. Thus, *if a function is continuous in an interval* (a,b)*, it is uniformly continuous in this interval*.

We again remark, that we assume $f(x)$ to be continuous, not only for all x inside the interval (a,b), but also for $x = a$ and $x = b$.

We shall further illustrate the property of uniform continuity by a simple example. We first rewrite the above inequality in another form, replacing the symbol $x^{/}$ by x, and $x^{//}$ by $(x+h)$. Now $x^{//} - x^{/} = h$ is the increment of the independent variable, and $f(x+h) - f(x)$ is
the corresponding increment of the function. The property of uniform continuity now becomes:

$$\left| f(x+h) - f(x) \right| < \varepsilon \quad \text{if} \quad |h| < \eta ,$$

where x and $(x+h)$ are any two points in the interval (a,b).

Take the example of the function:

$$f(x) = x^2 .$$

We now have:

$$f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2 .$$

For any given x, the expression $(2xh + h^2)$ for the increment of our function obviously tends to zero, as the increment of the independent variable tends to zero. This is a further confirmation that the function in question is continuous for every x. It will be continuous, for instance, in the interval $-1 < x < 2$. We show that it is uniformly continuous in this interval. We
have to satisfy the inequality:

$$\left| 2xh + h^2 \right| < \varepsilon \tag{11.3}$$

for suitable choice of η in the inequality $|h| < \eta$, where x and $(x+h)$ must lie in the interval $(-1,2)$. We have:

$$\left| 2xh + h^2 \right| \le |2xh| + h^2 = 2|x||h| + h^2 .$$

The maximum value of $|x|$ in the interval is two, and hence we can rewrite |the above inequality with greater force as:

$$\left|2xh+h^2\right| \le 4|h|+h^2.$$

We shall always take $|h|<1$. Then $h^2<|h|$, and we can write the above in the form:

$$\left|2xh+h^2\right| < 4|h|+|h|$$

or

$$\left|2xh+h^2\right| < 5|h|.$$

The inequality (11.3) will certainly be satisfied, if we take $|h|$ on the condition $5|h|<\varepsilon$. Thus, h must satisfy two inequalities:

$$|h|<1 \text{ and } |h|<\frac{\varepsilon}{5}.$$

We can thus take for η the least of the two numbers 1 and $\varepsilon/5$. For small ε (in fact, $\varepsilon<5$), we must take $\eta=\varepsilon/5$, and this η is evidently the same, for a given ε, for all x in the interval $(-1,2)$.

The property mentioned cannot obtain in the case of discontinuous functions, or those continuous only inside an interval. Take the function, the graph of which is shown in Fig. 9.2. It is defined in the interval $(-1,+1)$ and has a discontinuity at $x=0$. It has values as close as desired to unity, but it does not take the value unity, or values greater than unity. There is thus no maximum among the values of this function. Similarly, there is no minimum. The elementary function $y=x$ does not take either a maximum or minimum value inside the interval $(0,1)$. If it is considered in the closed interval $(0,1)$, it reaches its minimum value at $x=0$, and its maximum at $x=1$. Take another function, $f(x)=\sin(1/x)$ continuous in the interval $0<x<1$, open on the left. As x tends to zero, the argument $1/x$ increases indefinitely, and $\sin(1/x)$ oscillates between (—1) and (+ 1), having no limit as $x\to+0$. We show that this function is not uniformly continuous in the interval $0<x<1$. We take two values: $x'=1/n\pi$ and $x''=2/(4n+1)\pi$, where n is a positive integer. Both values lie in the interval for any choice of n. Further, we have:

$$f(x')=\sin n\pi=0;$$

$$f(x'')=\sin\left(2n\pi+\frac{1}{2}\pi\right)=1.$$

Thus:

$$f(x'')-f(x')=1$$

and

$$x'' - x' = \frac{2}{(4n+1)\pi} - \frac{1}{n\pi}.$$

As the positive integer n tends to infinity, the difference $x'' - x'$ tends to zero, whilst $f(x'') - f(x')$ remains equal to 1. It is thus evident that there does not exist a positive η, such that, in the interval $0 < x < 1$, (11.2) implies $\left| f(x'') - f(x') \right| < 1$; this corresponds to choosing $\varepsilon = 1$ in formula (11.1).

Take the function $f(x) = x\sin(1/x)$. The first term of the product tends to zero as $x \to +0$, whilst the absolute value of the second, $\sin(1/x)$, does not exceed unity; hence , $f(x) \to 0$ as $x \to +0$. The second term has no meaning for $x = 0$, but if we complete the definition of our function by taking $f(0) = 0$, i.e. if we take $f(x) = x\sin(1/x)$ for $0 < x < 1$ and $f(0) = 0$, we obtain a function continuous in the closed interval $(0,1)$. The functions $\sin(1/x)$ and $x\sin(1/x)$ are evidently continuous for any x, excepting zero.

12. COMPARISON OF INFINITESIMALS AND OF INFINITELY LARGE MAGNITUDES

If α and β are two magnitudes, simultaneously tending to zero, the theorem regarding the limit of a quotient cannot be used for finding the limit of the ratio β / α. We shall assume that the variables α and β, whilst tending to zero, do not take the value zero. If the ratio β / α tends to a finite limit, differing from zero, the ratio α / β will also tend to a finite limit, differing from zero. We say in this case that β and α are *infinitesimals of the same order*. If the ratio β / α has a limit at zero, we say that β is an infinitesimal of higher order in comparison with α, or that α is an infinitesimal of lower order in comparison with β. If the ratio β / α tends to infinity, α / β tends to zero, i.e. β is of lower order compared with α, and α of higher order compared with β. It is easy to show that, *if α and β are infinitesimals of the same order, and γ is an infinitesimal of higher order compared with α, γ is also of higher order as regards β.* By hypothesis $\gamma / \alpha \to 0$, and α / β has a finite limit, differing from zero. From the self evident equation $\gamma / \beta = \gamma / \alpha \cdot \alpha / \beta$, and using the theorem regarding the limit of a product, it follows at once that $\gamma / \beta \to 0$, which proves our statement.

We note an important particular case of infinitesimals of the same order. If $\alpha / \beta \to 1$ (so that also $\beta / \alpha \to 1$), infinitesimals α and β are referred to as equivalent. It follows at once from the equation

$$\frac{\beta - \alpha}{\alpha} = \frac{\beta}{\alpha} - 1,$$

that the *equivalence of* α *and* β *implies that the difference* $\beta - \alpha$ *is an infinitesimal of higher order than* α*.* It similarly follows from the equation

$$\frac{\beta - \alpha}{\beta} = 1 - \frac{\alpha}{\beta}$$

that their equivalence implies that $\beta - \alpha$ is an infinitesimal of higher order than β

If β / α^k, where k is a positive constant, tends to a finite limit, differing from zero, we say that β is an infinitesimal of order k with respect to α. If $\beta / \alpha \to c$, where c is a number, not zero, $\left| \beta / c \cdot \alpha^k \right| \to 1$, i.e. β and ca^k are equivalent infinitesimals, and therefore, $\gamma = \beta - ca^k$ is an infinitesimal of higher order than β (or than ca^k). If α is taken as the basic infinitesimal, the equation $\beta = ca^k + \gamma$, where γ is an infinitesimal of higher order than ca^k, represents the isolation from the infinitesimal β of the infinitesimal term ca^k (of the simplest form with respect to a), in such a way that the remainder is an infinitesimal γ of higher order than β (or than ca^k).

An analogous comparison can be made of the infinitely large magnitudes u and v. If v / u tends to a limit, finite and not zero, we say that u and v are infinitely large magnitudes of the same order. If $v / u \to 0$, then $u / v \to \infty$. We say in this case that v is of a lower order of greatness with respect to u, or that u is of a higher order of greatness with respect to v. If $v / u \to 1$, the infinitely large magnitudes are said to be equivalent. If v / u^k, where k is a positive constant, has a limit, which is finite and not zero, we say that v is of the *k*-th order of greatness with respect to u. All the above remarks about infinitesimals apply for infinitely large magnitudes.

We further remark, that if the ratio β / α or v / u has no limit at all, the corresponding infinitesimals or large order magnitudes are said to be incomparable.

13. THE NUMBER e

Our present example is important later on: we consider the variable taking the values

$$\left(1+\frac{1}{n}\right)^n,$$

where n tends to $+\infty$, increasing through positive integers. Using Newton's binomial formula, we obtain:

$$\left(1+\frac{1}{n}\right)^n = 1+\frac{n}{1}\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\dots$$

$$+\frac{n(n-1)(n-2)\dots(n-k+1)}{k!}\cdot\frac{1}{n^k}+\dots+\frac{n(n-1)(n-2)\dots 2\cdot 1}{n!}\cdot\frac{1}{n^n}=$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)+\dots+$$

$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right).$$

The sum written above contains $(n+1)$ positive terms. As the integer n increases, the number of terms increases and each term itself also increases, since in the expression for the general term:

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)$$ [1]

$k!$remains unchanged, whilst the differences in brackets increase with increasing n. We thus see that the variable in question increases with increasing n; so that it is sufficient to show that the variable I is bounded, in order to prove that its limit exists.

We replace all the differences appearing in the general term by I unity, and all the factors of $k!$, starting with 3, by 2. The general term is evidently now increased, and we shall have, on using the formula for the sum of the terms of a geometrical progression:

$$\left(1+\frac{1}{n}\right)^n<1+1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{k-1}}+\dots+\frac{1}{2^{n-1}}=1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=3-\frac{1}{2^{n-1}}<3,$$

i.e. the variable $(1+1/n)^n$ is bounded. We denote its limit by the letter e:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e \quad (n \text{ is a positive integer}). \qquad (13.1)$$

---

1 (The product $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{k-1}{n}\right)$ *is obtained from the fraction* $\frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}$ *,if; noting that there are* k *terms n in the denominator, each of the k terms of the product on top is divided by* n *.)*

This limit is evidently not greater than 3.

We now show that the expression $\left(1+1/x\right)^x$ tends to the same limit e, if x tends to $+\infty$, taking any values.

Let n be the greatest integer included in x, i.e.

$$n \le x < n+1.$$

The number n evidently tends to $+\infty$ along with x. On noting that a power term increases, both with increase of the positive base, greater than unity, and with increase of the exponent of the power, we can write:

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^{n+1} \qquad (13.2)$$

But by equation (13.1):

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{n\to+\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)} = \frac{e}{1} = e$$

and

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to+\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)\right] = e.$$

Thus, the extreme terms of inequality (13.2) tend to the limit e, and hence the middle term must tend to the same limit, i.e.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e. \qquad (13.3)$$

We now consider the case when x tends to $-\infty$.

We introduce a new variable y in place of x, putting

$$x = -1-y,$$

whence

$$y = -1-x.$$

It is evident from the last equation that y tends to $+\infty$ as x tends to $-\infty$.

On changing the variables in the expression $(1+1/x)^x$ and noting equation (13.3), we obtain:

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(\frac{-y}{-1-y}\right)^{-1-y} = \lim_{y\to+\infty}\left(\frac{1+y}{y}\right)^{1+y} =$$

$$= \lim_{y\to+\infty}\left[\left(1+\frac{1}{y}\right)^y\left(1+\frac{1}{y}\right)\right] = e\cdot 1 = e.$$

If x tends to ∞, with either sign, i.e. $|x| \to +\infty$, it follows from the above that here also:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \qquad (13.4)$$

We shall later give a suitable method for calculating e to any degree of accuracy. Clearly, it is an irrational number; we have, to an accuracy of seven decimal places: e = 2.7182818 . . .

We can now easily find the limit of $(1+k/x)^x$, where k is a given number. Using the continuity of a power function, we obtain:

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{1}{x/k}\right)^{x/k}\right]^k = \lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^k = e^k,$$

where y denotes x/k, and tends to infinity along with x.

An expression of the form $(1+k/n)^n$ is encountered in compound f interest theory.

We suppose that an increment of capital occurs annually. If capital a returns an interest annually of p per cent, the accumulated capital in the course of a year will be:

$$a(1+k),$$

where

$$k = \frac{p}{100};$$

after another year has elapsed, it will be:

$$a(1+k)^2;$$

and in general, after the lapse of m years, it will be:

$$a(1+k)^m.$$

We now suppose that the increment of capital takes place every $1/n$ of a year. The number k is now diminished n times, since the percentage interest is counted over a year, whilst the number of intervals of time is increased n times; so that the accumulated capital over m years will be:

$$a\left(1+\frac{k}{n}\right)^{mn}.$$

Finally, let n tend to infinity, i.e. an increment of capital occurs in every smallest possible interval of time, and in the limit, continuously. After the lapse of m years, the accumulated capital will be:

$$\lim_{n\to\infty} a\left(1+\frac{k}{n}\right)^{mn} = \lim_{n\to\infty} a\left[\left(1+\frac{k}{n}\right)^n\right]^m = ae^{km}.$$

The number e is used as a base of logarithms. These are referred to as *natural logarithms* and are here denoted by the simple sign **ln** or **log** without indicating the base.

For x tending to zero, both numerator and denominator in the expression $\frac{\ln(1+x)}{x}$ tend to zero. Let us examine this indeterminate form.

We introduce a new variable y, putting $x=\frac{1}{y}$, i.e. $y=\frac{1}{x}$, whence evidently, as $x\to 0$, y tends to infinity. Substituting the new variable, and making use of the continuity of a logarithm and formula (13.4), we obtain:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{y\to\infty} y\ln\left(1+\frac{1}{y}\right)=\lim_{y\to\infty}\ln\left(1+\frac{1}{y}\right)^{y}=\ln e=1.$$

The advantage of the present choice of a base of logarithms is clear from this. Just as, using radian measure of angles, the true value of $(\sin x)/x$ is unity for $x=0$*, in the case of natural logarithms the true value of* $\frac{\ln(1+x)}{x}$ *is also unity for* $x=0$.

The following relationship follows from the definition of logarithm:

$$N=a^{\log_a N}.$$

Taking logarithms to base e in this equation, we obtain:

$$\log N=\log_a N\cdot\log a \quad \text{or} \quad \log_a N=\log N\cdot\frac{1}{\log a}.$$

This relationship gives the logarithm of a number *N* to any base a in terms of its natural logarithm. The factor $M=1/\ln a$ is called the *modulus* of the system of logarithms to base a, and for $a=10$ it is given with an accuracy of seven decimal places by:

M = 0.4342945 . . .

Of course $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=M$

14. ON SOME INFINITESIMALS

1. Some other limits can be reduced to $\lim_{x\to 0}\frac{\sin x}{x}=1$ by substituting the new variable or with simple rearrangemant. For example

$$\lim_{x\to0}\frac{\sin^{-1}x}{x}=\left|t=\sin^{-1}x\right|=\lim_{x\to0}\frac{t}{\sin t}=\frac{1}{\lim\limits_{x\to0}\frac{\sin t}{t}}=1,$$

$$\lim_{x\to0}\frac{\tan x}{x}=\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{1}{\cos x}=1,$$

$$\lim_{x\to0}\frac{\tan^{-1}x}{x}=\left|t=\tan^{-1}x\right|=\lim_{x\to0}\frac{t}{\tan t}=\frac{1}{\lim\limits_{x\to0}\frac{\tan t}{t}}=1.$$

Limit $\lim\limits_{x\to0}\frac{e^x-1}{x}$ can be reduced to $\lim\limits_{x\to0}\frac{\ln(1+x)}{x}$ as the following

$$\lim_{x\to0}\frac{e^x-1}{x}=\left|\begin{matrix}t=e^x-1\\ x=\ln\left(t+1\right)\end{matrix}\right|=\lim_{t\to0}\frac{t}{\ln\left(t+1\right)}=\frac{1}{\lim\limits_{t\to0}\frac{\ln\left(t+1\right)}{t}}=1.$$

Therefore we get equivalent infinitesimals

$\sin x\sim\sin^{-1}x\sim\tan x\sim\tan^{-1}x\sim\ln(x+1)\sim e^x-1\sim x$.

2. Any infinitesimals in a quotient can be replaced with its equivalent, but difference of two infinitesimals of the same order is an infinitesimal of higher order.

For example, we saw above that

$$\lim_{x\to0}\frac{\sin x}{x}=1,$$

i.e. $\sin x$ and x are equivalent infinitesimals, and therefore $\sin x-x$ is an infinitesimal of higher order than x (it is incorrect to replace $\sin x$ with x !!). We see later, that this difference is equivalent to $-x^3/6$, i.e. it is an infinitesimal of the third order with respect to x.

3. We show that the difference $1-\cos x$ is an infinitesimal of the second order with respect to x. We have in fact, on using a well-known trigonometric formula and with a simple rearrangement,

$$\frac{1-\cos x}{x^2}=\frac{2\sin^2\frac{1}{2}x}{x^2}=\frac{1}{2}\left(\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}\right)^2.$$

If $x\to0$, $\alpha=x/2$ also tends to 0, and as we have shown:

$$\lim_{x\to0}\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}=\lim_{\alpha\to0}\frac{\sin\alpha}{\alpha}=1,$$

and hence,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2},$$

i.e. in fact, $1-\cos x$ is an infinitesimal of the second order with respect to x.

4. From the expression

$$\sqrt{1+x}-1=\frac{x}{\sqrt{1+x}+1}$$

we have:

$$\frac{\sqrt{1+x}-1}{x}=\frac{1}{\sqrt{1+x}+1}$$

whence

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2},$$

i.e. $\sqrt{1+x}-1$ and x are infinitesimals of the same order, $\sqrt{1+x}-1$ being equivalent to $x/2$.

5. We show that a polynomial of degree $m>1$ is an infinitely large magnitude of order m with respect to x. In fact,

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+...+a_{m-1}x+a_m}{x^m}=\lim_{x\to\infty}\left(a_0+\frac{a_1}{x}+...+\frac{a_{m-1}}{x^{m-1}}+\frac{a_m}{x^m}\right)=a_0.$$

15. APPLICATIONS TO PROBLEMS

Let us consider several applications the theory to certain problems.

1. Determine the following limit

$$\lim_{x\to\infty}\frac{4x^3-5x+3}{(5x+1)(6x+2)(7x+3)}$$

We can't apply the theorem concerning the limit of a quotient to the problem immediately, because the numerator and the denominator are infinitely large magnitudes, thus we have the indeterminate form $\left(\frac{\infty}{\infty}\right)$.

Therefore we need to remove the indeterminacy. Let factor out the highest power of x:

$$\lim_{x\to\infty}\frac{x^3\left(4-5\frac{1}{x^2}+3\frac{1}{x^3}\right)}{x^3\left(5+\frac{1}{x}\right)\left(6+\frac{2}{x}\right)\left(7+\frac{3}{x}\right)}$$

On canceling x^3 and taking into account that $\frac{1}{x^n} \to 0, n = 1, 2, 3.$ we get $\lim\limits_{x\to\infty} \frac{4}{5\cdot 6\cdot 7} = \frac{4}{210} = \frac{2}{105}$

We can arrive at the same result on keeping the highest power of x in every factor and on neglecting other.

$$\lim_{x\to\infty} \frac{4x^3 - 5x + 3}{(5x+1)(6x+2)(7x+3)} = \lim_{x\to\infty} \frac{4x^3}{(5x)(6x)(7x)} = \frac{2}{105}.$$

It can be easily seen that two polynomials of the same degree are infinitely large magnitudes of the same order, for $x \to \infty$. The limit of their ratio is the ratio of the coefficients of their highest terms.

If the two polynomials are of different degree, the one of higher degree is an infinitely large magnitude of higher order with respect to the other, for $x \to \infty$.

2. It's impossible to apply theorem concerning the limit of a quotient to the limit $\lim\limits_{x\to 2} \frac{x^3 - 8}{x^2 - 7x + 10}$ immediately, because the numerator and the denominator take value 0, thus we have the indeterminate form $\left(\frac{0}{0}\right)$.

This means the polynomials in the numerator and in the denominator posses the root at the point $x = 2$. If a polynomial posses root x_0 it can be divided by $x - x_0$. Lets factorize the numerator and the denominator.

$$\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 7x + 10} = \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x-5)}$$

On canceling $(x-2)$ we avoid the indeterminacy and get immediately

$$\lim_{x\to 2} \frac{(x^2 + 2x + 4)}{(x-5)} = \frac{12}{-3} = -4$$

3. We can't apply theorem concerning the limit of a sum to the limit $\lim\limits_{n\to\infty} \sqrt{n^2 + 5n - 8} - \sqrt{n^2 - 3n + 5}$ immediately, because we have the indeterminacy $(\infty - \infty)$.

We can reduce it to indeterminacy $\left(\frac{\infty}{\infty}\right)$ instead:

$$\lim_{n\to\infty} \sqrt{n^2 + 5n - 8} - \sqrt{n^2 - 3n + 5} = (\infty - \infty) =$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+5n-8}-\sqrt{n^2-3n+5}\right)\left(\sqrt{n^2+5n-8}+\sqrt{n^2-3n+5}\right)}{\sqrt{n^2+5n-8}+\sqrt{n^2-3n+5}} =$$

$$= \lim_{n\to\infty} \frac{\left(\left(n^2+5n-8\right)-\left(n^2-3n+5\right)\right)}{\sqrt{n^2+5n-8}+\sqrt{n^2-3n+5}} = \lim_{n\to\infty} \frac{8n-13}{\sqrt{n^2+5n-8}+\sqrt{n^2-3n+5}}.$$

On dividing by n we get:

$$\lim_{n\to\infty} \frac{8-\frac{13}{n}}{\sqrt{1+\frac{5}{n}-\frac{8}{n^2}}+\sqrt{1-\frac{3}{n}+\frac{5}{n^2}}} = \frac{8}{2} = 4.$$

4. The limit $\lim_{x\to 0} \frac{\sin 3x}{\ln\left(1+5x\right)}$ is another example of indeterminacy $\left(\frac{0}{0}\right)$

It can be found in two ways.

A)

With the following simple rearrangement we can reduce it to the product of 3 known limits. Then we apply the limit of product theorem, and get:

$$\lim_{x\to 0} \frac{\sin 3x}{\ln\left(1+5x\right)} = \lim_{x\to 0} \frac{3x\cdot \sin 3x \cdot 5x}{3x\cdot 5x\cdot \ln\left(1+5x\right)}$$

$$= \lim_{x\to 0} \frac{\sin 3x}{3x}\cdot \lim_{x\to 0}\frac{3x}{5x}\cdot \lim_{x\to 0}\frac{5x}{\ln\left(1+5x\right)} = 1\cdot\frac{3}{5}\cdot 1 = \frac{3}{5},$$

B) By using equivalent infinitesimals $\sin 3x \sim 3x,\ \ln\left(1+5x\right) \sim 5x$ as $x \to 0$ we get immediately $\lim_{x\to 0} \frac{\sin 3x}{\ln\left(1+5x\right)} = \lim_{x\to 0}\frac{3x}{5x} = \frac{3}{5}$

5. Limits $\lim_{x\to a-0}$, $\lim_{x\to a+0}$ (named one-sided limits from the negative and the positive direction respectively) allow we to investigate discontinuity of a function. There exist 3 cases:

1) $\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) = A$, where A is certain finite number. Under this condition point $x = a$ is either point of continuity (if $f(x) = A$) or a point of so-called *a removable discontinuity (if $f(x) \neq A$).* A is called *the true value of the function*
2) $\lim_{x\to a-0} f(x) = A \neq \lim_{x\to a+0} f(x) = B$, where A, B are certain finite number. In this case function $f(x)$ takes *a discontinuity of the 1st kind* or *jump discontinuity* at $x = a$

3) If at least one of these one-side limits does not equal a finite number, function $f(x)$ takes *a discontinuity of the 2nd kind* or *essential discontinuity* at $x = a$

Let us take for example $f\left(x\right) = e^{\frac{1}{x-5}}$ at point 5.

$\lim\limits_{x\to 5-0} e^{\frac{1}{x-5}} = 0$, because x increases and tends to 5, i.e. $\frac{1}{x-5} < 0$ and tends to $-\infty$. At the same time $\lim\limits_{x\to 5+0} e^{\frac{1}{x-5}} = +\infty$ because x decreases and tends to 5, i.e. $\frac{1}{x-5} > 0$ and tends to ∞.

The point $x_0 = 5$ is discontinuity of the 2nd kind.

EXERCISES

1. $\lim\limits_{n\to\infty} \frac{2+n-n^2+3n^3}{12-7n+5n^3}$. Answer: $\frac{3}{5}$.
2. $\lim\limits_{n\to\infty} \frac{\left(2n+1\right)^4 - \left(n-1\right)^4}{\left(2n+1\right)^4 + \left(n-1\right)^4}$. Answer: $\frac{15}{17}$.
3. $\lim\limits_{n\to\infty} \frac{\left(n+2\right)! + \left(n+1\right)!}{\left(n+2\right)! - \left(n+1\right)!}$ Answer: 1 .
4. $\lim\limits_{n\to\infty} \frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$ Answer: 1.
5. $\lim\limits_{n\to\infty}\left(\sqrt{\left(n+1\right)\left(n+2\right)} - \sqrt{\left(n-1\right)n}\right)$ Answer: 2.
6. $\lim\limits_{x\to 7} \frac{2x^2-11x-21}{x^2-9x+14}$ Answer: $\frac{17}{5}$.
7. $\lim\limits_{x\to 1} \frac{x^4-3x+2}{x^5-4x+3}$ Answer: 1.
8. $\lim\limits_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$ Answer: $\frac{4}{3}$.
9. $\lim\limits_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$ Answer: $\frac{12}{5}$.

10. $\lim\limits_{x\to\infty}\dfrac{\left(2x^3+7x-1\right)^6}{\left(2x^6-13x^2+x\right)^3}$ Answer: 8.

11. $\lim\limits_{x\to 0}\dfrac{\tan 4x}{\sin x}$ Answer: 4.

12. $\lim\limits_{x\to 0}\left(x\cot 5x\right)$ Answer: $\dfrac{1}{5}$.

13. $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}$ Answer: $\dfrac{\alpha}{\beta}$.

14. $\lim\limits_{x\to\pi}\dfrac{\sin 5x}{\sin 2x}$ Answer: $-\dfrac{5}{2}$.

15. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sin 6x-\sin 7x}$ Answer: -1.

16. $\lim\limits_{x\to 0}\left(\dfrac{2}{\sin 2x\sin x}-\dfrac{1}{\sin^2 x}\right)$ Answer: $\dfrac{1}{2}$.

17. $\lim\limits_{x\to 0}\dfrac{\cos 3x^3-1}{\sin^6 2x}$ Answer: $-\dfrac{9}{128}$.

18. $\lim\limits_{x\to 1}\dfrac{\sin 7\pi x}{\sin 2\pi x}$ Answer: $-\dfrac{7}{2}$.

19. $\lim\limits_{x\to\pi}\dfrac{\sin x}{\pi^2-x^2}$ Answer: $\dfrac{1}{2\pi}$.

20. $\lim\limits_{x\to\infty}\left(x\sin\dfrac{\pi}{x}\right)$ Answer: π .

21. $\lim\limits_{x\to\infty}\left(x^2\left(\cos\dfrac{1}{x}-\cos\dfrac{3}{x}\right)\right)$ Answer: 4 .

22. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2\sin 3x}-\sqrt{1-4\sin 5x}}{\sin 6x}$ Answer: $\dfrac{13}{6}$.

23. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}$ Answer: $\dfrac{1}{2}$.

24. $\lim\limits_{x\to 2}\left(2x-3\right)^{\frac{x}{x-2}}$ Answer: e^4.

25. $\lim\limits_{x\to+\infty}\left(\dfrac{5x^3+2}{5x^3}\right)^{\sqrt{x}}$ Answer: 1.

26. $\lim_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1+x}}$ Answer: 1.

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