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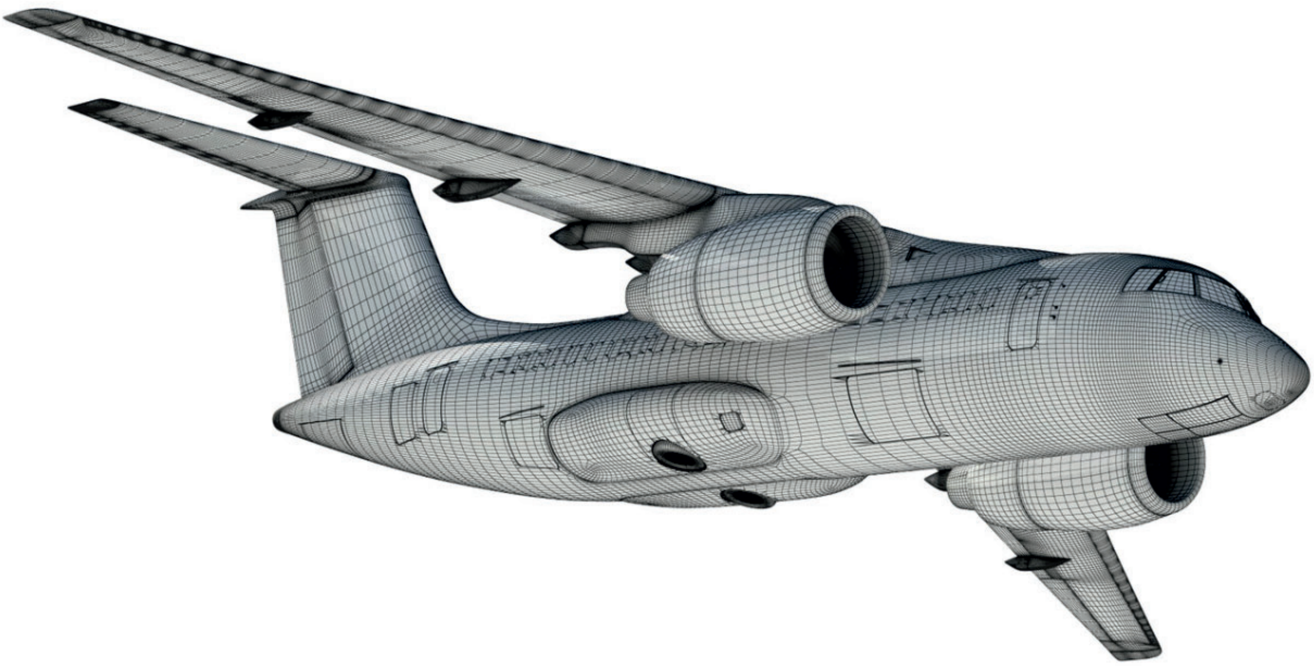
STRENGTH THEORIES. COMBINED LOADING



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MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National Aerospace University
«Kharkiv Aviation Institute»

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STRENGTH THEORIES. COMBINED LOADING

Textbook

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Розглянуто основні теорії міцності. Викладено методику розрахунку на міцність елементів конструкцій, що перебувають у складному напруженому стані. Описано окремі випадки складного опору – позацентрове розтягнення-стиснення і косий згин. Наведено таблиці довідкових даних, приклади розв'язання задач і рекомендації до виконання домашнього завдання з цієї теми.

Для здобувачів освіти, які вивчають курси «Опір матеріалів» і «Механіка матеріалів і конструкцій» під час самостійної роботи.

Reviewers: Dr. of Technical Sciences, Prof. K. V. Korytchenko,
Dr. of Technical Sciences, Prof. N. V. Smetankina

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The main strength theories are considered. A methodology for strength analysis of structural elements in complex stress states is presented. Particular cases of combined loading are described, such as eccentric tension–compression and oblique bending. Reference tables of data, examples of problem solutions, and recommendations for homework assignments on this topic are included.

For students studying the courses Strength of Materials and Mechanics of Materials and Structures in the context of independent work.

Figs 159. Tables 8. Bibliogr.: 6 items

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LIST OF SYMBOLS

α, β, γ	–	angles, numerical coefficients;
ε	–	strain;
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	–	strains in the system of principal axes;
μ	–	Poisson's ratio;
σ	–	normal stress;
$\sigma_1, \sigma_2, \sigma_3$	–	principal stresses;
$[\sigma]$	–	allowable normal stress;
σ_{ult}	–	ultimate stress;
σ_{yield}	–	yield stress;
σ_{eq}	–	equivalent stress;
τ	–	shear stress;
$[\tau]$	–	allowable shear stress;
a, b, c, d, l	–	lengths of segments, spans;
a	–	side length of a square;
b, h	–	width and height of a rectangle;
d	–	diameter;
E	–	modulus of elasticity (Young’s modulus);
F	–	cross-sectional area;
I_ρ	–	polar moment of inertia of the cross-section;
I_y, I_z	–	axial moments of inertia of the cross-section;
I_{yz}	–	centrifugal moment of inertia of the cross-section;
i_y, i_z	–	radii of gyration of the cross-section;
n	–	safety factor;
M	–	concentrated moment;
$M_x, M_{torsional}$	–	torsional moment;
M_y, M_z	–	bending moments about the y- and z-axes;
N_x	–	axial force;
P	–	concentrated force;
Q_y, Q_z	–	shear forces;
q	–	distributed load;
R	–	reaction;
U	–	potentional strain energy;
V	–	volume;
W_ρ	–	polar cross-section modulus;
W_y, W_z	–	cross-section modulus about the y- and z-axes;
x, y, z	–	Cartesian coordinates.

1. THEORIES OF STRENGTH

1.1. Problem Statement and Basic Definitions

1.1.1. Concept of the Limiting Stress State

The most crucial problem of engineering analysis is the assessment of a structural component's strength based on its known stress state.

The stress state at a point is entirely defined:

– in a simple (uniaxial) stress state – by one principal stress,

– in a plane (biaxial) stress state – by two,

– in a volumetric (triaxial) stress state – by three.

If the external loads do not exceed a certain value that depends on the material and the type of stress state, then the material remains in the elastic state. With an increase in external load, the principal stresses will also increase, and at certain values noticeable residual deformations or local cracks may appear. Such a ***stress state*** is called the ***limiting state***.

Thus, the ***limiting state*** is understood as a ***complex stress state*** in which the following occurs:

а) in a ductile material, residual (plastic) strain begins to develop;

б) in a brittle material, fracture begins.

1.1.2. Necessity of Creating Strength Theories and their Purpose

If the ***limiting stress state*** is known, then the strength analysis is reduced to ***determining the stress state at the critical point*** (or at all potentially critical points) of the body under investigation and comparing it with the ***limiting value***.

In the case of a ***simple (uniaxial) stress state***, it is quite easy to experimentally determine the ***limiting stress state*** through a tension or compression test.

The following are accepted as the ***limiting values***:

– the yield strength of a ductile material ($\sigma_{limit} = \sigma_{yield}$);

– the ultimate strength of a brittle material ($\sigma_{limit} = \sigma_{ult}$).

In these cases, the ***safety factors*** are given by:

$$n_{yield} = \frac{\sigma_{yield}}{\sigma}, \qquad n_{ult} = \sigma_{ult}\frac{\sigma_{ult}}{\sigma} ,$$

where σ is the stress acting at the critical point.

For a ***complex*** (biaxial or triaxial) ***stress state*** it is practically impossible to conduct tests ***for all possible ratios*** between σ_1, σ_2 and σ_3, since:

1) the number of possible relationships between the components of a complex stress state is infinite, therefore the number of experiments needed to determine the limiting states corresponding to these combinations of principal stresses is also infinite;

2) for many types of complex stress states, it is technically difficult, and sometimes impossible, to carry out an experiment to determine the limiting stress state. That is, current experimental techniques do not have the capability to realize tests for the majority of complex stress states.

   Such experiments, which require the use of exceptionally sophisticated equipment for both loading the specimen and recording its behaviour under load, have so far been conducted in research laboratories for only a very limited number of types of complex stress states.

   The interpretation of the results of such experiments is highly complicated and often contradictory, since during these tests it is practically impossible to meet the most important requirement of such experiments ***to ensure the homogeneity of the stress-strain state within the gage section of the specimen***.

Therefore, there arises the necessity of developing ***strength theories (also called strength hypotheses or limit state theories)***, i.e., general methods of strength analysis for any type of complex stress state, based on the mechanical properties of materials obtained from a limited number of the simplest mechanical tests.

> ***Theories of strength*** are designed for performing strength analyses under a ***complex stress state***.

1.1.3. The Concept of Equally Critical (Equally Strong) Stress States

Strength theories are based on the assumption that two stress states are considered equally strong if, when the principal stresses are proportionally increased, they simultaneously reach the limiting state. In that case, the safety factor for both stress states will be identical.

> Two stress states are called ***equally critical (equally strong)*** if they have the ***same safety factor***.

> ***The safety factor*** is a number indicating how many times all components of a complex stress state must be ***simultaneously*** increased for it to become the limiting state.

Let us consider an example. Suppose that for two identical elements made of the same material, a uniaxial stress state is realized for the first, and a complex stress state for the second (Fig. 1.1).

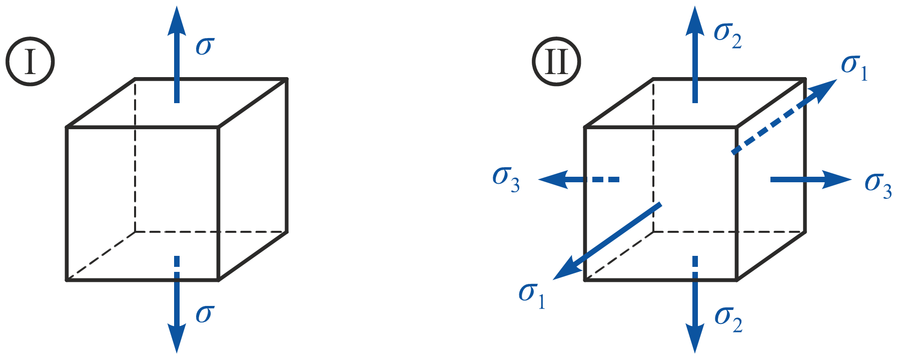

Fig. 1.1

Let us assume that element **I** began to deform plastically (or to fracture if it is brittle) at $\sigma = 280\ MPa$.

This value of σ should be considered the ***limiting*** one in the case of a uniaxial (one-dimensional) stress state:

$$\sigma_{limit} = 280\ MPa.$$

Element **II** began to deform plastically (or to fracture in a brittle manner) at:

$$\sigma_1 = 170\ MPa; \qquad \sigma_2 = 110\ MPa; \qquad \sigma_3 = 60\ MPa.$$

This combination of stresses should be considered as the ***limiting stress state*** for the given ratio between σ_1, σ_2, and σ_3, i.e.

$$\sigma_{1\ limit} = 170\ MPa; \qquad \sigma_{2\ limit} = 110\ MPa; \qquad \sigma_{3\ limit} = 60\ MPa.$$

Suppose that for elements **I** and **II** it is necessary to ensure the ***same safety factor*** with a coefficient of safety $n = 2$.

Then the uniaxial tension with a stress of:

$$\sigma = \frac{\sigma_{limit}}{n} = \frac{280}{2} = 140\ MPa$$

and the volumetric stress state with:

$$\sigma_1 = \frac{\sigma_{1\ limit}}{n} = \frac{170}{2} = 85\ MPa;$$

$$\sigma_2 = \frac{\sigma_{2\ limit}}{n} = \frac{110}{2} = 55\ MPa;$$

$$\sigma_3 = \frac{\sigma_{3\ limit}}{n} = \frac{60}{2} = 30\ MPa$$

are ***equally critical***, or ***equally strong***.

1.1.4. The Concept of Equivalent Stress

The comparison of stress states for a given material can be performed using equivalent stresses σ_{eq}.

> ***Equivalent stress*** (σ_{eq}) – is such a stress that must be created in a ***tensile specimen*** (i.e., under a linear (uniaxial) stress state), so that its stress state is ***equally critical*** to a given ***complex stress state*** (i.e., has the same ***safety factor***).

The problem and purpose of all theories of strength is to relate the equivalent stresses in two equally strong states by a specific relationship based on an analysis of the causes of fracture or transition of the material into the limiting state. This means determining the form of the function:

$$\sigma_{eq} = f(\sigma_1, \sigma_2, \sigma_3). \tag{1.1}$$

Thus, by means of strength theories, a transition is carried out from a complex stress state to an equally critical uniaxial one. That is, the equivalent stress is determined, which is then compared with the results of the simplest mechanical tests. In this way, a conclusion is drawn about the degree of strength exhaustion under a complex stress state.

1.2. Main Theories of Strength

All theories (criteria, hypotheses) of strength can be divided into two types:

a) theories constructed on hypotheses, i.e., based on logically justified assumptions;

б) theories based on a phenomenological approach, i.e., relying on the logical systematization of experimental research results.

1.2.1. Theory of Maximum Normal Stresses (the First Strength Theory)

By historical tradition, the theory (hypothesis) of maximum normal stresses is called the ***First Theory of Strength***. It was formulated in 1638 by ***Galileo Galilei***[1]. The supporters of this theory included G. Leibniz, G. Lamé, A. Clebsch, and M. Rankine. In English and American literature, it is known as ***Rankine's theory***.

This theory is based on the following assumption (hypothesis):

> The strength of an element subjected to a complex stress state is considered to be exhausted (i.e., its limiting stress state is reached), if the magnitude of the ***largest*** of the ***principal stresses*** reaches the ***limiting value*** determined from simple tension or compression tests.

Thus, the condition for the strength exhaustion takes the form:

$$\sigma_{max} = \sigma_1 = \sigma_{limit_t} \qquad or \qquad \sigma_{max} = |\sigma_3| = \sigma_{limit_c}, \tag{1.2}$$

where σ_{limit_t} and σ_{limit_c} are the limiting stresses determined from simple tensile and compression tests, respectively.

The condition of ensuring strength with a safety factor of n has the form:

$$\sigma_{eq}^{\mathrm{I}} = \sigma_1 \le [\sigma]_t \qquad or \qquad \sigma_{eq}^{\mathrm{I}} = |\sigma_3| \le [\sigma]_c, \tag{1.3}$$

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[1] **Galileo Galilei** (Italian: *Galileo Galilei*; February 15, 1564, Pisa – January 8, 1642, Arcetri) – Italian physicist, mechanic, astronomer, philosopher, and mathematician, who exerted significant influence on the science of his time. He was the first to employ the telescope for celestial observations and made a series of outstanding astronomical discoveries. Galileo was the founder of experimental physics and laid the foundation of classical mechanics.

where $[\sigma]_t = \frac{\sigma_{limit\,t}}{n}$ and $[\sigma]_c = \frac{\sigma_{limit\,c}}{n}$ – allowable tensile and compressive stresses, respectively.

As experimental verification has shown, this theory of strength:

a) does not reflect the conditions for a material's transition into the plastic state, meaning it cannot be used for the strength analysis of parts made from ductile materials;

б) allows to obtain satisfactory results for brittle materials (quartz, rocks, ceramics, tool steels, etc.) under a very limited number of stress state types.

At present, it is rarely applied.

1.2.2. Theory of Maximum Linear Strains (the Second Strength Theory)

The ***Theory of Maximum Linear Strains*** was proposed by ***Edme Mariotte***[2] in 1682. Supporters of this theory included L. Navier and V. Saint-Venant.

It is based on the following hypothesis:

> The strength of an element subjected to a complex stress state is considered to be exhausted (i.e., its limiting stress state is reached), if the magnitude of the ***maximum strain (relative elongation)*** of this element reaches the ***limiting value*** determined from simple tension or compression tests.

Thus, the strength exhaustion will occur when the condition is fulfilled

$$\varepsilon_{max} = \varepsilon_1 = \varepsilon_{limit_t} \quad \text{or} \quad \varepsilon_{max} = |\varepsilon_3| = |\varepsilon_{limit_c}|, \tag{1.4}$$

where ε_{limit_t} and ε_{limit_c} – are the maximum strains determined from simple tension and compression tests, respectively.

According to the generalized Hooke's law for a complex stress state

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[2] **Edme Mariotte** (French: *Edme Mariotte*; 1620, Dijon (Burgundy) – May 12, 1684, Paris) – French physicist of the 17th century, one of the founders (1666) and the first members of the Paris Academy of Sciences. His scientific work pertained to mechanics, heat, and optics. He served as prior of the Saint-Martin Monastery near Dijon.

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]; \qquad \varepsilon_3 = \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)].$$

For a uniaxial stress state:

$$\varepsilon_{limit_t} = \frac{\sigma_{limit_t}}{E}; \qquad \varepsilon_{limit_c} = \frac{\sigma_{limit_c}}{E}.$$

This makes it possible to rewrite the limit state equation (1.4) in terms of stresses:

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \sigma_{limit_t}; \qquad \sigma_3 - \mu(\sigma_1 + \sigma_2) = \sigma_{limit_c}. \tag{1.5}$$

Strength with a safety factor of n will be ensured under the condition:

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \frac{\sigma_{limit_t}}{n} = [\sigma]_t; \qquad \sigma_3 - \mu(\sigma_1 + \sigma_2) = \frac{\sigma_{limit_c}}{n} = [\sigma]_c.$$

Therefore, the strength condition according to the second strength theory finally takes the form:

$$\begin{aligned} \sigma_{\text{eq}}^{\text{II}} &= \sigma_1 - \mu(\sigma_2 + \sigma_3) \le [\sigma]_t; \\ \sigma_{\text{eq}}^{\text{II}} &= \sigma_3 - \mu(\sigma_1 + \sigma_2) \le [\sigma]_c, \end{aligned} \tag{1.6}$$

where $[\sigma]_t = \frac{\sigma_{limit_t}}{n}$ and $[\sigma]_c = \frac{\sigma_{limit_c}}{n}$ – allowable tensile and compressive stresses, respectively.

Experimental verification of this strength theory has led to results similar to those of the first strength theory.

Thus, the first and second theories of strength are of historical rather than practical interest.

1.2.3. Theory of Maximum Shear Stresses (the Third Strength Theory)

The ***Theory of Maximum Shear Stress*** was proposed by ***C. Coulomb***[3] in 1773. This theory was further developed in the works of J. Guest, A. Tresca, and J. Bauschinger.

It is based on the following hypothesis:

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[3] **Charles-Augustin de Coulomb** (French: *Charles-Augustin de Coulomb*, June 14, 1736 – August 23, 1806) – French military engineer and physicist, researcher of electromagnetic and mechanical phenomena; member of the Paris Academy of Sciences.

The strength of an element in a complex stress state is considered exhausted (i.e., its limiting stress state is reached), if the magnitude of the ***maximum shear stress*** reaches the ***limiting value*** determined from simple tension tests.

Consequently, the condition of strength exhaustion takes the form:

$$\tau_{max} = \tau_{limit}. \tag{1.7}$$

In case of complex stress state (Fig. 1.2, a)

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

In case of uniaxial stress state (Fig. 1.2, b)

$$\sigma_3 = 0 \qquad \text{and} \qquad \tau_{max} = \frac{\sigma_1}{2},$$

hence,

$$\tau_{limit} = \frac{\sigma_{limit}}{2}.$$

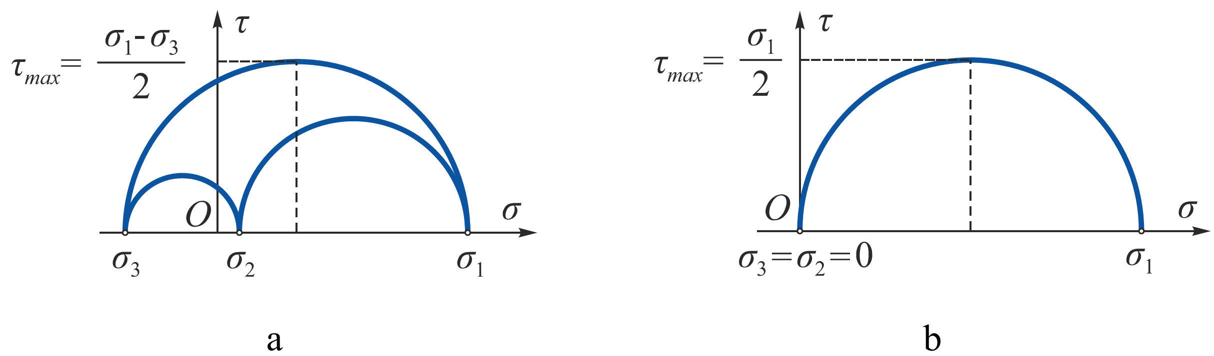

Fig. 1.2

This makes it possible to represent the condition for strength exhaustion (1.7) in terms of principal stresses:

$$\sigma_1 - \sigma_3 = \sigma_{limit}.$$

The strength with a safety factor n will be ensured if

$$\sigma_1 - \sigma_3 = \frac{\sigma_{limit}}{n} = [\sigma].$$

The strength condition according to this theory of strength takes the form

$$\sigma_{eq}^{\mathrm{III}} = \sigma_1 - \sigma_3 \leq [\sigma], \tag{1.8}$$

where $[\sigma] = \frac{\sigma_{limit}}{n}$ – allowable stress.

This theory of strength is confirmed by experiments as a theory of the transition of a material into the plastic state. Thus, in essence, it is a theory of plasticity and is widely used for the strength analysis of parts made from ductile materials, i.e., materials that resist tension and compression equally.

A drawback of the third strength theory is that it does not take into account the intermediate principal stress σ_2, which, as experiments show, also has some (although minor) influence on the strength of materials (the discrepancy between theoretical calculations and experimental data reaches 10–15%).

1.2.4. Energy Theory of Strength (the Fourth Theory of Strength, Distortion Energy Hypothesis)

In 1885, Italian mathematician E. Beltrami suggested that the specific potential strain energy (U_0) is responsible for strength exhaustion. However, experiments did not confirm this assumption.

Therefore, in 1904, M. T. Huber[4] proposed dividing U_0 into two parts: the specific potential volumetric strain energy ($U_{0\,vol}$) and the specific potential deviatoric (or distortional) strain energy ($U_{0\,d}$). He assumed that only the distortion energy is responsible for strength exhaustion.

Further development of this limit state theory was made in the works of R. von Mises[5] (1913), H. Hencky (1925).

The fourth strength theory is most often called the von Mises criterion.

It is based on the following hypothesis:

> The strength of an element in a complex stress state is considered exhausted (i.e., the limiting stress state occurs) if the ***specific potential deviatoric strain energy*** reaches the ***limiting value*** determined from simple tension tests.

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[4] **Maksymilian Tytus Huber** (Polish: *Maksymilian Tytus Huber*, January 4, 1872 – December 9, 1950) was a Polish scientist in the field of theoretical and applied mechanics, and the founder of the Polish school of mechanics.

[5] **Richard Edler von Mises** (German: *Richard Edler von Mises*, April 19, 1883, Lemberg, Austro-Hungarian Empire [now Lviv, Ukraine] – July 14, 1953, Boston, USA) was a mathematician and mechanician of Austrian origin. His works were devoted to aerodynamics, applied mechanics, fluid mechanics, aeronautics, statistics, and probability theory.

Thus, the condition of strength exhaustion takes the form:

$$U_{0\,d} = U_{0\,d\,limit}, \tag{1.9}$$

where $U_{0\,d}$ is the specific potential deviatoric strain energy spent to change the shape of the element when reaching a given complex stress state;

$U_{0\,d\,limit}$ is the limiting value of specific potential deviatoric strain energy determined from a simple tension test, i.e., in the case of a uniaxial stress state.

Determining the potential strain energy of an elementary volume

In an ideal elastic material, the potential strain energy accumulated in an elementary volume during its deformation is numerically equal to the sum of the work done by the forces applied to the faces of this volume.

In each of the coordinate directions, on the faces of an infinitesimal volume $dV = dxdydz$ (Fig. 1.3), normal forces act:

$$dP_1 = \sigma_1 dydz;$$

$$dP_2 = \sigma_2 dxdz;$$

$$dP_3 = \sigma_3 dxdy.$$

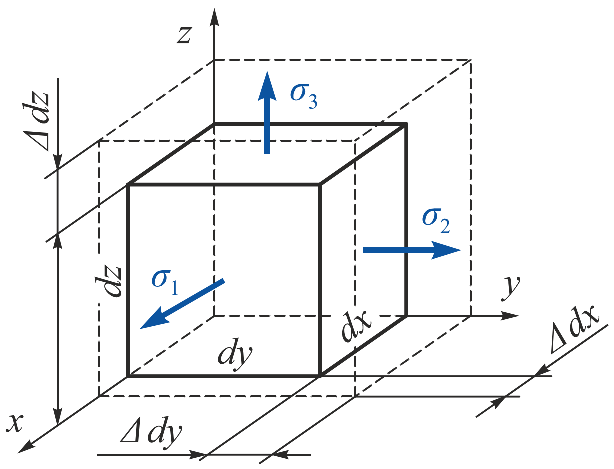


Fig. 1.3

The application of these forces leads to the development of principal strains:

$$\varepsilon_1 = \frac{\Delta dx}{dx}; \qquad \varepsilon_2 = \frac{\Delta dy}{dy}; \qquad \varepsilon_3 = \frac{\Delta dz}{dz}. \tag{1.10}$$

Then the displacements of the points of applied forces are:

$$\Delta dx = \varepsilon_1 dx; \qquad \Delta dy = \varepsilon_2 dy; \qquad \Delta dz = \varepsilon_3 dz.$$

By virtue of the validity of Hooke's law (a directly proportional relationship between forces and displacements), the total work of these elementary forces on the corresponding displacements Δdx, Δdy, and Δdz can be determined by the formula:

$$dW = dU = \frac{1}{2}(\sigma_1 dydz)\Delta dx + \frac{1}{2}(\sigma_2 dxdz)\Delta dy + \frac{1}{2}(\sigma_3 dydx)\Delta dz. \tag{1.11}$$

Here dU is the elementary potential strain energy of elastic deformation accumulated in the infinitesimal volume dV.

Let's introduce the concept of specific potential strain energy of elastic deformation, i.e., the energy accumulated in a unit volume:

$$\frac{dU}{dV} = \frac{dU}{dxdydz} = U_0.$$

Dividing the left and right parts of equation (1.11) by dV, we obtain:

$$U_0 = \frac{1}{2}\left(\sigma_1 \frac{\Delta dx}{dx} + \sigma_2 \frac{\Delta dy}{dy} + \sigma_3 \frac{\Delta dz}{dz}\right) \tag{1.12}$$

or taking into account expressions (1.10)

$$U_0 = \frac{1}{2}(\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3). \tag{1.13}$$

Let's substitute in (1.13) the values of ε_1, ε_2, ε_3 from the generalized Hooke's law:

$$\begin{cases} \varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)]; \\ \varepsilon_2 = \frac{1}{E}[\sigma_2 - \mu(\sigma_1 + \sigma_3)]; \\ \varepsilon_3 = \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)] \end{cases}$$

and after simple transformations we obtain

$$U_0 = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]. \tag{1.14}$$

The potential strain energy accumulated by an elastic body is expended on changing its shape and volume. Let's represent the specific potential strain energy U_0 as the sum of the specific potential volumetric strain energy and the specific potential deviatoric (distortional) strain energy:

$$U_0 = U_{0_{vol}} + U_{0_d}, \tag{1.15}$$

where U_{0_d} is the specific potential strain energy spent on shape change;

$U_{0_{vol}}$ is the specific potential strain energy spent on volumetric change.

Using the superposition principle, we transform the initial stress state (Fig. 1.4) and divide U_0 into two summands in accordance with expression (1.15).

Fig. 1.4 shows that

$$\left.\begin{aligned}\sigma_1 &= P + \sigma_1'\\ \sigma_2 &= P + \sigma_2'\\ \sigma_3 &= P + \sigma_3'\end{aligned}\right\} \Longrightarrow \left.\begin{aligned}\sigma_1' &= \sigma_1 - P\\ \sigma_2' &= \sigma_2 - P\\ \sigma_3' &= \sigma_3 - P\end{aligned}\right\}. \tag{1.16}$$

It follows that the first summand actually determines ***only the change in volume***, i.e., it describes the deformation of uniform (hydrostatic) tension. The second summand complements this stress state to the specified one.

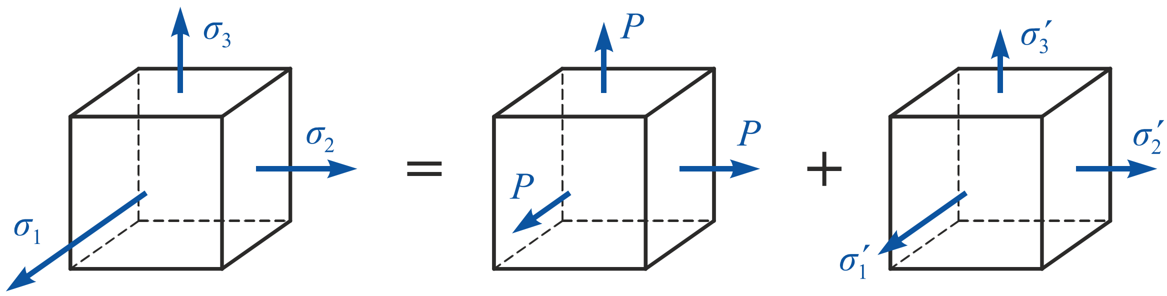


Fig. 1.4

Let's find the values of stresses σ_1', σ_2', and σ_3' from the condition that this part of the stress state does not participate in the change of volume, i.e.

$$\varepsilon_V = \frac{1-2\mu}{E}(\sigma_1' + \sigma_2' + \sigma_3') = 0.$$

Since

$$\frac{1-2\mu}{E} \neq 0,$$

then

$$\sigma_1' + \sigma_2' + \sigma_3' = 0.$$

Substituting in this expression the values σ_1', σ_2', σ_3' from formula (1.16), we obtain

$$P = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \tag{1.17}$$

where P is the average normal stress at the point.

Thus, the value P can always be determined unambiguously and in such a way that no change of volume occurs in the second stress state.

Since there is no mutual work in such a division of the initial stress state, this division is valid.

Let's determine the specific potential strain energy spent on volume change $U_{0\,vol}$.

For this, we substitute in (1.14) the value P instead of σ_1, σ_2, and σ_3. As a result, we obtain

$$U_{0\,vol} = \frac{1}{2E}(3P^2 - 2\mu 3P^2) = \frac{1-2\mu}{2E}3P^2. \tag{1.18}$$

Substituting into this expression the value of P from (1.17), we finally obtain

$$U_{0\,vol} = \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2. \tag{1.19}$$

Subtracting from U_0 (1.14) the value of $U_{0\,vol}$ (1.19), after performing the transformations, we obtain

$$U_{0\,d} = \frac{1+\mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]. \tag{1.20}$$

In the case of the uniaxial stress state this expression takes the form

$$U_{0\,d\ limit} = \frac{1+\mu}{6E} \cdot 2\sigma_{limit}^2. \tag{1.21}$$

By substituting the value of $U_{0\,d}$ from the equation (1.20) and $U_{0\,d\ limit}$ from the expression (1.21) into the strength exhaustion condition (1.9), we obtain

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{limit}^2,$$

or

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sigma_{limit}.$$

Strength with a safety factor n will be ensured under the condition:

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{\sigma_{limit}}{n} = [\sigma].$$

The strength condition finally takes the form

$$\sigma_{eq}^{\mathrm{IV}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le [\sigma], \tag{1.22}$$

where $[\sigma] = \dfrac{\sigma_{limit}}{n}$ is an allowable stress.

The fourth strength theory, like the third one, is well confirmed experimentally as a theory of material transition to plastic state and, along with the third strength theory, is widely used to analyse the strength of parts made from ductile materials.

The occurrence of small plastic strains in the material according to the fourth strength theory is determined more accurately than according to the third theory.

1.2.5. The Mohr's Theory of the Strength

Unlike the theories discussed above, the Mohr's[6] theory is not founded on hypotheses, but is constructed upon a logical systematization of experimental results.

The main assumption underlying this theory is that the strength exhaustion is determined only by the quantities σ_1 and σ_3, and does not depend on σ_2, which is fairly well confirmed by experiment.

The relationship between the strength properties of a material and the type of stress state is derived and justified by means of Mohr's circles.

Suppose that it is possible to test specimens of a given material under any arbitrary complex stress state. Let us select a stress state with a fixed ratio between σ_1 and σ_3 and, by proportionally increasing these stress state components, bring the specimen either to fracture or to the onset of plastic yielding. This stress state will be the limit state. On the $\sigma - \tau$ plane, we draw the largest of the three Mohr's circles. Next, we conduct analogous tests on specimens of the same material at different ratios between σ_1 and σ_3. Each such ratio corresponds to its own limiting Mohr's circle. Next, the envelope of all limiting Mohr's circles is constructed. This envelope essentially represents a mechanical ***characteristic of the material*** under a ***complex stress state***, just as under a ***uniaxial stress state***, just as under a uniaxial stress state the principal mechanical strength characteristics are the yield strength σ_y or the ultimate tensile strength σ_u, determined from tensile or compressive testing (Fig. 1.5).

If the envelope of the limit Mohr's circles for a given material has been obtained experimentally, then, in order to determine whether a stress state characterized by the principal stresses σ_1, σ_2, and σ_3 is limiting, and to assess the material's strength, a stress circle for σ_1 and σ_3 should be constructed at the critical location. Strength is ensured if this circle lies entirely within the area of the envelope.

---

[6] **Christian Otto Mohr** (German: *Christian Otto Mohr*; 8 October 1835, Wesselbüren – 2 October 1918, Dresden) was a German engineer and scholar in the field of theoretical mechanics and the mechanics of materials. He studied at the Polytechnic School in Hanover. Beginning in 1855, he worked on the construction of railways and bridges in Hanover and Oldenburg. From 1867 he served as professor, first in Stuttgart and later in Dresden. His research focused on problems of the mechanics of materials, particularly their graphical representation. In 1882, he developed a graphical method of stress analysis, known as Mohr's circle.

To determine the safety factor, it is necessary to establish by what multiple σ_1 and σ_3 must be simultaneously increased so that the largest Mohr's circle touches the limit envelope. The number indicating by how many times the values of σ_1 and σ_3 are scaled is equal to the safety factor.

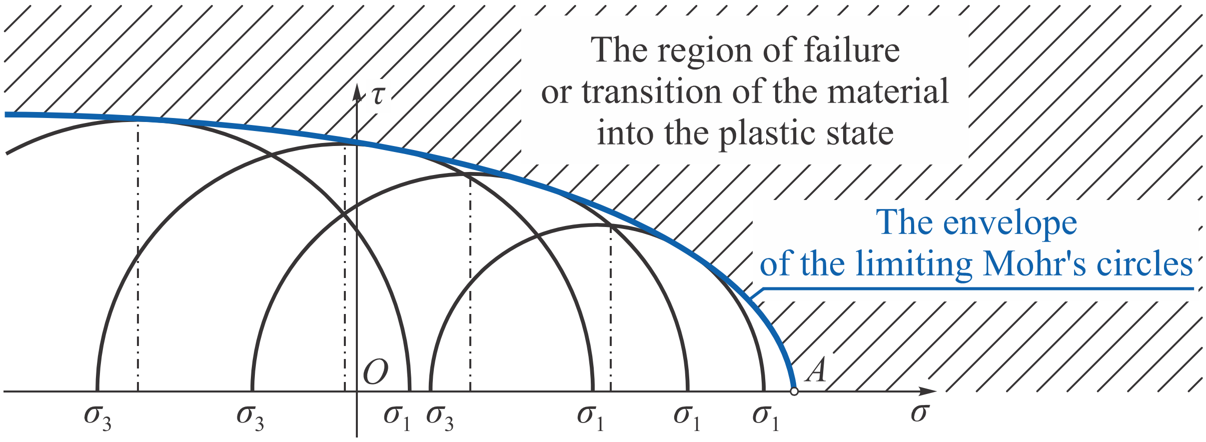


Fig. 1.5

To construct the actual envelope of the limiting Mohr's circles, it would be necessary to experimentally investigate all possible stress states. his is an unfeasible problem; therefore, the question arises of how to construct the envelope of the limiting Mohr's circles using only a limited number of sufficiently simple tests, the technical implementation of which is possible. Three such limiting circles can be constructed in a relatively simple way (Fig. 1.6):

1st circle: by a simple tension test;

2nd circle: by a simple compression test;

3d circle: by a torsion test of a thin-walled tube, in which a state of pure shear is realized at all points of the test specimen (the tube).

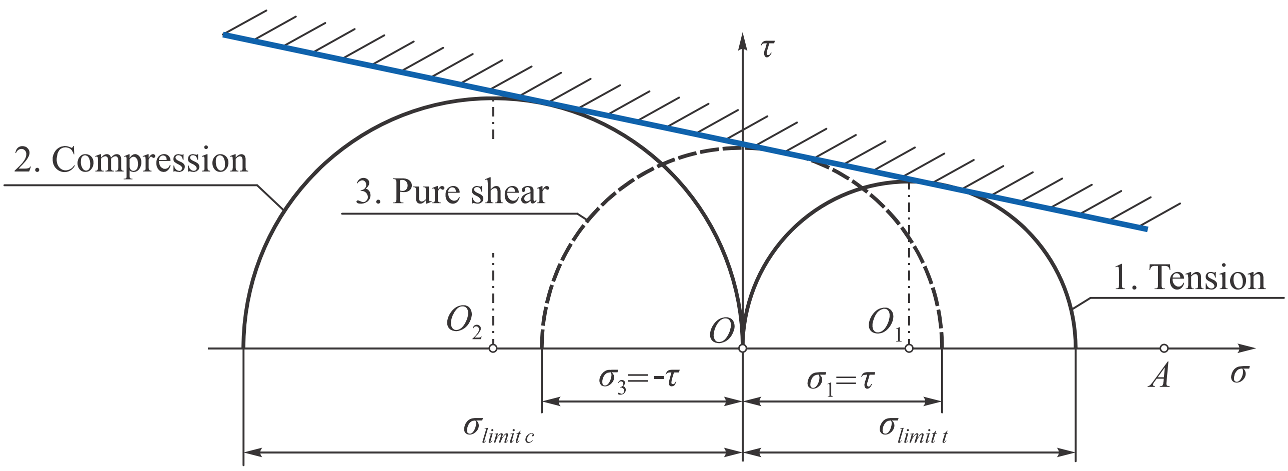


Fig. 1.6

For the ductile materials (see Fig. 1.6)

$$\sigma_{limit_t} = \left|\sigma_{limit_c}\right| = \sigma_{yield_t} = \sigma_{yield_c};$$

for the brittle materials

$$\sigma_{limit_t} = \sigma_{ult_t}; \quad \sigma_{limit_c} = \sigma_{ult_c}.$$

Point ***A*** on the circular diagram characterizes the state of hydrostatic uniform triaxial tension.

To obtain relationships suitable for practical strength analysis, the envelope of the limiting Mohr's circles is approximated by a tangent line to the tensile and compressive circular diagrams. This approximation yields sufficiently accurate results if the center of the stress circle lies between points O_1 (the center of the circle of pure tension) and O_2 (the center of the circle of pure compression) (Fig. 1.7),

where σ_{limit_t} and σ_{limit_c} denote the limiting stresses obtained from tests under pure tension and pure compression, respectively. In the derivation, the absolute value of σ_{limit_c} is used;

σ_1 and σ_3 are the principal stresses of a complex stress state for which the Mohr's circle becomes limiting.

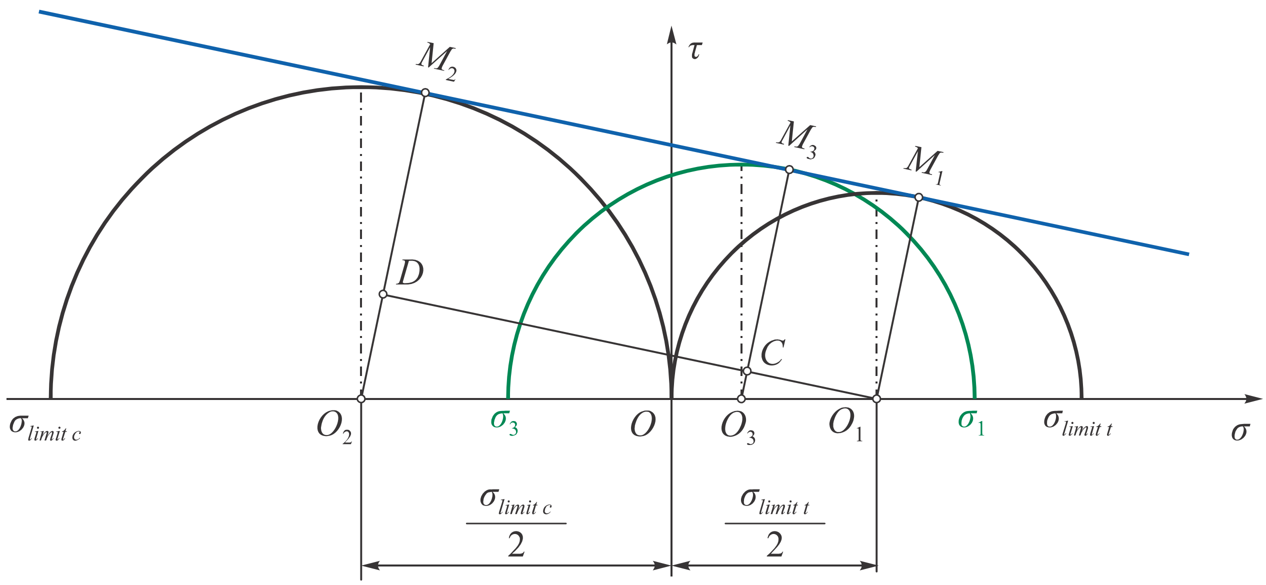


Fig. 1.7

From the geometric relationships, we obtain the strength condition for an intermediate stress state with principal stresses σ_1, σ_3 and the limiting Mohr's circle centered at point O_3 (Fig. 1.7).

Let's draw the straight lines O_1M_1, O_2M_2, O_3M_3 connecting the centres of the limiting Mohr's circles with their tangent points on the limiting line, as well as the straight line OD parallel to M_1M_2. Since the triangles O_1O_3C and O_1O_2D are similar, then

$$\frac{\overline{O_3C}}{\overline{O_2D}} = \frac{\overline{O_1O_3}}{\overline{O_1O_2}},$$

where

$$\overline{O_3C} = \overline{O_3M_3} - \overline{O_1M_1} = \frac{\sigma_1 - \sigma_3}{2} - \frac{\sigma_{limit\,t}}{2};$$

$$\overline{O_2D} = \overline{O_2M_2} - \overline{O_1M_1} = \frac{\sigma_{limit\,c}}{2} - \frac{\sigma_{limit\,t}}{2};$$

$$\overline{O_1O_3} = \overline{OO_1} - \overline{OO_3} = \frac{\sigma_{limit\,t}}{2} - \frac{\sigma_1 + \sigma_3}{2};$$

$$\overline{O_1O_2} = \overline{OO_1} + \overline{OO_2} = \frac{\sigma_{limit\,t}}{2} + \frac{\sigma_{limit\,c}}{2}.$$

Then

$$\frac{(\sigma_1 - \sigma_3) - \sigma_{limit\,t}}{\sigma_{limit\,c} - \sigma_{limit\,t}} = \frac{\sigma_{limit\,t} - (\sigma_1 + \sigma_3)}{\sigma_{limit\,t} + \sigma_{limit\,c}}.$$

Dividing both the numerator and the denominator of the previous expression by $\sigma_{limit\,c}$ and denoting

$$k = \frac{\sigma_{limit\,t}}{\sigma_{limit\,c}}, \tag{1.23}$$

we get

$$\frac{\frac{(\sigma_1 - \sigma_3)}{\sigma_{limit\,c}} - k}{1 - k} = \frac{k - \frac{\sigma_1 + \sigma_3}{\sigma_{limit\,c}}}{k + 1}.$$

Let's transform this equality to the form

$$\sigma_1 - k\sigma_3 = k\sigma_{limit\,c}.$$

Taking into account expression (1.23), the strength exhaustion condition (with a safety factor of one) is finally obtained in form

$$\sigma_{eq}^{\mathrm{M}} = \sigma_1 - k\sigma_3 = \sigma_{limit\,t}.$$

The strength with a safety factor of n will be ensured provided that the following inequality holds:

$$\sigma_{eq}^{\mathrm{M}} = \sigma_1 - k\sigma_3 \leq [\sigma] \tag{1.24}$$

or

$$\sigma_{eq}^{\mathrm{M}} = \sigma_1 - \frac{\sigma_{limit\,t}}{\sigma_{limit\,c}}\sigma_3 \leq [\sigma], \tag{1.25}$$

where $[\sigma] = \dfrac{\sigma_{limit}}{n}$ is an allowable stress;

$k = \sigma_{yield_t} / \sigma_{yield_c}$ for ductile materials;

$k = \sigma_{ult_t} / \sigma_{ult_c}$ for brittle materials.

For ductile materials, since

$$\sigma_{limit_t} = \sigma_{yield_t} = \sigma_{limit_c} = \sigma_{yield_c}$$

this relation degenerates into the strength criterion according to the maximum shear stress theory (the third strength theory).

The Mohr's strength theory can be regarded as the primary theory recommended for the design of parts made from brittle and semi-brittle materials, i.e., materials that resist tension and compression differently ($[\sigma]_t \neq [\sigma]_c$).

Currently, the applicability of the Mohr`s theory of strength is limited because experimental data are practically absent in the regions of hydrostatic tension (at $\sigma_1 > 0$ and $\sigma_3 > 0$) and hydrostatic compression (at $\sigma_1 < 0$ and $\sigma_3 < 0$). Nevertheless, such stress states occur relatively rarely. The Mohr`s theory provides the most reliable results for mixed stress states (at $\sigma_1 > 0$ and $\sigma_3 < 0$).

1.2.6. The Strength Condition According to the Third and Fourth Theories of Strength under a Particular Case of Plane Stress State

Let us consider a particular case of plane stress state.

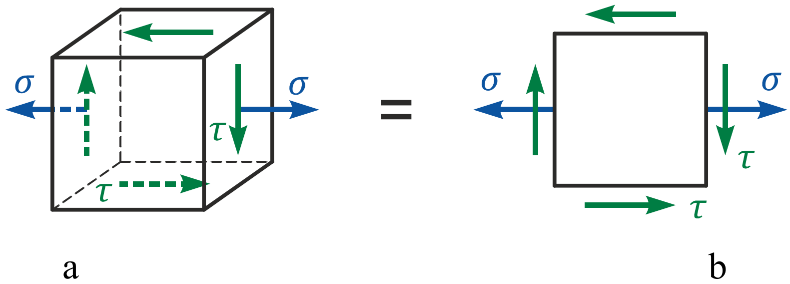


Fig. 1.8

This case of plane stress state is most frequently realized at critical points of bar-type parts under combined bending with torsion, as well as under the “plane transverse bending” deformation mode. Two variants of this case of plane stress state are shown in Fig. 1.8. In the design practice, the simpler variant (Fig. 1.8, b) is most often used. The stresses in Fig. 1.8 are given without indices, since specific problems may be considered in various coordinate systems.

The principal stresses for this case of plane stress state are determined from the following relationships:

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2};$$
$$\sigma_2 = 0; \quad (1.26)$$
$$\sigma_3 = \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}.$$

The indices are assigned to the principal stresses in this manner because, regardless of the sign of σ

$$\frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} > 0 \quad \text{and} \quad \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} < 0$$

with any non-zero value of τ.

Let us consider the strength conditions:

a) according to the third strength theory

$$\sigma_{eq}^{\mathrm{III}} = \sigma_1 - \sigma_3 \leq [\sigma].$$

Substituting the values of σ_1 and σ_3 from equation (1.26) into the previous expression, we obtain

$$\sigma_1 - \sigma_3 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} - \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \sqrt{\sigma^2 + 4\tau^2}.$$

Thus, finally:

$$\sigma_{eq}^{\mathrm{III}} = \sqrt{\sigma^2 + 4\tau^2} \leq [\sigma]; \quad (1.27)$$

b) according to the fourth strength theory with $\sigma_2 = 0$ the relation (1.22) will take the form

$$\sigma_{eq}^{\mathrm{IV}} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} \leq [\sigma].$$

Substituting the values of σ_1 and σ_3 from equation (1.26) into previous expression, we obtain

$$\sqrt{\frac{\sigma^2}{4} + 2\frac{\sigma}{2}\cdot\frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} + \frac{1}{4}(\sigma^2 + 4\tau^2) + \frac{\sigma^2}{4} - 2\frac{\sigma}{2}\cdot\frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} + \frac{1}{4}(\sigma^2 + 4\tau^2) - \left(\frac{\sigma^2}{4} - \frac{1}{4}(\sigma^2 + 4\tau^2)\right)} = \sqrt{\sigma^2 + 3\tau^2}.$$

Thus, finally:

$$\sigma_{eq}^{\mathrm{IV}} = \sqrt{\sigma^2 + 3\tau^2} \leq [\sigma]. \quad (1.28)$$

Remark | Relations (1.27) and (1.28), depending on which of the strength theories is adopted as the working one, allow verification of strength at characteristic points of a beam cross-section under transverse bending.

1.3. Problem-Solving Examples

Example 1.1

Compare the equivalent stresses for the stress states shown in Fig. 1.9. Calculate the equivalent stresses using the fourth (energy) strength theory. The stress values are given in *MPa*.

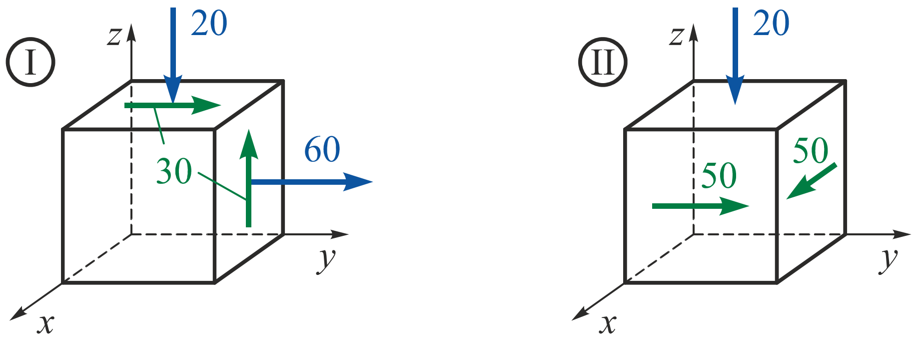


Fig. 1.9

Solution

Determine the principal stresses for the first case:

$$\sigma_{1,\,2,\,(3)} = \frac{\sigma_y + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{yz}^2} = \frac{60 - 20}{2} \pm \sqrt{\left(\frac{60 + 20}{2}\right)^2 + 30^2} =$$

$$= 20 \pm 50;$$

$$\sigma_1 = 70\ MPa; \quad \sigma_2 = 0; \quad \sigma_3 = -30\ MPa.$$

Equivalent stress according to the fourth strength theory

$$\sigma_{eq}^{\mathrm{IV}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} =$$

$$= \frac{1}{\sqrt{2}}\sqrt{(70 - 0)^2 + \big(0 - (-30)\big)^2 + (-30 - 70)^2} = 88.9\ MPa.$$

Determine the principal stresses for the second case:

$$\sigma_{\max,\,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{0 - 0}{2} \pm \sqrt{\left(\frac{0 + 0}{2}\right)^2 + 50^2} = 0 \pm 50;$$

$$\sigma_1 = 50\ MPa; \quad \sigma_2 = -20\ MPa; \quad \sigma_3 = -50\ MPa.$$

Equivalent stress according to the fourth strength theory

$$\sigma_{eq}^{\mathrm{IV}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} =$$

$$= \frac{1}{\sqrt{2}}\sqrt{(50 + 20)^2 + \big(-20 - (-50)\big)^2 + (-50 - 50)^2} = 88.9\ MPa.$$

Thus, the given stress states are equally critical.

Example 1.2

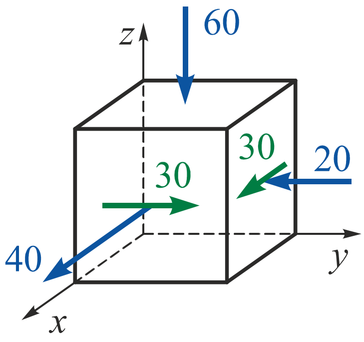


Fig. 1.10

In the critical cross-section of a part made of gray cast iron EN-GJL-200 ($\sigma_{ult_t} = 200\ MPa$; $\sigma_{ult_c} = 750\ MPa$; $\mu = 0.25$), an element is isolated, on the faces of which stresses (in MPa) act as shown in Fig. 1.10. It is necessary to verify the strength of the element.

Solution

Let us denote the stresses shown in Fig. 1.10 according to the xyz coordinate system:

$$\sigma_x = 40\ MPa; \qquad \sigma_y = -20\ MPa; \qquad \sigma_z = -60\ MPa;$$

$$\tau_{yx} = 30\ MPa; \qquad \tau_{xy} = -30\ MPa.$$

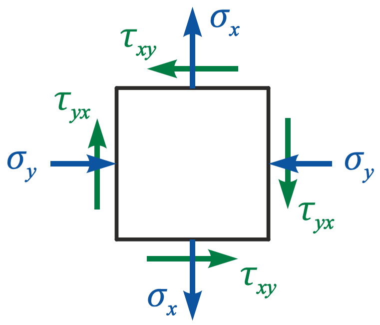


Fig. 1.11

The plane whose normal is parallel to the z-axis is principal, since no shear stresses act on it.

Let us show the stress state on the other two planes in the xOy plane (Fig. 1.11).

Determine the principal stresses:

$$\sigma_{max,\ min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{40 - 20}{2} \pm \sqrt{\left(\frac{40 - (-20)}{2}\right)^2 + 30^2};$$

$$\sigma_{max} = 10 + 31.46 = 41.46\ MPa; \quad \sigma_{min} = 10 - 31.46 = -21.46\ MPa.$$

Assign indices to the principal stresses in accordance with the condition $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\sigma_1 = 41.46\ MPa; \quad \sigma_2 = -21.46\ MPa; \quad \sigma_3 = -60\ MPa.$$

Let us verify the calculation results using the property of normal stresses invariance:

$$\sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 = const;$$

$$40 - 20 - 60 = 41.46 - 21.46 - 60 = -40.$$

Let's check the strength of the element. Assign the allowable stresses, choosing a safety factor of $[n] = 3$, which is recommended for brittle materials that resist tension and compression differently:

$$[\sigma]_t = \frac{\sigma_{ult_t}}{n} = \frac{200}{3} = 66.67\ MPa; \qquad [\sigma]_c = \frac{\sigma_{ult_c}}{n} = \frac{750}{3} = 250\ MPa.$$

According to the ***first strength theory***:

$$\sigma_{eq}^{\mathrm{I}} = \sigma_1 = 41.46\ MPa \le [\sigma]_t = 66.67\ MPa;$$

$$\sigma_{eq}^{\mathrm{I}} = |\sigma_3| = 60\ MPa \le [\sigma]_c = 250\ MPa.$$

Strength is ensured.

According to the ***second strength theory***:

$$\sigma_{eq}^{\mathrm{II}} = \sigma_1 - \mu(\sigma_2 + \sigma_3) \le [\sigma]_t;$$

$$41.46 - 0.25(-21.46 - 60) = 61.825\ MPa \le [\sigma]_t = 66.67\ MPa;$$

$$\sigma_{eq}^{\mathrm{II}} = \sigma_3 - \mu(\sigma_1 + \sigma_2) \le [\sigma]_c;$$

$$|-60 - 0.25(41.46 - 21.46)| = 65\ MPa \le [\sigma]_c = 250\ MPa.$$

Strength is ensured.

According to the ***third strength theory***:

$$\sigma_{eq}^{\mathrm{III}} = \sigma_1 - \sigma_3 = 41.46 - (-60) = 100.46\ MPa \ge [\sigma]_t = 66.67\ MPa.$$

Strength is insufficient.

According to the ***fourth strength theory***:

$$\sigma_{eq}^{\mathrm{IV}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le [\sigma]_t;$$

$$\frac{1}{\sqrt{2}}\sqrt{\big(41.46 - (-21.46)\big)^2 + \big(-21.46 - (-60)\big)^2 + (-60 - 41.46)^2} =$$

$$= 102.2\ MPa \ge [\sigma]_t = 66.67\ MPa.$$

Strength is insufficient.

According to ***Mohr's strength theory***

$$\sigma_{eq}^{\mathrm{M}} = \sigma_1 - \frac{\sigma_{ult_t}}{\sigma_{ult_c}}\sigma_3 = 41.46 - \frac{200}{750}(-60) = 51.46\ MPa \le [\sigma]_t = 66.67\ MPa.$$

Strength is ensured.

This example demonstrates the use of different strength theories for a verification analysis of a part made from a ***brittle material***. The use of the third and fourth strength theories, which are applied for ductile materials, led to a negative result.

2. CONSTRUCTION OF DIAGRAMS OF INTERNAL FORCES AND MOMENTS FOR ARBITRARILY LOADED BROKEN BARS

2.1. Diagrams and Fundamental Rules of Their Construction

All six internal forces and moments may act in the cross-sections of a cranked bar: N_x, Q_z, Q_y, M_x, M_y and M_z. All the rules used for constructing diagrams in beams and planar frames also apply to cranked bars.

> A ***diagram*** (historically on French *"épure" – sketch*) is a graphical representation showing the variation of an internal force or an internal moment along the longitudinal axis of the bar.

The graphical representation of a function is highly illustrative, making it easy to evaluate its key features. In the context of mechanics of materials and structural analysis, this means the ability to identify the critical section. This is the main purpose of constructing diagrams.

Basic rules for constructing diagrams

1. The diagram's base-axis is drawn parallel to the longitudinal axis of the bar. If the bar's axis is curved, the diagram's axis is also curved (or cranked).
2. The value of the internal force or moment acting in a cross-section of the bar is plotted ***to scale along the normal to the base-axis*** at the point corresponding to that cross-section.
3. Each diagram must indicate the name of the internal force or moment, the units of measurement, the sign convention, numerical values at characteristic points, and be hatched perpendicularly to the diagram's axis.

Rules for dividing a structural element into segments

1. The law of external load application (including support reactions) remains unchanged within a single segment. That is, segment boundaries are defined by the cross-sections where concentrated forces (P) or concentrated force couples (moment M) are applied, or where the action of an external distributed load (q) begins or ends.
2. The geometry of the cross-section does not change within a segment; alternatively, boundaries occur where the cross-sectional area changes abruptly.

3. The boundaries of segments in a frame and a cranked bar are also nodal points (points of bending).
4. The material from which the bar is made does not change within a segment.

Sign conventions for constructing diagrams

1. Axial force N_x

An external force acting on either side of the cross-section $n-n$ makes a positive contribution to the magnitude of the axial force N_x if it causes tension (directed away from that cross-section) and negative if it causes compression (directed toward that cross-section) (Fig. 2.1).

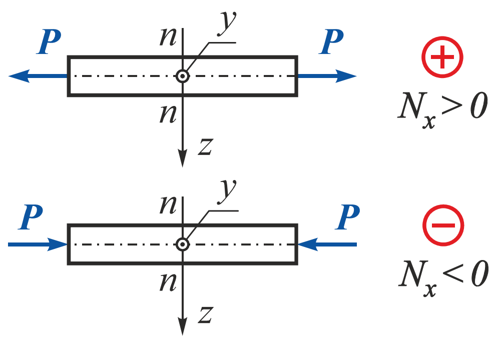


Fig. 2.1

2. Torsional moment M_x

When constructing torsional moment diagrams, an arbitrary sign convention is used. In this text, we will use the following rule: an external torque acting on either side of the cross-section $n-n$ makes a positive contribution to the magnitude of the torsional moment M_x, if, when viewed from the direction of the outward normal to the considered cross-section, it is directed counter-clockwise, and negative if it is directed clockwise (Fig. 2.2)

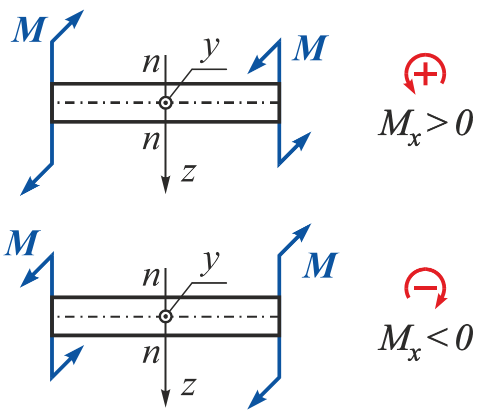


Fig. 2.2

3. Shear forces Q_z and Q_y

An external transverse force acting on either side of the cross-section $n-n$ makes a positive contribution to the magnitude of the shear force Q_z (or Q_y) acting in that cross-section if it tends to rotate the considered piece of a bar clockwise relative to the principal central axis of inertia y (or z), and negative if it rotates it counter-clockwise (Fig. 2.3).

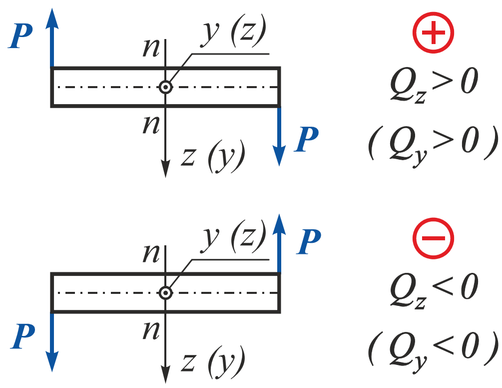


Fig. 2.3

4. Bending moments M_y and M_z

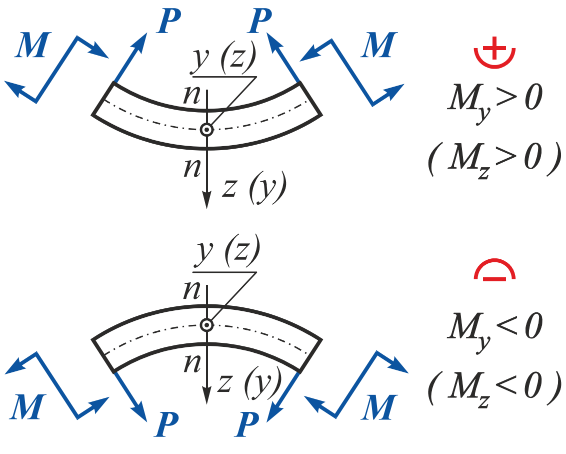


Fig. 2.4

An external force or a concentrated moment acting on either side of the cross-section $n - n$ makes a positive contribution to the magnitude of the bending moment M_y (or M_z), acting in that cross-section if this external force or concentrated moment causes compression of the top fibers and tension of the bottom fibers of the bar (causing a convex-downward bend), and negative if it causes tension of the top fibers and compression of the bottom fibers (causing a convex-upward bend) (Fig. 2.4).

The bending moment diagrams M_y and M_z are drawn on the side of the bar\`s tensile fibers.

2.2. Construction of Internal Forces and Moments Diagrams for a Planar Cranked Bar with Out-of-Plane Loading

A ***planar cranked bar with out-of-plane loading*** is a cranked bar whose elements are rigidly connected at the nodes (fixed joints) and lie in a single plane, but the external loads act in arbitrary directions.

When constructing diagrams, the external forces are represented by ***their projections*** onto the accepted ***coordinate axes***.

In engineering practice, two main methods are used for constructing diagrams of internal forces and moments in cranked bars.

2.2.1. The First Method for Constructing Diagrams

The essence of this method is as follows: after determining the internal forces and moments in the first segment, all external loads (concentrated forces and moments, distributed loads) acting within that segment are resolved and transferred, in accordance with the theorems of statics, to the initial cross-section of the second segment. After determining the internal forces and moments in the second segment, all external forces acting within that segment are resolved to the initial cross-section of the third segment, and so on. With such approach, each segment of the cranked bar is treated as a cantilever.

Example 2.1

Construct the diagrams of internal forces and moments for the given cranked bar (Fig. 2.5).

Given: $P_1 = 15\ kN;\ P_2 = 20\ kN;\ P_3 = 10\ kN;$
$a = 2\ m;\ b = 3\ m;\ c = 4\ m.$

It is necessary *to construct the diagrams of*

$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$

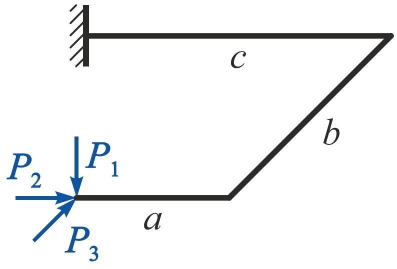


Fig. 2.5

Solution

1. Let us draw the cranked bar to scale and divide it into three segments: I, II, and III.

In an arbitrary cross-section of each segment, at a distance x from its beginning, we will place the local xyz coordinate system such that the x-axis coincides with the bar's longitudinal axis, the z-axis is directed downwards, and the horizontal y-axis, together with the first two axes, forms a right-handed orthogonal basis (Fig. 2.6).

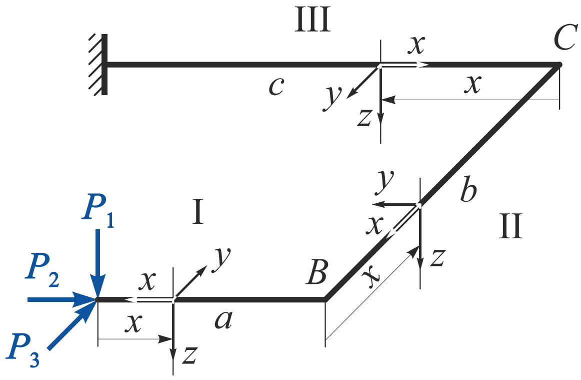


Fig. 2.6

Remark | If the cranked bar is "unfolded" along the shortest angular path into a straight line, the directions of the x, y and z axes must coincide across all segments.

2. Using the method of sections, we'll write the equations for the internal forces and moments for each segment.

Let's consider **segment I** (Fig. 2.7) $(0 \le x \le a,\ \ a = 2\ m)$.

$N_x^I = -P_2 = -20\ kN;$

$Q_z^I = P_1 = 15\ kN;$

$Q_y^I = -P_3 = -10\ kN;$

$M_x^I = 0;$

$M_y^I = -P_1 x = 15x \Big|_{x=0} = 0 \Big|_{x=a=2\ m} =$
$= -30\ kN{\cdot}m;$

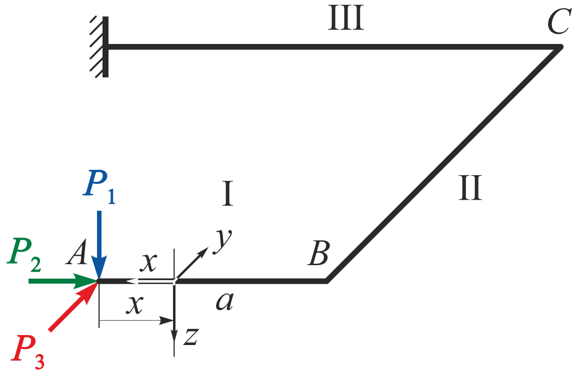


Fig. 2.7

$M_z^I = -P_3 x = -10x \Big|_{x=0} = 0 \Big|_{x=a=2\ m} = -20\ kN{\cdot}m.$

Remark | The moment created by a non-zero force relative to a certain axis is zero if the force's line of action intersects or is parallel to that axis.

Let's consider **segment II** $(0 \le x \le b,\ \ b = 3\ m)$.

Let us transfer the forces P_1, P_2, and P_3 to the initial cross-section of segment II (point B) (Fig. 2.8).

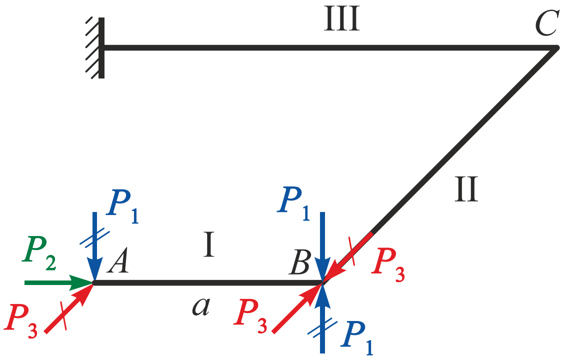


Fig. 2.8

Since the ***line of action*** of force P_2 ***passes through*** point B, according to the principles of statics, the application point of force P_2 can be simply transferred to point B.

To resolve force P_1 to point B, we apply a ***statically equivalent system*** (a static zero) at this point, consisting of two equal and oppositely directed forces P_1 whose lines of action coincide.

Thus, the action of force P_1 applied at point A is statically equivalent to the combined action of force P_1 and a moment $P_1 a$ applied at point B.

The same procedure is used to transfer force P_3.

The calculation scheme for segment **II** is shown in Fig. 2.9.

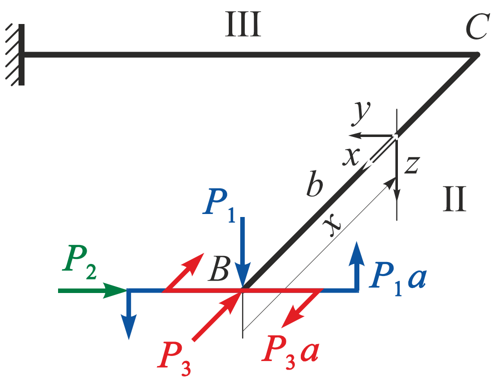


Fig. 2.9

$N_x^{II} = -P_3 = -10\ kN;$

$Q_z^{II} = P_1 = 15\ kN;$

$Q_y^{II} = P_2 = 20\ kN;$

$M_x^{II} = P_1 a = 15 \cdot 2 = 30\ kN{\cdot}m;$

$$M_y^{II} = -P_1 x = -15x \ \Big|_{x=0} = 0 \ \Big|_{x=b=3\ m} = -45\ kN{\cdot}m;$$

$$M_z^{II} = -P_3 a + P_2 x = -10 \cdot 2 + 20x \ \Big|_{x=0} = -20\ kN{\cdot}m \ \Big|_{x=b=3\ m} = 40\ kN{\cdot}m.$$

Let's consider **segment III** $(0 \le x \le c, \;\; c = 4\ m)$.

We now resolve the system of external forces acting on segment **II** to the initial cross-section of segment **III** (point C), in the same manner as discussed above.

The calculation scheme of segment **III** is shown in Fig. 2.10.

Remark As shown in static, a moment is considered as a free load and may be relocated within its plane or between parallel planes without changing its magnitude or direction.

$N_x^{III} = P_2 = 20\ kN;$

$Q_z^{III} = P_1 = 15\ kN;$

$Q_y^{III} = P_3 = 10\ kN;$

$M_x^{III} = P_1 b = 15 \cdot 3 = 45\ kN{\cdot}m;$

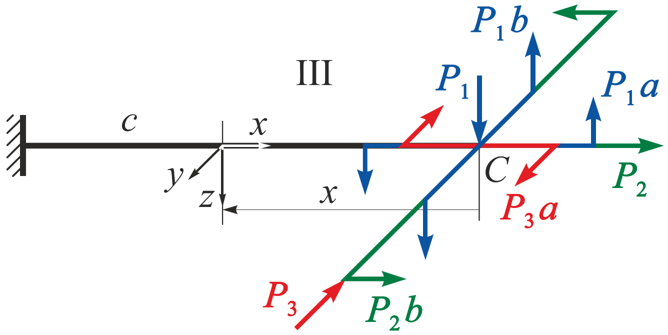


Fig. 2.10

$$M_y^{III} = P_1 a - P_1 x = 15 \cdot 2 - 15x \;\Big|_{x=0} = 30\ kN{\cdot}m \;\Big|_{x=c=4\ m} = -30\ kN{\cdot}m;$$

$$M_z^{III} = P_2 b - P_3 a + P_3 x = 20 \cdot 3 - 10 \cdot 2 + 10x \;\Big|_{x=0} = 40 \;\Big|_{x=c=4\ m} = 80\ kN{\cdot}m.$$

Remarks 1. To correctly establish the sign of the axial force N_x created by the external load P_2 (P_3) in the cross-sections of Segment II (III), this force is to be regarded as applied in the same direction at the initial cross-section of the corresponding segment (see Fig. 2.6 – 2.10).

2. To correctly determine the sign of the shear force Q_z (Q_y) created by the external load P_1 (P_2 and P_3) in the cross-sections of Segments II and III, this force is to be regarded as applied in the same direction at the initial cross-sections of the corresponding segments II and III (see Fig. 2.6 – 2.10).

3. Let us construct the diagrams (Fig. 2.11).

Remark When constructing internal forces and moments diagrams for a cranked bar, the following must be taken into account:

a) the diagrams of N_x and M_x can be drawn in any plane;

b) the diagrams of Q_z, Q_y, M_y, M_z must be drawn only ***in their respective planes of action***;

c) the diagrams of M_y and M_z are drawn ***on the side of the tensile fibres*** of the bar.

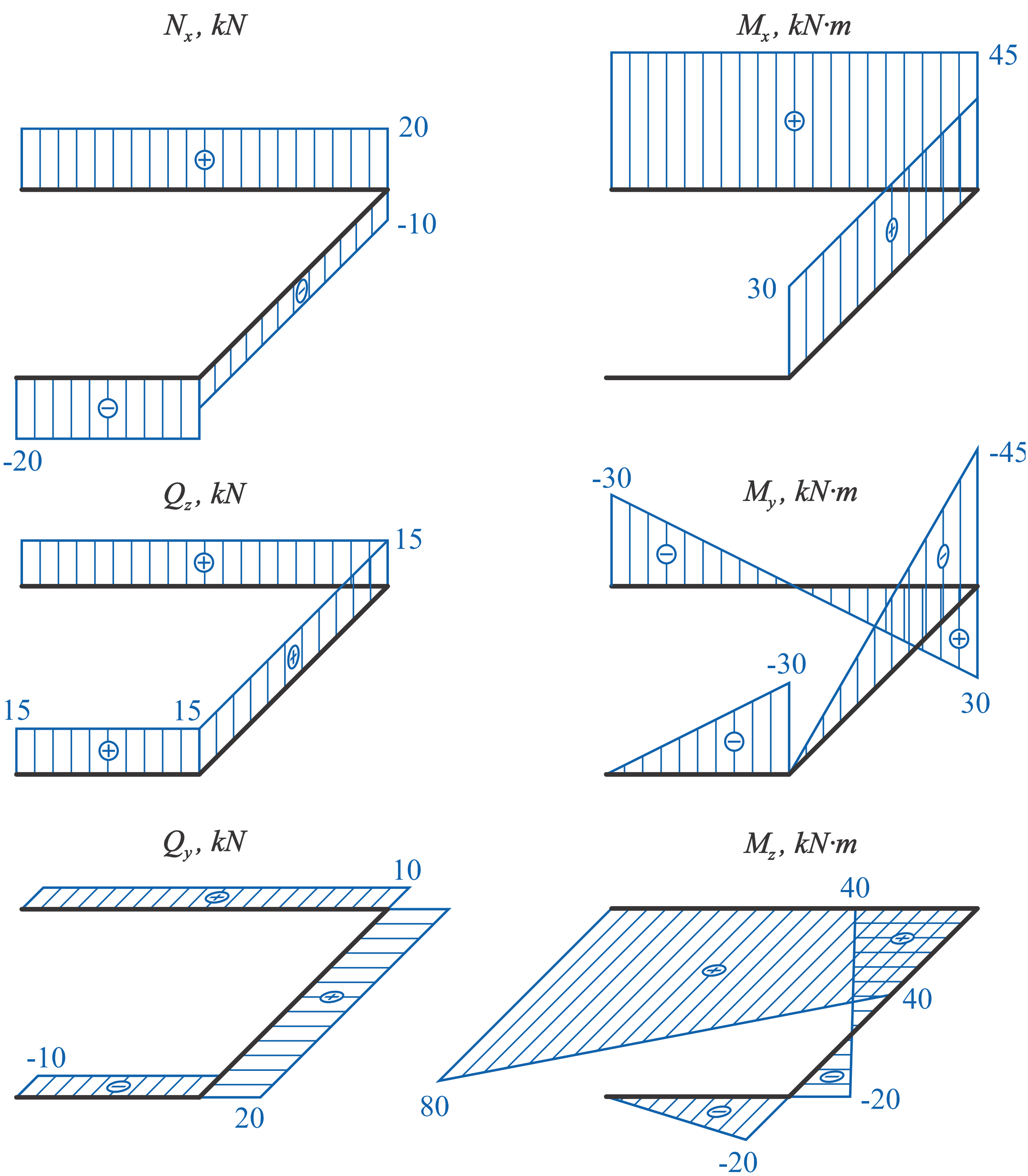


Fig. 2.11

4. Let us check the correctness of the diagram construction.

To this end, infinitesimal elements of the cranked bar are isolated at the junctions of its parts (nodes *B* and *C*), and their equilibrium is analysed under the action of internal and external loads applied within these nodes (Fig. 2.12).

In Fig. 2.12, all internal forces and moments are shown in their true directions.

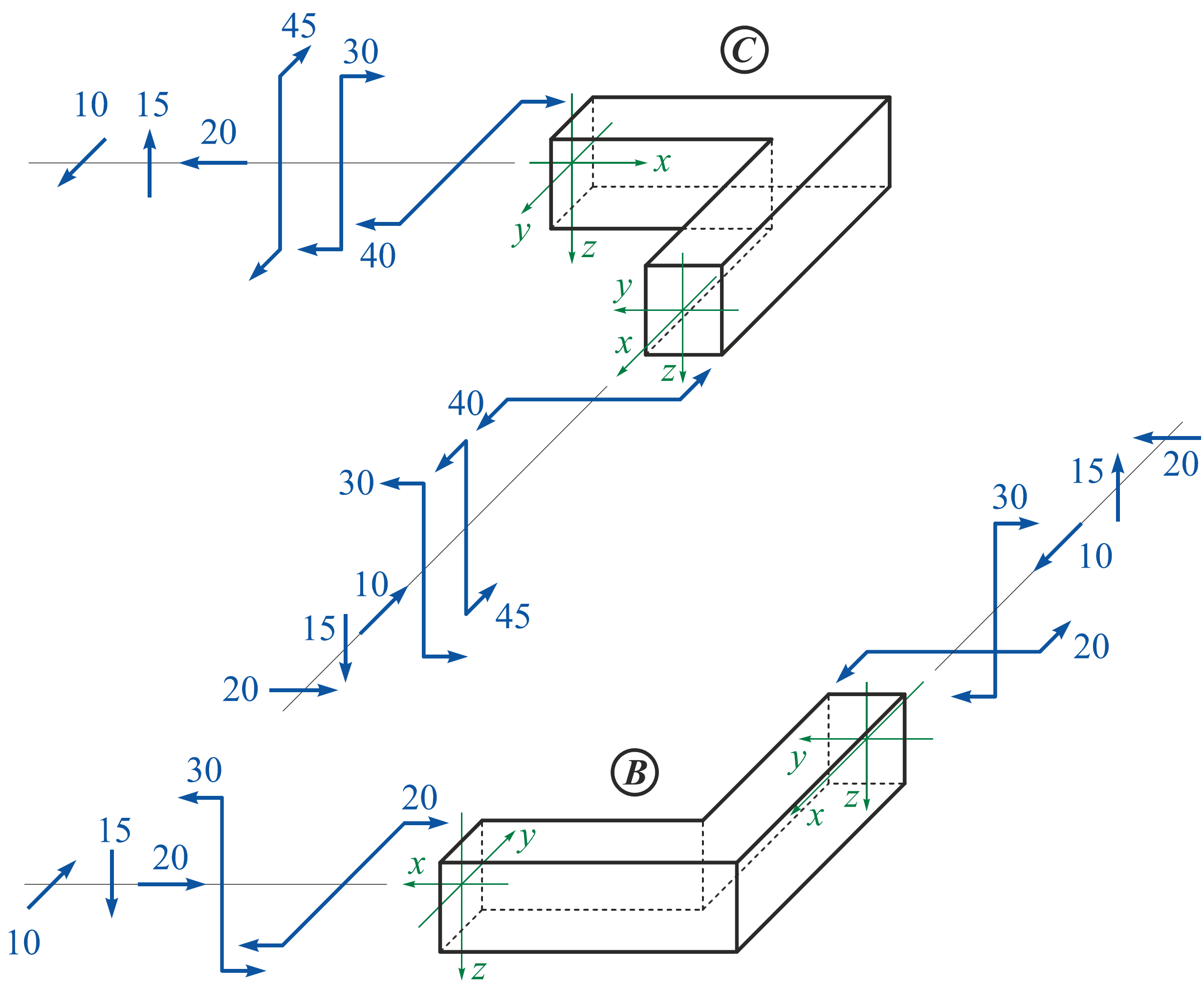


Fig. 2.12

Equilibrium equations for node B:

$\sum P_x = 20 - 20 = 0$; $\sum P_y = 10 - 10 = 0$; $\sum P_z = 15 - 15 = 0$;

$\sum M_x = 0$; $\sum M_y = 30 - 30 = 0$; $\sum M_z = 20 - 20 = 0$.

Equilibrium equations for node C:

$\sum P_x = 20 - 20 = 0$; $\sum P_y = 10 - 10 = 0$; $\sum P_z = 15 - 15 = 0$;

$\sum M_x = 45 - 45 = 0$; $\sum M_y = 30 - 30 = 0$; $\sum M_z = 40 - 40 = 0$.

Remark When constructing internal forces and moments diagrams for a cranked bar, each of its elements must be considered as a rod in tension-compression, a shaft in torsion, and a beam in transverse bending in two planes. In this process, all sign conventions for N_x, Q_z, Q_y, M_x, M_y, M_z, and all rules for constructing diagrams are preserved.

Example 2.2

Construct the diagrams of internal forces and moments for the given cranked bar (Fig. 2.13).

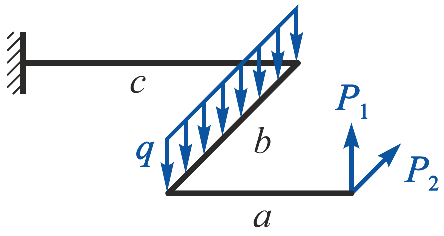


Fig. 2.13

Given: $P_1 = 20\ kN;\ P_2 = 10\ kN;\ q = 15\ kN/m;$
$a = 2\ m;\ b = 2\ m;\ c = 3\ m.$

It is necessary to construct the diagrams of

$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$

Solution

1. Let us draw the cranked bar to scale and divide it into three segments: I, II, and III.

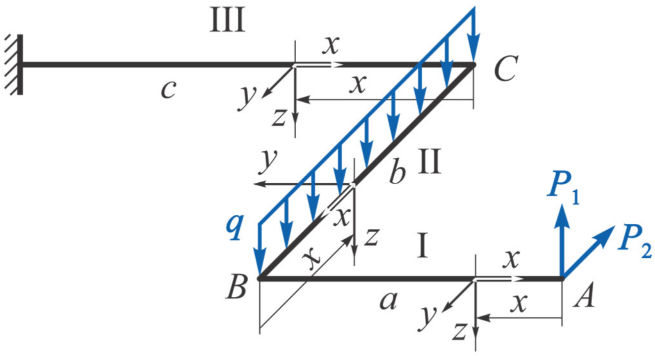


Fig. 2.14

In an arbitrary cross-section of each segment at a distance x from its beginning, we will place a local xyz coordinate system (Fig. 2.14).

2. Using the method of sections, we'll write the equations for the internal forces and moments for each segment.

Let's consider **segment I** (Fig. 2.15) $(0 \le x \le a,\ \ a = 2\ m)$.

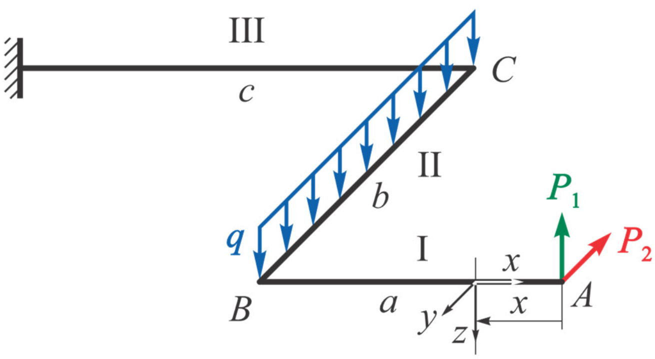


Fig. 2.15

$N_x^I = 0;$

$Q_z^I = -P_1 = -20\ kN;$

$Q_y^I = P_2 = 10\ kN;$

$M_x^I = 0;$

$$M_y^I = P_1 x = 20x \Big|_{x=0} = 0 \Big|_{x=a=2\ m} = 40\ kN{\cdot}m;$$

$$M_z^I = P_2 x = 10x \Big|_{x=0} = 0 \Big|_{x=a=2\ m} = 20\ kN{\cdot}m.$$

Let's consider **segment II** $(0 \le x \le b, \;\; b = 2\ m)$.

Let us transfer the forces P_1 and P_2 to the initial cross-section of segment **II** (point B) (Fig. 2.16).

$N_x^{II} = -P_2 = -10\ kN;$

$Q_z^{II} = qx - P_1 = 15x - 20 \quad \Big|_{x=0} =$

$= -20\ kN \quad \Big|_{x=b=2\ m} = 10\ kN.$

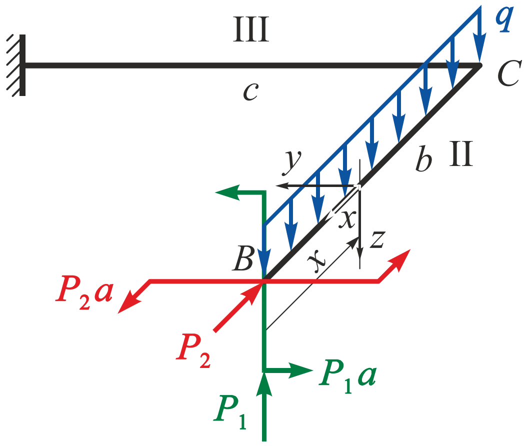


Fig. 2.16

Since the shear force Q_z changes sign within Segment II, it is necessary to determine the point x_e, at which $Q_z^{II} = 0$:

$$qx_e - P_1 = 0 \;\Rightarrow\; x_e = \frac{P_1}{q} = \frac{20}{15} = 1.33\ m;$$

$Q_y^{II} = 0;$

$M_x^{II} = P_1 a = 20 \cdot 2 = 40\ kN{\cdot}m;$

$$M_y^{II} = P_1 x - \frac{qx^2}{2} = 20x - \frac{15x^2}{2}\Bigg|_{x=0} = 0 \quad \Big|_{x=b=2\ m} = 10\ kN{\cdot}m \quad \Big|_{x=x_e=1{,}33\ m} =$$

$= 13{,}33\ kN{\cdot}m;$

$M_z^{II} = P_2 a = 10 \cdot 2 = 20\ kN{\cdot}m.$

Let's consider **segment III** $(0 \le x \le c, \;\; c = 3\ m)$.

Let us replace the distributed load acting within the second segment with a resultant force (Fig. 2.17) and resolve the system of external forces on segment II to the initial cross-section of segment III (point C).

The calculation scheme for segment **III** is shown in Fig. 2.18.

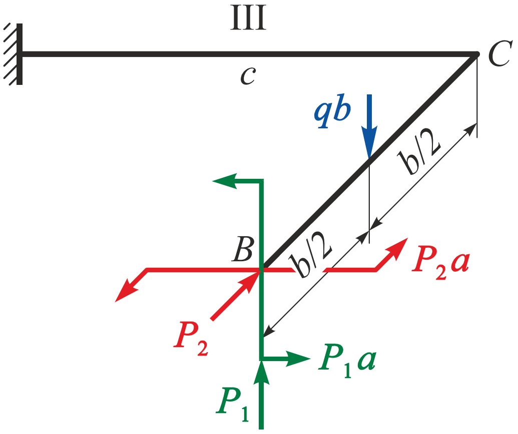


Fig. 2.17

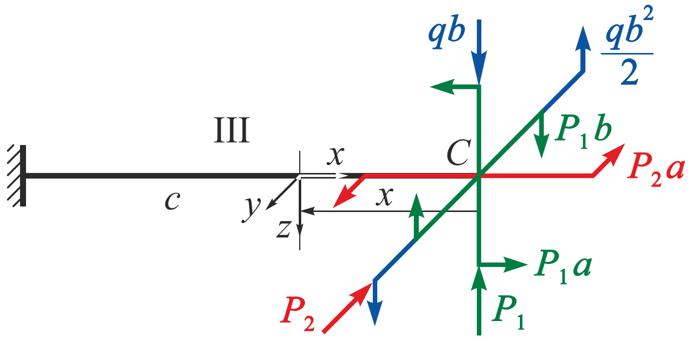


Fig. 2.18

$N_x^{III} = 0;$

$Q_z^{III} = -P_1 + qb = -20 + 15 \cdot 2 = 10 \; kN;$

$Q_y^{III} = P_2 = 10 \; kN;$

$$M_x^{III} = -P_1 b + \frac{qb^2}{2} = -20 \cdot 2 + \frac{15 \cdot 2^2}{2} = -10 \; kN{\cdot}m;$$

$$M_y^{III} = P_1 a + P_1 x - qbx = 20 \cdot 2 + 20x - 15 \cdot 2x \; \Big|_{x=0} =$$

$$= 40 \; kN{\cdot}m \; \Big|_{x=c=3\,m} = 10 \; kN{\cdot}m;$$

$$M_z^{III} = P_2 x + P_2 a = 10 \cdot x + 10 \cdot 2 \; \Big|_{x=0} = 20 \; kN{\cdot}m \; \Big|_{x=c=3\,m} = 50 \; kN{\cdot}m.$$

3. Let us construct the diagrams (Fig. 2.19).

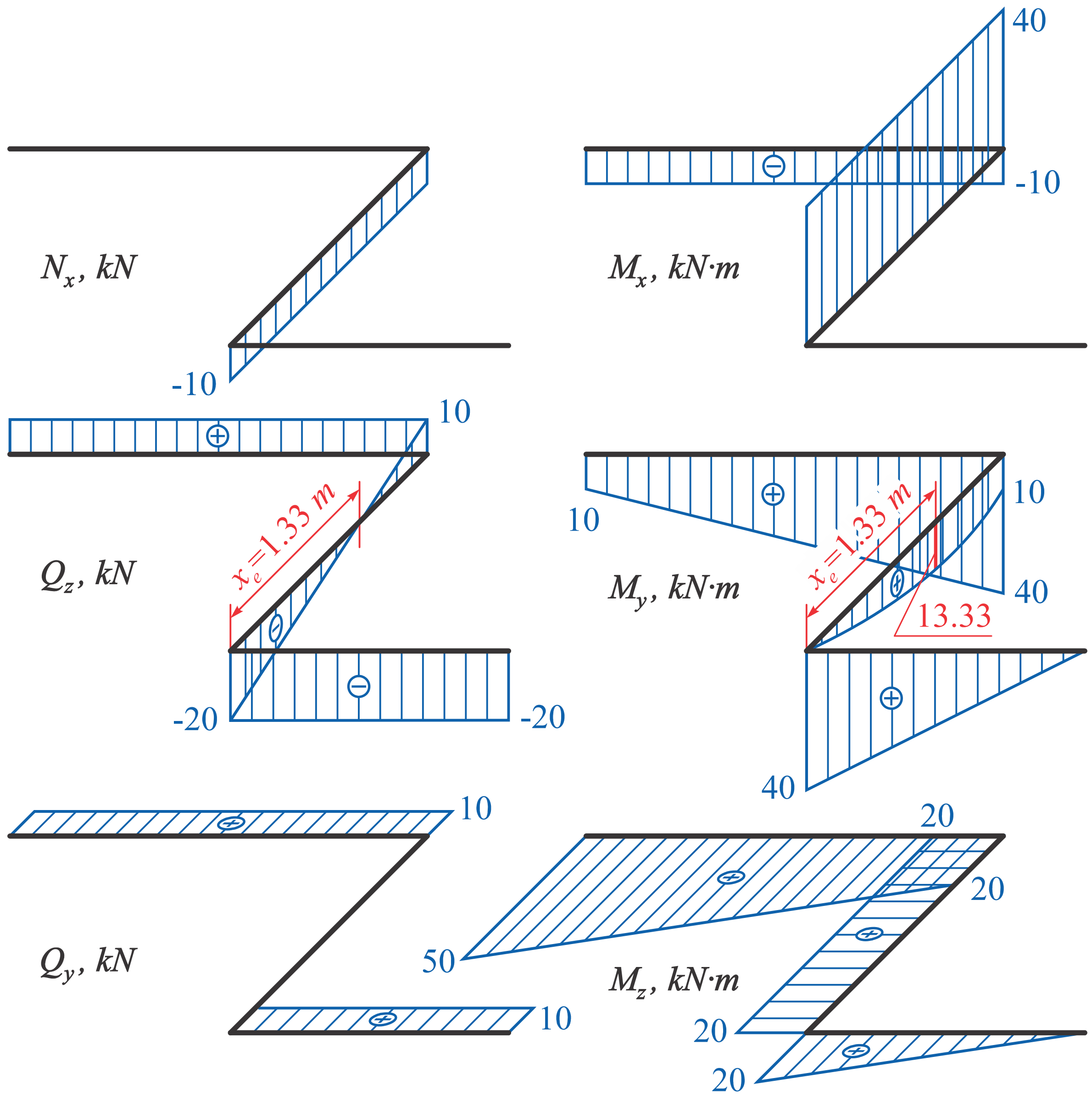


Fig. 2.19

4. Let us check the correctness of the diagram construction.

To this end, infinitesimal elements of the cranked bar are isolated at the junctions of its parts (nodes *B* and *C*), and their equilibrium is analysed under the action of internal and external loads applied within these nodes (Fig. 2.20).

In Fig. 2.20, all internal forces and moments are shown in their true directions.

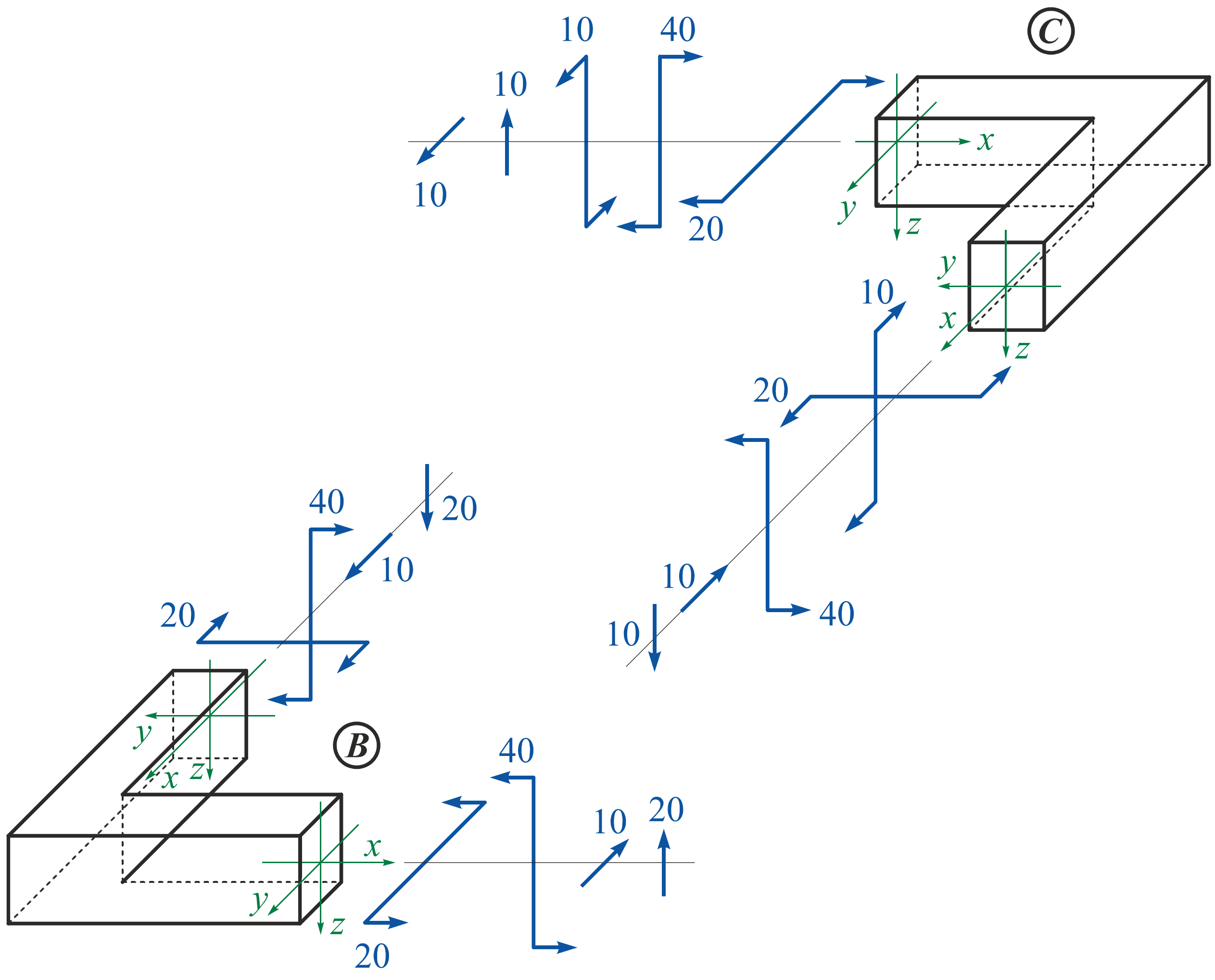


Fig. 2.20

Equilibrium equations for node *B*:

$\sum P_x = 0;$ $\sum P_y = 10 - 10 = 0;$ $\sum P_z = 20 - 20 = 0;$

$\sum M_x = 0;$ $\sum M_y = 40 - 40 = 0;$ $\sum M_z = 20 - 20 = 0.$

Equilibrium equations for node *C*:

$\sum P_x = 0;$ $\sum P_y = 10 - 10 = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 10 - 10 = 0;$ $\sum M_y = 40 - 40 = 0;$ $\sum M_z = 20 - 20 = 0.$

2.2.2. The Second Method for Constructing Diagrams

When using the second method, the operation of resolving the system of external forces acting on the considered segment into the initial section of the subsequent segment is excluded, which allows the solution to be presented in a more compact form.

Example 2.3

Construct the diagrams of internal forces and moments for the given cranked bar (Fig. 2.21).

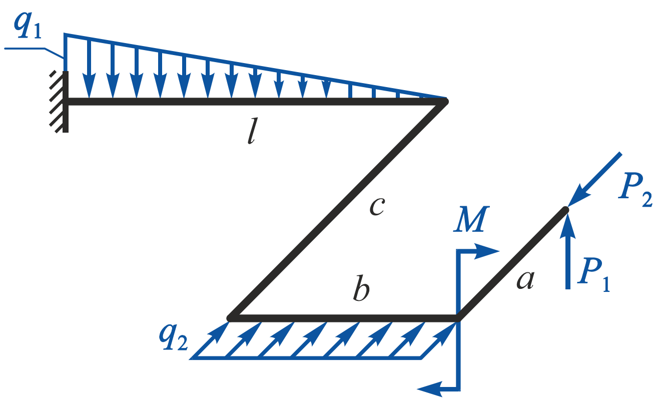


Fig. 2.21

Given: $P_1 = 5\ kN$; $P_2 = 30\ kN$;
$q_1 = 15\ kN/m$; $q_2 = 20\ kN/m$;
$M = 20\ kN{\cdot}m$; $a = 3\ m$;
$b = 2\ m$; $c = 2\ m$; $l = 3\ m$.

It is necessary *to construct the diagrams of*

$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$

Solution

1. Let us draw a cranked bar to scale and divide it into segments. In an arbitrary cross-section of each segment, at a distance x from its beginning, let us introduce a local xyz coordinate system so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downwards, and the horizontal y-axis, together with the first two axes, forms a right-handed orthogonal basis (Fig. 2.22).

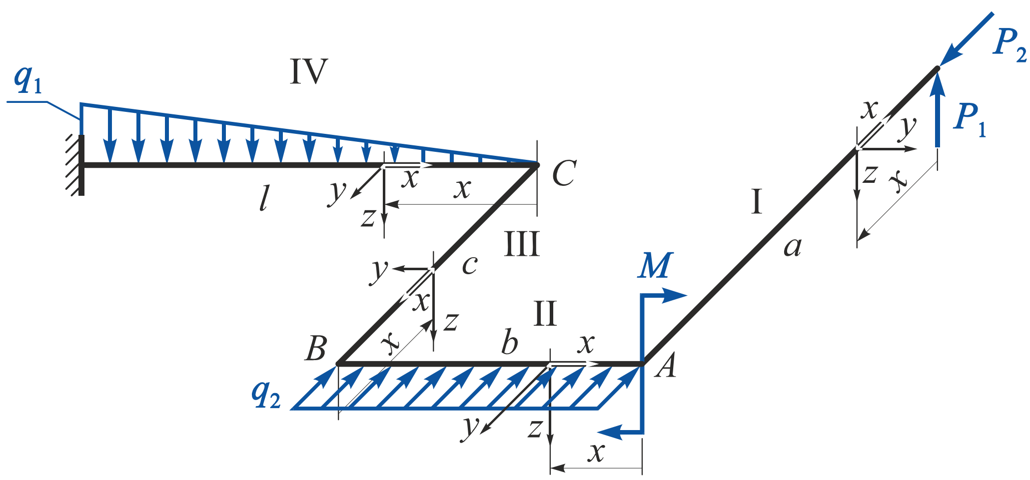


Fig. 2.22

Remarks | 1. To obtain a formally ordered sign convention for the internal forces and moments across all segments, it is preferable to derive the coordinate system for segment II by a simple translation, that is, by rotating the coordinate system of segment I through 90° about the z-axis, and so on.
2. If the cranked bar is straightened along the shortest angular path into a single line, the directions of the x, y and z axes in all segments must coincide.

2. Using the method of sections, write the equilibrium equations for the internal forces and moments on each segment.

Segment I $(0 \le x \le a,\ \ a = 3\ m)$.

$N_x^I = -P_2 = -30\ kN$; $\quad Q_z^I = -P_1 = -5\ kN$; $\quad Q_y^I = 0$;

$M_x^I = 0$;

$$M_y^I = P_1 x = 5 \cdot x\ \Big|_{x=0} = 0\ \Big|_{x=a=3\ m} = 15\ kN{\cdot}m;$$

$M_z^I = 0.$

Segment II $(0 \le x \le b,\ \ b = 2\ m)$.

$N_x^{II} = 0$;

$Q_z^{II} = -P_1 = -5\ kN$;

$$Q_y^{II} = q_2 x - P_2 = 20 \cdot x - 30\ \Big|_{x=0} = -30\ kN\ \Big|_{x=b=2\ m} = 10\ kN.$$

Since the shear force Q_y changes sign within segment **II**, it is necessary to determine the point x_e at which $Q_y^{II} = 0$:

$$q_2 x_e - P_2 = 0 \quad \Rightarrow \quad x_e = \frac{P_2}{q_2} = \frac{30}{20} = 1.5\ m;$$

$M_x^{II} = P_1 a = 5 \cdot 3 = 15\ kN{\cdot}m$;

$$M_y^{II} = P_1 x - M = 5 \cdot x - 20 = \ \Big|_{x=0} = -20\ kN{\cdot}m\ \Big|_{x=b=2\ m} = -10\ kN{\cdot}m;$$

$$M_z^{II} = \frac{q_2 x^2}{2} - P_2 x = \frac{20x^2}{2} - 30 \cdot x\ \Big|_{x=0} = 0\ \Big|_{x=b=2\ m} =$$

$$= -20\ kN{\cdot}m\ \Big|_{x=x_e=1.5\ m} = -22.5\ kN{\cdot}m.$$

Segment III $(0 \le x \le c,\ \ c = 2\ m)$.

Let us construct a separate calculation scheme by replacing the distributed load acting within the second segment with a resultant concentrated force (Fig. 2.23).

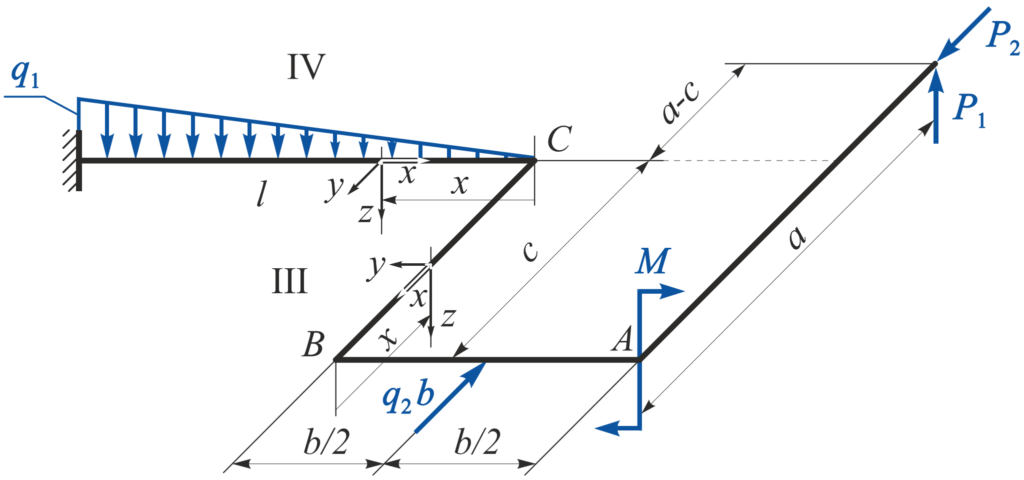


Fig. 2.23

$N_x^{III} = P_2 - q_2 b = 30 - 20 \cdot 2 = -10\ kN;$

$Q_z^{III} = -P_1 = -5\ kN;$

$Q_y^{III} = 0;$

$M_x^{III} = -M + P_1 b = -20 + 5 \cdot 2 = -10\ kN{\cdot}m;$

$M_y^{III} = P_1(x - a) = 5(x - 3)\ \Big|_{x=0} = -15\ kN{\cdot}m\ \Big|_{x=c=2\,m} = -5\ kN{\cdot}m;$

$M_z^{III} = q_2 b \dfrac{b}{2} - P_2 b = 20 \cdot 2 \cdot \dfrac{2}{2} - 30 \cdot 2 = -20\ kN{\cdot}m.$

Segment IV $(0 \le x \le l,\ \ l = 3\ m)$.

$N_x^{IV} = 0;$

$Q_z^{IV} = \dfrac{q_1 x^2}{2l} - P_1 = \dfrac{15x^2}{2 \cdot 3} - 5\ \Big|_{x=0} = -5\ kN\ \Big|_{x=l=3\,m} = 17.5\ kN.$

Since the shear force Q_y changes sign within segment IV, it is necessary to determine the point x_e at which $Q_z^{IV} = 0$:

$$\frac{q_1 x_e^2}{2l} - P_1 = 0 \quad \Longrightarrow \quad x_e = \sqrt{\frac{2P_1 l}{q_1}} = \sqrt{\frac{2 \cdot 5 \cdot 3}{15}} = 1.414\ m;$$

$Q_y^{IV} = q_2 b - P_2 = 20 \cdot 2 - 30 = 10\ kN;$

$M_x^{IV} = P_1(a - c) = 5 \cdot (3 - 2) = 5\ kN{\cdot}m;$

$$M_y^{IV} = P_1(b+x) - \frac{q_1 x^3}{6l} - M = 5 \cdot (2+x) - \frac{15 \cdot x^3}{6 \cdot 3} - 20 =$$

$$= \Big|_{x=0} = -10\ kN{\cdot}m \quad \Big|_{x=l=3\ m} = -17.5\ kN{\cdot}m \quad \Big|_{x=x_e=1.414\ m} = -9.998\ kN{\cdot}m;$$

$$M_z^{IV} = q_2 b\left(\frac{b}{2}+x\right) - P_2(b+x) = 20 \cdot 2 \cdot (1+x) - 30 \cdot (2+x) =$$

$$= \Big|_{x=0} = -20\ kN{\cdot}m \quad \Big|_{x=l=3\ m} = 10\ kN{\cdot}m.$$

3. Let us construct the diagrams (Fig. 2.24).

Remarks

1. When constructing internal forces and moments diagrams for a cranked bar, each of its elements must be considered as a rod in tension-compression, a shaft in torsion, and a beam in transverse bending in two planes. In this process, all sign conventions for N_x, Q_z, Q_y, M_x, M_y, M_z, and all rules for constructing diagrams are preserved:

a) the diagrams of N_x and M_x can be drawn in any plane;

b) the diagrams of Q_z, Q_y, M_y, and M_z must be drawn only ***in their respective planes of action***;

c) the diagrams of M_y and M_z are drawn ***on the side of the tensile fibres*** of the bar.

2. At right-angled corners (fixed joints) of a planar cranked bar, there occurs a mutual transition of M_x into M_y, as well as of N_x into Q_y, and vice versa.

3. For ***parallel segments*** of a planar cranked bar with out-of-plane loading, the following rules hold true, the following rules hold true, provided there are no concentrated moments acting perpendicular to the plane of the bar at the nodes of these segments:

a) for the ***coincident directions of the paths*** (see Fig. 2.22, segments II and IV), the value of M_y at the end of one segment (segment II) must be equal to the value of M_y at the beginning of the next segment (segment IV) (see Fig. 2.24);

b) for the ***opposite (counter) directions of the paths*** (see Fig. 22, segments I and III), the value of M_y at the end of one segment (segment I) must be equal in magnitude and opposite in sign to the value of M_y at the beginning of the following segment (segment III) (see Fig. 2.24).

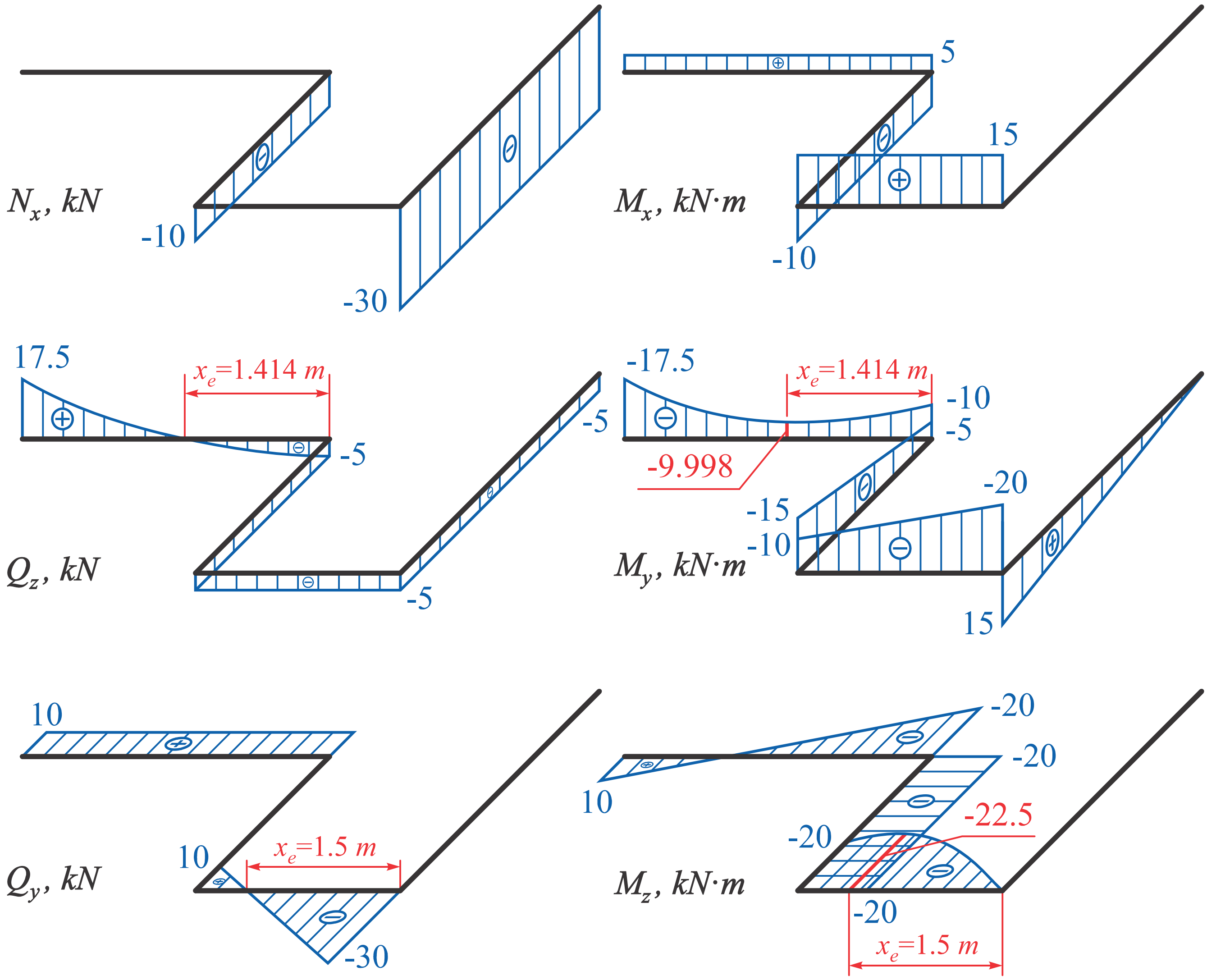


Fig. 2.24

4. Let us check the correctness of the diagram construction.

To this end, infinitesimal elements of the cranked bar are isolated at the junctions of its parts (nodes *A*, *B* and *C*), and their equilibrium is analysed under the action of internal and external loads applied within these nodes (Fig. 2.25).

In Fig. 2.25, all internal forces and moments are shown in their true directions.

Equilibrium equations for node *A*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 20 - 20 = 0;$

$\sum M_x = 0;$ $\sum M_y = 30 - 30 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node *B*:

$\sum P_x = 0;$ $\sum P_y = 0;$ $\sum P_z = 20 - 20 = 0;$

$\sum M_x = 0;$ $\sum M_y = 40 - 40 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node *C*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 15 - 10 - 5 = 0;$

$\sum M_x = 30 - 30 = 0;$ $\sum M_y = 30 + 10 - 40 = 0;$ $\sum M_z = 30 - 30 = 0.$

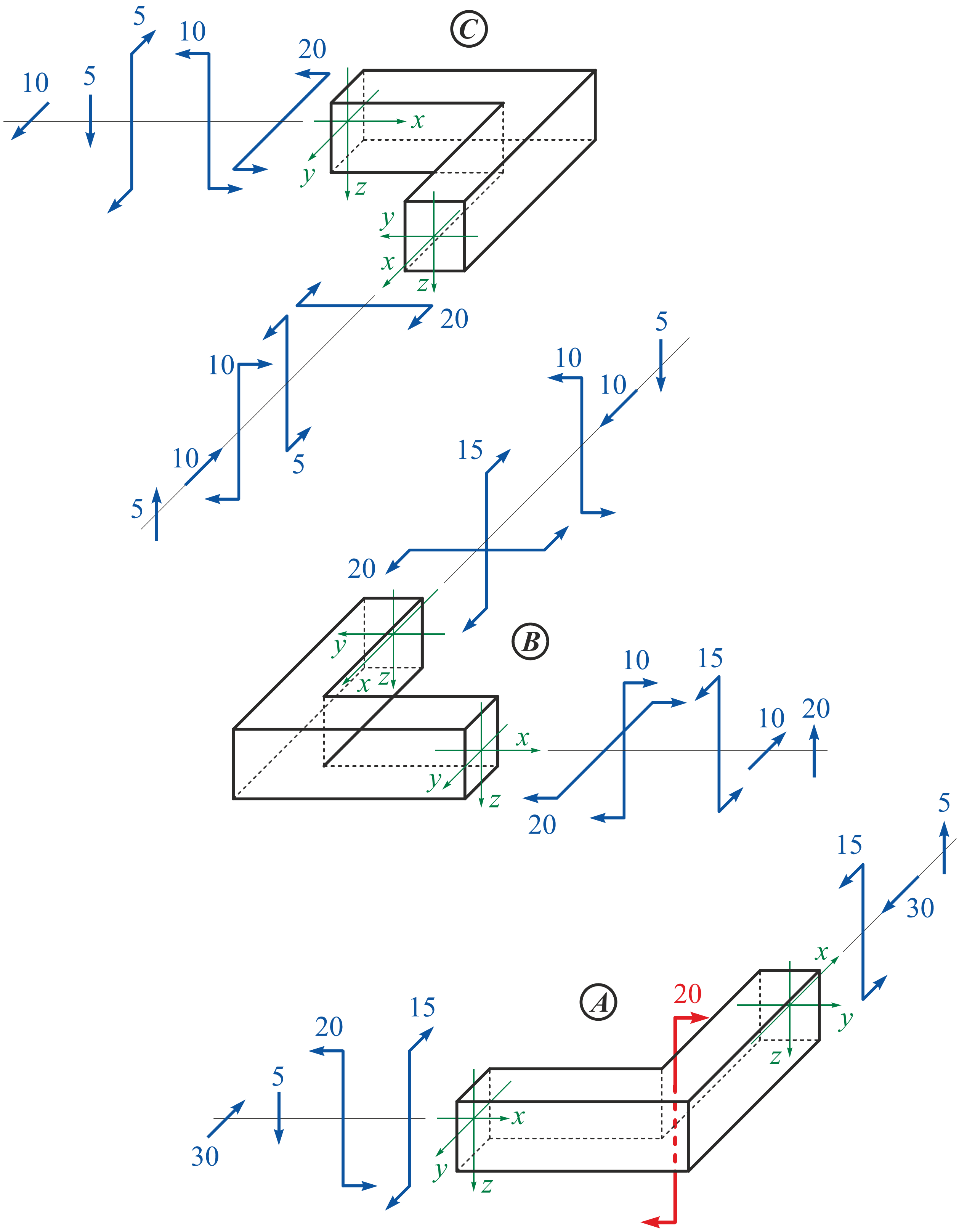


Fig. 2.25

Example 2.4

Construct the diagrams of internal forces and moments for the given cranked bar (Fig. 2.26).

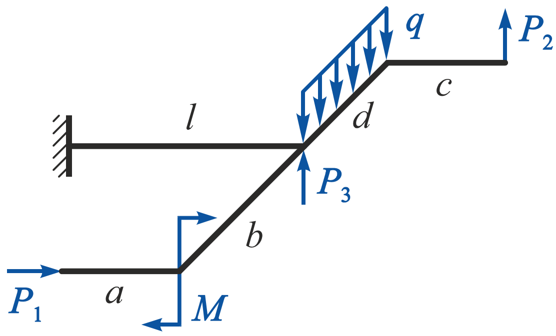


Fig. 2.26

Given: $P_1 = 10\ kN$; $P_2 = 20\ kN$; $P_3 = 5\ kN$;
$M = 30\ kN{\cdot}m$; $q = 5\ kN/m$;
$a = 2\ m$; $b = 3\ m$; $c = 2\ m$;
$d = 2\ m$; $l = 4\ m$.

It is necessary to construct the diagrams of

$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$

Solution

1. Let us draw a cranked bar to scale and divide it into segments. In an arbitrary cross-section of each segment, at a distance x from its beginning, let us introduce a local xyz coordinate system so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downward, and the horizontal y-axis, together with the first two axes, forms a right-handed orthogonal basis (Fig. 2.27).

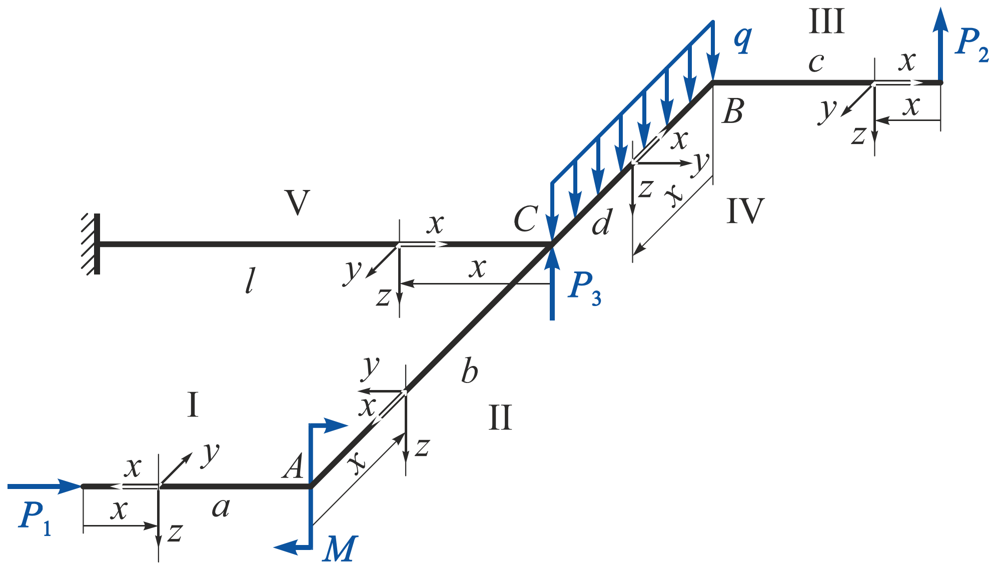


Fig. 2.27

2. Using the method of sections, write the equilibrium equations for the internal forces and moments on each segment.

Segment I $(0 \le x \le a,\ \ a = 2\ m)$.

$N_x^I = -P_1 = -10\ kN$; $Q_z^I = 0$; $Q_y^I = 0$;

$M_x^I = 0$; $M_y^I = 0$; $M_z^I = 0$.

Segment II $(0 \le x \le b,\ \ b = 3\ m)$.

$N_x^{II} = 0;$ $\quad Q_z^{II} = 0;$ $\quad Q_y^{II} = P_1 = 10\ kN;$

$M_x^{II} = -M = -30\ kN{\cdot}m;$ $\quad M_y^{II} = 0;$

$$M_z^{II} = P_1 x = 10 \cdot x \ \Big|_{x=0} = 0 \ \Big|_{x=b=3\ m} = 30\ kN{\cdot}m.$$

Segment III $(0 \le x \le c,\ \ c = 2\ m)$.

$N_x^{III} = 0;$ $\quad Q_z^{III} = -P_2 = -20\ kN;$ $\quad Q_y^{III} = 0;$

$$M_x^{III} = 0; \quad M_y^{III} = P_2 x = 20x \ \Big|_{x=0} = 0 \ \Big|_{x=c=3\ m} = 40\ kN{\cdot}m; \quad M_z^{III} = 0.$$

Segment IV $(0 \le x \le d,\ \ d = 2\ m)$.

$N_x^{IV} = 0;$

$$Q_z^{IV} = qx - P_2 = 5x - 20 \ \Big|_{x=0} = -20\ kN \ \Big|_{x=d=2\ m} = -10\ kN;$$

$Q_y^{IV} = 0;$ $\quad M_x^{IV} = -P_2 c = -20 \cdot 2 = -40\ kN{\cdot}m;$

$$M_y^{IV} = P_2 x - \frac{qx^2}{2} = 20x - \frac{5x^2}{2} \ \Big|_{x=0} = 0 \ \Big|_{x=d=2\ m} = 30\ kN{\cdot}m;$$

$M_z^{IV} = 0.$

Segment V $(0 \le x \le l,\ \ l = 4\ m)$.

Let us construct a separate calculation scheme by replacing the distributed load acting within the fourth segment with a resultant concentrated force (Fig. 2.28).

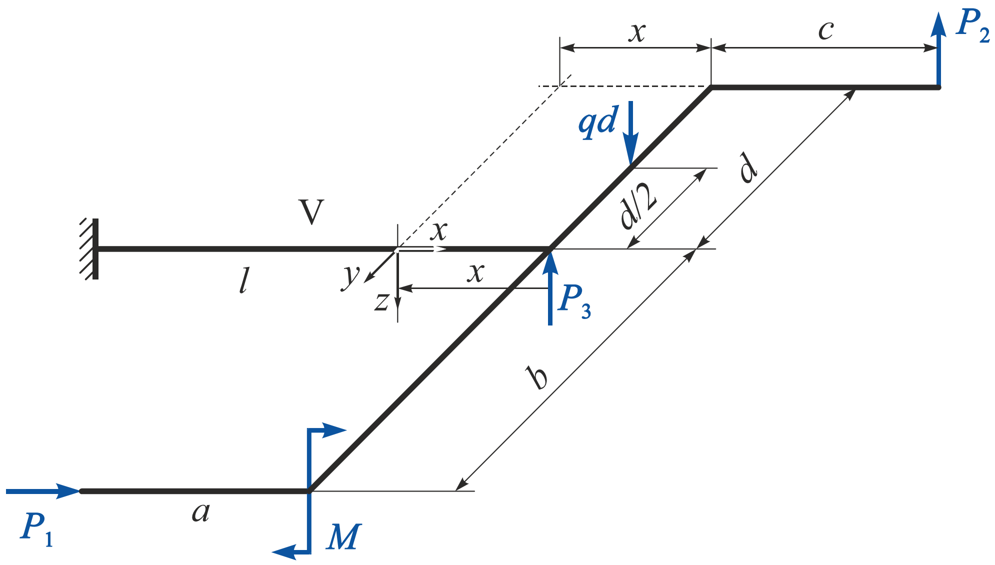


Fig. 2.28

$N_x^V = P_1 = 10\ kN;$ $Q_z^V = qd - P_2 - P_3 = 5 \cdot 2 - 20 - 5 = -15\ kN;$

$Q_y^V = 0;$ $M_x^V = -qd\dfrac{d}{2} + P_2 d = -5 \cdot 2 \cdot \dfrac{2}{2} + 20 \cdot 2 = 30\ kN{\cdot}m;$

$$M_y^V = P_3 x + P_1(c + x) - qdx - M = 5x + 20 \cdot (2 + x) - 5 \cdot 2 \cdot x - 30 =$$

$$= \Big|_{x=0} = 10\ kN{\cdot}m \quad \Big|_{x=l=4\ m} = 70\ kN{\cdot}m;$$

$M_z^V = P_1 b = 10 \cdot 3 = 30\ kN{\cdot}m.$

3. Let us construct the diagrams (Fig. 2.29).

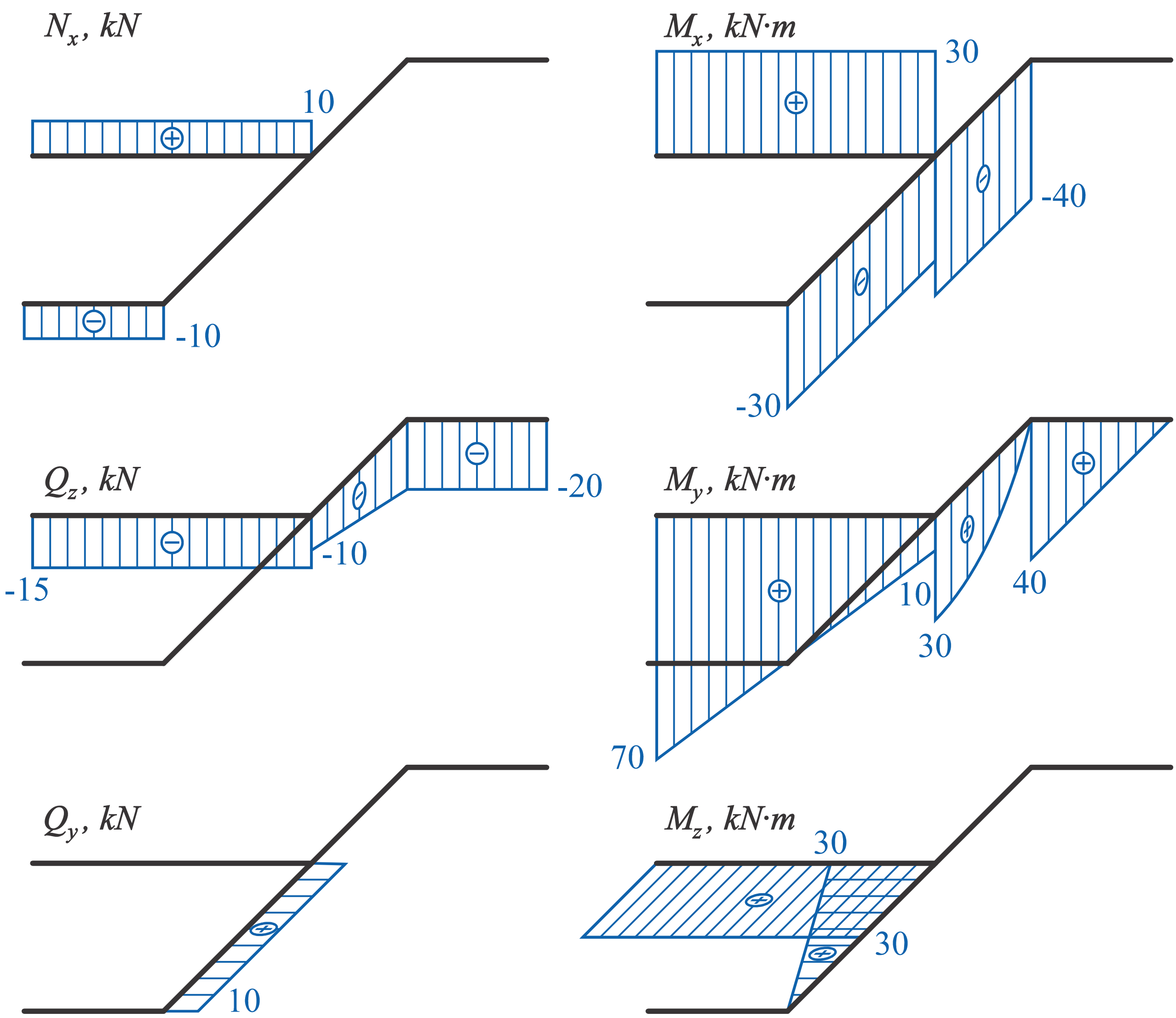


Fig. 2.29

4. Let us check the correctness of the diagram construction.

To this end, infinitesimal elements of the cranked bar are isolated at the junctions of its parts (nodes *A*, *B* and *C*), and their equilibrium is analysed under the action of internal and external loads applied within these nodes (Fig. 2.30).

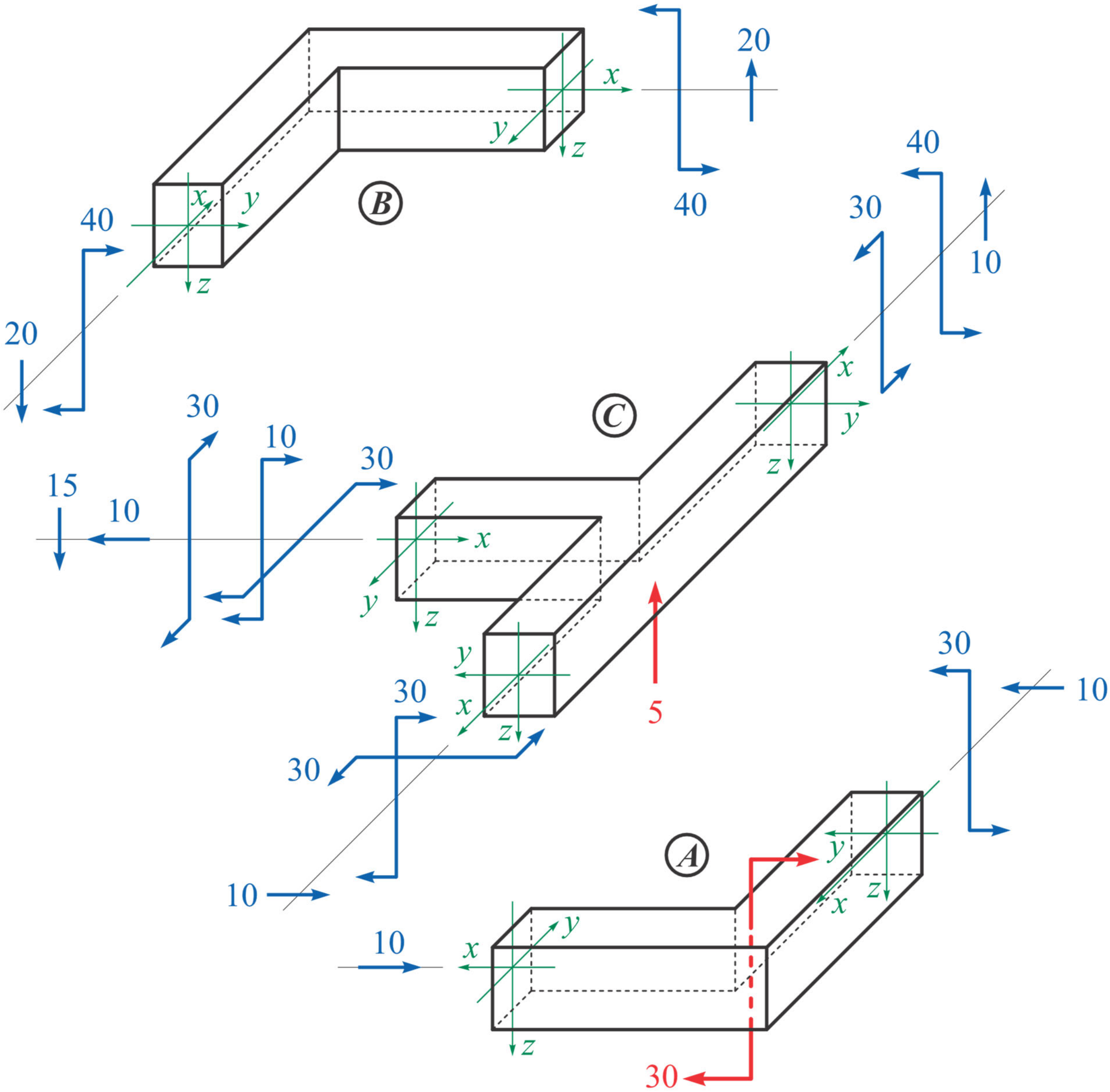


Fig. 2.30

Equilibrium equations for node *A*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 20 - 20 = 0;$

$\sum M_x = 0;$ $\sum M_y = 30 - 30 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node *B*:

$\sum P_x = 0;$ $\sum P_y = 0;$ $\sum P_z = 20 - 20 = 0;$

$\sum M_x = 0;$ $\sum M_y = 40 - 40 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node *C*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 15 - 10 - 5 = 0;$

$\sum M_x = 30 - 30 = 0;$ $\sum M_y = 30 + 10 - 40 = 0;$ $\sum M_z = 30 - 30 = 0.$

2.3. Construction of Diagrams of Internal Forces and Moments for a Spatial Cranked Bar

A cranked bar is called ***spatial*** if all its elements are rigidly connected at the nodes (fixed joints) and their longitudinal axes do not lie in a single plane.

Example 2.5

Construct the diagrams of internal forces and moments for the given spatial cranked bar (Fig. 2.31).

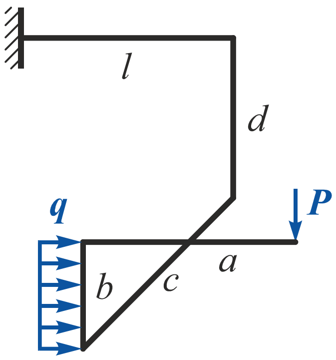


Fig. 2.31

Given: $P = 10\ kN;\quad q = 15\ kN/m;$
$a = 2\ m;\quad b = 1\ m;\quad c = 2\ m;$
$d = 1.5\ m;\quad l = 2\ m.$

It is necessary to construct the diagrams of

$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$

Solution

1. Let us draw a cranked bar to scale and divide it into segments.

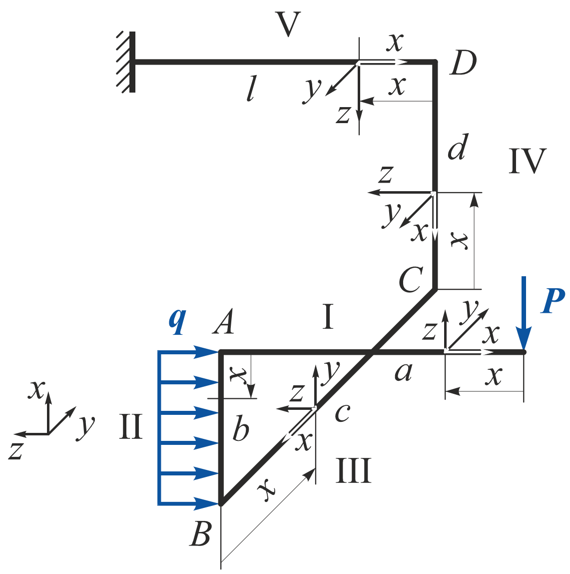


Fig. 2.32

In an arbitrary cross-section of segment V, at a distance x from its beginning, let us introduce a local xyz coordinate system so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downward, and the horizontal y-axis, together with the first two axes, forms a right-handed orthogonal basis (Fig. 2.32). On segment IV, the coordinate system is obtained by a 90° rotation about the y-axis; on segment III, by a 90° rotation about the z-axis; on segment II, also by a 90° rotation about the z-axis; and on segment I, by a 90° rotation about the y-axis.

2. Using the method of sections, write the equilibrium equations for the internal forces and moments on each segment.

Segment I (see Fig. 2.15) $(0 \leq x \leq a, \;\; a = 2\ m)$.

$N_x^I = 0;$

$Q_z^I = -P = -10\ kN;$

$Q_y^I = 0;$

$M_x^I = 0;$

$M_y^I = Px = 10x \Big|_{x=0} = 0 \Big|_{x=a=2\ m} = 20\ kN{\cdot}m;$

$M_z^I = 0.$

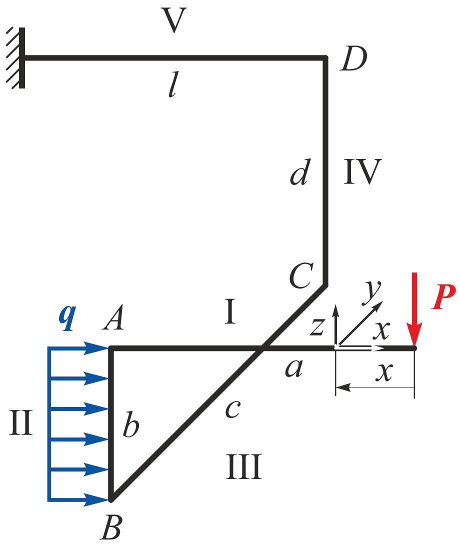


Fig. 2.33

Segment II $(0 \leq x \leq b, \;\; b = 1\ m)$.

Resolve the load P to the initial cross-section of segment II (point A). (Fig. 2.34).

$N_x^{II} = -P = -10\ kN;$

$Q_z^{II} = -qx = -15x \Big|_{x=0} =$

$= 0 \Big|_{x=b=1\ m} = -15\ kN;$

$Q_y^{II} = 0;$

$M_x^{II} = 0;$

$M_y^{II} = Pa + \dfrac{qx^2}{2} = 10 \cdot 2 + \dfrac{15x^2}{2}\Big|_{x=0} =$

$= 20\ kN{\cdot}m \Big|_{x=b=1\ m} = 27.5\ kN{\cdot}m;$

$M_z^{II} = 0.$

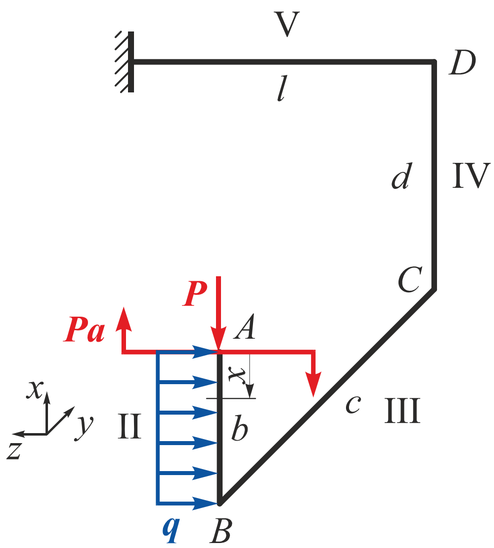


Fig. 2.34

Segment III $(0 \leq x \leq c, \;\; c = 2\ m)$.

Replace the distributed load acting within segment II with an equivalent concentrated force, and resolve the system of external forces on segment II to the initial cross-section of segment III (point B).

The calculation scheme of segment III is shown in Fig. 2.35.

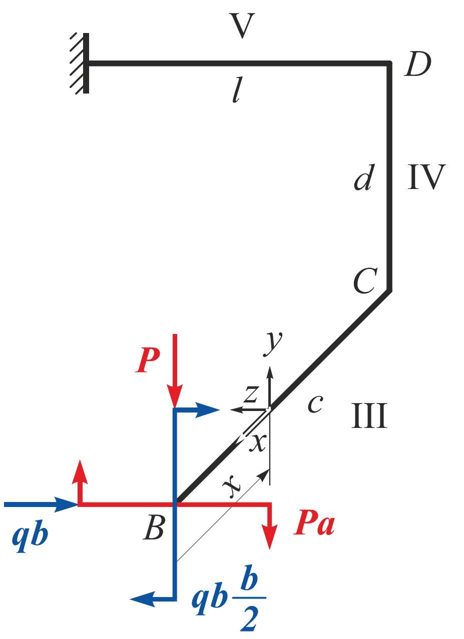


Fig. 2.35

$$N_x^{III} = 0;$$

$$Q_z^{III} = -qb = -15\ kN;$$

$$Q_y^{III} = P = 10\ kN;$$

$$M_x^{III} = -Pa - qb\frac{b}{2} = -10 \cdot 2 - 15 \cdot 1 \cdot \frac{1}{2} = = -27.5\ kN{\cdot}m;$$

$$M_y^{III} = qbx = 15 \cdot 1 \cdot x \Big|_{x=0} = = 0 \Big|_{x=c=2\ m} = 30\ kN{\cdot}m;$$

$$M_z^{III} = Px = 10x \Big|_{x=0} = 0 \Big|_{x=c=2\ m} = 20\ kN{\cdot}m.$$

Segment IV $(0 \le x \le d,\ \ d = 1{,}5\ m)$.

Resolve the system of loads acting on segment III to the initial cross-section of segment IV (point C) (Fig. 2.36).

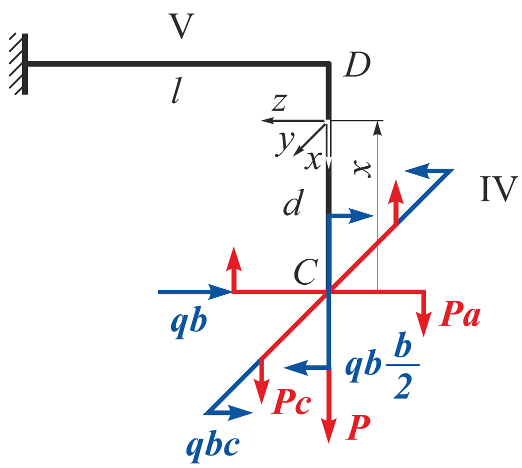


Fig. 2.36

$$N_x^{IV} = P = 10\ kN;$$

$$Q_z^{IV} = -qb = -15\ kN;$$

$$Q_y^{IV} = 0;$$

$$M_x^{IV} = -qbc = -15 \cdot 1 \cdot 2 = -30\ kN{\cdot}m;$$

$$M_y^{IV} = -Pa - qb\frac{b}{2} + qbx = -10 \cdot 2 - 15 \cdot 1 \cdot \frac{1}{2} + 15 \cdot 1 \cdot x \Big|_{x=0} = = -27.5\ kN{\cdot}m \Big|_{x=d=1.5\ m} = -5\ kN{\cdot}m;$$

$$M_z^{IV} = Pc = 20\ kN{\cdot}m.$$

Segment V $(0 \le x \le l,\ \ l = 2\ m)$.

Resolve the system of loads acting on segment IV to the initial cross-section of segment V (point *D*) (Fig. 2.37).

$N_x^V = qb = 15\ kN;$

$Q_z^V = P = 10\ kN;$

$Q_y^V = 0;$

$M_x^V = Pc = 10 \cdot 2 = 20\ kN{\cdot}m;$

$M_y^V = qbd - qb\dfrac{b}{2} - Pa - Px =$

$= 15 \cdot 1 \cdot 1.5 - 15 \cdot 1 \cdot \dfrac{1}{2} - 10 \cdot 2 - 10x =$

$= \Big|_{x=0} = -5\ kN{\cdot}m \quad \Big|_{x=l=2\ m} = -25\ kN{\cdot}m;$

$M_z^V = qbc = 15 \cdot 1 \cdot 2 = 30\ kN{\cdot}m.$

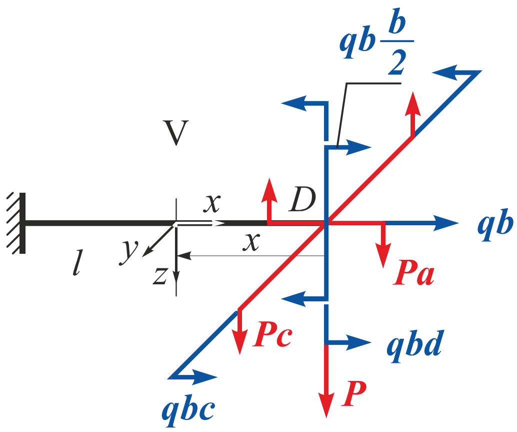


Fig. 2.37

3. Let us construct the diagrams (Fig. 2.38).

Remarks

1. When constructing diagrams of internal forces and moments for a cranked bar, the following should be taken into account:
   a) the diagrams of N_x and M_x can be drawn in any plane;
   b) the diagrams of Q_z, Q_y, M_y, and M_z must be drawn only ***in their respective planes of action***;
   c) the diagrams of M_y and M_z are drawn ***on the side of the tensile fibres*** of the bar.
2. ***If***, for the members of a cranked bar lying in parallel planes, the same orientation of the *xyz* axes is used, ***then*** on all these segments one obtains a formally ordered system of signs for the internal ***shear force*** Q_z and bending moment M_y. For example, on segment I – the same as on segment V, on segment II – the same as on segment IV (see Fig. 2.32), and then

   $Q_z^I = P = 10\ kN;$

   $M_y^I = -Px = \Big|_{x=0} = 0 \quad \Big|_{x=a=2\ m} = -20\ kN{\cdot}m;$

   $Q_z^{II} = qx = \Big|_{x=0} = 0 \quad \Big|_{x=b=1\ m} = 15\ kN;$

   $M_y^{II} = -Pa - \dfrac{qx^2}{2} = \Big|_{x=0} = -20\ kN{\cdot}m \quad \Big|_{x=b=1\ m} = -27.5\ kN{\cdot}m.$

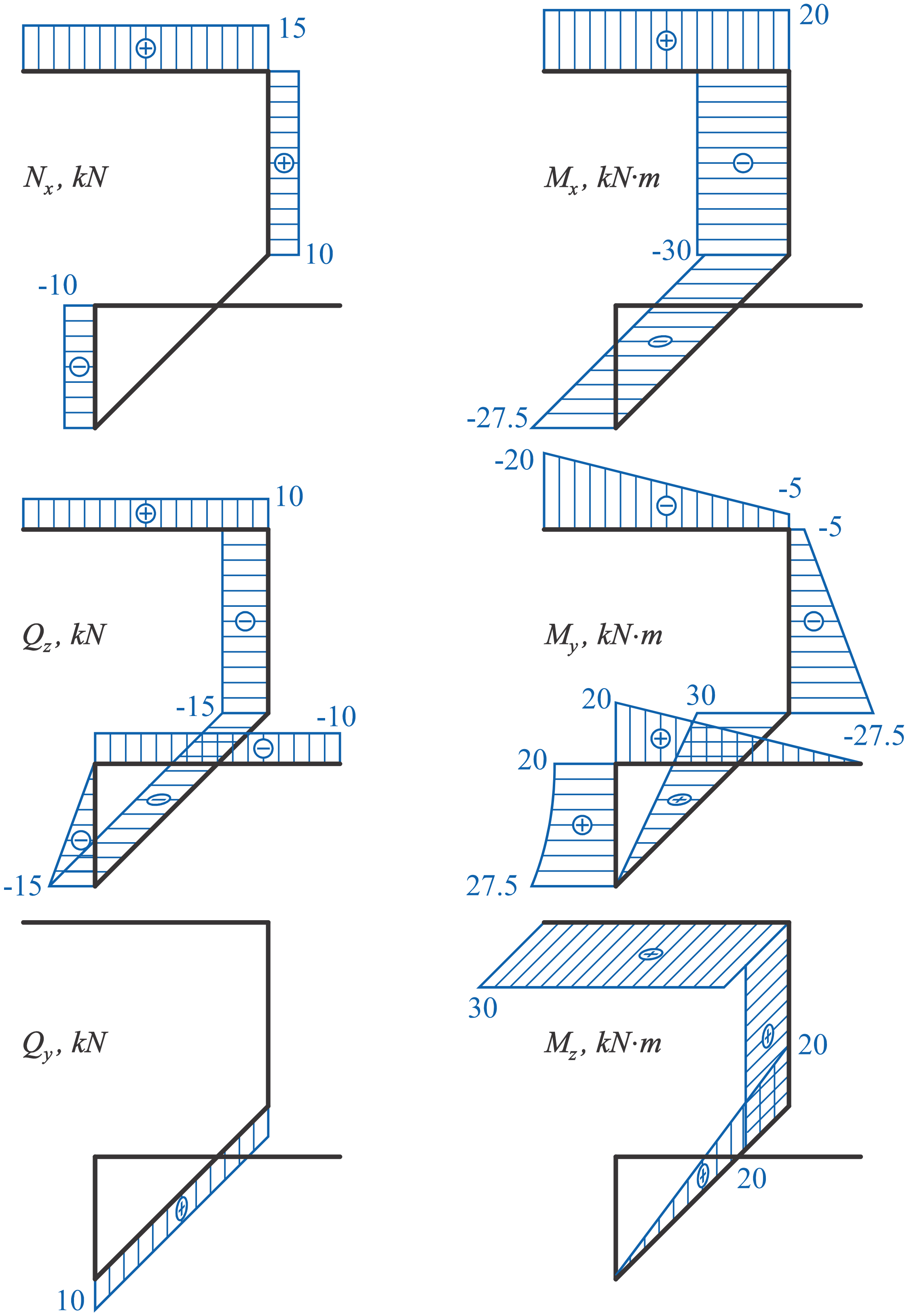


Fig. 2.38

4. Let us check the correctness of the diagram construction.

To this end, we will consider the equilibrium of nodes *A, B, C*, and *D* under the action of internal forces and moments and external forces applied within these nodes (Fig. 2.39).

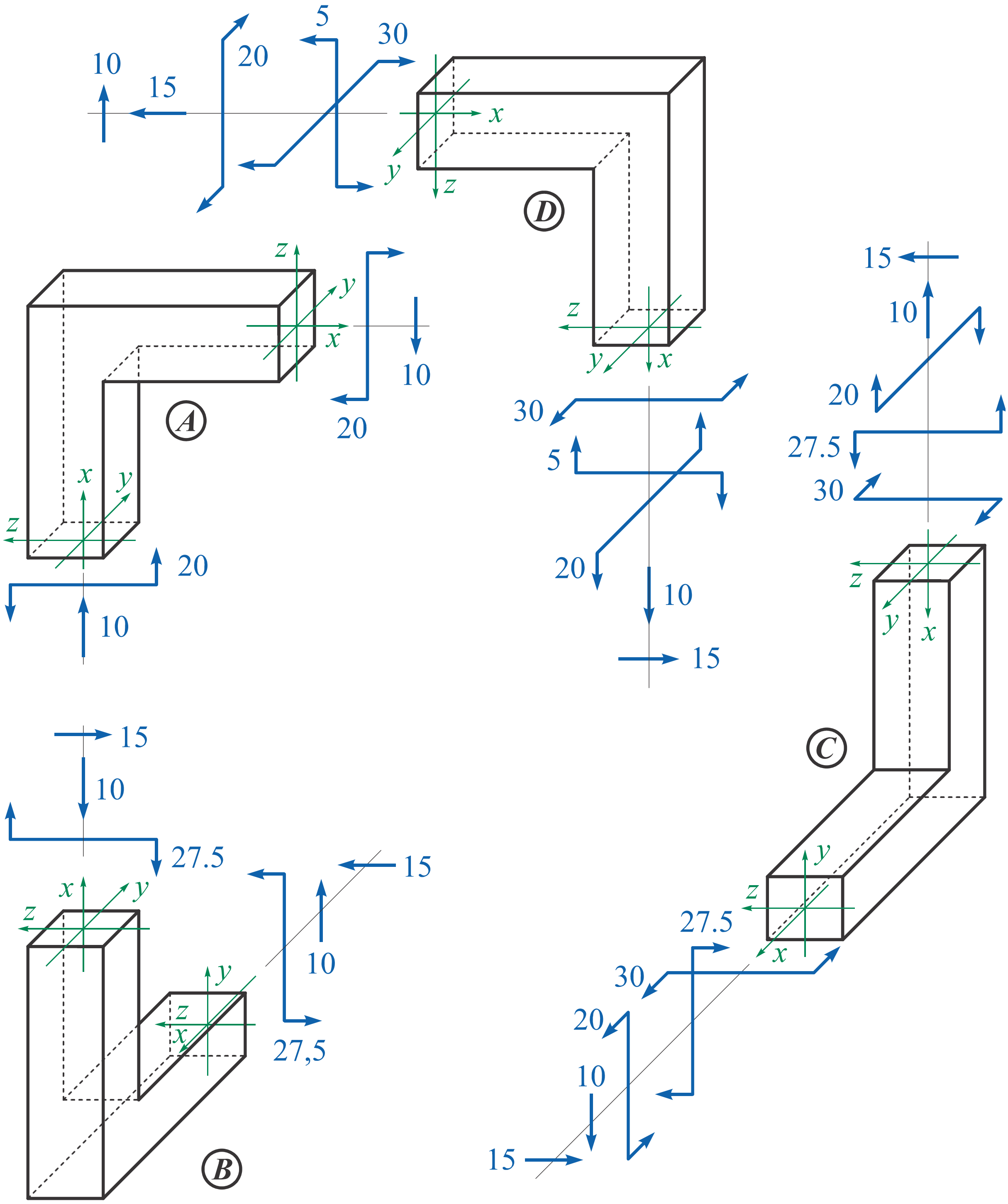


Fig. 2.39

Equilibrium equations for node A:

$\sum P_x = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 0;$ $\sum M_y = 20 - 20 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node B:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 0;$ $\sum M_y = 27.5 - 27.5 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node C:

$\sum P_x = 15 - 15 = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 20 - 20 = 0;$ $\sum M_y = 27.5 - 27.5 = 0;$ $\sum M_z = 30 - 30 = 0.$

Equilibrium equations for node D:

$\sum P_x = 15 - 15 = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 20 - 20 = 0;$ $\sum M_y = 5 - 5 = 0;$ $\sum M_z = 30 - 30 = 0.$

Example 2.6

Construct the diagrams of internal forces and moments for the given spatial cranked bar (Fig. 2.40).

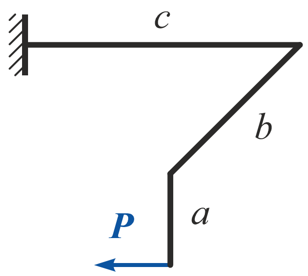


Fig. 2.40

Given: $P = 10\ kN;\quad a = 1\ m;\quad b = 2\ m;\quad l = 3\ m.$

It is necessary to construct the diagrams of

$$N_x,\ Q_z,\ Q_y,\ M_x,\ M_y,\ M_z.$$

Solution

1. Let us draw a cranked bar to scale and divide it into segments.

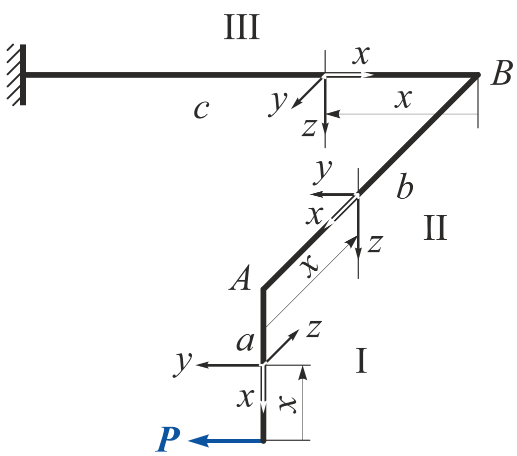


Fig. 2.41

In an arbitrary cross-section of segment III, at a distance x from its beginning, let us introduce a local xyz coordinate system so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downward, and the horizontal y-axis, together with the first two axes, forms a right-handed orthogonal basis (Fig. 2.41). On segment II, the coordinate system is obtained by a 90° rotation about the z-axis of the coordinate system of segment III; on segment I, by a 90° rotation about the y-axis of the coordinate system of segment II.

2. Using the method of sections, write the equilibrium equations for the internal forces and moments on each segment.

Segment I $(0 \le x \le a,\ \ a = 1\ m)$.

$N_x^I = 0;$ $Q_z^I = 0;$ $Q_y^I = P = 10\ kN;$

$M_x^I = 0;$ $M_y^I = 0;$

$$M_z^I = -Px = -10 \cdot x \ \Big|_{x=0} = 0 \ \Big|_{x=a=1\ m} = -10\ kN{\cdot}m.$$

Segment II $(0 \le x \le b,\ \ b = 2\ m)$.

$N_x^{II} = 0;$ $\qquad Q_z^{II} = 0;$ $\qquad Q_y^{II} = P = 10\ kN;$

$M_x^{II} = -Pa = -10 \cdot 1 = -10\ kN{\cdot}m;$ $\qquad M_y^{II} = 0;$

$$M_z^{II} = -Px = -10 \cdot x \Big|_{x=0} = 0 \Big|_{x=b=2\ m} = -20\ kN{\cdot}m.$$

Segment III $(0 \le x \le c,\ \ c = 3\ m)$.

$N_x^{III} = -P = -10\ kN;$ $\qquad Q_z^{III} = 0;$ $\qquad Q_y^{III} = 0;$

$M_x^{III} = 0;$ $\qquad M_y^{III} = -Pa = -10\ kN{\cdot}m;$ $\qquad M_z^{III} = -Pb = -20\ kN{\cdot}m.$

3. Let us construct the diagrams (Fig. 2.42).

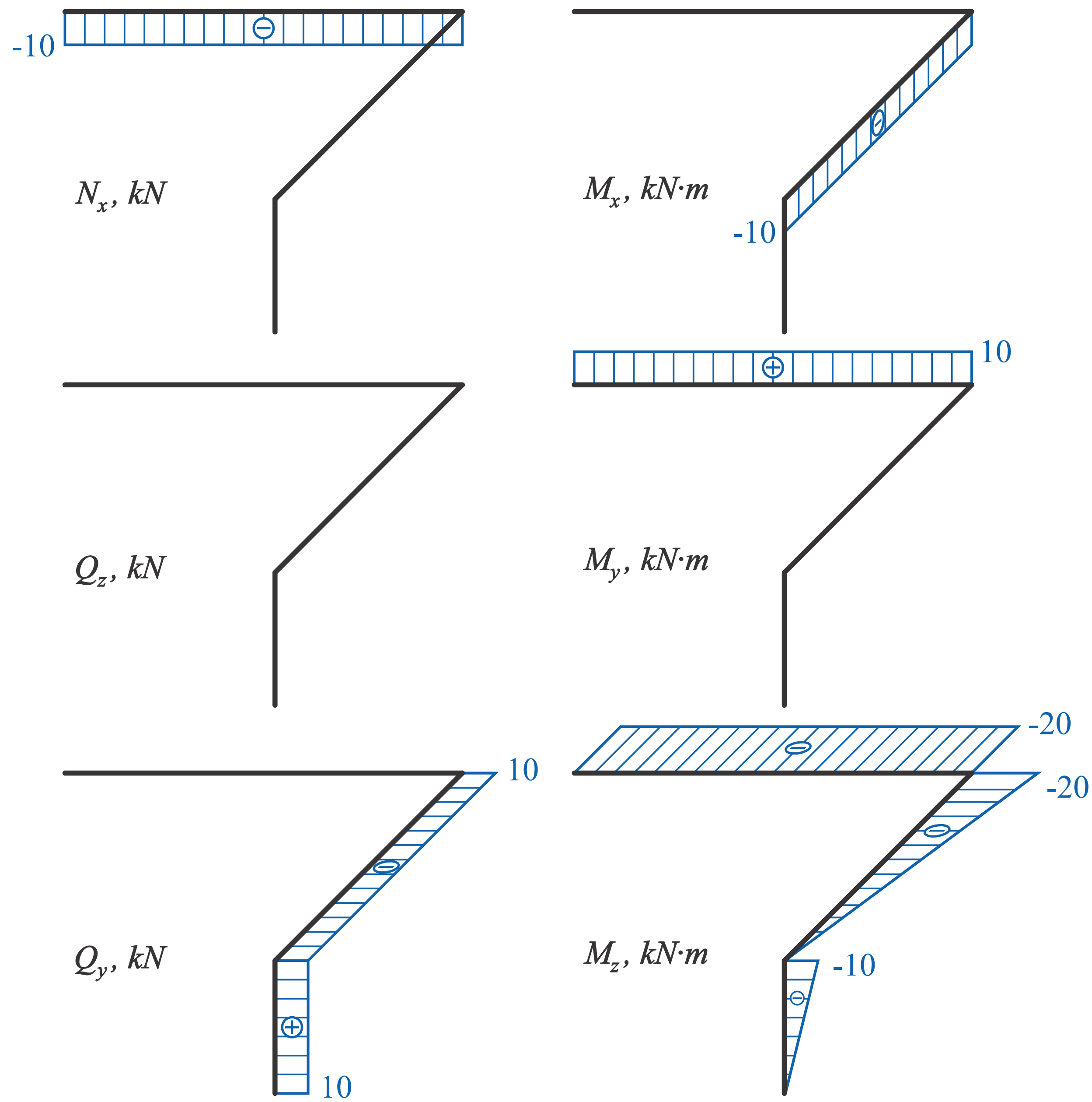


Fig. 2.42

4. Let us check the correctness of the diagram construction.

To this end, infinitesimal elements of the cranked bar are isolated at the junctions of its parts (nodes *A* and *B*), and their equilibrium is analysed under the action of internal forces and moments and external forces applied within these nodes (Fig. 2.43).

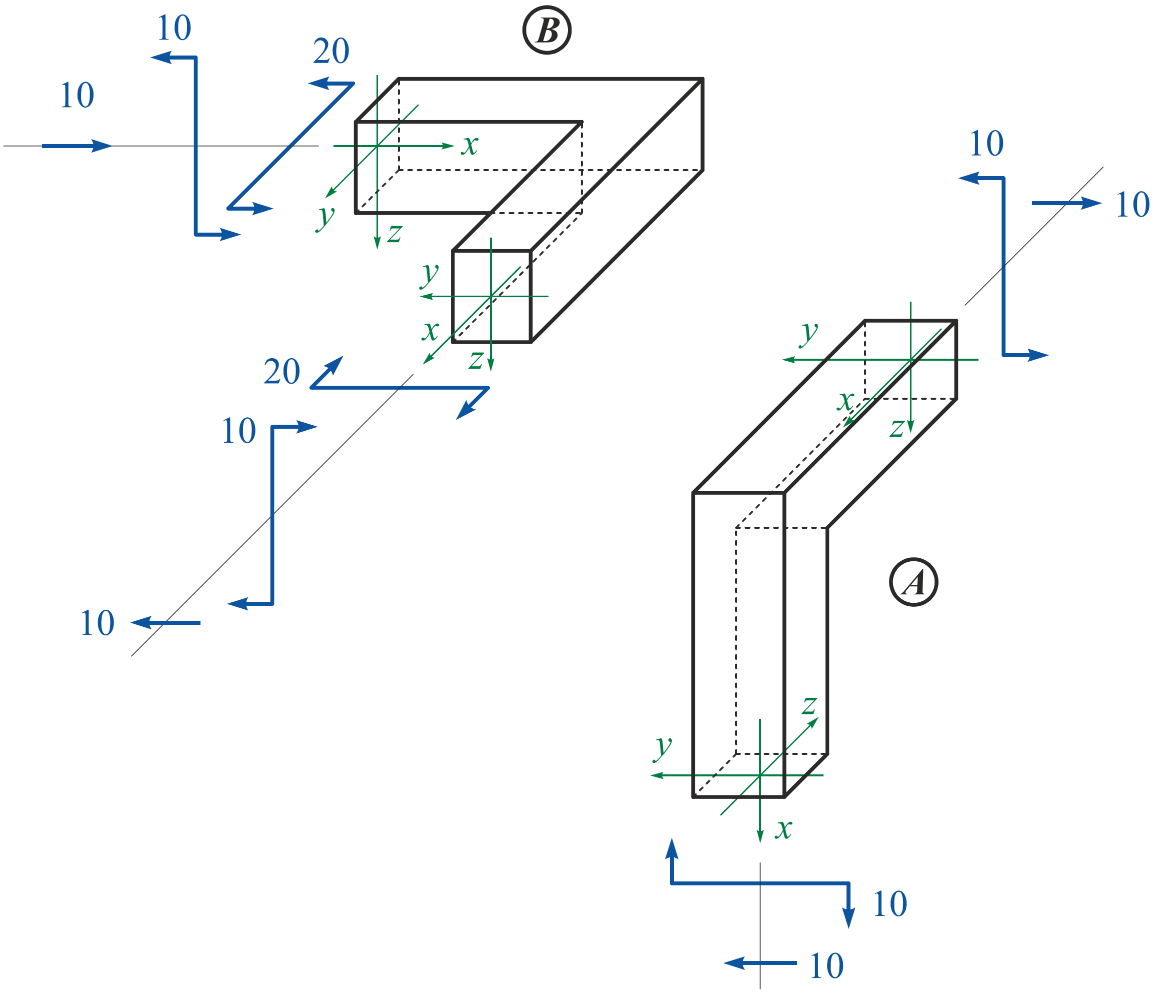


Fig. 2.43

Equilibrium equations for node *A*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 0;$

$\sum M_x = 0;$ $\sum M_y = 10 - 10 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node *B*:

$\sum P_x = 10 - 10 = 0;$ $\sum P_y = 0;$ $\sum P_z = 0;$

$\sum M_x = 0;$ $\sum M_y = 10 - 10 = 0;$ $\sum M_z = 20 - 20 = 0.$

3. COMBINED LOADING

3.1. General Provisions

By ***combined loading*** we shall understand such a type of deformation in the cross section of a bar when two or more internal forces and moments act simultaneously in this section.

In the general case, under arbitrary loading, six internal forces and moments can act in the cross section of a bar (Fig. 3.1):

N_x is an axial force;
Q_y and Q_z are shear forces;
M_x is a torsional moment;
M_y and M_z are bending moments.

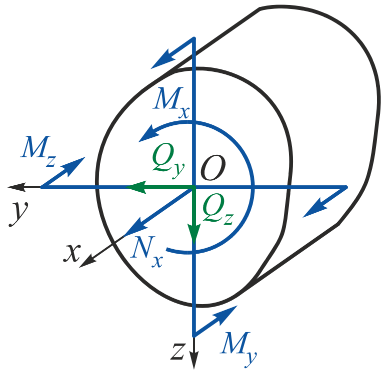


Fig. 3.1

The realization of ***tension-compression*** (presence of N_x) or ***bending*** (presence of M_y or M_z) creates ***normal stresses*** σ_x at points within the bar's cross-section, whereas the realization of ***torsion*** (presence of M_x) or ***shear*** (presence of Q_y or Q_z) creates ***shear stresses*** τ.

In ***strength analyses*** for sufficiently long bars, the ***shear forces*** Q_z and Q_y are usually neglected, since the shear stresses they induce are significantly smaller compared with the shear stresses caused by a ***torsional moment*** M_x and the normal stresses caused by bending moments M_y and M_z.

The stresses acting in the cross sections of a bar under ***combined loading*** will be determined using the ***principle of superposition***, which is valid if the structures are:

a) ***physically linear***, i.e., obeying Hooke's law ($\sigma < \sigma_p$, here σ_p is a proportional limit)

б) ***geometrically linear***, when under the action of loads all displacements remain much smaller than the characteristic dimensions of the structure (the hypothesis of relative rigidity holds). This allows one, when compiling the static equilibrium equations to determine support reactions and internal forces and moments, not to take into account changes in linear and angular dimensions of the structure.

The methodology of strength analysis depends on the cross-section shape of the bar (rectangular, circular).

3.2. The Rectangular Cross-Section

3.2.1. Bending with Torsion and Tension-Compression of a Rectangular Cross-Section Bar

As practical calculations show, in a rectangular cross-section:

$$\tau_{max}(Q) \ll \tau_{max}(M_{tortional}) \quad \text{and} \quad \tau_{max}(Q) \ll \sigma_{max}(M_{bending}),$$

therefore, shear stresses caused by shear forces are neglected in this case.

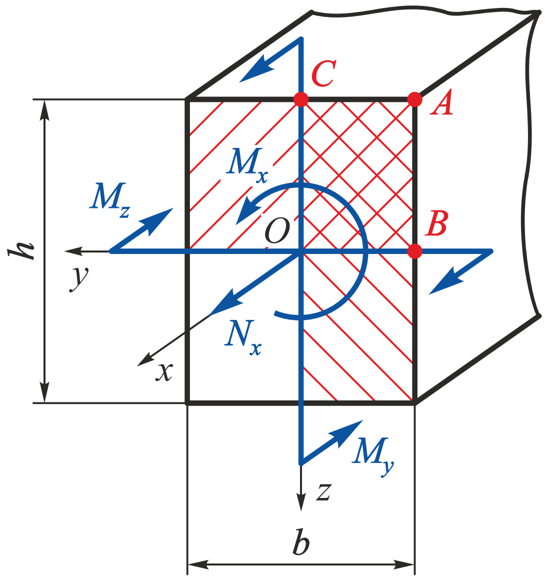


Fig. 3.2

Let us consider a rectangular cross section (Fig. 3.2) with dimensions $h \times b$ (for definiteness, assume $h > b$), subjected to:

а) axial force N_x;

б) torsional moment M_x;

в) bending moment M_y acting in the vertical plane;

г) bending moment M_z acting in the horizontal plane.

We will construct the stress diagrams due to each internal force or moment separately, and by applying the principle of superposition, we will analyze their combined effect (Fig. 3.3 – 3.6):

1. Only the ***axial force*** $\boldsymbol{N_x}$ (Fig. 3.3) is applied:

$$\sigma_x(N_x) = \frac{N_x}{F}, \tag{3.1}$$

where $F = bh$ is the cross-sectional area.

2. Only the ***torsional moment*** $\boldsymbol{M_x}$ (Fig. 3.4) is applied:

$$\tau_{max} = \tau_B = \tau_{B'} = \frac{M_x}{W_t}; \tag{3.2}$$

$$\tau'_{max} = \tau_C = \tau_{C'} = \gamma\tau_{max}; \tag{3.3}$$

$$\tau_A = \tau_{A'} = \tau_D = \tau_{D'} = 0, \tag{3.4}$$

where $W_t = \alpha h b^2$ is the torsional section modulus for a rectangular cross-section;

α, γ are coefficients depending on the ratio h/b (see Appendix).

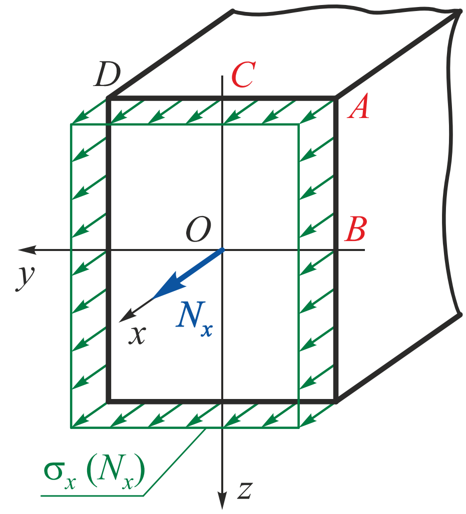


Fig. 3.3

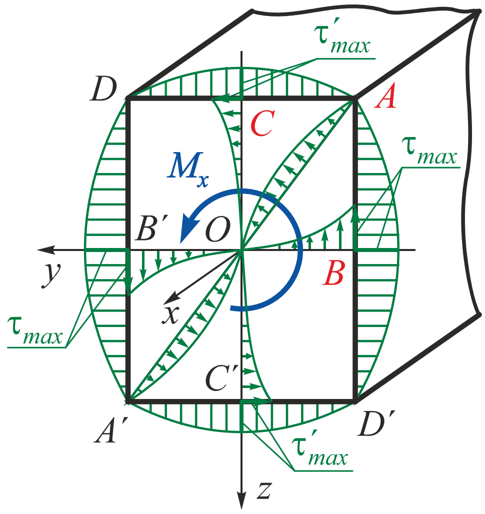


Fig. 3.4

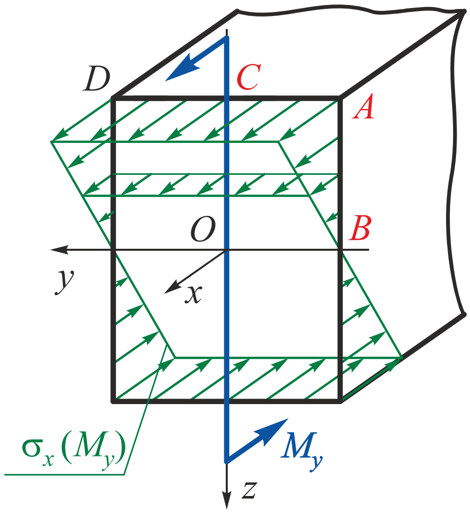


Fig. 3.5

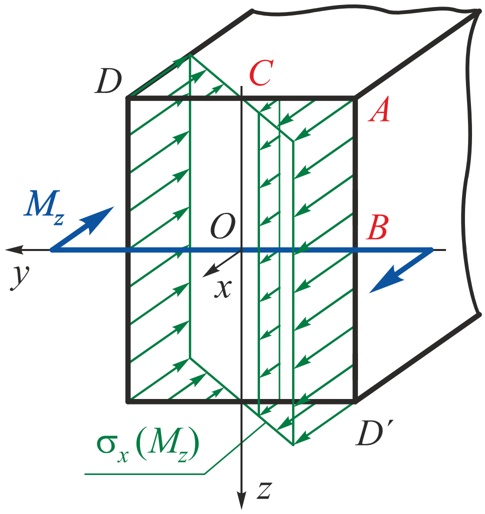


Fig. 3.6

3. Only the ***bending moment*** $\boldsymbol{M_y}$ (see Fig. 3.5) is applied:

$$\sigma_{x\,max}(M_y) = \sigma_{x\,A}(M_y) = \sigma_{x\,C}(M_y) = \sigma_{x\,D}(M_y) = \frac{M_y}{W_y};$$
$$\sigma_{x\,B}(M_y) = 0, \tag{3.5}$$

where $W_y = \dfrac{bh^2}{6}$ is the section modulus with respect to the y-axis under bending.

For points of the section that are symmetric about the centroid to points A, C, and D, the stresses are equal in magnitude and opposite in sign.

4. Only the ***bending moment M_z*** (see Fig. 3.6) is applied:

$$\sigma_{x\,max}(M_z) = \sigma_{x\,A}(M_z) = \sigma_{x\,B}(M_z) = \sigma_{x\,D'}(M_z) = \frac{M_z}{W_z};$$
$$\sigma_{x\,C}(M_z) = 0, \tag{3.6}$$

where $W_z = \dfrac{hb^2}{6}$ is the section modulus with respect to the z-axis under bending.

For points of the section that are symmetric about the centroid to points A, B, and D', the stresses are equal in magnitude and opposite in sign.

From the analysis of the diagrams in Fig. 3.3–3.6 it follows that it is not possible to unambiguously identify an ***only critical point***. Therefore, all ***potentially critical points*** of the section must be considered.

Remark	The situation at points D and D' is typically not considered, since at these points the normal stresses due to bending moments M_y and M_z have opposite signs, which means they cancel each other out.

Based on the diagrams, the following conclusions can be formulated:

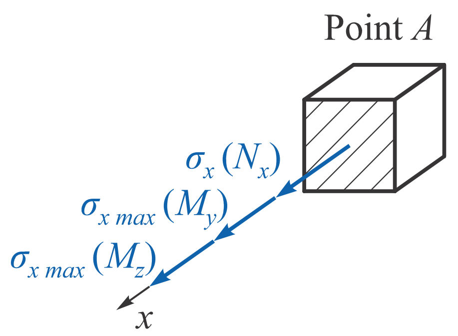


Fig. 3.7

1. At point A of the section, a ***uniaxial (simple) stress state*** is realized (Fig. 3.7), and at this point, the maximum normal stress occurs (the shaded area coincides with the cross-sectional plane; stresses on hidden faces are not shown).

The strength condition at this point is as follows:

$$\sigma_{x\,A} = \sigma_{x_{max}} = \frac{|N_x|}{F} + \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \le [\sigma]. \tag{3.7}$$

Remark	A uniaxial stress state is also realized at all other corner points.

2. Other potentially critical points are *B* (midpoint of the longer side) and *C* (midpoint of the shorter side). The stress state at these points is shown in Fig. 3.8 (the shaded areas coincide with the cross-sectional plane; stresses on hidden faces are not shown).

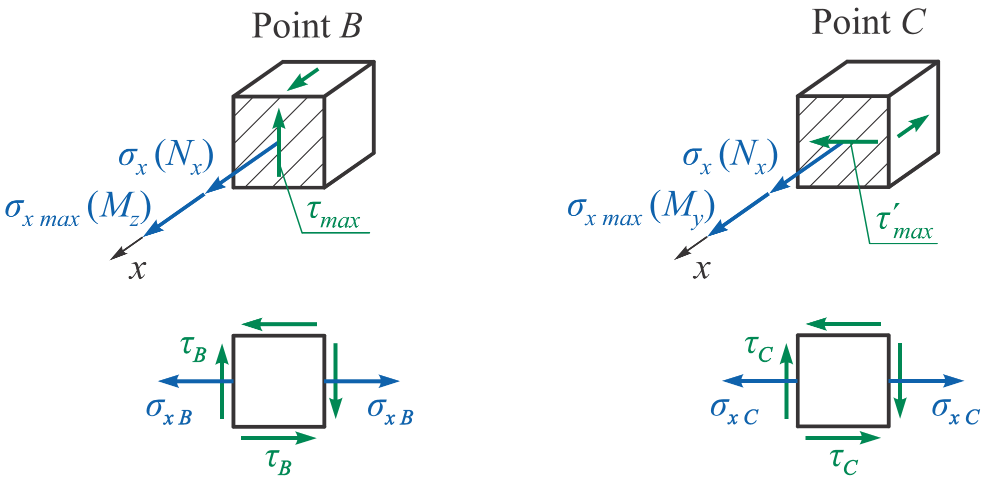


Fig. 3.8

At these points, a ***plane*** (combined) ***stress state*** of a particular type is realized, which was considered in Clause 1.2.6. Therefore, the strength analysis at these potentially critical points should be performed according to one of the strength theories.

The **strength conditions** at points B and C, according to formulas (1.27) and (1.28), have the following form:

a) according to the ***third strength theory***

$$\sigma_{eq\,B}^{\mathrm{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_z|}{W_z}\right)^2 + 4\left(\frac{M_x}{W_t}\right)^2} \le [\sigma]; \tag{3.8}$$

$$\sigma_{eq\,C}^{\mathrm{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_y|}{W_y}\right)^2 + 4\left(\gamma\frac{M_x}{W_t}\right)^2} \le [\sigma]; \tag{3.9}$$

b) according to the ***fourth strength theory***

$$\sigma_{eq\,B}^{\mathrm{IV}} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_z|}{W_z}\right)^2 + 3\left(\frac{M_x}{W_t}\right)^2} \le [\sigma]; \tag{3.10}$$

$$\sigma_{eq\,C}^{\mathrm{IV}} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_y|}{W_y}\right)^2 + 3\left(\gamma\frac{M_x}{W_t}\right)^2} \le [\sigma]. \tag{3.11}$$

3.2.2. Procedure for Determining the Dimensions of a Rectangular Cross-Section

Given: N_x, M_x, M_y, M_z, $h/b = k$, $[\sigma]$ (see Fig. 3.2).
Determine: h, b.

1. A quarter of the section (quadrant), which is under ***triaxial tension*** (for $N_x > 0$) or ***triaxial compression*** (for $N_x < 0$), is identified (selected), and its three corner points *A, B,* and *C* are marked as potentially critical points. The fourth corner point O, which coincides with the section centroid, is not considered, since at this point

$$\tau_O(M_x) = \sigma_{x\,O}(M_y) = \sigma_{x\,O}(M_z) = 0.$$

2. The dimensions of the rectangular cross-section should be determined successively for the potentially critical points *A*, *B* and *C*.

The strength conditions at these points are written ***in the first approximation*** by neglecting N_x, since usually $\sigma_x(N_x) \ll \sigma_x(M_y, M_z)$. The calculation begins with the corner point *A*, which in most cases proves to be the critical one in practice.

Strength condition at point A

$$\sigma_{x\,A} = \sigma_{x_{max}} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \leq [\sigma]. \tag{3.12}$$

Here $W_y = \dfrac{bh^2}{6}$; $W_z = \dfrac{hb^2}{6}$.

Taking into account that

$$h/b = k \quad \Rightarrow \quad h = kb,$$

the expressions for the section moduli determination under bending can be rewritten in the form

$$W_y = \frac{k^2 b^3}{6}; \qquad W_z = \frac{kb^3}{6}. \tag{3.13}$$

By substituting equations (3.13) into the equation (3.12), we obtain

$$\frac{6M_y}{k^2 b^3} + \frac{6M_z}{kb^3} \leq [\sigma],$$

from which we determine

$$b \geq \sqrt[3]{\frac{\dfrac{6M_y}{k^2} + \dfrac{6M_z}{k}}{[\sigma]}} = \sqrt[3]{\frac{6M_y + 6kM_z}{k^2[\sigma]}}. \tag{3.14}$$

Strength conditions at point B, located at the midpoint of the long side:

$$\sigma_{eq\,B}^{\mathrm{III}} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{W_t}\right)^2} \leq [\sigma]; \qquad \sigma_{eq\,B}^{\mathrm{IV}} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 3\left(\frac{M_x}{W_t}\right)^2} \leq [\sigma]. \tag{3.15}$$

Here $W_t = \alpha h b^2$.

Taking into account $h = kb$, it follows that

$$W_t = \alpha k b^3. \tag{3.16}$$

By substituting equations (3.13) and (3.16) into the equation (3.15), after transformations we obtain

$$b^{\text{III}} \geq \sqrt[6]{\frac{\left(\frac{6M_z}{k}\right)^2 + 4\left(\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}}; \qquad b^{\text{IV}} \geq \sqrt[6]{\frac{\left(\frac{6M_z}{k}\right)^2 + 3\left(\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}}. \tag{3.17}$$

Strength conditions at point C, located at the midpoint of the short side:

$$\sigma_{eq\,C}^{\text{III}} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\gamma\frac{M_x}{W_t}\right)^2} \leq [\sigma]; \quad \sigma_{eq\,C}^{\text{IV}} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 3\left(\gamma\frac{M_x}{W_t}\right)^2} \leq [\sigma]. \tag{3.18}$$

By substituting equations (3.13) and (3.16) into the equation (3.18), after transformations we obtain

$$b^{\text{III}} \geq \sqrt[6]{\frac{\left(\frac{6M_y}{k^2}\right)^2 + 4\left(\gamma\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}}; \qquad b^{\text{IV}} \geq \sqrt[6]{\frac{\left(\frac{6M_y}{k^2}\right)^2 + 3\left(\gamma\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}}. \tag{3.19}$$

Remark | α, γ are coefficients depending on the ratio h/b (see Appendix).

3. Choose the larger dimension b. Then find $h = kb$.

Remark | The cross-sectional point for which the larger pair of dimensions was selected will be the most critical point of the section.

4. If an **axial force** N_x acts in the cross-section, ***in the second approximation*** the strength is verified at the critical point (*A*, *B* or *C*) taking into account the presence of N_x (at point *A*, according to condition (3.7); at point *B,* according to expressions (3.8), (3.10); and at point *C*, by formulas (3.9), (3.11), depending on which strength theory has been adopted as the governing one).

If the strength verification confirms fulfillment of the strength conditions with an accuracy of $\Delta\sigma \leq 5$ %, then the calculation is completed.

If the strength condition at the critical point is ***not satisfied***, it is necessary ***to increase*** the dimensions h and b using the method of successive approximations.

Remarks | 1. If ***one of the bending moments*** M_y or M_z ***is equal to zero***, then the ***corner points*** of the cross-section ***are excluded*** from the category of critical points.
2. If the ***torsional moment*** M_x is equal to zero, then the ***corner point*** becomes the only critical point of the cross-section.

3.3. The Circular Cross-Section

3.3.1. Bending with Torsion of a Circular Cross-Section Bar

As practical calculations show, in a circular cross-section:

$$\tau_{max}(Q) \ll \tau_{max}(M_{torsional}) \quad \text{and} \quad \tau_{max}(Q) \ll \sigma_{max}(M_{bending}),$$

therefore, the shear stresses resulting from shear forces are neglected in this case.

Since all axes passing through the centroid of a circular cross-section are the principal central axes of inertia of that section, it is not necessary to consider bending separately in the coordinate planes (horizontal and vertical).

Usually, bending of a circular cross-section bar is considered under the action of the resultant bending moment (Fig. 3.9)

$$M_{bending} = \sqrt{M_y^2 + M_z^2}. \tag{3.20}$$

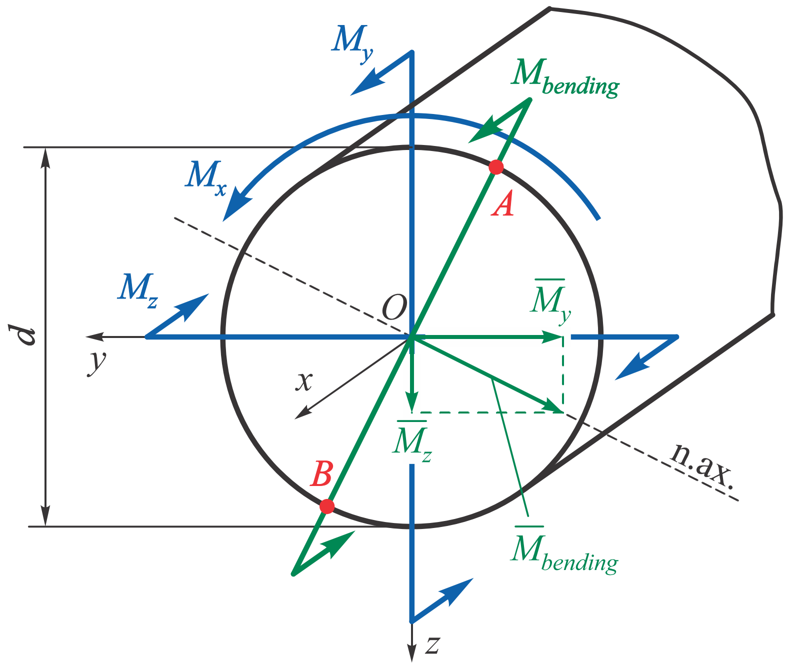


Fig. 3.9

Then, the maximum normal stresses σ_x acting at points A and B, which are most distant from the neutral axis (n.ax.), are determined by the formula:

$$\sigma_{x\,max\,A} = |\sigma_{x\,max\,B}| = \frac{M_{bending}}{W_{\text{n.ax.}}} = \frac{\sqrt{M_y^2 + M_z^2}}{W_{\text{n.ax.}}}. \tag{3.21}$$

Shear stresses τ caused by the torsional moment M_x reach their maximum value at the peripheral points of the section, i.e. on the circumference, including at points *A* and *B*:

$$\tau_{max} = \tau_A = \tau_B = \frac{M_x}{W_\rho}. \tag{3.22}$$

Consequently, points *A* and *B* are the most critical points of the entire cross-section, since both σ_{max} and τ_{max} act at these points. As the strength conditions (1.27) and (1.28) are independent of the signs of σ and τ, points *A* and *B* are ***equally critical***.

Since the stress state at considered points *A* and *B* is ***plane*** (Fig. 3.10) (the shaded areas coincide with the cross-sectional plane; stresses on hidden faces are not shown), the strength analysis must be carried out according to one of the strength theories (the third or the fourth).

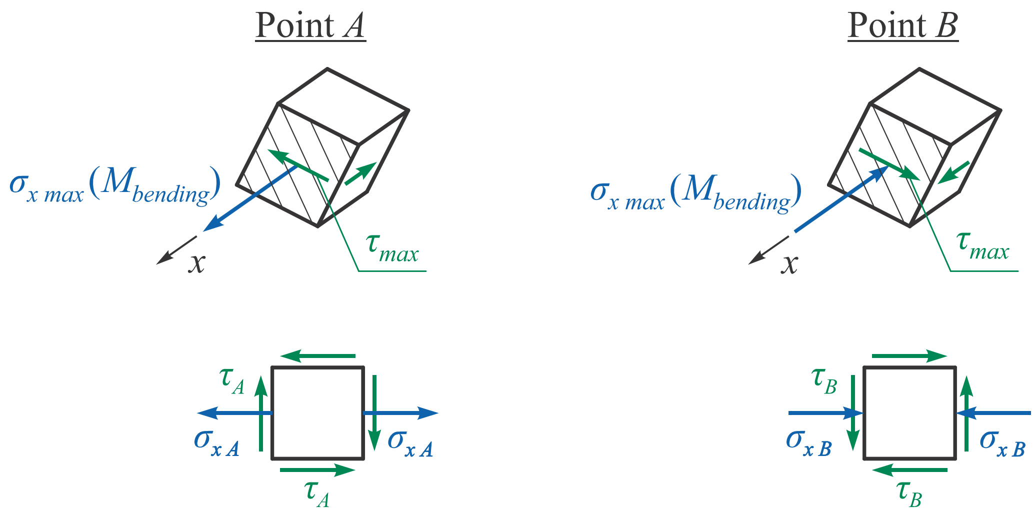


Fig. 3.10

The strength conditions at points *A* and *B* are as follows:

a) ***according to the third strength theory:***

$$\sigma^{III}_{eq\ A(B)} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_{bending}}{W_{n.ax.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \le [\sigma],$$

where $W_\rho = \dfrac{\pi d^3}{16}$ is a polar section modulus;

$W_{n.ax.} = \dfrac{\pi d^3}{32}$ is the section modulus with respect to the neutral axis.

Since for a circular cross-section:

$$W_\rho = 2W_{n.ax.},$$

it follows that

$$\sigma^{\text{III}}_{eq_{A\,(B)}} = \sqrt{\frac{M^2_{bending}}{W^2_{\text{n.ax.}}} + \frac{M^2_x}{W^2_{\text{n.ax.}}}} = \frac{\sqrt{M^2_{bending} + M^2_x}}{W_{\text{n.ax.}}}.$$

Let us introduce the notation:

$$M^{\text{III}}_{design} = \sqrt{M^2_{bending} + M^2_x} = \sqrt{M^2_y + M^2_z + M^2_x}, \quad (3.23)$$

where M^{III}_{design} is the design (or equivalent) moment according to the fourth strength theory.

Then, finally:

$$\sigma^{\text{III}}_{eq_{A\,(B)}} = \frac{M^{\text{III}}_{design}}{W_{\text{n.ax.}}} \leq [\sigma]. \quad (3.24)$$

b) according to the fourth strength theory:

$$\sigma^{\text{IV}}_{eq\;A(B)} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{M_{bending}}{W_{\text{n.ax.}}}\right)^2 + 3\left(\frac{M_x}{W_\rho}\right)^2} \leq [\sigma].$$

Then

$$\sigma^{\text{IV}}_{eq_{A(B)}} = \sqrt{\frac{M^2_{bending}}{W^2_{\text{n.ax.}}} + \frac{3}{4}\frac{M^2_x}{W^2_{\text{n.ax.}}}} = \frac{\sqrt{M^2_{bending} + 0.75M^2_x}}{W_{\text{n.ax.}}}.$$

Let's introduce the notation:

$$M^{\text{IV}}_{design} = \sqrt{M^2_{bending} + 0.75M^2_x} = \sqrt{M^2_y + M^2_z + 0.75M^2_x}, \quad (3.25)$$

where M^{IV}_{design} is the design (or equivalent) moment according to the fourth strength theory.

Then, finally:

$$\sigma^{\text{IV}}_{eq_{A\,(B)}} = \frac{M^{\text{IV}}_{design}}{W_{\text{n.ax.}}} \leq [\sigma]. \quad (3.26)$$

Since

$$W_{\text{n.ax.}} = \frac{\pi d^3}{32},$$

then, using the strength condition (3.24) or (3.26), the design problem can be solved, i.e., the diameter of the cross-section can be determined:

$$d^{\text{III}} = \sqrt[3]{\frac{32 M_{design}^{\text{III}}}{\pi[\sigma]}} \tag{3.27}$$

or

$$d^{\text{IV}} = \sqrt[3]{\frac{32 M_{design}^{\text{IV}}}{\pi[\sigma]}}. \tag{3.28}$$

3.3.2. Bending with Torsion and Tension-Compression of a Circular Cross-Section Bar

This calculation case differs from the previous one by the presence of an axial force N_x.

Of the two equally critical points considered earlier, only one becomes the critical one. This is the point, in which the stresses from the action of the axial force N_x and the bending moment $M_{bending}$ are summed up.

The strength conditions at the critical point are:

a) ***according to the third strength theory:***

$$\sigma_{eq}^{\text{III}} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{n.ax.}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \leq [\sigma]; \tag{3.29}$$

b) ***according to the fourth strength theory:***

$$\sigma_{eq}^{\text{IV}} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{n.ax.}}\right)^2 + 3\left(\frac{M_x}{W_\rho}\right)^2} \leq [\sigma]. \tag{3.30}$$

In the design calculation of structures, in this case, in the first approximation, the diameter d is determined by neglecting the axial force N_x, according to formula (3.27) or (3.28), since in most cases:

$$\sigma_x(N_x) \ll \sigma_x(M_{bending}),$$

and in the second approximation, the strength verification at the critical point (A or B, depending on the direction of N_x) is performed according to condition (3.29) or (3.30).

3.4. Problem-Solving Examples

Example 3.1

For the cranked bar, diagrams of internal forces and moments have been constructed (see Example 2.1). With an allowable stress of $[\sigma] = 220\ MPa$, it is necessary to design:

a) the dimensions of a circular cross-section;

b) the dimensions of a rectangular cross-section with $k = h/b = 2$;

c) to construct diagrams of the distribution of normal and shear stresses from the action of N_x, M_x, M_y, M_z for the rectangular cross-section;

d) to show the stress state at the critical points of the rectangular and circular cross-sections;

e) to use the third and fourth strength theories for determining the circular cross-section, and the third strength theory for the rectangular one;

f) to compare the weights of the resulting bars.

Solution

1. Determine the critical cross-section.

From the analysis of the internal force and moment diagrams (see Example 2.1, Fig. 2.12), it is evident that the ***most critical section*** is at the fixed-end section, where the following internal forces and moments act:

$$N_x = 20\ kN; \quad M_x = 45\ kN{\cdot}m; \quad M_y = -30\ kN{\cdot}m; \quad M_z = 80\ kN{\cdot}m.$$

2. Determine the diameter of the circular cross-section.

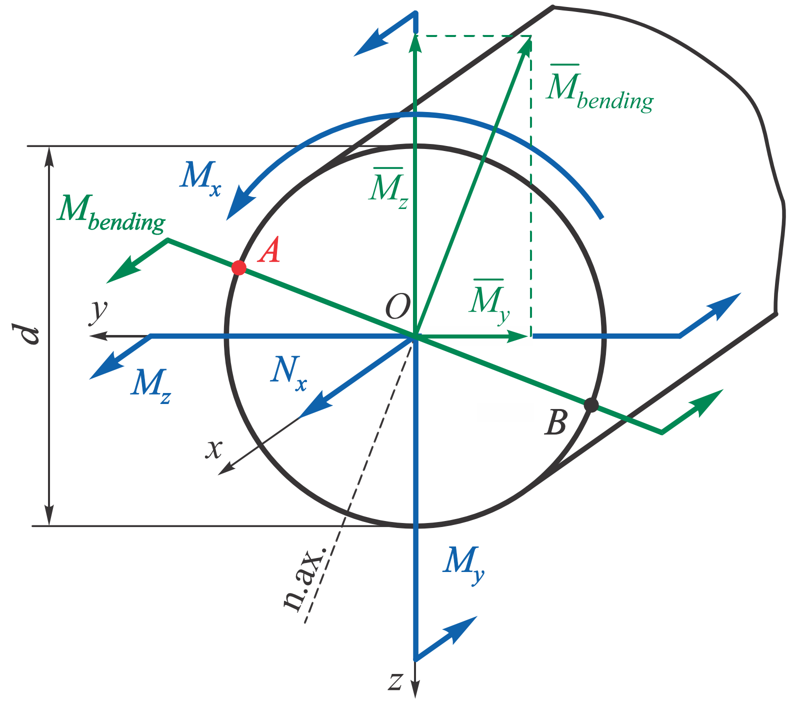


Fig. 3.11

The plane of action of the resultant bending moment $M_{bending} = \sqrt{M_y^2 + M_z^2}$ the position of the points of maximum bending normal stresses (points A and B) (Fig. 3.11). Due to the presence of compressive normal stresses caused by the axial force N_x, the maximum normal stress occurs at point A, where the stresses from the axial force and the bending moment are summed.

At the same time, this point is also the location of maximum shear stresses resulting from torsion, since it lies on the circumference of the cross-section.

Thus, point A is the only critical point of the circular cross-section.

Since the stress state at point A is plane, the strength calculation must be carried out according to one of the strength theories (the third or the fourth) (Fig. 3.12).

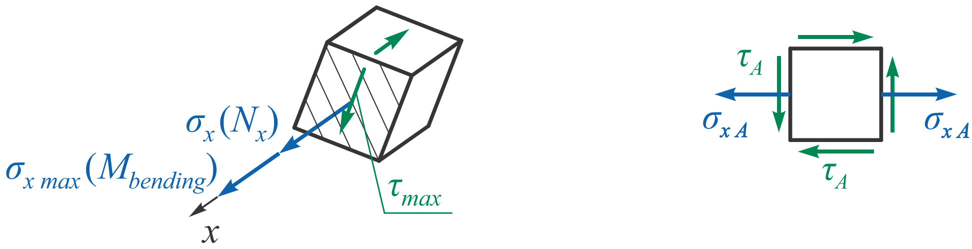


Fig. 3.12

The third strength theory

In the first approximation, to determine the diameter d, we write the strength condition at point A, neglecting the influence of the axial force N_x (3.24):

$$\sigma_{eq_A}^{\text{III}} = \frac{M_{design}^{\text{III}}}{W_{\text{n.ax.}}} = \left\{ \text{since } W_{\text{n.ax.}} = \frac{\pi d^3}{32} \right\} = \frac{32 M_{design}^{\text{III}}}{\pi d^3} \leq [\sigma],$$

from which we obtain:

$$d^{\text{III}} \geq \sqrt[3]{\frac{32 M_{design}^{\text{III}}}{\pi [\sigma]}},$$

where, in accordance with equation (3.23),

$$M_{design}^{\text{III}} = \sqrt{M_y^2 + M_z^2 + M_x^2} = \sqrt{30^2 + 80^2 + 45^2} = 96.566\ kN{\cdot}m.$$

Then,

$$d^{\text{III}} \geq \sqrt[3]{\frac{32 \cdot 96.566 \times 10^3}{\pi \cdot 220 \times 10^6}} = 0.1647\ m.$$

We use formula (3.29) to determine the actual design stresses at the critical point, taking into account the axial force N_x:

$$\sigma_{eq\ A}^{\text{III}} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{\text{n.ax.}}} \right)^2 + 4 \left(\frac{M_x}{W_\rho} \right)^2},$$

where $M_{bending} = \sqrt{M_y^2 + M_z^2} = \sqrt{30^2 + 80^2} = 85.44\ kN{\cdot}m$;

$$F = \frac{\pi d^2}{4} = \frac{\pi \cdot 0.1647^2}{4} = 2.1305 \times 10^{-2}\ m^2;$$

$$W_{\text{n.ax.}} = \frac{\pi d^3}{32} = \frac{\pi \cdot 0.1647^3}{32} = 4.3861 \times 10^{-4}\ m^3;$$

$$W_\rho = \frac{\pi d^3}{16} = 2W_{\text{n.ax.}} = \frac{\pi \cdot 0.1647^3}{16} = 8.7722 \times 10^{-4}\ m^3.$$

We get:

$$\sigma_{eq\,A}^{\text{III}} = \sqrt{\left(\frac{20 \times 10^3}{2.1305 \times 10^{-2}} + \frac{85.44 \times 10^3}{4.3861 \times 10^{-4}}\right)^2 + 4\left(\frac{45 \times 10^3}{8.7722 \times 10^{-4}}\right)^2} =$$

$$= 220.995\ MPa.$$

The ***overstress*** is:

$$\Delta\sigma\ \% = \frac{\sigma_{eq\,A}^{\text{III}} - [\sigma]}{[\sigma]} \cdot 100\ \% = \frac{220.995 - 220}{220} \cdot 100\ \% = 0.45\ \% < 5\ \%.$$

Thus, the strength of the bar`s cross-section is ensured.

Remark | Since $\sigma_x(N_x) \ll [\sigma]$, the contribution of the axial force N_x to the normal stress at the critical point can be estimated as

$$\sigma_x(N_x) = \frac{N_x}{F} = \frac{20 \times 10^3}{2.1305 \times 10^{-2}} = 0.939\ MPa.$$

The fourth strength theory

In the first approximation, to determine the diameter d, we write the strength condition at point *A*, neglecting the influence of the axial force N_x (see (3.26)):

$$\sigma_{eq_A}^{\text{IV}} = \frac{M_{design}^{\text{IV}}}{W_{\text{n.ax.}}} = \left\{\text{since } W_{\text{n.ax.}} = \frac{\pi d^3}{32}\right\} = \frac{32 M_{design}^{\text{IV}}}{\pi d^3} \le [\sigma],$$

from which we obtain:

$$d^{\text{IV}} \ge \sqrt[3]{\frac{32 M_{design}^{\text{IV}}}{\pi[\sigma]}},$$

where, in accordance with equation (3.25),

$$M_{design}^{\text{IV}} = \sqrt{M_y^2 + M_z^2 + 0.75 \cdot M_x^2} = \sqrt{30^2 + 80^2 + 0.75 \cdot 45^2} = 93.908\ kN{\cdot}m.$$

Then

$$d^{\text{IV}} \geq \sqrt[3]{\frac{32 \cdot 93.908 \times 10^3}{\pi \cdot 220 \times 10^6}} = 0.1632\ m.$$

We use formula (3.30) to determine the actual design stresses at the critical point, taking into account the axial force N_x:

$$\sigma_{eq\ A}^{\text{IV}} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{\text{n.ax.}}}\right)^2 + 3\left(\frac{M_x}{W_\rho}\right)^2},$$

where $M_{bending} = \sqrt{M_y^2 + M_z^2} = \sqrt{30^2 + 80^2} = 85.44\ kN{\cdot}m$;

$$F = \frac{\pi d^2}{4} = \frac{\pi \cdot 0.1632^2}{4} = 2.0918 \times 10^{-2}\ m^2;$$

$$W_{\text{n.ax.}} = \frac{\pi d^3}{32} = \frac{\pi \cdot 0.1632^3}{32} = 4.2674 \times 10^{-4}\ m^3;$$

$$W_\rho = \frac{\pi d^3}{16} = 2W_{\text{n.ax.}} = \frac{\pi \cdot 0.1632^3}{16} = 8.5378 \times 10^{-4}\ m^3.$$

We get:

$$\sigma_{eq\ A}^{\text{IV}} = \sqrt{\left(\frac{20 \times 10^3}{2.0918 \times 10^{-2}} + \frac{85.44 \times 10^3}{4.2674 \times 10^{-4}}\right)^2 + 3\left(\frac{45 \times 10^3}{8.5378 \times 10^{-4}}\right)^2} =$$

$$= 220.916\ MPa.$$

The ***overstress*** is:

$$\Delta\sigma\ \% = \frac{\sigma_{eq\ A}^{\text{IV}} - [\sigma]}{[\sigma]} \cdot 100\ \% = \frac{220.916 - 220}{220} \cdot 100\ \% = 0.42\ \% < 5\ \%.$$

Thus, the strength of the bar`s cross-section is ensured.

3. Determine the dimensions of the rectangular cross-section.

Since $M_z > M_y$, we orient the section horizontally to ensure the cross-section strength with smaller dimensions.

The section with the applied internal loads is shown in Fig. 3.13. The internal forces and moments are applied in accordance with the adopted sign conventions:

- a positive axial force N_x means tension;
- a positive torsional moment M_x means counter-clockwise rotation;
- a negative bending moment M_y means tension in the top fibers and compression in the bottom fibers;

– a positive bending moment M_z means tension in the left fibers and compression in the right fibers.

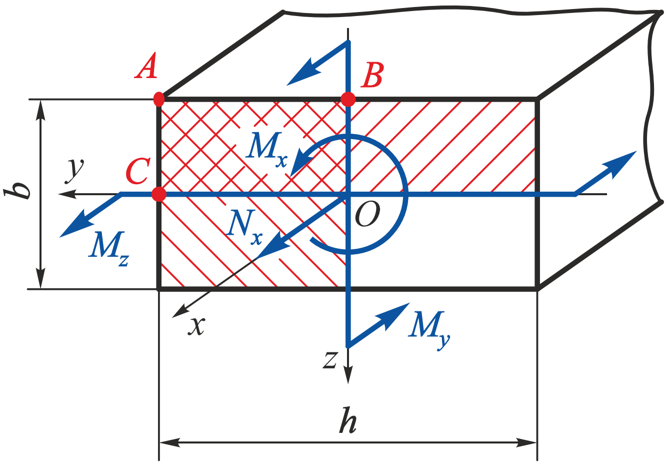


Fig. 3.13

Let's determine the potentially critical points of the section. We identify the triaxially tensioned quarter (since $N_x > 0$) of the section (hatched area in Fig. 3.13) and mark its three corner points: A, B, and C. These will be the potentially critical points.

Let us construct (draw) the diagrams of normal and shear stresses distributions across the section (Fig. 3.14 – 3.17).

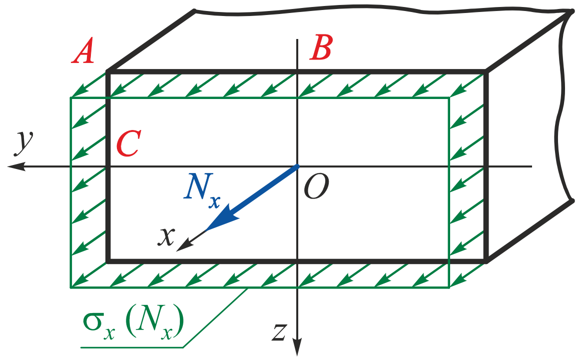


Fig. 3.14

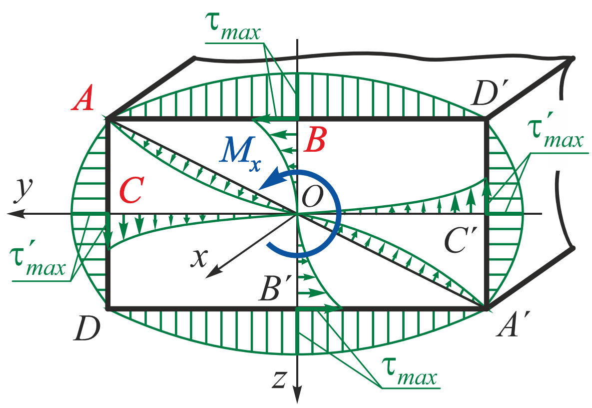


Fig. 3.15

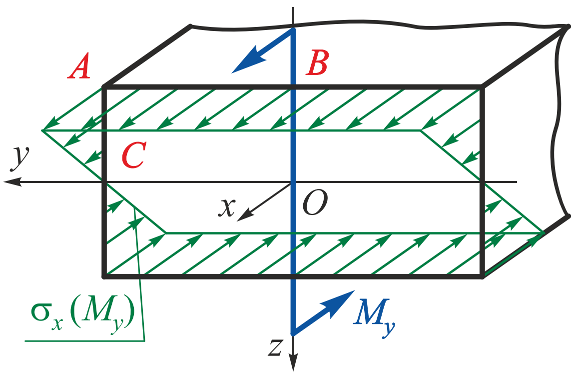


Fig. 3.16

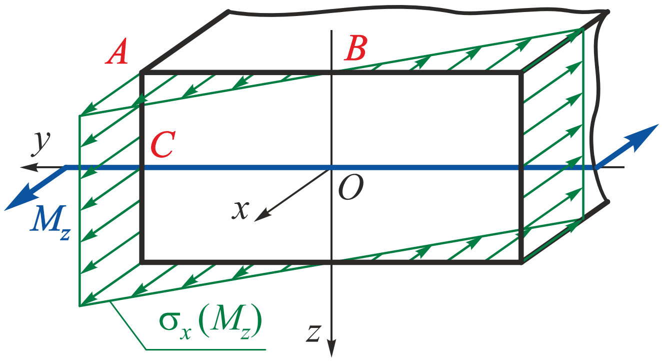


Fig. 3.17

At each of the three potentially critical points of the cross-section, let us present the type of stress state and formulate the strength conditions (without taking into account the influence of the axial force N_x).

Point *A*

At point *A* of the cross-section, a ***uniaxial stress state*** is realized (Fig. 3.18).

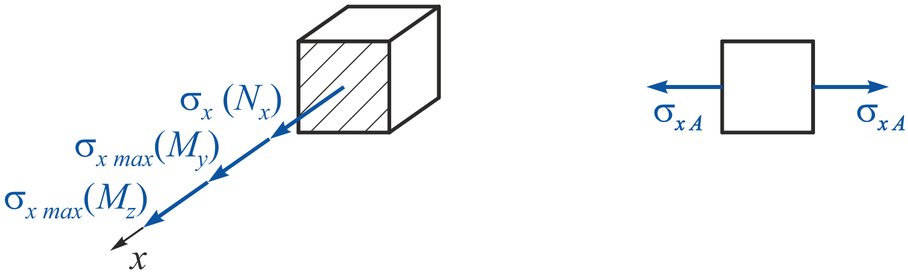


Fig. 3.18

The strength condition at this point has the form:

$$\sigma_{x\,A} = \sigma_{x_{max}} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \leq [\sigma],$$

where $W_y = \dfrac{hb^2}{6} = \left\{ \text{since } k = \dfrac{h}{b} \Rightarrow h = kb \right\} = \dfrac{kb^3}{6}$;

$$W_z = \frac{bh^2}{6} = \left\{ \text{since } k = \frac{h}{b} \Rightarrow h = kb \right\} = \frac{k^2 b^3}{6}.$$

Substituting the values of W_y and W_z into the strength condition and performing the transformations, we obtain:

$$b \geq \sqrt[3]{\frac{6k|M_y| + 6|M_z|}{k^2[\sigma]}} = \sqrt[3]{\frac{6 \cdot 2 \cdot |-30 \times 10^3| + 6 \cdot 80 \times 10^3}{2^2 \cdot 220 \times 10^6}} = 0.0985\ m.$$

Point *B*

At point B of the cross-section, a ***plane stress state*** is realized (Fig. 3.19).

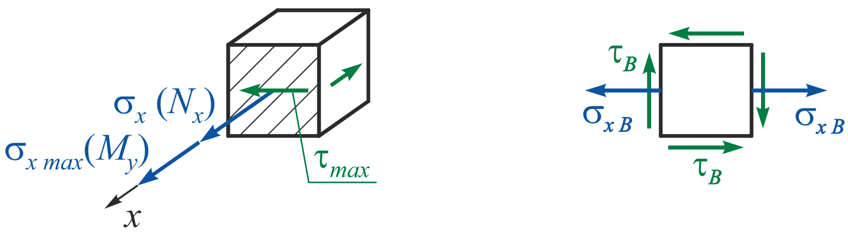


Fig. 3.19

We write the strength condition using the third strength theory:

$$\sigma^{III}_{eq\,B} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\frac{M_x}{W_{torsional}}\right)^2} \leq [\sigma],$$

where $W_y = \frac{hb^2}{6} = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \frac{kb^3}{6}$;

$W_{torsional} = \alpha hb^2 = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \alpha kb^3$;

$\alpha = 0.246$ is a coefficient that depends on the ratio $h/b = 2$ (see Appendix).

Substituting the values of W_y and $W_{torsional}$ into the strength condition, we get:

$$b^{\text{III}} \geq \sqrt[6]{\frac{\left(\frac{6M_y}{k}\right)^2 + 4\left(\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{\left(\frac{6 \cdot |-30 \times 10^3|}{2}\right)^2 + 4\left(\frac{45 \times 10^3}{0.246 \cdot 2}\right)^2}{(220 \times 10^6)^2}} = 0.0975\ m.$$

Point *C*

At point C of the cross-section, a plane stress state is realized (Fig. 3.20).

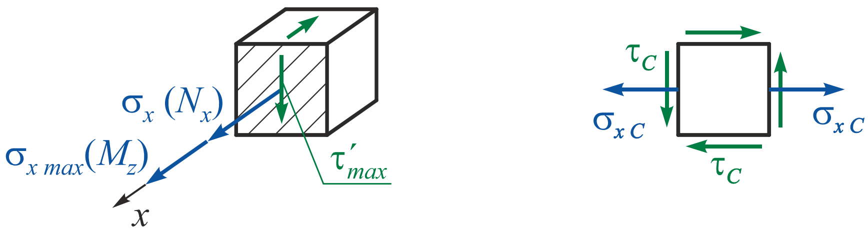


Fig. 3.20

The strength condition, using the third strength theory, is:

$$\sigma_{eq\ C}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 4\left(\gamma \frac{M_x}{W_{torsional}}\right)^2} \leq [\sigma],$$

where $W_z = \frac{bh^2}{6} = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \frac{k^2 b^3}{6}$;

$W_{torsional} = \alpha hb^2 = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \alpha kb^3$;

$\alpha = 0.246$, $\gamma = 0.795$ are coefficients that depend on the ratio $h/b = 2$ (see Appendix).

Substituting the values of W_z and $W_{torsional}$ into the strength condition, we get:

$$b^{\text{III}} \geq \sqrt[6]{\frac{\left(\frac{6M_z}{k^2}\right)^2 + 4\left(\gamma \frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{\left(\frac{6 \cdot (80 \times 10^3)}{2^2}\right)^2 + 4\left(0.795 \cdot \frac{45 \times 10^3}{0.246 \cdot 2}\right)^2}{(220 \cdot 10^6)^2}} =$$

$$= 0.09499\ m.$$

Let us select the largest of the three values of b:

$$\begin{cases} b_A \geq 0.0985\ m; \\ b_B \geq 0.0975\ m; \\ b_C \geq 0.09499\ m. \end{cases}$$

Thus, point A is the most critical point of the cross-section.

The design dimensions of the rectangular cross-section and its geometric characteristics are:

$$b = 0.0985\ m;$$

$$h = kb = 2 \cdot 0.0985 = 0.197\ m;$$

$$F = bh = 0.0985\ \cdot 0.197 = 0.0194\ m^2;$$

$$W_y = \frac{hb^2}{6} = \frac{0.197 \cdot 0.0985^2}{6} = 3.1856 \times 10^{-4}\ m^3;$$

$$W_z = \frac{bh^2}{6} = \frac{0.0985 \cdot 0.197^2}{6} = 6.3711 \times 10^{-4}\ m^3;$$

$$W_{\text{torsional}} = \alpha hb^2 = 0.246 \cdot 0.197 \cdot 0.0985\ ^2 = 4.7019 \times 10^{-4}\ m^3.$$

Since a uniaxial stress state is realized at point A, we determine the contribution of the axial force N_x to the total normal stress using the formula:

$$\sigma_{x\,A}(N_x) = \frac{N_x}{F} = \frac{20 \times 10^3}{0.0194} = 1.031\ MPa.$$

The ***overstress*** is:

$$\Delta\sigma\ \% = \frac{\sigma_{x\,A}(N_x)}{[\sigma]} \cdot 100\ \% = \frac{1.031}{220} \cdot 100\ \% = 0.47\ \% < 5\ \%.$$

Thus, the strength of the bar`s cross-section is ensured.

4. Compare weights of the circular and rectangular cross-section bars found using the third strength theory:

$$\frac{G^{\bigcirc}}{G^{\square}} = \frac{F^{\bigcirc}}{F^{\square}} = \frac{2.1305 \times 10^{-2}}{0.0194} = 1.098.$$

Therefore, for the given combination of internal forces and moments and aspect ratio $k = h/b = 2$ for the rectangle, it is more advantageous to use a rectangular cross-section to reduce the weight of the structure.

However, the largest overall dimension of the rectangular cross-section is larger than the diameter of the circular one:

$$h = 0.197\ m > d = 0.1647\ m.$$

Example 3.2

In the critical cross-section of a bar, a bending moment $M_y = 5\ kN{\cdot}m$ and a torsional moment $M_x = 20\ kN{\cdot}m$ are acting (Fig. 3.21). It is required, according to the fourth strength theory, to select the diameter, d, of a circular cross-section, the side size, a, of a square cross-section, and the dimensions b and h (with the ratio $k = h/b = 1.75$) if the allowable stress is $[\sigma] = 160\ MPa$. Show the stress state at the critical points of the cross-sections and compare the bars by weight.

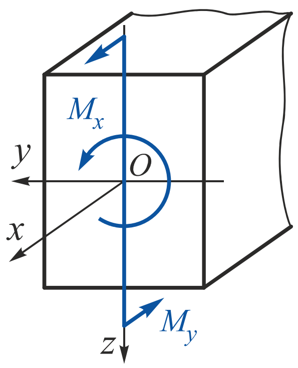


Fig. 3.21

Given: $M_y = -5\ kN{\cdot}m$; $M_x = 20\ kN{\cdot}m$;

$[\sigma] = 160\ MPa$; $k = h/b = 1.75$;

For a square cross-section: $\alpha = 0.208$; $\gamma = 1$;

For a rectangular cross-section: $\alpha = 0.239$; $\gamma = 0.820$.

It is necessary to determine d, a, b, and h; to show the stress state at the critical points of the cross-sections; and to compare the bars by weight.

Solution

1. Let us determine the diameter of the circular cross-section.

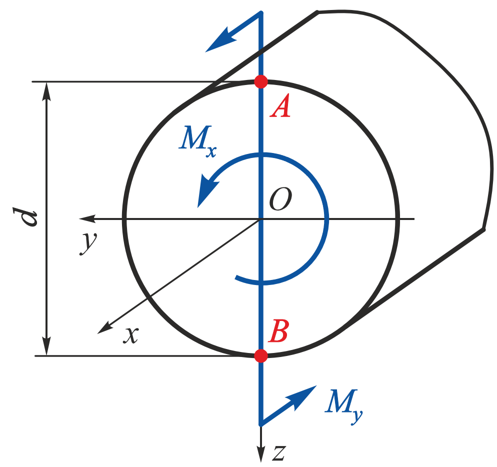


Fig. 3.22

In the circular cross-section, under the specified loading parameters, the maximum normal stresses from the action of the bending moment M_y occur at points A and B of the cross-section, which are the most distant from the neutral axis y (Fig. 3.22).

At the same time, these points are also the locations of maximum shear stresses from the action of the torsional moment M_x, since they lie on the circumference of the cross-sectionч.

Consequently, points A and B are the most critical points of the entire section, since both the maximum normal stress σ_{max} and the maximum shear stress τ_{max} act at these points. Because the strength conditions (1.27) and (1.28) are insensitive to the signs of σ and τ, points A and B are ***equally critical***. Therefore, we will consider the stress state only at point A.

There is a plane stress state at point A, therefore, the strength analysis must be performed according to the fourth strength theory (Fig. 3.23).

Fig. 3.23

To determine the diameter d, we write the strength condition (3.26)

$$\sigma_{eq_A}^{\mathrm{IV}} = \frac{M_{design}^{\mathrm{IV}}}{W_{\text{n.ax.}}} = \left\{\text{since } W_{\text{n.ax.}} = \frac{\pi d^3}{32}\right\} = \frac{32 M_{design}^{\mathrm{IV}}}{\pi d^3} \leq [\sigma],$$

from which we get:

$$d^{\mathrm{IV}} = \sqrt[3]{\frac{32 M_{design}^{\mathrm{IV}}}{\pi[\sigma]}},$$

where, in accordance with equation (3.25),

$$M_{design}^{\mathrm{IV}} = \sqrt{M_y^2 + M_z^2 + 0.75 \cdot M_x^2} = \sqrt{5^2 + 0^2 + 0.75 \cdot 20^2} = 18.028\ kN{\cdot}m.$$

Then

$$d^{\mathrm{IV}} = \sqrt[3]{\frac{32 \cdot 18.028 \times 10^3}{\pi \cdot 160 \times 10^6}} = 0.1047\ m.$$

2. Let us determine the side size of the square cross-section.

In a square cross-section under the specified loading parameters (Fig. 3.24), there are two potentially ***critical*** and ***equally critical points***, C and C'. The maximum normal and maximum shear stresses act at these points.

The corner points of the section A, A', D, and D' are excluded from the category of potentially critical points because $\sigma_{x\,A} = \sigma_{x\,A'} = \sigma_{x\,D} = \sigma_{x\,D'} = \sigma_{x\,max}$, while $\tau_A = \tau_{A'} = \tau_D = \tau_{D'} = 0$.

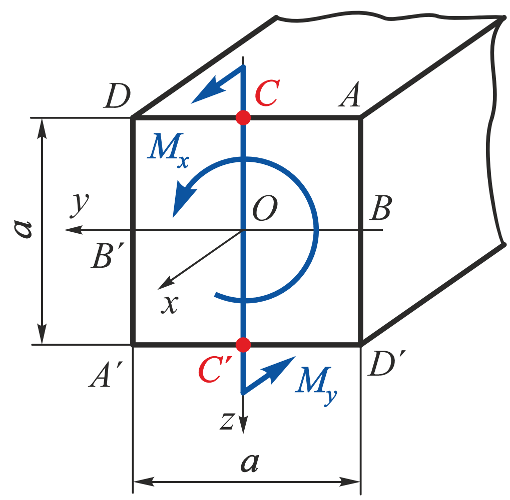


Fig. 3.24

Points B and B' are also excluded from the category of potentially critical points, since $\sigma_{x\,B} = \sigma_{x\,B'} = 0$, while $\tau_B = \tau_{B'} = \tau_{max}$.

Since points C and C' are equally critical, we will consider only one of them, point C. A plane stress state is realized at this point (Fig. 3.25).

Fig. 3.25

We write the strength condition using the fourth strength theory:

$$\sigma_{eq\,C}^{\mathrm{IV}} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 3\left(\frac{M_x}{W_{torsional}}\right)^2} \leq [\sigma],$$

where $W_y = \dfrac{hb^2}{6} = \left\{\text{since } h = b = a \implies k = \dfrac{a}{b} = 1\right\} = \dfrac{a^3}{6};$

$W_{torsional} = \alpha hb^2 = \left\{\text{since } h = b = a \implies k = \dfrac{a}{b} = 1\right\} = \alpha a^3;$

$\alpha = 0.239$ is a coefficient that depends on the ratio $h/b = 1$ (see Appendix).

Substituting the values of W_y and $W_{torsional}$ into the strength condition, we get:

$$a^{\mathrm{IV}} \geq \sqrt[6]{\frac{\left(6M_y\right)^2 + 3\left(\frac{M_x}{\alpha}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{(6\cdot|-5\times10^3|)^2 + 3\left(\frac{20\times10^3}{0.208}\right)^2}{(160\times10^6)^2}} = 0.0974\ m.$$

3. Let us determine the dimensions of the rectangular cross-section.

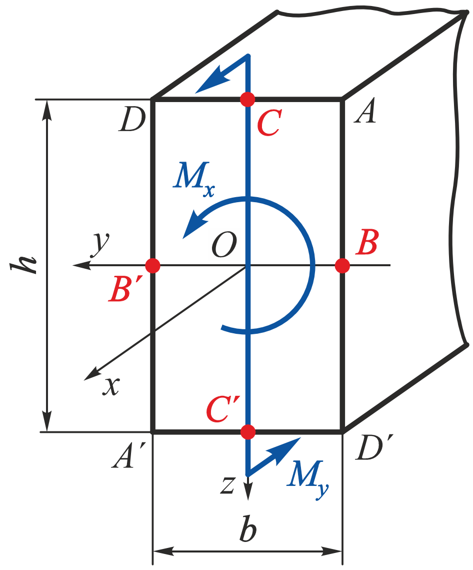


Fig. 3.26

In a rectangular cross-section under the specified loading parameters (Fig. 3.26), the potentially critical points are the equally critical points B and B', at which stresses

$\sigma_{x\,B} = \sigma_{x\,B'} = 0$ and $\tau_B = \tau_{B'} = \tau_{max}$ act,

as well as the equally critical points C and C', where stresses

$\sigma_{x\,C} = \sigma_{x\,C'} = \sigma_{x\,max}$ and $\tau_C = \tau_{C'} = \tau'_{max}$ act.

The corner points A, A', D, and D' of the section are excluded from the category of potentially critical points because

$$\sigma_{x\,A} = \sigma_{x\,A'} = \sigma_{x\,D} = \sigma_{x\,D'} = \sigma_{x\,max},$$

while

$$\tau_A = \tau_{A'} = \tau_D = \tau_{D'} = 0.$$

Since points B and B' are equally critical, we will consider only one of them, point B. A state of pure shear is realized at this point (Fig. 3.27).

Fig. 3.27

The strength condition at point B has the form:

$$\tau_B = \tau_{max} = \frac{|M_x|}{W_{torsional}} \leq [\tau] \approx 0.5 \cdot [\sigma],$$

where $W_{torsional} = \alpha h b^2 = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \alpha k b^3;$

$\alpha = 0.239$ is a coefficient that depends on the ratio $h/b = 1.75$ (see Appendix).

Substituting the value of $W_{torsional}$ into the strength condition, we get:

$$b_B \geq \sqrt[3]{\frac{|M_x|}{\alpha k \cdot 0.5 \cdot [\sigma]}} = \sqrt[3]{\frac{20 \times 10^3}{0.239 \cdot 1.75 \cdot 0.5 \cdot 160 \times 10^6}} = 0.0842\ m.$$

Since points C and C' are equally critical, we will consider only one of them, point C. A plane stress state is realized at this point (Fig. 3.28).

Fig. 3.28

We write the strength condition using the fourth strength theory:

$$\sigma_{eq\ C}^{IV} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 3\left(\gamma \frac{M_x}{W_{torsional}}\right)^2} \leq [\sigma],$$

where $W_y = \frac{bh^2}{6} = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \frac{k^2 b^3}{6};$

$W_{torsional} = \alpha h b^2 = \left\{ \text{since } k = \frac{h}{b} \implies h = kb \right\} = \alpha k b^3;$

$\alpha = 0.239$, $\gamma = 0.820$ are coefficients that depend on the ratio $h/b = 1.75$ (see Appendix).

Substituting the values of W_y and $W_{torsional}$ into the strength condition, we get:

$$b_C^{\mathrm{IV}} \geq \sqrt[6]{\frac{\left(\frac{6M_y}{k^2}\right)^2 + 3\left(\gamma\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{\left(\frac{6\cdot|-5\times10^3|}{1.75^2}\right)^2 + 3\left(0{,}820\cdot\frac{20\times10^3}{0.239\cdot1.75}\right)^2}{(160\times10^6)^2}} =$$

$$= 0.0753\ m.$$

Let us select the largest of the two values of b:

$$\begin{cases} b_B \geq 0.0842\ m; \\ b_C \geq 0.0753\ m. \end{cases}$$

Thus, point *B* is the most critical point of the cross-section.

The final dimensions of the rectangular cross-section are:

$$b = 0.0842\ m; \qquad h = kb = 1.75\cdot0.0842 = 0.1474\ m.$$

4. *Compare the weight* of the circular, square, and rectangular cross-section bars.

Let us calculate the areas of the
circular cross-section:

$$F^{\bigcirc} = \frac{\pi d^2}{4} = \frac{\pi\cdot0.1047^2}{4} = 8.6096\times10^{-3}\ m^2;$$

square cross-section:

$$F^{\square} = a^2 = 0.0974^2 = 9.4868\times10^{-3}\ m^2;$$

rectangular cross-section:

$$F^{▯} = bh = 0.0842\cdot0.1474 = 12.4111\times10^{-3}\ m^2.$$

Then

$$\frac{G^{\square}}{G^{\bigcirc}} = \frac{F^{\square}}{F^{\bigcirc}} = \frac{9.4868\times10^{-3}}{8.6096\times10^{-3}} = 1.102;$$

$$\frac{G^{▯}}{G^{\bigcirc}} = \frac{F^{▯}}{F^{\bigcirc}} = \frac{12.4111\times10^{-3}}{8.6096\times10^{-3}} = 1.442.$$

Thus, for the specified combination of internal **forces and moments** and the aspect ratio of the rectangle $k = h/b = 1.75$, in order to reduce the weight of the structure, it is more advantageous to use a circular cross-section, since it is lighter than the square one by 10.2 % and lighter than the rectangular one by 44.2 %.

In this case, the circular cross-section is also preferable in terms of overall dimensions.

3.5. Special Cases of Combined Loading

3.5.1. Oblique Bending

Oblique bending, like plane bending, is subdivided into pure and transverse.

Pure oblique bending occurs when only a bending moment acts in the cross-section of a beam, and the plane of its action does not contain any of the principal central axes of inertia of the cross-section. In this case, it does not matter whether the principal central axes of inertia of the section are its axes of symmetry or not.

Transverse oblique bending occurs under the condition that the cross-section of the beam has ***two axes of symmetry***, and the transverse loads act in different sections and in different planes containing the longitudinal axis of the beam, or the transverse loads act in a single force plane that contains the longitudinal axis of the beam, but does not coincide with any of the planes of symmetry of the beam (Fig. 3.29).

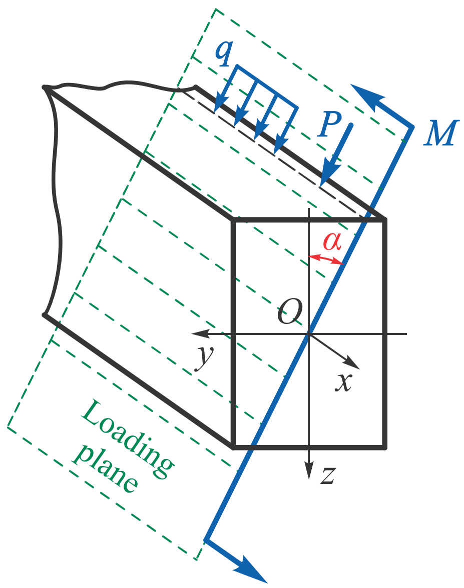


Fig. 3.29

If the beam's cross-section has one axis of symmetry or no axes of symmetry at all, then the transverse loads must act in planes that contain not the beam's longitudinal axis, but rather the line connecting the ***shear centers*** of the beam's cross-sections.

Taking into account that in most standard courses on Mechanics of Materials and Structures the concept of the shear centre of beam`s cross-sections is not considered, and given the fact that the overwhelming majority of real beams operate under conditions of transverse bending, we will only consider beams whose cross-sections have two axes of symmetry. This somewhat narrows the application field of the calculation relations obtained in this section; however, their practical significance is quite high due to the widespread use of beams with rectangular, box, cross-shaped, I-beam, and other sections, including composite and multi-cell sections, that have two axes of symmetry.

Determination of normal stresses in the cross-section of a beam

Let us consider a beam operating under conditions of oblique bending (Fig. 3.30).

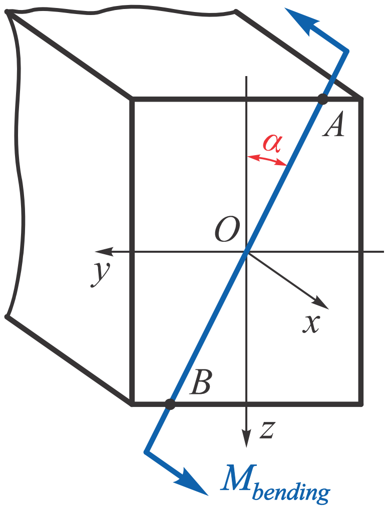


Fig. 3.30

In this figure, O is the centroid of the cross-section, axes y and z are the principal central axes of inertia of the section; $M_{bending}$ is the bending moment acting in the cross-section of the bar at an angle α to the z-axis; the shear force is not shown, since the normal stress σ_x is determined only by the presence of bending moments.

The action plane of the bending moment, i.e. the plane of loading, contains the x-axis, but does not contain either the y or z-axis.

Relying on the principle of superposition, oblique bending can be considered as simultaneous pure plane bending in two principal planes: xOz and xOy. Therefore, oblique bending is a special case of combined loading.

In Fig. 3.31, the moment $M_{bending}$ is shown in its usual and vector forms according to the rule adopted in the Theoretical Mechanics course. From this figure, it is clear that:

$$\left.\begin{aligned} M_y &= M_{bending} \cos\alpha\,; \\ M_z &= M_{bending} \sin\alpha\,. \end{aligned}\right\} \tag{3.31}$$

In Fig. 3.32, the diagram of the internal forces and moments action in the section is shown, transformed in accordance with formula (3.31).

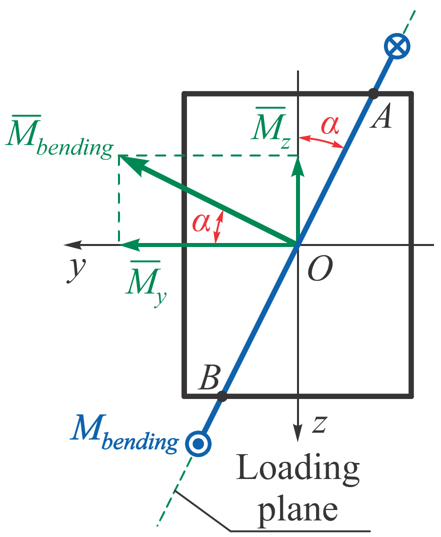


Fig. 3.31

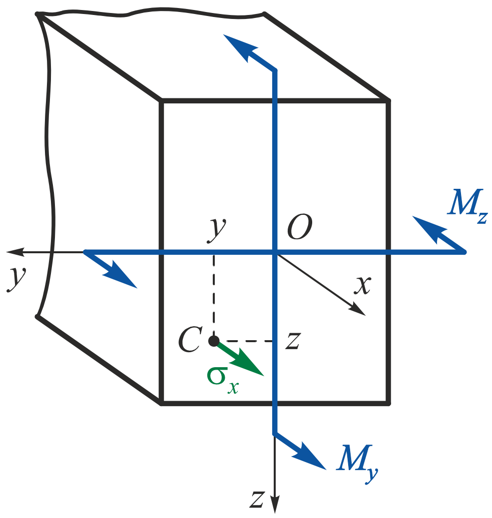


Fig. 3.32

The normal stress σ_x in an arbitrary point C is determined, using the principle of independence of force action (superposition), as follows:

$$\sigma_x = \sigma_x(M_y) + \sigma_x(M_z) = \frac{M_y z}{I_y} + \frac{M_z y}{I_z}. \quad (3.32)$$

By substituting the values of M_y and M_z from relations (3.31) into formula (3.32), we obtain:

$$\sigma_x = M_{bending}\left(\frac{z}{I_y}\cos\alpha + \frac{y}{I_z}\sin\alpha\right), \quad (3.33)$$

where y, z are the coordinates of an arbitrary point C;
$M_{bending}$ is the bending moment acting in the cross-section;
I_y and I_z are inertia moments of the cross-section;
α is the angle of inclination of the loading plane to the z-axis.

Remark Sometimes it is convenient to work directly with the given bending moments M_y and M_z, acting in two arbitrarily chosen perpendicular planes xOz and xOy.
Then, the formulas for determining the normal stresses caused by these moments will have the form:

$$\sigma_x(M_y) = \frac{M_y}{I_y I_z - I_{yz}^2}\left(I_z z - I_{yz} y\right);$$

$$\sigma_x(M_z) = \frac{M_z}{I_y I_z - I_{yz}^2}\left(I_{yz} z - I_y y\right).$$

These formulas are especially convenient for designing beams where the web and flanges are parallel to the y and z axes.

Determination of the cross-section neutral axis position

When performing strength analyses, the strength condition is written for the critical point of the cross-section, i.e., for the point, at which the normal stresses reach their maximum values. The most stressed point of an arbitrary shape cross-section is the point most distant from the neutral axis, which separates the tension and compression zones of the section.

Using equation (3.33), we determine the position of the neutral axis of the section from the condition that, at the points belonging to the neutral axis, $\sigma_x = 0$. Since $M_{bending} \neq 0$, it follows that

$$\frac{z}{I_y}\cos\alpha + \frac{y}{I_z}\sin\alpha = 0.$$

From this, we obtain the equation of a straight line passing through the origin:

$$z = \left(-\frac{I_y}{I_z}\frac{\sin\alpha}{\cos\alpha}\right)y = \left(-\frac{I_y}{I_z}\operatorname{tg}\alpha\right)y \tag{3.34}$$

or

$$z = k_1 y,$$

where

$$k_1 = \operatorname{tg}\varphi = -\frac{I_y}{I_z}\operatorname{tg}\alpha\,. \tag{3.35}$$

Expression (3.35) allows us to find the ***inclination angle of the neutral axis*** to the y-axis, and the minus sign indicates that the loading plane and the neutral axis in oblique bending pass through opposite quadrants. If the angle $\varphi > 0$, then it is measured counterclockwise from the y-axis; if the angle $\alpha > 0$, then it is measured clockwise from the z-axis.

The angle φ does not depend on the magnitude of the force P, but only on the angle of inclination of the loading plane to the z-axis and on the shape of the cross-section.

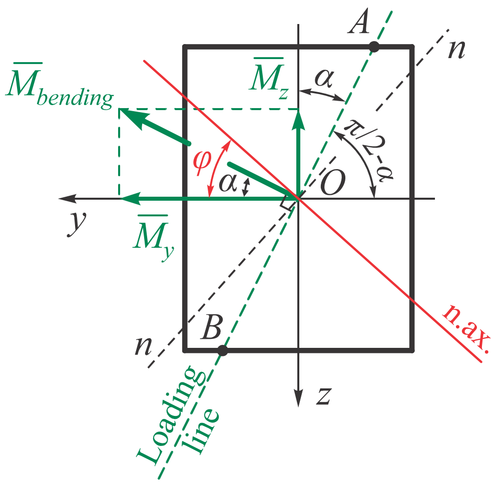


Fig. 3.33

Let us determine the orientation of the neutral axis relative to the action plane of $M_{bending}$, since it cannot be uniquely assumed that they are perpendicular (Fig. 3.33).

The equation of the loading line, i.e., the line of intersection of the action plane of $M_{bending}$ and the cross-sectional plane (line AB in Figs 3.30 and 3.32), in the coordinate system zOy has the form:

$$z = k_2 y,$$

where

$$k_2 = \operatorname{tg}\left(\frac{\pi}{2} - \alpha\right) = \operatorname{ctg}\alpha = \frac{1}{\operatorname{tg}\alpha}. \tag{3.36}$$

From the course of analytic geometry, it is known that the condition of perpendicularity of two straight lines is

$$k_1 = -\frac{1}{k_2}. \tag{3.37}$$

By comparing the values of k_1 from equation (3.35) and k_2 from (3.36), it is evident that condition (3.37) is not fulfilled in this case:

$$-\frac{I_y}{I_z}\operatorname{tg}\alpha \neq -\operatorname{tg}\alpha\,.$$

This means that, in the general case, the neutral line of the cross-section is not perpendicular to the action plane of the bending moment $M_{bending}$ (the loading line).

In the special case of cross-sections for which $I_y = I_z = I_{max} = I_{min}$ (for example, square or circular), all axes passing through the cross-section centroid are principal central axes of inertia. For such sections, according to the definition, realization of oblique bending is impossible, because the neutral axis is perpendicular to the loading plane.

During oblique bending, according to formulas (3.31), the ratio of the bending moments M_y and M_z is constant along the entire length of the bar ($M_z/M_y = \text{tg}\,\alpha$). Therefore, the inclination angle φ of the neutral line is also constant. This means that the cross-sections of the bar, remaining plane, rotate around neutral axes parallel to each other, as in the case of simple plane bending. The curvature of the bar axis occurs in a single plane $n - n$, normal to the direction of the neutral axis (see Fig. 3.33), which defines the name of this type of deformation. This plane is called the ***bending plane***.

Remark | If the loading line in a bar with a rectangular cross-section passes along one of the diagonals, the neutral axis will pass along the other diagonal:

$$\text{tg}\,\varphi = -\frac{I_y}{I_z}\text{tg}\,\alpha = -\frac{bh^3}{12}\cdot\frac{12}{hb^3}\cdot\frac{b}{h} = -\frac{h}{b}.$$

The strength condition in oblique bending

Expression (3.33) in the σ_x, y, z coordinate system represents the equation of a plane that intersects the beam's cross-section along the neutral axis. Consequently, the normal stresses (tensile and compressive) reach maximum values at those points of the cross-section that are most distant from its neutral axis.

If the coordinates of the points most distant from the neutral axis are known (y^* and z^*), then the strength condition takes the form:

$$\sigma_{x\,max} = M_{bending}\left(\frac{z^*}{I_y}\cos\alpha + \frac{y^*}{I_z}\sin\alpha\right) \le [\sigma]. \quad (3.38)$$

For determining the critical points of complex shape sections, it is necessary to construct tangents to the contour of the section parallel to the neutral axis. The points of tangency will then be the critical points.

Remark | For cross-sections with protruding corners, in which both principal axes of inertia are axes of symmetry (rectangular, box-type, I-beam, etc.), the critical points are located at the corners of these sections, i.e., they can be found without determining the position of the neutral axis:

$$\sigma_{x\,max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} = |M_{bending}|\left(\frac{\cos\alpha}{W_y} + \frac{\sin\alpha}{W_z}\right) \le [\sigma]. \quad (3.39)$$

3.5.2. Problem-Solving Examples

Example 3.3

Select the dimensions of a rectangular cross-section of a cantilever beam (Fig. 3.34); construct the diagram of normal stresses in the critical cross-section; determine the position of the neutral axis.

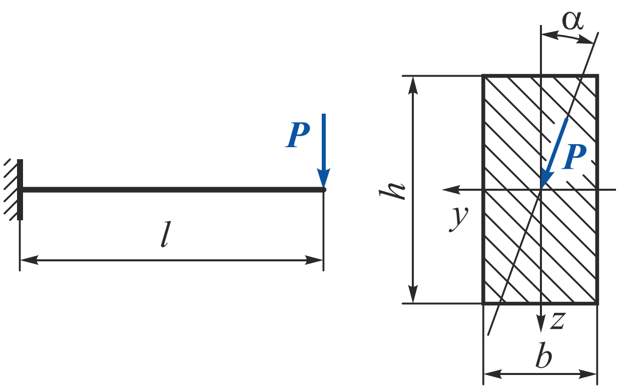


Fig. 3.34

Given: $P = 20\ kN;\ l = 2\ m;$

$\alpha = 20°;\ [\sigma] = 120\ MPa;$

$k = h/b = 2.$

It is necessary to determine b and h; to construct the σ_x diagram in the critical section; and to find the position of the neutral axis.

Solution

1. From Fig. 3.34, it is seen that the maximum bending moment acts at the fixed end:

$$M_{bending} = -Px = -20x \Big|_{x=0} = 0 \Big|_{x=l=2\ m} = -40\ kN{\cdot}m.$$

Let us represent the maximum bending moment in projections onto the principal central axes of inertia of the beam's cross-section (the y and z-axes) (Fig. 3.35):

$$M_y = M_{bending} \cos\alpha = -40 \cdot \cos 20^{\circ} = -37.588\ kN{\cdot}m;$$

$$M_z = M_{bending} \sin\alpha = -40 \cdot \sin 20^{\circ} = -13.681\ kN{\cdot}m.$$

2. Show the scheme of internal forces and moments acting in the critical cross-section (Fig. 3.36).

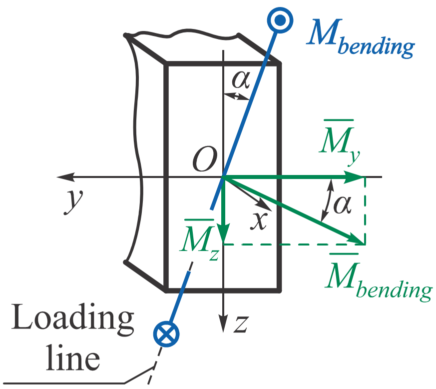


Fig. 3.35

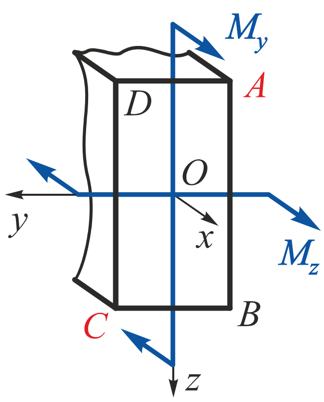


Fig. 3.36

From this figure, it is evident that the most critical points of the cross-section are the corner points *A* (with the maximum tensile stress) and *C* (with the maximum compressive stress).

Since the beam is made of a ductile material, $[\sigma]_t = [\sigma]_c = [\sigma]$, and consequently, points *A* and *C* are equally critical.

Let us write the strength condition for oblique bending at the critical point *A*

$$\sigma_{x\,max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} = |M_{bending}|\left(\frac{\cos\alpha}{W_y} + \frac{\sin\alpha}{W_z}\right) \le [\sigma].$$

We can rewrite this formula in another form:

$$\sigma_{x\,max} = \frac{|M_{bending}|}{W_y}\left(\cos\alpha + \frac{W_y}{W_z}\sin\alpha\right) = \frac{|M_{bending}|}{W_y}(\cos\alpha + k\sin\alpha) \le [\sigma],$$

where $W_y = \frac{bh^2}{6} = \left\{\text{since } k = \frac{h}{b} \Rightarrow h = kb\right\} = \frac{k^2b^3}{6}$;

$$W_z = \frac{hb^2}{6} = \left\{\text{since } k = \frac{h}{b} \Rightarrow h = kb\right\} = \frac{kb^3}{6};$$

$$\frac{W_y}{W_z} = \frac{k^2b^3}{6}\frac{6}{kb^3} = k.$$

Then

$$b \ge \sqrt[3]{\frac{6|M_{bending}|}{k^2[\sigma]}(\cos\alpha + k\sin\alpha)} =$$

$$= \sqrt[3]{\frac{6\cdot 40\times10^3}{2^2\cdot120\times10^6}(\cos 20^\circ + 2\cdot\sin 20^\circ)} = 0.0933\ m;$$

$$h = kb = 2\cdot 0.0977 = 0.1866\ m;$$

$$W_y = \frac{bh^2}{6} = \frac{0.0933\cdot0.1866^2}{6} = 541.444\times10^{-6}\ m^3;$$

$$W_z = \frac{hb^2}{6} = \frac{0.1866\cdot0.0933^2}{6} = 270.722\times10^{-6}\ m^3.$$

3. Let us determine the acting stresses at the corner points of the cross-section and construct the diagram of normal stress distribution acting in the critical cross-section (Fig. 3.37):

$$\sigma(M_y) = \frac{|M_y|}{W_y} = \frac{37.588\times10^3}{541.444\times10^{-6}} = 69.422\ MPa;$$

$$\sigma(M_z) = \frac{|M_z|}{W_z} = \frac{13.681\times10^3}{270.722\times10^{-6}} = 50.535\ MPa;$$

$$\sigma_A = \sigma(M_y) + \sigma(M_z) = 69.422 + 50.535 = 119.957\ MPa;$$

$$\sigma_B = -\sigma(M_y) + \sigma(M_z) = -69.422 + 50.535 = -18.887\ MPa;$$

$$\sigma_C = -\sigma(M_y) - \sigma(M_z) = -69.422 - 50.535 = -119.957\ MPa;$$

$$\sigma_D = \sigma(M_y) - \sigma(M_z) = 69.422 - 50.535 = 18.887\ MPa.$$

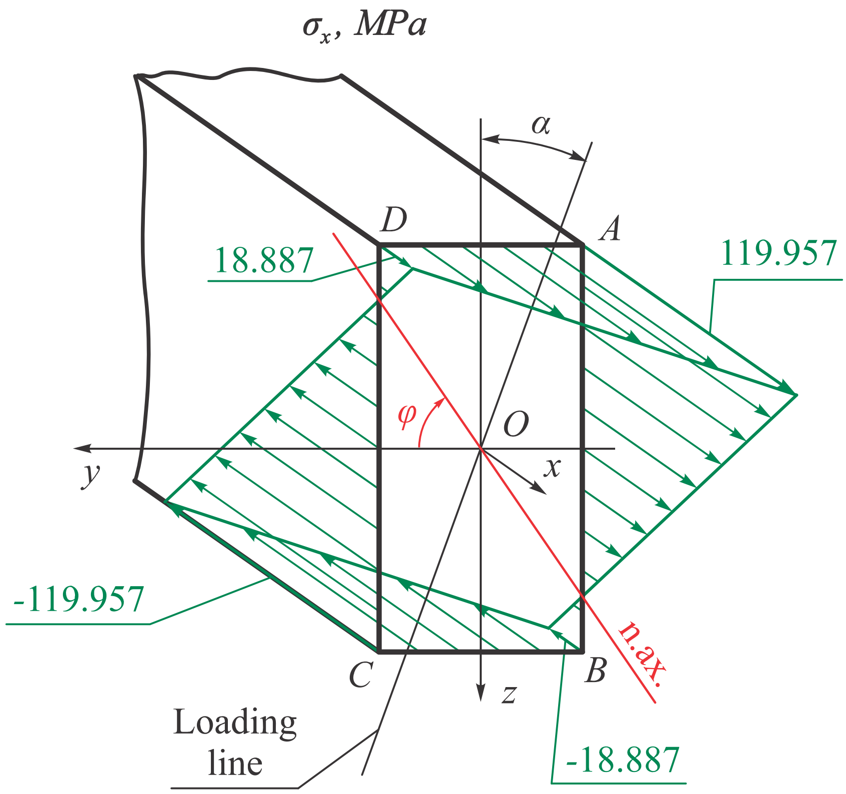


Fig. 3.37

4. We determine analytically the position of the neutral axis

$$\operatorname{tg}\varphi = -\frac{I_y}{I_z}\operatorname{tg}\alpha = -\frac{bh^3}{12}\frac{12}{hb^3}\operatorname{tg}\alpha = -k^2\operatorname{tg}\alpha = -2^2 \cdot \operatorname{tg}20^{\circ} = -1.4559;$$

$$\varphi = -55.52°.$$

5. Let us compare the stresses in oblique and plane bending (with $\alpha = 0$):

$$\sigma_{x\,max(\alpha=0)} = \frac{M_y}{W_y} = \frac{40 \times 10^3}{541.444 \times 10^{-6}} = 73.877\ MPa;$$

$$\frac{\sigma_A}{\sigma_{\alpha=0}} = \frac{119.957}{73.877} = 1.62.$$

The maximum stresses in oblique bending are greater than in plane bending by a factor of 1.62, i.e., oblique bending is more dangerous than plane bending.

Example 3.4

When installing an I-beam (No. 20, $W_y = 184 \times 10^{-6}\ m^3$, $W_z = 23.1 \cdot 10^{-6}\ m^3$) on supports, intended to operate in bending in the vertical plane coinciding with the web plane, an error was made: the web of the I-beam deviated from the vertical by an angle $\alpha = 1°$. Determine the increase in the maximum normal stresses associated with this deviation.

Solution

The deviation of the I-beam's axis (the z-axis) from the vertical leads to occurrence of oblique bending (Fig. 3.38) and the appearance of bending moments M_y and M_z.

Let's represent the caused by the action of force P in projections onto the principal central axes of inertia of the beam's cross-section (the y and z-axes):

$$M_y = M_{bending}\cos\alpha = M_{bending}\cdot\cos 1^{\circ} = \\ = 0{,}99985\cdot M_{bending};$$

$$M_z = M_{bending}\sin\alpha = M_{bending}\cdot\sin 1^{\circ} = \\ = 0{,}01745\cdot M_{bending}.$$

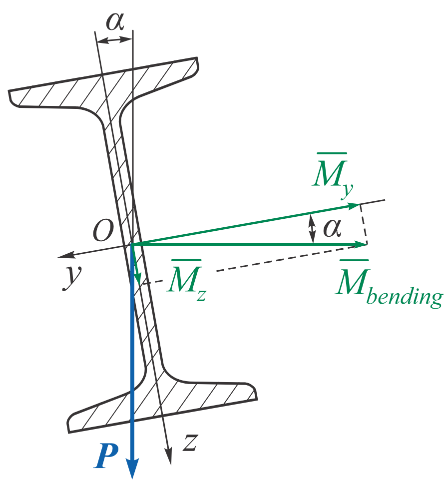


Fig. 3.38

The maximum stress during oblique bending is:

$$\sigma_{x\,max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} = \frac{M_{bending}\cos\alpha}{W_y} + \frac{M_{bending}\sin\alpha}{W_z} = \frac{M_{bending}}{W_y}\left(\cos\alpha + \frac{W_y}{W_z}\sin\alpha\right).$$

Then

$$\sigma_{x\,max} = \frac{M_{bending}}{W_y}\left(\cos 1^{\circ} + \frac{184\times 10^{-6}}{23.1\times 10^{-6}}\sin 1^{\circ}\right) = 1.139\cdot\frac{M_{bending}}{W_y}.$$

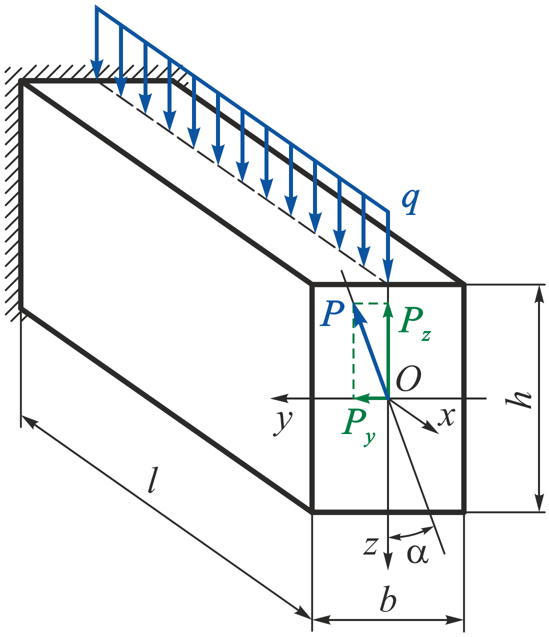
In the case of correct installation of the beam, the force P would coincide with the vertical z-axis, and simple (plane) bending would occur; the bending moment would be equal to $M_{bending}$, and the maximum normal stress would be

$$\sigma_{x\,max(\alpha=0)} = \frac{M_{bending}}{W_y}.$$

Thus, the maximum stresses during oblique bending due to such a minor deviation from the vertical increase by 13.9%.

Example 3.5

A cantilever beam of rectangular cross-section is loaded by a concentrated force P and a uniformly distributed (rectangular) load q (Fig. 3.39). Determine the maximum stress acting in the critical cross-section.



Given: $P = 12\ kN$; $q = 14\ kN/m$;
$b = 0.1\ m$; $k = h/b = 1.5$;
$l = 1.5\ m$; $\alpha = 20°$.

It is necessary to define $\sigma_{x\,max}$.

Fig. 3.39

Solution

1. Let's represent all external loads acting on the beam in projections onto the principal central axes of inertia of the beam's cross-section (the y and z-axes).

In this case:

$P_z = P\cos\alpha = 12 \cdot \cos 20° = 11.276\ kN$; $q_z = q = 14\ kN/m$;

$P_y = P\sin\alpha = 12 \cdot \sin 20° = 4.104\ kN$; $q_z = 0$.

2. Construct the bending moment diagram acting in the vertical plane (xOz) (Fig. 3.40):

$$M_y = P_z x - \frac{qx^2}{2} = 11.276 \cdot x - \frac{14x^2}{2} \Big|_{x=0} = 0 \Big|_{x=l=1{,}5\ m} = 1.164\ kN{\cdot}m.$$

Let's determine the extreme value of M_y:

$$\frac{dM_y}{dx} = P_z - qx_e = 0 \quad \Longrightarrow \quad x_e = \frac{P_z}{q} = \frac{11.276}{14} = 0.805\ m \quad \text{and} \quad M_y^e = 4.541\ kN{\cdot}m.$$

3. Construct the bending moment diagram acting in the horizontal plane (xOy) (see Fig. 3.40):

$$M_z = -P_y x = -4.104 \cdot x \Big|_{x=0} = 0 \Big|_{x=l=1{,}5\ m} = -6.156\ kN{\cdot}m.$$

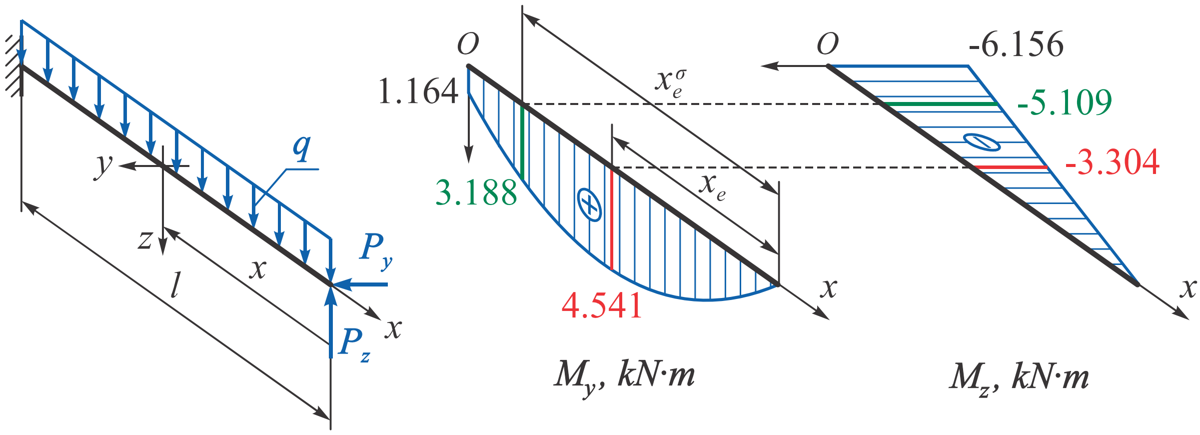


Fig. 3.40

4. Let us calculate the stress acting at the critical point of the critical cross-section:

In any cross-section, the maximum stress acts at one of the points most distant both from the y-axis and from the z-axis. Therefore,

$$\sigma_{x\,max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} = \frac{P_z x - \frac{qx^2}{2}}{W_y} + \frac{P_y x}{W_z}.$$

Introduce the notation: $m = \frac{W_y}{W_z} = \frac{bh^2}{6} \cdot \frac{6}{hb^2} = \frac{h}{b} = k = 1.5.$

Then

$$\sigma_{x\,max} = \frac{1}{W_y}\left(P_z x - \frac{qx^2}{2} + mP_y x\right).$$

We determine the maximum value of the maximum stress.

$$\frac{d(\sigma_{x\,max})}{dx} = \frac{1}{W_y}\left(P_z - x_e^\sigma + mP_y\right) = 0, \text{ from where}$$

$$x_e^\sigma = \frac{P_z}{q} + m\frac{P_y}{q} = \{\text{since } m = k = 1.5\} = \frac{11.276}{14} + \frac{1.5 \cdot 4.104}{14} = 1.245\ m.$$

Let us determine the values of the bending moments acting in the critical section:

$$M_{y_0} = 11.276 \cdot x - \frac{14x^2}{2} \Big|_{x=x_e^\sigma} = 3.188\ kN{\cdot}m;$$

$$M_{z_0} = -4.104 \cdot x \Big|_{x=x_e^\sigma} = -5.109\ kN{\cdot}m.$$

Thus, finally:

$$\sigma_{x\,max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} = \frac{3.188 \times 10^3}{0.000375} + \frac{|-5.109 \times 10^3|}{0.00025} = 28.937\ MPa.$$

Remark With oblique bending, the critical cross-section is not always the one where the bending moment has the greatest value. In this case, at the critical section:

$$M_{bending} = \sqrt{M_{y_0}^2 + M_{z_0}^2} = \sqrt{3.188^2 + 5.109^2} = 6.022\ kN{\cdot}m;$$

$$\sigma_{x\,max} = 28.937\ MPa;$$

at the fixed end:

$$M_{bending} = \sqrt{M_y^2 + M_z^2} = \sqrt{1.164^2 + 6.156^2} = 6.265\ kN{\cdot}m;$$

$$\sigma_{x\,max} = \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{1.164}{0.000375} + \frac{6.156}{0.00025} = 27.728\ MPa;$$

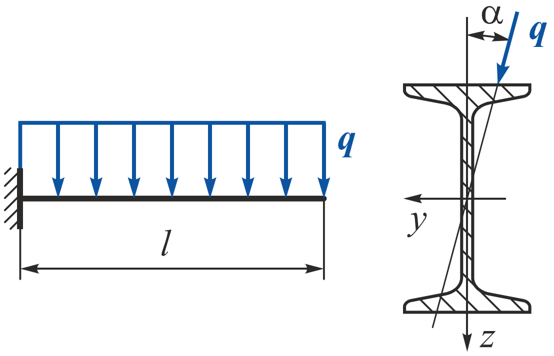
at the section with $x_e = 0.805\ m$:

$$M_{bending} = \sqrt{M_y^2 + M_z^2} = \sqrt{4.541^2 + 3.304^2} = 5.616\ kN{\cdot}m;$$

$$\sigma_{x\,max} = \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{4.541}{0.000375} + \frac{3.304}{0.00025} = 25.325\ MPa.$$

Example 3.6

A cantilever beam of I-section with a length $l = 1\ m$ is subjected to a uniformly distributed (rectangular) load with an intensity of $q = 5\ kN/m$ (Fig. 3.41). The loading plane forms an angle of $\alpha = 15°$ with the web plane of the I-beam. Design the I-beam section if $[\sigma] = 160\ MPa$.



Given: $l = 1\ m$; $q = 10\ kN/m$;

$\alpha = 15°$; $[\sigma] = 160\ MPa$.

It is required to select the I-beam number.

Fig. 3.41

Solution

From Fig. 3.41 it is clear that the maximum bending moment will act at the fixed end:

$$M_{bending} = -\frac{qx^2}{2} = -5x \Big|_{x=0} = 0 \Big|_{x=l=1\ m} = -5\ kN{\cdot}m.$$

The strength condition in this case (a section with protruding corners) has the form

$$\sigma_{x\,max} = |M_{bending}|\left(\frac{\cos\alpha}{W_y} + \frac{\sin\alpha}{W_z}\right) = \frac{|M_{bending}|}{W_y}\left(\cos\alpha + \frac{W_y}{W_z}\sin\alpha\right) \leq [\sigma],$$

from which we get:

$$W_y \geq \frac{|M_{bending}|}{[\sigma]}\left(\cos\alpha + \frac{W_y}{W_z}\sin\alpha\right) = \frac{5\times10^3}{160\times10^6}\left(\cos15° + \frac{W_y}{W_z}\sin15°\right) =$$

$$=31.25\times10^{-6}\left(0.966 + \frac{W_y}{W_z}0.259\right) = 30.188\times10^{-6} + \frac{W_y}{W_z}8.094\times10^{-6}\ m^3.$$

The section will be designed using the method of successive approximations. The right side of this expression contains the ratio W_y/W_z, which varies from 6.12 (I-beam No. 10) to 14.07 (I-beam No. 60).

As a first approximation, we assume $W_y/W_z = 10$, then

$$W_y \geq 30.188\times10^{-6} + 10\cdot8.094\times10^{-6} = 111.128\times10^{-6}\ m^3.$$

From the steel section tables, we select I-beam No. 16, which has

$$W_y = 109\times10^{-6}\ m^3; \quad W_z = 14.5\times10^{-6}\ m^3.$$

Let us check its strength:

$$\sigma_{x\,max} = \frac{|M_{bending}|}{W_y}\left(\cos\alpha + \frac{W_y}{W_z}\sin\alpha\right) = \frac{5\times10^3}{109\times10^{-6}}\left(0.966 + \frac{109}{14.5}\cdot0.259\right) =$$

$$= 133.62\ MPa.$$

Underload is given by $\Delta\sigma\ \% = \left|\frac{133.88-160}{160}\right| = 16.325\ \%.$

Then, we select from the steel section tables I-beam No. 14, which has:

$$W_y = 81.7\times10^{-6}\ m^3; \quad W_z = 11.5\times10^{-6}\ m^3.$$

Let us check its strength

$$\sigma_{x\,max} = \frac{5\times10^3}{81.7\times10^{-6}}\left(0.966 + \frac{81.7}{11.5}\cdot0.259\right) = 171.73\ MPa.$$

Overload is given by $\Delta\sigma\ \% = \left|\frac{171.73-160}{160}\right| = 7.33\ \% > 5\ \%.$

Such an overload is not permissible; therefore, we finally select I-beam No. 16.

Let us compare the maximum stresses under oblique and plane bending (at $\alpha = 0$):

$$\sigma_{x\,max\ (\alpha=0)} = \frac{M_y}{W_y} = \frac{5\times10^3}{109\times10^{-6}} = 45.872\ MPa; \qquad \frac{\sigma_A}{\sigma_{\alpha=0}} = \frac{133.62}{45.872} = 2.91.$$

Thus, the maximum stresses under oblique bending are greater than under simple bending by a factor of 2.91.

3.5.3. Eccentric Tension-Compression

Eccentric tension-compression occurs in a bar's cross-sections, in the case when the bar is loaded by a force whose action line is parallel to the longitudinal axis of the bar but does not coincide with it.

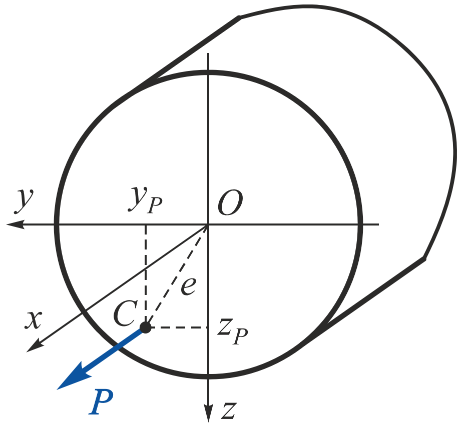


Fig. 3.42

Let a force P act on a bar, parallel to the longitudinal axis and applied at point C of the cross-section. The coordinates of this point in the principal axes system are denoted as y_P and z_P, and the distance from this point to the x-axis, which is called the eccentricity, is e (Fig. 3.42).

If the force P is transferred parallel to itself from point C to the centroid of the cross-section, then the eccentric tension can be represented as the sum of three simple deformations: tension and bending in two planes.

Then, in all cross-sections of the bar, the following internal forces and moments will act:

$$N_x = P;$$
$$M_y = P \cdot z_P; \qquad (3.40)$$
$$M_z = P \cdot y_P.$$

Thus, eccentric tension-compression can be considered, based on the principle of superposition, as the result of the combined action of ***pure tension-compression*** and ***oblique*** or ***plane pure bending***. That is, it is a special case of ***combined loading***.

Determining the acting stresses

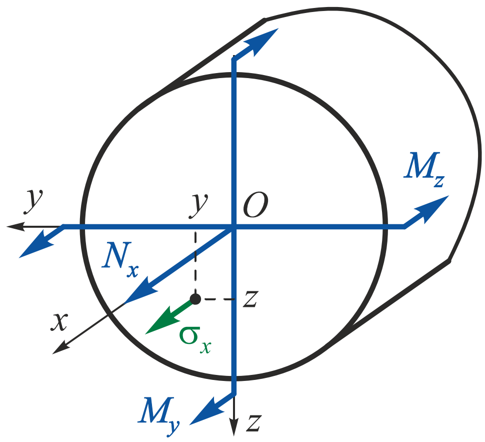


Fig. 3.43

In Fig. 3.43 a diagram of internal forces and moments action in the bar`s cross-section is shown. The absence of a shear force in the bar's section means that ***pure bending*** (***plane or oblique***) occurs. This removes the limitations that must be satisfied in the case of ***transverse oblique bending***.

Using the principle of superposition, we determine the normal stresses at an arbitrary point of the cross-section located in the first quadrant, with coordinates $y,\ z$ (see Fig. 3.43). We choose the direction of the y and z axes (i.e. the position of the first quadrant) such that N_x, M_y, and M_z create positive stresses in this quadrant. Then:

$$\sigma_x = \sigma_x(N_x) + \sigma_x(M_y) + \sigma_x(M_z) = \frac{N_x}{F} + \frac{M_y}{I_y} z + \frac{M_z}{I_z} y, \tag{3.41}$$

where F is the area of the cross-section;

I_y, I_z are the inertia moments of the cross-section.

Substitute into this relation the values of N_x, M_y, and M_z from formulas (3.40):

$$\sigma_x = \frac{P}{F} + \frac{P z_P}{I_y} z + \frac{P y_P}{I_z} y = \frac{P}{F}\left(1 + \frac{z_P}{\frac{I_y}{F}} z + \frac{y_P}{\frac{I_z}{F}} y\right). \tag{3.42}$$

Let us introduce the notation:

$$i_y = \sqrt{\frac{I_y}{F}}, \qquad i_z = \sqrt{\frac{I_z}{F}}, \tag{3.43}$$

where i_y, i_z are the radii of gyration of the bar's cross-section relative to its principal central axes of inertia.

Taking into account relations (3.43), we rewrite expression (3.42) in the following form:

$$\sigma_{x_E} = \frac{P}{F}\left(1 + \frac{z_P}{i_y^2} z + \frac{y_P}{i_z^2} y\right), \tag{3.44}$$

where z_P, y_P are the coordinates of the P-force application point;

y, z are the coordinates of the point at which the stress is determined.

The obtained expression makes it possible to find the stress at any point of the cross-section in any quadrant.

Determining of the neutral axis position

To find the critical point (or points) of the section, it is necessary to determine the position of the neutral axis of the section.

The equation of the neutral axis is obtained from the condition $\sigma_x = 0$, by equating the right side of expression (3.44) to zero. Since $P/F \neq 0$, it follows that:

$$1 + \frac{z_P}{i_y^2} z + \frac{y_P}{i_z^2} y = 0. \tag{3.45}$$

This is the equation of a straight line not passing through the origin.

We transform equation (3.45) into the intercept form on the coordinate axes

$$\frac{z}{\left(-\frac{i_y^2}{z_P}\right)} + \frac{y}{\left(-\frac{i_z^2}{y_P}\right)} = 1 \tag{3.46}$$

or

$$\frac{z}{a_z} + \frac{y}{a_y} = 1. \tag{3.47}$$

Consequently, the intercepts cut off by the neutral axis on the y and z-axes can be determined from the relations

$$a_y = -\frac{i_z^2}{y_P}, \qquad a_z = -\frac{i_y^2}{z_P}. \tag{3.48}$$

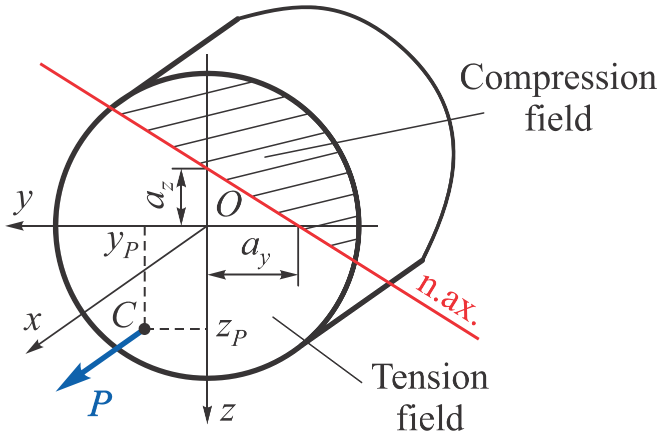


Fig. 3.44

From these relations, it follows that a_y and y_P, and a_z and z_P always have mutually opposite signs. That is, the point of application of the force (point C) and the neutral axis always lie on opposite sides of the centroid of the section (Fig. 3.44).

Let's consider some characteristic features related to the behaviour of the neutral axis with different positions of the P-force application point:

1. The neutral axis position does not depend on the magnitude or sign of force P;
2. Under eccentric tension-compression, the neutral axis can either intersect the cross-section or lie outside its boundaries. In the first case, stresses of different signs arise in the cross-section: one part of the section is in tension, and other part is in compression. In the second case, the stresses at all points of the cross-section will have the same sign.
3. The neutral axis position depends on the coordinates of load application point. The closer the force is applied to the centroid of the cross-section (the smaller the eccentricity), the farther away the neutral axis is located from it.
4. If force P is applied at a point on the y-axis ($z_P = 0$), then the neutral axis will be parallel to the z-axis, since $a_z = -i_y^2/z_P = -i_y^2/0 = -\infty$.

If force P is applied at a point on the z-axis ($y_P = 0$), then the neutral axis will be parallel to the y-axis, since $a_y = -\, i_z^2/y_P = -\, i_z^2/0 = -\infty$.

Strength condition for eccentric tension-compression

Expression (3.44) in the σ_x, y, z coordinate system represents the equation of a plane. Consequently, $\sigma_x = \sigma_{max}$ will occur at those points of the cross-section most distant from its neutral axis.

If the coordinates of the critical point are defined and equal to y^* and z^*, then the strength condition takes the form

$$\sigma_{x\,max} = \frac{P}{F}\left(1 + \frac{z_P}{i_y^2} z^* + \frac{y_P}{i_z^2} y^*\right) \leq [\sigma]. \qquad (3.49)$$

For cross-sections of complex shapes, the coordinates of the critical points can be determined by drawing tangents to the contour of the cross-section parallel to the neutral axis. The points of tangency whose distance to the neutral axis is maximal are the critical points of the cross-section.

Remark For cross-sections with protruding corners, where both principal axes of inertia are axes of symmetry (e.g., rectangular, box, I-beam, etc.), the critical points are located at the corners of these sections. That is, they can be found without determining the position of the neutral axis:

$$\sigma_{x\,max} = \frac{|N_x|}{F} + \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \leq [\sigma]. \qquad (3.50)$$

3.5.4. Cross-Section Core

When designing bars made of materials with poor resistance to tension (e.g., cast iron, brickwork, concrete), it is desirable to ensure that the entire cross-section works only in compression. This is achieved by preventing the P-force application point from moving too far from the centroid of the section, thereby limiting the eccentricity.

It is also desirable to know in advance what eccentricity may be allowed for a selected type of cross-section without risking the occurrence of opposite signs stresses in the section of the bar. To this end, it is necessary to establish the region of possible positions of the force application point, within which the stresses at all points of the cross-section will have the same sign. This region is called the ***core of the cross-section***.

The cross-section core is the region located in the vicinity of the cross-sectioncentroid, within which the application of a tensile or compressive force results in stresses of the same sign at all points of the cross-section.

From this definition, it follows that if a tensile or compressive force is applied on the boundary of the cross-section core, the neutral axis touches the contour of the cross-section.

To construct the boundary contour of the cross-section core, it is necessary to consider various positions of the neutral axis tangent to the cross-section contour and compute the coordinates of the corresponding P-force application points using formulas derived from relations (3.46):

$$y_P = -\frac{i_z^2}{a_y}; \qquad z_P = -\frac{i_y^2}{a_z}. \tag{3.51}$$

The calculated coordinates determine points lying on the boundary of the cross-section core.

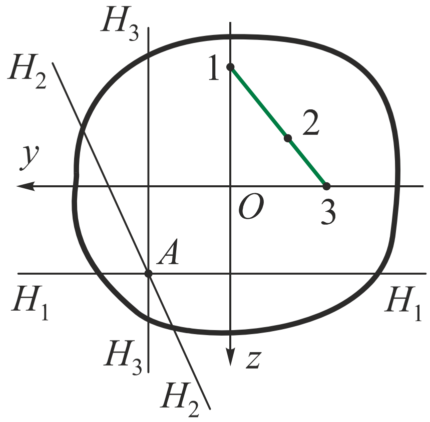


Fig. 3.45

To facilitate the construction of the cross-section core, we use a property of the neutral axis: when the neutral axis rotates around some fixed-point A on the section contour, the point of application of the force moves along a certain straight line (Fig. 3.45).

In Fig. 3.45 three positions of the P-force application point are shown on a certain line 1–3, and correspondingly, three positions of the neutral axis.

To prove this property, it is sufficient to substitute the coordinates of point A $(y_A,\ z_A)$, which lies on the neutral axis, into equation (3.45).

We obtain:

$$1 + \frac{z_P}{i_y^2} z_A + \frac{y_P}{i_z^2} y_A = 0. \tag{3.52}$$

Indeed, expression (3.52) for $z_A = const$ represents the equation of a straight line with respect to the coordinates of the points of force application P – $(y_P,\ z_P)$.

Thus, to construct the cross-section core for a given shape, it is necessary to draw several positions of the neutral axis that coincide with the section sides and also touch its protruding points. Then, the coordinates of the points lying on the boundary of the cross-section core are calculated.

3.5.5. Problem-Solving Examples

Example 3.7

For a bar with a rectangular cross-section (Fig. 3.46), find the acting stresses at the section characteristic points; construct the diagram of normal stress distribution in this section; and analytically determine the position of the neutral axis.

Given: $P = 40\ kN$; $b = 0.08\ m$; $h = 0.04\ m$.

It is necessary to:
1) determine the acting stresses at the section characteristic points;
2) construct the diagram of σ_x distribution;
3) determine the position of the neutral axis.

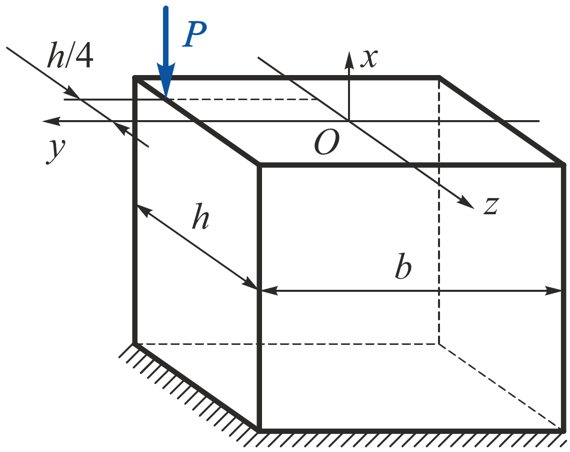


Fig. 3.46

Solution

1. We reduce the external force P to the cross-section centroid as a statically equivalent system (Fig. 3.47). Then, in an arbitrary cross-section of the bar, the following internal forces and moments will act:

$$|N_x| = P = 40\ kN;$$

$$|M_y| = P \cdot z_P = |-40 \cdot 0.01| = 0.4\ kN{\cdot}m;$$

$$|M_z| = P \cdot y_P = 40 \cdot 0.04 = 1.6\ kN{\cdot}m,$$

where

$$y_P = \frac{b}{2} = \frac{0.08}{2} = 0.04\ m;$$

$$z_P = -\frac{h}{4} = -\frac{0.04}{4} = -0.01\ m.$$

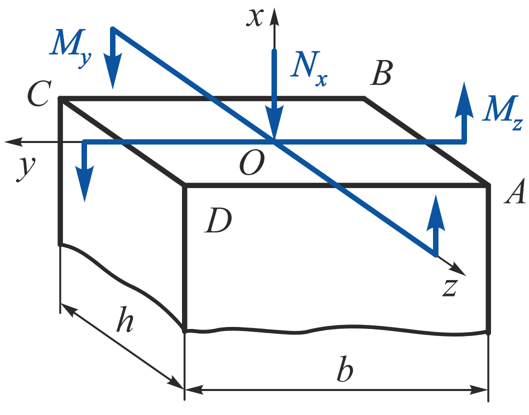


Fig. 3.47

2. Let us determine the stresses in the corner points of the cross-section using formula (3.48) and construct the distribution diagram of the normal stresses acting in the section (Fig. 3.48):

$$\sigma_{x\,max} = \sigma_x(N_x) + \sigma_{x\,max}(M_y) + \sigma_{x\,max}(M_z) = \frac{|N_x|}{F} + \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z},$$

where $F = bh = 0.08 \cdot 0.04 = 0.0032\ m^2$;

$$W_y = \frac{bh^2}{6} = \frac{0.08 \cdot 0.04^2}{6} = 21.33 \times 10^{-6}\ m^3;$$

$$W_z = \frac{hb^2}{6} = \frac{0.04 \cdot 0.08^2}{6} = 42.67 \times 10^{-6}\ m^3;$$

$$\sigma_x(N_x) = \frac{|N_x|}{F} = \frac{40 \cdot 10^3}{0.0032} = 12.5\ MPa;$$

$$\sigma_{x\,max}(M_y) = \frac{|M_y|}{W_y} = \frac{0.4 \times 10^3}{21.33 \times 10^{-6}} = 18.753\ MPa;$$

$$\sigma_{x\,max}(M_z) = \frac{|M_z|}{W_z} = \frac{1{,}6 \times 10^3}{42.67 \times 10^{-6}} = 37.497\ MPa.$$

Then

$$\sigma_A = -\sigma(N_x) + \sigma(M_y) + \sigma(M_z) = -12.5 + 18.753 + 37.497 = 43.75\ MPa;$$

$$\sigma_B = -\sigma(N_x) - \sigma(M_y) + \sigma(M_z) = -12.5 - 18.753 + 37.497 = 6.244\ MPa;$$

$$\sigma_C = -\sigma(N_x) - \sigma(M_y) - \sigma(M_z) = -12{,}5 - 18.753 - 37.497 = -68.75\ MPa;$$

$$\sigma_D = -\sigma(N_x) + \sigma(M_y) - \sigma(M_z) = -12.5 + 18.753 - 37.497 = -31.244\ MPa.$$

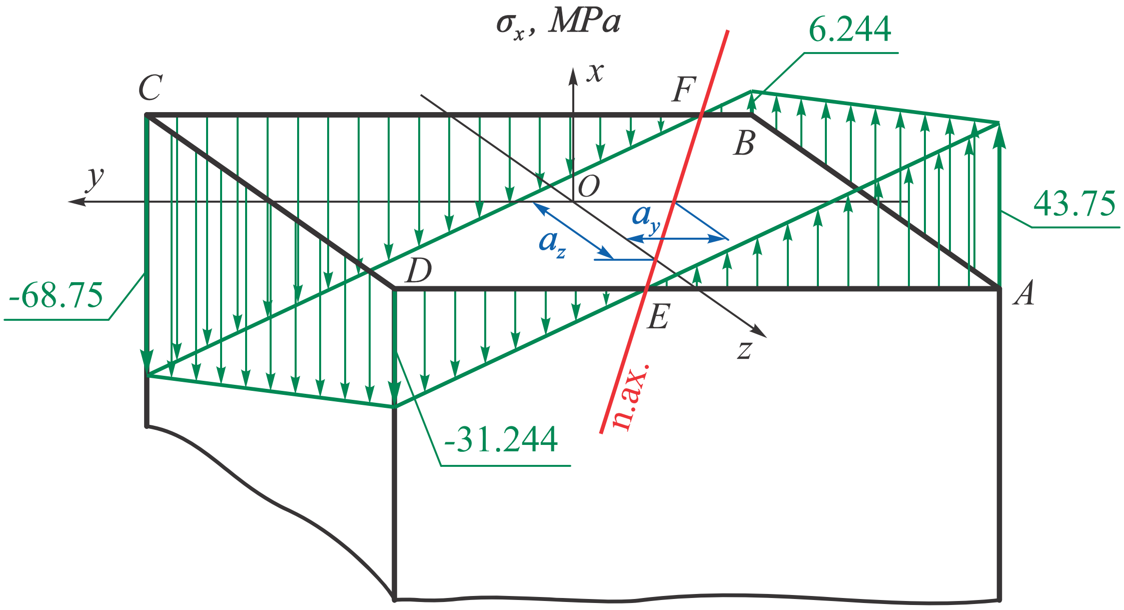


Fig. 3.48

Let us draw the neutral axis through the intersection points of the normal stresses diagram with the cross-sectional plane (points E and F). From Fig. 3.48, it is clear that the force application point and the neutral axis lie on opposite sides of the centroid of the cross-section.

3. We analytically determine the neutral axis position, i.e., the intercepts cut by the neutral axis on the y and z axes, using relations (3.48), and we compare them with the intercepts obtained by the graphical method (Fig. 3.48):

$$a_y = -\frac{i_z^2}{y_P} = -\frac{(2.309 \times 10^{-2})^2}{0.04} = -0.01333\ m;$$

$$a_z = -\frac{i_y^2}{z_P} = -\frac{(1.155 \times 10^{-2})^2}{-0.01} = 0.01334\ m,$$

where

$$I_y = \frac{bh^3}{12} = \frac{0.08 \cdot 0.04^3}{12} = 42.67 \times 10^{-8}\ m^4;$$

$$I_z = \frac{hb^3}{12} = \frac{0.04 \cdot 0.08^3}{12} = 170.67 \times 10^{-8}\ m^4;$$

$$i_y = \sqrt{\frac{I_y}{F}} = \sqrt{\frac{42.67 \times 10^{-8}}{0.0032}} = 1.155 \cdot 10^{-2}\ m;$$

$$i_z = \sqrt{\frac{I_z}{F}} = \sqrt{\frac{170.67 \times 10^{-8}}{0.0032}} = 2.309 \times 10^{-2}\ m.$$

Example 3.8

A crack appeared on the edge of a steel strip loaded with a tensile force (Fig. 3.49). To prevent the crack from propagating, a fillet was milled out in its place. Determine the amount by which the stress in the strip increased as a result. Neglect stress concentration.

Given: $P = 40\ kN$; $b = 50\ mm$; $t = 8\ mm$; $a = 5\ mm$.

It is necessary to compare the maximum stresses acting in the original strip and the strip with the fillet.

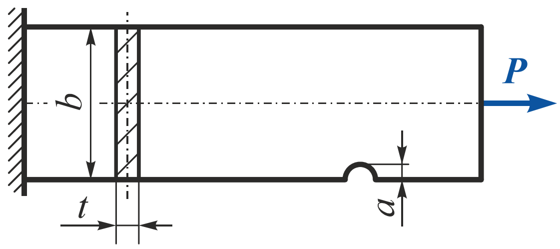


Fig. 3.49

Solution

In the cross-sections of the strip without the fillet, pure tension is realized. Therefore, the normal stresses are uniformly distributed across the section and are determined from the relation:

$$\sigma_x = \frac{P}{bt} = \frac{40 \times 10^3}{50 \times 10^{-3} \cdot 8 \times 10^{-3}} = 100\ MPa.$$

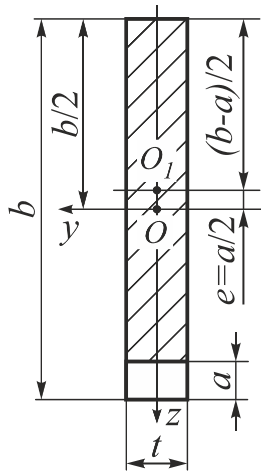


Fig. 3.50

In the section with the fillet, eccentric tension is realized. That is, the axial force and the bending moment act:

$$N_x = P,$$

$$M_y = Pe = \left\{\text{since } \ e = \frac{b}{2} - \frac{b-a}{2} = \frac{a}{2}\right\} = P\frac{a}{2},$$

where e is the eccentricity of the force application (Fig. 3.50).

In Fig. 3.50:
O is the P-force application point;
O_1 is the centroid of the cross-section with the fillet.

The maximum normal stress will act in the lower part of the cross-section with the fillet:

$$\sigma_{x\,max} = \frac{P}{F} + \frac{M_y}{W_y} = \frac{P}{t(b-a)} + \frac{P(a/2)\cdot 6}{t(b-a)^2} = \frac{P}{t(b-a)}\left(1 + \frac{3a}{b-a}\right) =$$

$$= \frac{40\times 10^3}{8\times 10^{-3}\cdot(50\times 10^{-3} - 5\times 10^{-3})}\left(1 + \frac{3\cdot 5\times 10^{-3}}{50\times 10^{-3} - 5\times 10^{-3}}\right) = 148.15\ MPa.$$

The stress increased by the value:

$$\Delta\sigma_x = \frac{\sigma_{x\,max} - \sigma_x}{\sigma_x} = \frac{148.15 - 100}{100}\cdot 100\ \% = 48.15\ \%.$$

Remark If the same fillet is cut out symmetrically from the opposite side of the strip, then central (axial) tension will occur in this section. Taking into account the reduction in the cross-sectional area, we get:

$$\sigma_x = \frac{P}{t(b-2a)} = \frac{40\times 10^3}{8\times 10^{-3}\cdot(50\times 10^{-3} - 2\cdot 5\times 10^{-3})} = 125\ MPa.$$

Let us consider the distribution of normal stresses along the height of the section with the fillet:

$$\sigma_x = \frac{P}{F} + \frac{M_y}{I_y}z = \frac{P}{(b-a)t} + \frac{Pa\cdot 12}{2t(b-a)^3}\cdot z = \frac{P}{t(b-a)}\left(1 + \frac{6a}{(b-a)^2}\cdot z\right) =$$

$$= 111.11\times 10^6\cdot(1 + 14.815\cdot z)\ \Big|_{z_1} = 148.15\ MPa\ \Big|_{z_2} = 74.07\ MPa,$$

where $z_1 = \dfrac{b-a}{2} = \dfrac{50\times 10^{-3} - 5\times 10^{-3}}{2} = 22.5\cdot 10^{-3}\ m;$

$z_2 = -z_1 = -22.5\times 10^{-3}\ m.$

The diagrams of the normal stress distribution along the height for the section without the fillet and with the fillet are shown in Fig. 3.51.

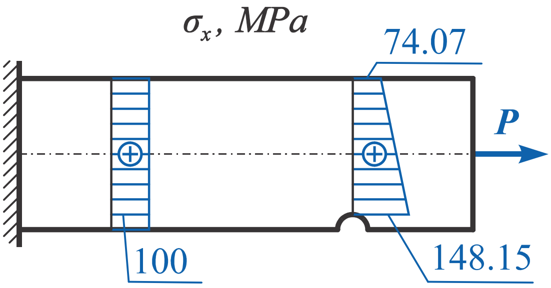


Fig. 3.51

Example 3.9

Determine the cross-section dimension h of the clamp (Fig. 3.52) if the compression force on the parts is $P = 10\ kN$.

Given: $P = 10\ kN$; $a = 80\ mm$;

$b = 14\ mm$; $[\sigma] = 100\ MPa$.

It is necessary to determine h.

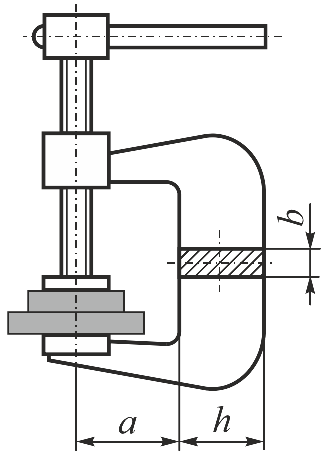


Fig. 3.52

Solution

In the cross-section of the clamp, eccentric tension occurs (Fig. 3.53), for which the strength condition has the form:

$$\sigma_{x\,max} = \frac{N_x}{F} + \frac{M_y}{W_y} \le [\sigma],$$

where $F = bh$ is the cross-sectional area;

$N_x = P$ is the longitudinal force;

$M_y = Pe = P(a + h/2)$ is the bending moment relative to the y-axis;

$W_y = bh^2/6$ is the section modulus relative to the y-axis.

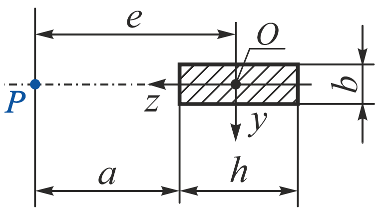


Fig. 3.53

Then

$$\sigma_{x\,max} = \frac{P}{bh} + \frac{P(a + h/2) \cdot 6}{bh^2} = \frac{P}{bh} + \frac{6Pa}{bh^2} + \frac{3P}{bh} = \frac{4P}{bh} + \frac{6Pa}{bh^2} \le [\sigma].$$

After transformations, we obtain a quadratic equation with respect to h:

$$[\sigma]bh^2 - 4Ph - 6Pa = 0.$$

Its solution is

$$h \geq \frac{-(-4P) \pm \sqrt{(4P)^2 - 4 \cdot [\sigma] \cdot b \cdot (-6Pa)}}{2 \cdot [\sigma] \cdot b} = \frac{40 \times 10^3}{2 \cdot 100 \times 10^6 \cdot 14 \times 10^{-3}} \pm$$

$$\pm \frac{\sqrt{(4 \cdot 10 \times 10^3)^2 + 4 \cdot 100 \times 10^6 \cdot 14 \times 10^{-3} \cdot 6 \cdot 10 \times 10^3 \cdot 80 \times 10^{-3}}}{2 \cdot 100 \times 10^6 \cdot 14 \times 10^{-3}} =$$

$$= \frac{40 \times 10^3 \pm 168 \times 10^3}{2.8 \times 10^6} = 14.286 \times 10^{-3} \pm 60 \times 10^{-3}.$$

From which we get:

$$h \geq 74.56 \times 10^{-3} \; m.$$

Let us compare the contribution of bending and tension to the total stress:

$$\sigma_{x\,max} = \sigma_{x\,max}(N_x) + \sigma_{x\,max}(M_y) = \frac{P}{bh} + \frac{P\left(a + \frac{h}{2}\right) \cdot 6}{bh^2} =$$

$$= \frac{10 \times 10^3}{14 \times 10^{-3} \cdot 74.56 \times 10^{-3}} + \frac{10 \times 10^3 \cdot \left(80 \times 10^{-3} + \frac{74.56 \times 10^{-3}}{2}\right) \cdot 6}{14 \times 10^{-3} \cdot (74.56 \times 10^{-3})^2} =$$

$$= 9.58 \times 10^6 + 91.41 \times 10^6 = 99.99 \; MPa;$$

$$\frac{\sigma_{x\,max}(N_x)}{\sigma_{x\,max}} = \frac{9.58}{99.99} = 0.096; \qquad \frac{\sigma_{x\,max}(M_y)}{\sigma_{x\,max}} = \frac{90.41}{99.99} = 0.904.$$

Thus, $\sigma_{x\,max}(M_y)$ exceeds $\sigma_{x\,max}(N_x)$ by a factor of 9.42.

Remark The contribution of tensile stresses from the action of the bending moment M_y can be reduced by decreasing the eccentricity e. In practice, T or I-beam cross-sections are typically used, which shifts the centroid of the section O closer to the P-force action line and places more material in the region of tensile stresses, to which brittle materials are more sensitive (Fig. 3.54).

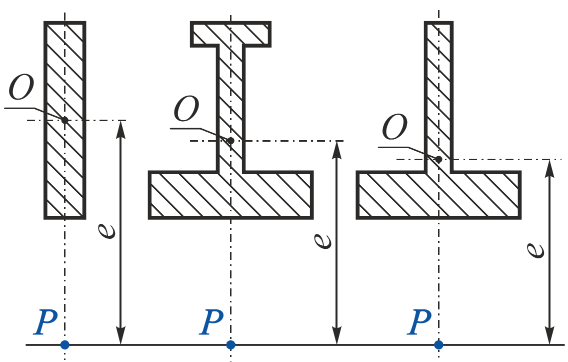


Fig. 3.54

Example 3.10

Construct the section core for a circular cross-section with diameter d (Fig. 3.55).

Solution

If the application point of the tensile or compressive stresses lies on the boundary of the cross-section core, then the neutral axis touches the cross-section without intersecting it.

Assume that the neutral axis $1-1$ is tangent to the circle at point A and is parallel to the z-axis (see Fig. 3.55). In this case, the intercepts cut by the neutral axis on the coordinate axes are:

$$a_y = -r = -\frac{d}{2}; \qquad a_z = \infty.$$

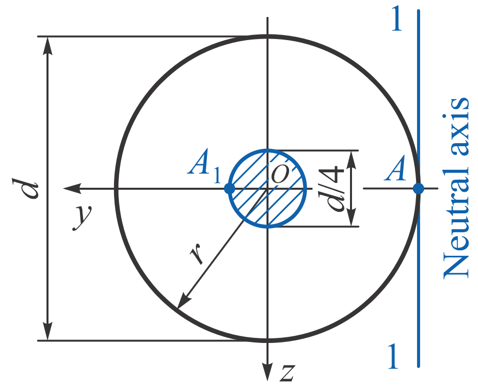


Fig. 3.55

The coordinates of the tensile or compressive force application point are determined by formulas (3.51):

$$y_{я\,(A_1)} = y_P = -\frac{i_z^2}{a_y} = -\frac{I_z}{Fa_y} = -\frac{\pi d^4}{64}\frac{4}{\pi d^2}\left(-\frac{2}{d}\right) = \frac{d}{8} = \frac{r}{4};$$

$$z_{я\,(A_1)} = z_P = -\frac{i_y^2}{a_z} = -\frac{I_y}{Fa_z} = -\frac{\pi d^4}{64}\frac{4}{\pi d^2}\left(\frac{1}{\infty}\right) = 0,$$

where F is the cross-sectional area;

i_y, i_z are the radii of gyration relative to the y and z-axes;

I_y, I_z are the moments of inertia relative to the y and z-axes.

Thus, in order for the neutral axis to touch the cross-section at point A, it is necessary that the tensile or compressive force be applied at point $A_1\{r/4\,;0\}$.

Due to the symmetry of the cross-section with respect to any axes passing through the geometric centre of the circle, it follows that for other positions of the neutral axis on the circumference of diameter d, the points of the cross-section core form a concentric circle with a diameter of $d/4$.

Example 3.11

Construct the section core for a rectangular cross-section with side dimensions $b \times h$ (Fig. 3.56).

Solution

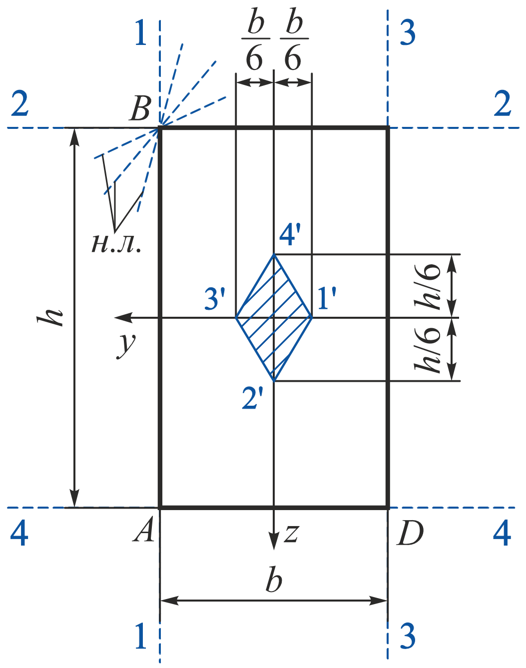


Fig. 3.56

We align the neutral axis with the side of the rectangle AB (position 1– 1). Then, the intercepts cut by the neutral axis on the coordinate axes are:

$$a_y = \frac{b}{2}; \qquad a_z = \infty.$$

According to equations (3.51), the coordinates of the corresponding point $(1')$ of the cross-section core are:

$$y_{(1')} = y_P = -\frac{i_z^2}{a_y} = -\frac{I_z}{Fa_y} = -\frac{hb^3}{12}\frac{1}{bh}\left(\frac{2}{b}\right) = -\frac{b}{6};$$

$$z_{(1')} = z_P = -\frac{i_y^2}{a_z} = -\frac{I_y}{Fa_z} = -\frac{bh^3}{12}\frac{1}{bh}\left(\frac{1}{\infty}\right) = 0,$$

where F is the cross-sectional area;
i_y, i_z are the radii of gyration relative to the y and z-axes;
I_y, I_z are the moments of inertia relative to the y and z-axes.

Now, let's align the neutral axis with the side BC (position 2– 2). Then, the intercepts cut by the neutral axis on the coordinate axes are:

$$a_y = \infty; \qquad a_z = -\frac{h}{2},$$

and the coordinates of the corresponding point $(2')$ of the cross-section core will have the values:

$$y_{(2')} = y_P = -\frac{i_z^2}{a_y} = -\frac{I_z}{Fa_y} = -\frac{hb^3}{12}\frac{1}{bh}\left(\frac{1}{\infty}\right) = 0;$$

$$z_{(2')} = z_P = -\frac{i_y^2}{a_z} = -\frac{I_y}{Fa_z} = \frac{bh^3}{12}\frac{1}{bh}\left(-\frac{2}{h}\right) = \frac{h}{6}.$$

In a similar way, determine the coordinates of points $3'$ and $4'$ corresponding to neutral axis positions $3-3$ and $4-4$.

To construct the cross-section core, we use the following property of the neutral axis: when the neutral axis is rotated about some fixed point on the cross-section contour, the force application point moves along a straight line. In this case, when the neutral axis is rotated about the fixed point B (dashed lines in Fig. 3.56), the P-force application point moves along the straight line passing through points $1'$ and $2'$.

By connecting points $1', 2', 3'$, and $4'$ with straight lines, we get the contour of the cross-section core in the form of a rhombus with diagonals equal to $h/3$ and $b/3$.

Therefore, in a rectangular cross-section under eccentric tension or compression, the stress will be of a single sign if the force application point does not lie outside the middle third of the section side.

Let us consider a special case of eccentric compression when one of the eccentricities is zero ($z_P = 0$, $y_P = e$). Show the diagrams of normal stress distribution for a rectangular section with an eccentricity e that is zero, less than, equal to, and greater than one-sixth of the section width (Fig. 3.57).

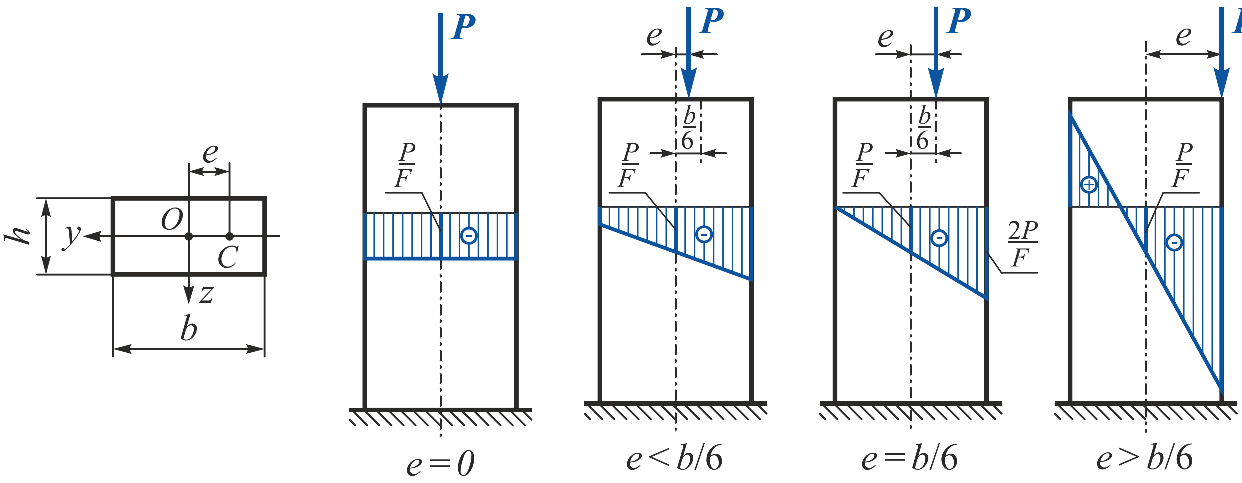


Fig. 3.57

From Fig. 3.57 it is evident that, for all positions of the force P, the stress at the centroid (point O) is the same and equal to P/F.

Example 3.12

Construct the cross-section core of a Channel section No. 10 (Fig. 3.58).

Solution

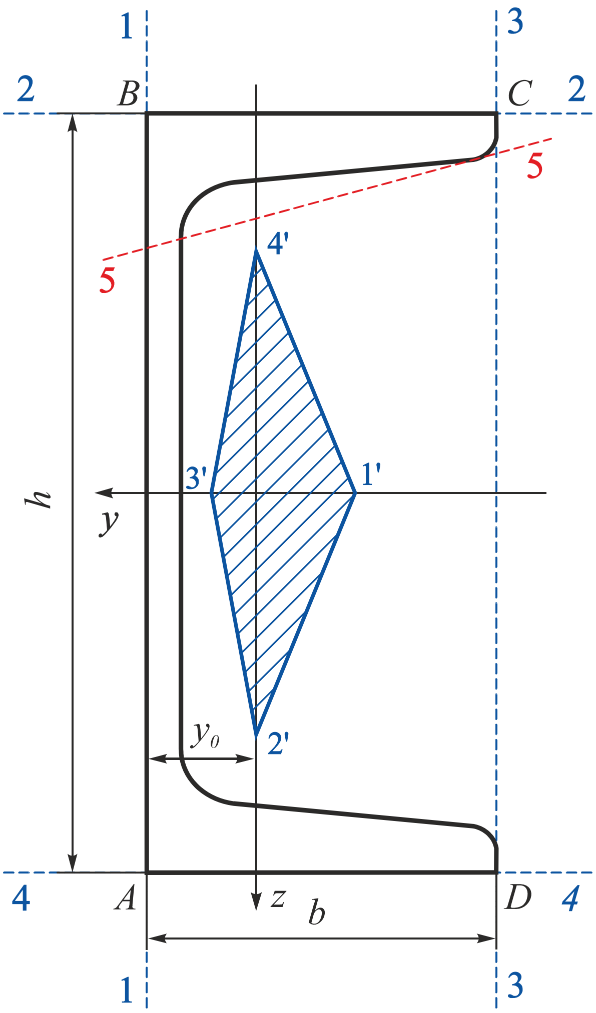


Fig. 3.58

1. From the steel section tables, let us write out all the geometric characteristics for channel section No. 10:

$h = 10 \times 10^{-2}\ m;$

$b = 4.6 \times 10^{-2}\ m;$

$i_y = 3.99 \times 10^{-2}\ m;$

$i_z = 1.37 \times 10^{-2}\ m;$

$y_0 = 1.44 \times 10^{-2}\ m.$

2. We make a drawing of the channel section to scale.

3. We align the neutral axis with the side AB (position $1-1$). Then we determine the intercepts that the neutral axis cuts on the coordinate axes:

$a_y = y_0 = 1.44 \times 10^{-2}\ m;$

$a_z = \infty.$

According to expression (3.51), the coordinates of the corresponding point $(1')$ of the cross-section core are:

$$y_{(1')} = y_P = -\frac{i_z^2}{a_y} = -\frac{(1.37 \times 10^{-2})^2}{1.44 \times 10^{-2}} = -1.30 \times 10^{-2}\ m;$$

$$z_{(1')} = z_P = -\frac{i_y^2}{a_z} = -\frac{(3.99 \times 10^{-2})^2}{\infty} = 0,$$

where i_y, i_z are the radii of gyration relative to the y and z-axes.

4. Now we align the neutral axis with the side BC (position $2-2$). We get the intercepts that the neutral axis cuts on the coordinate axes:

$$a_y = \infty; \qquad a_z = -\frac{h}{2} = -\frac{10 \times 10^{-2}}{2} = -5 \times 10^{-2}\ m,$$

and the corresponding point coordinates $(2')$ of the section core will have the values:

$$y_{(2')} = y_P = -\frac{i_z^2}{a_y} = -\frac{(1.37 \times 10^{-2})^2}{\infty} = 0;$$

$$z_{(2')} = z_P = -\frac{i_y^2}{a_z} = -\frac{(3.99 \times 10^{-2})^2}{-5 \times 10^{-2}} = 3.18 \times 10^{-2}\ m.$$

5. Now we align the neutral axis with the side CD (position 3– 3). Next, let us determine the intercepts cut off by the neutral axis on the coordinate axes:

$$a_y = -(b - y_0) = -(4.6 \times 10^{-2} - 1.44 \times 10^{-2}) = -3.16 \times 10^{-2}\ m; \qquad a_z = \infty,$$

and the corresponding point coordinates $(3')$ of the section core will have the values:

$$y_{(3')} = y_P = -\frac{i_z^2}{a_y} = -\frac{(1.37 \times 10^{-2})^2}{-3.16 \times 10^{-2}} = 0.59 \times 10^{-2}\ m;$$

$$z_{(3')} = z_P = -\frac{i_y^2}{a_z} = -\frac{(3.99 \times 10^{-2})^2}{\infty} = 0,$$

6. Now we align the neutral axis with the side DA (position 4– 4). Then the intercepts that the neutral axis cuts on the coordinate axes are:

$$a_y = \infty; \qquad a_z = \frac{h}{2} = \frac{100 \times 10^{-2}}{2} = 50 \times 10^{-2}\ m,$$

and the corresponding point coordinates $(4')$ of the section core will have the values:

$$y_{(4')} = y_P = -\frac{i_z^2}{a_y} = -\frac{(1.37 \times 10^{-2})^2}{\infty} = 0;$$

$$z_{(4')} = z_P = -\frac{i_y^2}{a_z} = -\frac{(3.99 \times 10^{-2})^2}{5 \times 10^{-2}} = -3.18 \times 10^{-2}\ m.$$

By connecting points $1'$, $2'$, $3'$ and $4'$ with straight lines, we obtain the contour of the section core, which is a quadrilateral that is asymmetrical with respect to the z-axis.

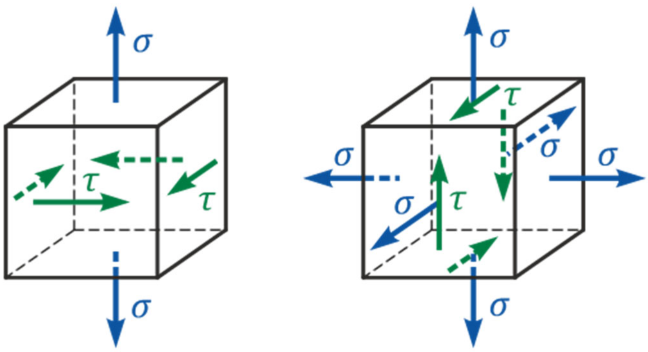
Remark

1. The position and shape of the cross-section core depend only on the shape and dimensions of the cross-section, but not on the magnitude of the applied force.

2. The neutral axis, when rolled along the cross-section contour, has to avoid intersecting the section at any position (position 5– 5 in Fig. 3.58 is unacceptable).

4. SOLVING NONSTANDARD PROBLEMS

Example 4.1



Case A Case B

Fig. 4.1

Determine, according to the ***maximum shear stress theory***, which of the stress states (Fig. 4.1, Cases A and B) is more dangerous if $\sigma = \tau$.

Solution

Using the principle of superposition (principle of force action independence), we reduce the given stress state of the elements to the following forms:

Case A – to a triaxial stress state with the components of the principal normal stresses (Fig. 4.2):

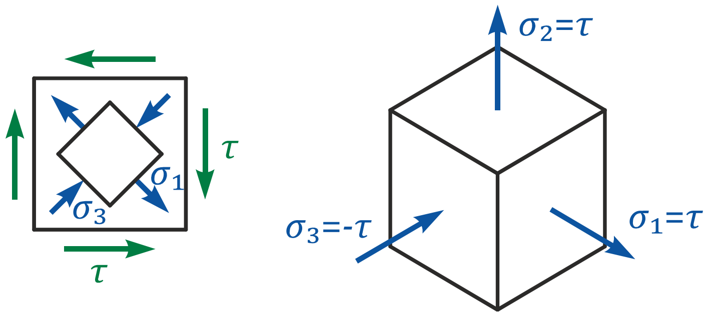


Fig. 4.2

$$\sigma_1 = \tau,$$

$$\sigma_2 = \tau,$$

$$\sigma_3 = -\tau.$$

In doing so:

$$\sigma_{eq}^{\text{III}} = \sigma_1 - \sigma_3 = 2\tau.$$

Case B – to hydrostatic tension and pure shear (Fig. 4.3):

$$\sigma_1 = \tau, \qquad \sigma_3 = -\tau.$$

Since no shear stresses arise on any of the cutting planes under hydrostatic tension, in this case as well:

$$\sigma_{eq}^{\text{III}} = \sigma_1 - \sigma_3 = 2\tau.$$

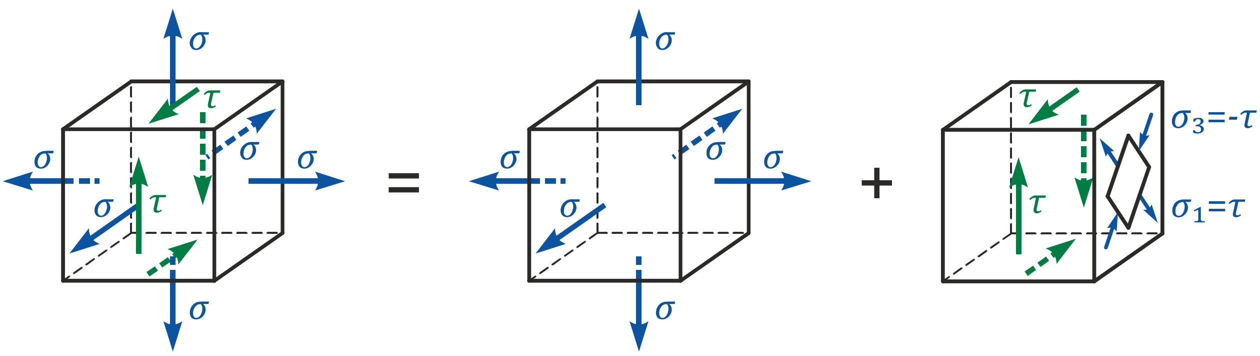


Fig. 4.3

Thus, according to the maximum shear stress theory, stress states A and B are equally critical.

Example 4.2

The stress state shown in Fig. 4.4 is supplemented by a hydrostatic compression (stresses on the invisible faces are not shown). As a result, all the potential strain energy is related only to the change in shape. Determine the safety factor with respect to yielding stress of $\sigma_{yield} = 240\ MPa$. Use the maximum shear stress theory.

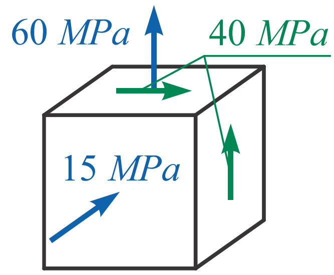


Fig. 4.4

Solution

Since the element volume (Fig. 4.5) does not change, then

$$\sigma_x + \sigma_y + \sigma_z = -(15 + \sigma) - \sigma + (60 - \sigma) = 0.$$

Hence

$$-3\sigma + 45 = 0;$$
$$\sigma = 15\ MPa.$$

Then

$$\sigma_x = -30\ MPa;\quad \sigma_y = -15\ MPa;\quad \sigma_z = 45\ MPa.$$

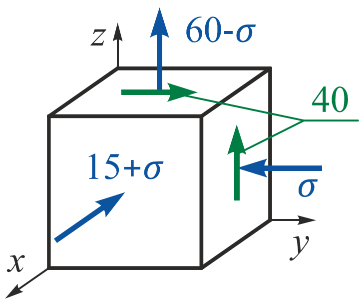


Fig. 4.5

Let us find the principal stresses:

$$\sigma_{1,\,2,\,(3)} = \frac{\sigma_y + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_y - \sigma_z}{2}\right)^2 + \tau_{zx}^2} = \frac{-15 + 45}{2} \pm \sqrt{\left(\frac{-15 - 45}{2}\right)^2 + 40^2} =$$
$$= 15 \pm 50;$$

$$\sigma_1 = 65\ MPa;\qquad \sigma_2 = -30\ MPa;\qquad \sigma_3 = -35\ MPa.$$

The equivalent stress according to the maximum shear stress theory is

$$\sigma_{eq}^{III} = \sigma_1 - \sigma_3 = 65 - (-35) = 100\ MPa.$$

Thus, the safety factor with respect to yielding stress is

$$n_{yield} = \frac{\sigma_{yield}}{\sigma_{eq}^{III}} = \frac{240}{100} = 2.4.$$

Example 4.3

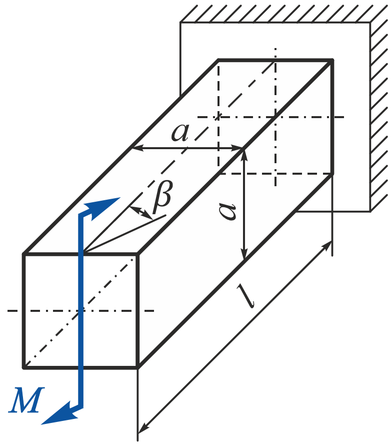


Fig. 4.6

A **cantilever bar** with a square cross-section of side a and length l is loaded by a moment M (Fig. 4.6). Which position of the moment M (angle β) is the most critical according to the strength condition? Use the maximum shear stress theory for the solution.

Solution

Let us decompose the moment M into its components (Fig. 4.7):

$$M_y = M\cos\beta \quad \text{и} \quad M_x = M\sin\beta.$$

In the most critical point of the cross-section (A), the following normal and shear stresses act (see Fig. 4.7):

$$\sigma_{x\,(A)} = \frac{M_y}{W_y} = \left\{\text{since} \;\; W_y = \frac{a^3}{6}\right\} = \frac{6M\cos\beta}{a^3};$$

$$\tau_{(A)} = \frac{M_x}{W_{torsional}} = \{\text{since} \;\; W_{torsional} = 0.208a^3\} = \frac{M\sin\beta}{0.208a^3}.$$

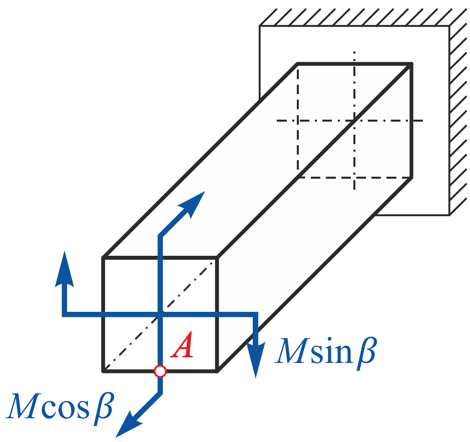


Fig. 4.7

The equivalent stress at point A:

$$\sigma^{\text{III}}_{eq\,(A)} = \sqrt{\sigma^2 + 4\tau^2} =$$

$$= \frac{M}{a^3}\sqrt{36\cos^2\beta + 4\frac{\sin^2\beta}{0.208^2}}.$$

From the condition

$$\frac{d\sigma^{\text{III}}_{eq\,(A)}}{d\beta} = 0$$

we obtain $\beta = 0$ and $\beta = \pi/2$.

At $\quad \beta = 0$: $\quad \sigma^{\text{III}}_{eq\,(A)} = 6\frac{M}{a^3}.$

At $\quad \beta = \pi/2$: $\quad \sigma^{\text{III}}_{eq\,(A)} = \frac{2M}{0.208a^3} = 9.615\frac{M}{a^3}.$

Thus, the most critical case according to the strength condition is pure torsion at $\beta = \pi/2$.

Example 4.4

Based on the condition of equally critical stress states, compare the weights of two bars (Fig. 4.8). Use the maximum shear stress theory.

Given: moment M, material of the bars is steel.

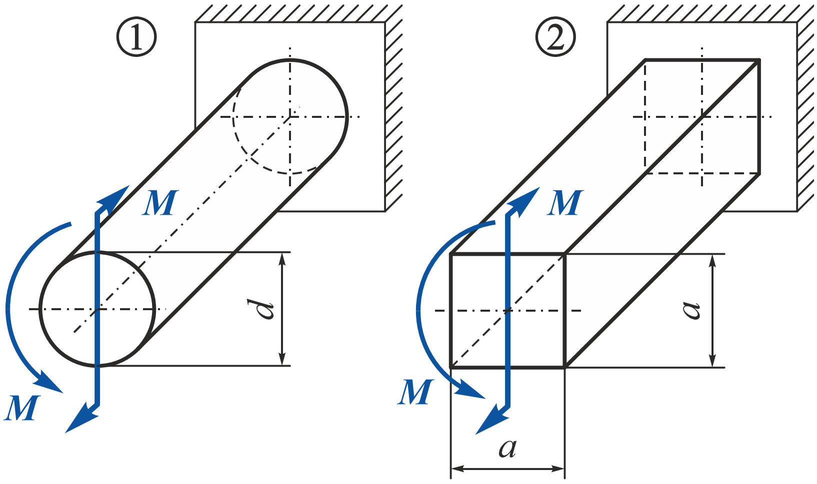


Fig. 4.8

Solution

The stress states at the critical points of the first and second bars are plane. The equivalent stresses in the first bar are:

$$\sigma_{eq\,(1)}^{\mathrm{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M}{W_{\mathrm{n.ax.}}}\right)^2 + 4\left(\frac{M}{W_\rho}\right)^2}\,.$$

Considering that for a circular cross-section:

$$W_{n.ax.} = \frac{\pi d^3}{32}, \quad W_\rho = \frac{\pi d^3}{16}, \quad \text{we get} \quad \sigma_{eq\,(1)}^{\mathrm{III}} = 14.405\frac{M}{d^3}.$$

The equivalent stresses in the second bar are:

$$\sigma_{eq\,(2)}^{\mathrm{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M}{W_{\mathrm{n.ax.}}}\right)^2 + 4\left(\frac{M}{W_{torsion}}\right)^2}\,.$$

Considering that for a square cross-section:

$$W_{n.ax.} = \frac{a^3}{6}, \quad W_{torsional} = 0.208a^3, \quad \text{we obtain} \quad \sigma_{eq\,(2)}^{\mathrm{III}} = 11.334\frac{M}{a^3}.$$

From the condition of equally critical stress states $\sigma_{eq\,(1)}^{\mathrm{III}} = \sigma_{eq\,(2)}^{\mathrm{III}}$, we obtain

$$14.405\frac{M}{d^3} = 11.334\frac{M}{a^3} \quad \Rightarrow \quad a = 0.923d.$$

Then the weight of the bar with circular cross-section amounts to

$$\frac{\pi d^2}{4 \cdot (0.923d)^2} = 0.922$$

of the weight of the bar with square cross-section.

Example 4.5

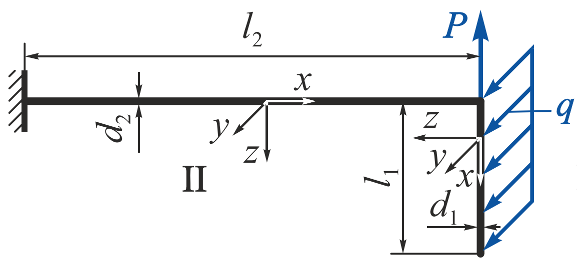


Fig. 4.9

A cranked bar is loaded with a rectangular distributed load q and a concentrated force P (Fig. 4.9). The values of q, the lengths of the segments l_1, $l_2 = 3l_1$, and the diameters of the cross-sections d_1, $d_2 = d_1\sqrt[3]{19}$ are known.

Determine the magnitude of the force P, if the maximum equivalent stresses (according to the maximum shear stress theory) are equal in the segments of the cranked bar.

Solution

At the end of the first segment, the maximum bending moment arises from the action of the distributed load

$$M_{z\,max}^{I} = \frac{ql_1^2}{2}.$$

The normal stresses acting in this cross-section are

$$\sigma_{max\,I} = \frac{M_{z\,max}^{I}}{W_{\text{n.ax.}}^{I}} = \left\{\text{taking into account}\quad W_{\text{n.ax.}}^{I} = \frac{\pi d_1^3}{32}\right\} = \frac{ql_1^2}{2}\cdot\frac{32}{\pi d_1^3} = \frac{16\cdot ql_1^2}{\pi d_1^3}.$$

At the second segment, there is a combined action of the bending moments M_y, M_z and the torsional moment M_x. At the end of the second segment:

$$M_{bending\ max} = \sqrt{M_{y\,max}^2 + M_{z\,max}^2} = \sqrt{(ql_1l_2)^2 + (Pl_2)^2}; \qquad M_x = \frac{ql_1^2}{2};$$

$$M_{design\,max}^{\text{III}} = \sqrt{M_{bending\,max}^2 + M_x^2} = \sqrt{(3\cdot ql_1^2)^2 + (3\cdot Pl_1)^2 + \left(\frac{ql_1^2}{2}\right)^2}.$$

The equivalent stresses acting in this cross-section:

$$\sigma_{max\,II} = \sigma_{eq\,\text{II}}^{\text{III}} = \frac{M_{design\,max}^{\text{III}}}{W_{\text{n.ax.}}^{II}},$$

where $W_{\text{n.ax.}}^{II} = \frac{\pi d_2^3}{32} = \frac{\pi\left(d_1\sqrt[3]{19}\right)^3}{32} = \frac{19\cdot\pi d_1^3}{32}.$

Using the condition of the problem $\sigma_{max\,I} = \sigma_{max\,II}$, after substitution and transformation, we obtain

$$P = 3ql_1.$$

Example 4.6

Determine the magnitude of force P for which the neutral axis in the bar cross-section at the fixed-end will coincide with the diagonal of the rectangular section $b \times h$ (Fig. 4.10).

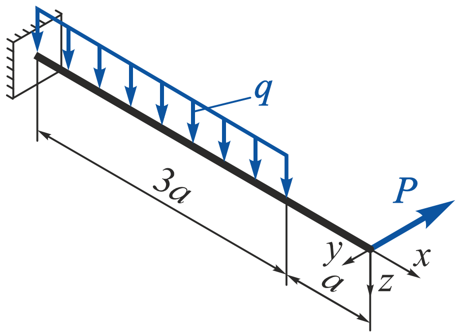


Fig. 4.10

Solution

The position of the neutral axis under oblique bending is determined by the formula

$$\operatorname{tg}\varphi = -\frac{I_y}{I_z}\operatorname{tg}\alpha = \left\{\text{taking into account that } \operatorname{tg}\alpha = \frac{M_z}{M_y}\right\} = -\frac{I_y}{I_z}\frac{M_z}{M_y}.$$

The bending moments in the support cross-section of the bar

$$M_y = -\frac{q(3a)^2}{2} = -\frac{9qa^2}{2}; \qquad M_z = 4Pa.$$

The axial inertia moments of the rectangular cross-section Осевые моменты инерции прямоугольного сечения

$$I_y = \frac{bh^3}{12}; \qquad I_z = \frac{hb^3}{12}.$$

According to the problem statement, the inclination angle of the neutral axis must coincide with the diagonal of the rectangular cross-section, i.e.:

$$\operatorname{tg}\varphi = \frac{h}{b}.$$

We equate the tangents of the neutral axis and rectangle's diagonal inclination angles. After substituting the expressions for the bending moments and moments of inertia, we get:

$$\frac{h}{b} = \frac{bh^3}{12} \cdot \frac{12}{hb^3} \cdot \frac{4Pa \cdot 2}{9qa^2};$$

$$\frac{h}{b} = \frac{h^2}{b^2} \cdot \frac{8P}{9qa}.$$

Finally

$$P = \frac{9bqa}{8h}.$$

Example 4.7

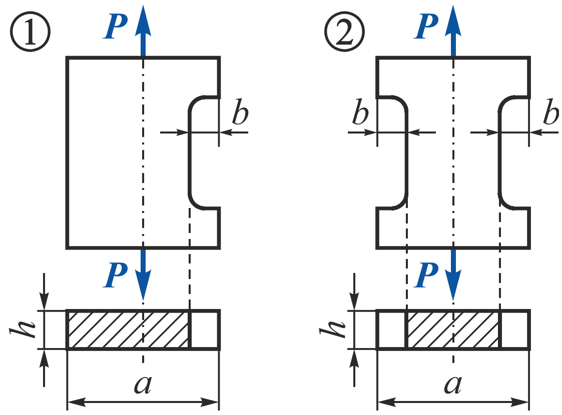


Fig. 4.11

At what $\lambda = b/a$ value will the maximum normal stress in the first bar become greater than in the second one (Fig. 4.11)? Neglect stress concentration.

Solution

The first bar is subjected to eccentric tension, while the second bar is under central (pure) tension.

In the weakened cross-section of the first bar, the force P is applied not at the centroid of the section, and therefore creates a bending moment $M_{bending} = P\,b/2$.

We write the expressions for determining the maximum normal stresses for the first and second bars, considering that $b = \lambda a$:

$$\sigma_{max}^{(1)} = \sigma(P) + \sigma(M_{bending}) = \frac{P}{F^{(1)}} + \frac{M_{bending}}{W_{\text{n.ax.}}} = \frac{P}{(a-b)h} + \frac{Pb \cdot 6}{2 \cdot h(a-b)^2} =$$

$$= \frac{P}{ah}\left(\frac{1}{1-\lambda} + \frac{3\lambda}{(1-\lambda)^2}\right);$$

$$\sigma_{max}^{(2)} = \sigma(P) = \frac{P}{F^{(2)}} = \frac{P}{(a-2b)h} = \frac{P}{ah}\left(\frac{1}{1-2\lambda}\right).$$

Let us consider the extreme case $\sigma_{max}^{(1)} = \sigma_{max}^{(2)}$, then

$$\frac{1}{1-\lambda} + \frac{3}{(1-\lambda)^2} = \frac{1}{1-2\lambda};$$

$$(1-\lambda)(1-2\lambda) + 3\lambda(1-2\lambda) - (1-\lambda)^2 = 0;$$

$$-5\lambda^2 + 2\lambda = 0.$$

The solution of this quadratic equation will be

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2/5 = 0.4.$$

The zero root is not admissible for physical reasons; therefore, at the ratio $\lambda = b/a \geq 2/5$ the maximum normal stress in the first bar becomes greater than in the second $\left(\sigma_{max}^{(1)} \geq \sigma_{max}^{(2)}\right)$.

Example 4.8

Determine at what k value the normal stress ratio on sections 1 and 2 will be the greatest (Fig. 4.12). Neglect stress concentration.

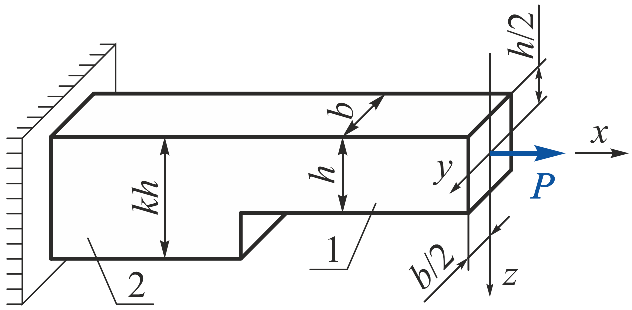


Fig. 4.12

Solution

The normal stresses in the thin part of the bar is

$$\sigma_{max}^{(1)} = \frac{P}{bh}.$$

The maximum normal stresses in the thickened part of the bar is

$$\sigma_{max}^{(2)} = \frac{P}{bkh} + \frac{P \cdot 0.5h(k-1)}{W_y} = \frac{P}{bkh} + \frac{P \cdot 0.5h(k-1) \cdot 6}{b(kh)^2} =$$

$$= \frac{P}{bh}\left(\frac{1}{k} + \frac{3(k-1)}{k^2}\right) = \frac{P}{bh}\frac{4k-3}{k^2}.$$

The ratio of the maximum normal stresses in segments 2 and 1 is

$$f(k) = \frac{\sigma_{max}^{(2)}}{\sigma_{max}^{(1)}} = \frac{4k-3}{k^2}.$$

To determine the extreme k value, we take the derivative of the function $f(k)$ and set it equal to zero:

$$\frac{df(k)}{dk} = 0.$$

After transformation, we obtain a quadratic equation:

$$-4k^2 + 6k = 0.$$

The solutions of this quadratic equation will be

$$k_1 = 0 \quad и \quad k_2 = \frac{3}{2} = 1.5.$$

The zero root is not suitable for physical reasons, therefore, at the ratio of $k = 3/2$ the maximum normal stress ratio on sections 2 and 1 will be the greatest:

$$f_{max}(k_2) = \frac{4 \cdot \frac{3}{2} - 3}{\left(\frac{3}{2}\right)^2} = \frac{12}{9} = \frac{4}{3}.$$

Example 4.9

How many times will the maximum stress increase in a square cross-section bar, fixed at one end and subjected to a tensile force P applied at the other end along its longitudinal axis, if the force, while remaining parallel to itself, moves: a) to the square side's midpoint and b) to the square corner's vertex? The self-weight is neglected.

Solution

Under axial tension, the normal stress in any point of the cross-section is

$$\sigma = \frac{P}{a^2},$$

where a is the square cross-section side length.

In a square, all centroidal axes are principal axes; therefore:

$$I = \frac{a^4}{12}.$$

When the force P is applied at the square side midpoint, the maximum normal stress is:

$$\sigma_{max\ (\mathrm{a})} = \frac{P}{a^2} + \frac{P \cdot \frac{a}{2}}{\frac{a^4}{12}} \cdot \frac{a}{2} = \frac{P}{a^2} + \frac{3P}{a^2} = \frac{4P}{a^2}.$$

Then

$$\frac{\sigma_{max\ (\mathrm{a})}}{\sigma} = \frac{4P}{a^2} \cdot \frac{a^2}{P} = 4,$$

that is, when the force is applied at the side midpoint of the square cross-section, the maximum normal stresses increase in 4 times.

When the force P is applied at the square corner point, the maximum normal stress is:

$$\sigma_{max\ (\mathrm{b})} = \frac{P}{a^2} + \frac{P \cdot \frac{a}{2}}{\frac{a^4}{12}} \cdot \frac{a}{2} + \frac{P \cdot \frac{a}{2}}{\frac{a^4}{12}} \cdot \frac{a}{2} = \frac{P}{a^2} + \frac{3P}{a^2} + \frac{3P}{a^2} = \frac{7P}{a^2}.$$

Then

$$\frac{\sigma_{max\ (\mathrm{b})}}{\sigma} = \frac{7P}{a^2} \cdot \frac{a^2}{P} = 7,$$

that is, when the force is applied at the vertex of the square cross-section, the maximum normal stresses increase in 7 times.

Example 4.10

For the bar shown in Fig. 4.13, determine and show the ***neutral surface (layer)*** position, if the weight of the bar is Q and the magnitudes of the forces are $P_1 = Q/2$ and $P_2 = Q/24$.

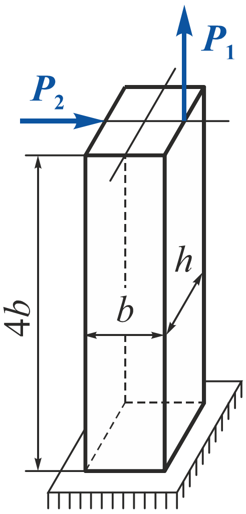


Fig. 4.13

Solution

In an arbitrary cross-section of the bar (Fig. 4.14) at a distance x, the following internal force and moment will act

$$N_x = P_1 - \frac{Qx}{6b} = \frac{Q}{2} - \frac{Qx}{6b} = \frac{Q}{2}\left(1 - \frac{x}{3b}\right);$$

$$M_y = P_1\frac{b}{2} - P_2 x = \frac{Q}{2}\cdot\frac{b}{2} - \frac{Q}{24}x = \frac{Q}{4}\left(b - \frac{x}{6}\right).$$

The normal stress at any arbitrary cross-section point is

$$\sigma = \frac{N_x}{F} + \frac{M_y}{I_y}z = \frac{Q}{2bh}\left(1 - \frac{x}{3b}\right) + \frac{Q}{4}\cdot\frac{12}{hb^3}\left(b - \frac{x}{6}\right)z.$$

At the ***neutral axis*** in an arbitrary section $\sigma = 0$, then

$$\frac{Q}{2bh}\left(1 - \frac{x}{3b}\right) + \frac{3Q}{hb^3}\left(b - \frac{x}{6}\right)z_{\text{n.ax.}} = 0.$$

After transformations we obtain

$$z_{\text{n.ax.}} = -b\frac{(3b - x)}{(6b - x)}.$$

To construct the neutral surface in the bar, we determine the position of the neutral axis in several cross-sections along the height of the bar (see Fig. 4.14):

at $x = 0$ $\quad z = -\frac{b}{2};$

$x = b$ $\quad z = -\frac{2b}{5};$

$x = 2b$ $\quad z = -\frac{b}{4};$

$x = 3b$ $\quad z = 0;$

$x = 4b$ $\quad z = \frac{b}{2}.$

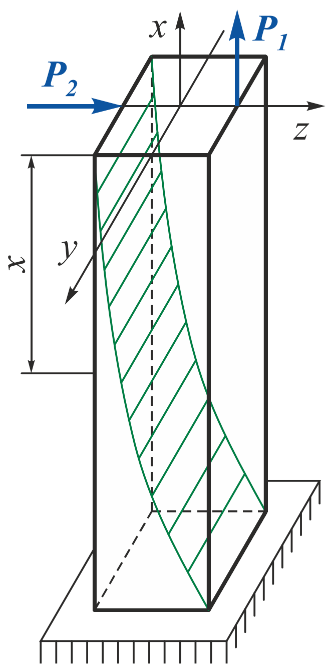


Fig. 4.14

Example 4.11

The wooden beam of a rectangular cross-section with the length $l = 2.4\,m$ (Fig. 4.15) is simply supported at its ends and loaded at mid-span with a concentrated force P. Strain gauges A and B with a 20 mm base and a magnification of 1000 times were installed in the beam critical section. They recorded the following reading changes: A – a 9 mm decrease, B – a 6 mm increase. Determine the magnitude and the direction (angle α) of the applied force P, as well as the value of the maximum normal stress in the beam, if $b = 120\ mm$, $h = 200\ mm$ and the modulus of longitudinal elasticity of wood is $E = 1 \times 10^4\ MPa$.

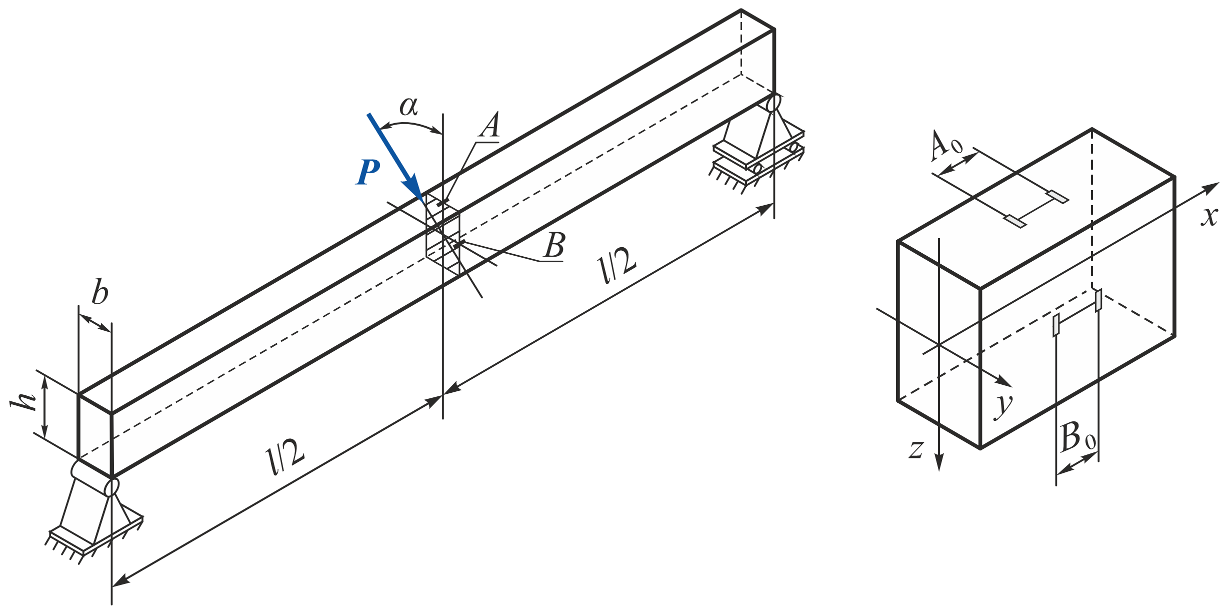


Fig. 4.15

Solution

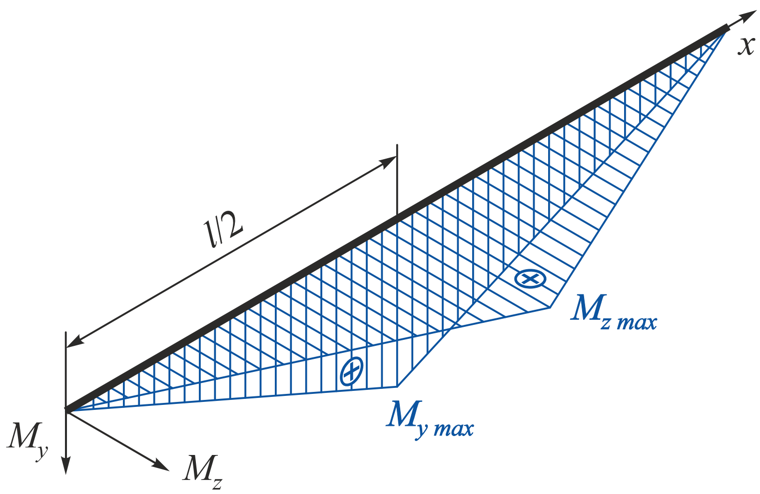


Fig. 4.16

We decompose the force P into its components and construct the bending moment diagrams (Fig. 4.16):

$$P_z = P\cos\alpha\,;$$

$$P_y = P\sin\alpha\,.$$

In the critical section, the internal moments are

$$M_y = M_{y\,max} = \frac{P_z}{2} \cdot \frac{l}{2} = \frac{Pl\cos\alpha}{4};$$

$$M_z = M_{z\,max} = -\frac{P_y}{2} \cdot \frac{l}{2} = -\frac{Pl\sin\alpha}{4}.$$

At A and B gauges, a uniaxial stress state is realized (Fig. 4.17). At these points stresses are

$$|\sigma_A| = \sigma_{A\,max}(M_y) = \frac{M_y}{W_y} = \frac{Pl\cos\alpha}{4} \cdot \frac{6}{bh^2} = \frac{3}{2} \cdot \frac{Pl\cos\alpha}{bh^2};$$

$$\sigma_B = \sigma_{B\,max}(M_y) = \frac{|M_z|}{W_z} = \frac{Pl\sin\alpha}{4} \cdot \frac{6}{hb^2} = \frac{3}{2} \cdot \frac{Pl\ \sin\alpha}{hb^2}.$$

According to Hooke's law for a uniaxial stress state:

$$|\sigma_A| = |E\varepsilon_A| = \left|E\frac{\Delta_A}{A_0K}\right|;$$

$$\sigma_B = E\varepsilon_B = E\frac{\Delta_B}{B_0K},$$

where Δ_A, Δ_B are the strain gauge reading changes;

A_0, B_0 are the strain gauge bases;

K is the magnification factor of the strain gauges.

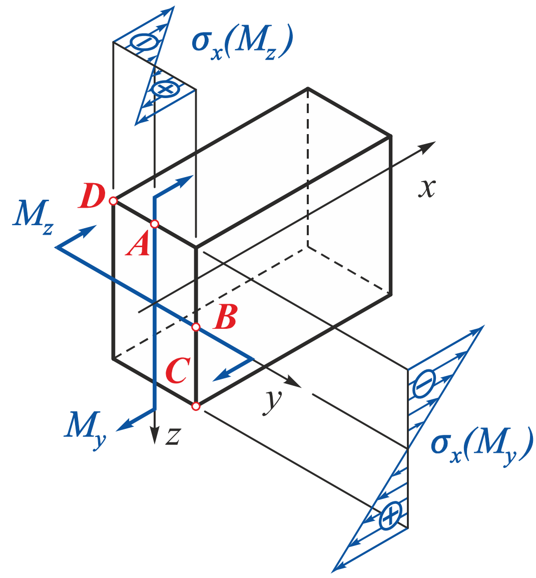


Fig. 4.17

We equate the normal stress values at points A and B, respectively:

$$\frac{3}{2} \cdot \frac{Pl\cos\alpha}{bh^2} = E\frac{\Delta_A}{A_0K}; \qquad (4.1)$$

$$\frac{3}{2} \cdot \frac{Pl\ \sin\alpha}{hb^2} = E\frac{\Delta_B}{B_0K}. \qquad (4.2)$$

We divide (4.2) by (4.1). After transformations, we get:

$$\alpha = \mathrm{arctg}\left(\frac{\Delta_B}{\Delta_A}\frac{b}{h}\right) = \mathrm{arctg}\left(\frac{6\times10^{-3}}{9\times10^{-3}} \cdot \frac{0.15}{0.2}\right) = 26.57°.$$

From (4.1) we obtain

$$P = \frac{\Delta_A E}{A_0 K} \cdot \frac{2bh^2}{3l \cdot \cos\varphi} = \frac{9 \times 10^{-3} \cdot 2 \times 10^{11}}{20 \times 10^{-3} \cdot 1000} \cdot \frac{2 \cdot 0.15 \cdot 0.2^2}{3 \cdot 2.4 \cdot \cos 26.57°} = 8.4\ kN.$$

The maximum stresses in the critical cross-section act at points C and D. For point C we obtain

$$\sigma_C = \sigma_{max} = \sigma\left(M_{y\,max}\right) + \sigma(M_{z\,max}) =$$

$$= \frac{3}{2} \cdot \frac{Pl\cos\alpha}{bh^2} + \frac{3}{2} \cdot \frac{Pl\ \sin\alpha}{hb^2} = \frac{3Pl}{2bh}\left(\frac{\cos\alpha}{h} + \frac{\sin\alpha}{b}\right) =$$

$$= \frac{3 \cdot 8.4 \times 10^3 \cdot 2.4}{2 \cdot 0.2 \cdot 0.15}\left(\frac{\cos 26.57°}{0.2} + \frac{\sin 26.57°}{0.15}\right) = 7.51\ MPa.$$

At point D the stress differs only by sign.

Example 4.12

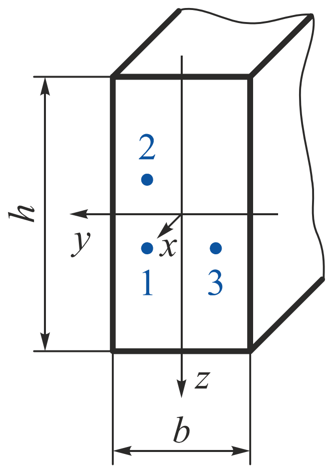


Fig. 4.18

In a rectangular cross-section of a bar, normal stresses σ_x arise from the action of internal forces N_x, M_y and M_z (Fig. 4.18). The known values of normal stresses in three points are $\sigma_{x\,1} = 9\ MPa$, $\sigma_{x\,2} = 6\ MPa$, $\sigma_{x\,3} = 12\ MPa$. The points coordinates are: $y_1 = 3\ cm$, $z_1 = 3\ cm$, $y_2 = 3\ cm$, $z_2 = -3\ cm$, $y_3 = -3\ cm$, $z_3 = 3\ cm$.

Determine the magnitudes of the internal forces and moments as well as the neutral axis position, if $b = 12\ cm$, $h = 24\ cm$.

Solution

The normal stress at an arbitrary point of the cross-section is:

$$\sigma_x = \frac{N_x}{F} + \frac{M_y \cdot z}{I_y} + \frac{M_z \cdot y}{I_z},$$

where N_x, M_y, M_z are the axial force and bending moments acting in the section;

y, z are the coordinates of an arbitrary point;

F is the area of the cross-section;

I_y, I_z are the axial moments of inertia of the cross-section relative to the y and z-axes.

We calculate the geometric characteristics of the rectangular section:

$$F = bh = 0.12 \cdot 0.24 = 2.88 \times 10^{-2}\ cm^2;$$

$$I_y = \frac{bh^3}{12} = \frac{0.12 \cdot 0.24^3}{12} = 1.3824 \times 10^{-4}\ cm^4;$$

$$I_z = \frac{hb^3}{12} = \frac{0.24 \cdot 0.12^3}{12} = 0.3456 \times 10^{-4}\ cm^4.$$

We write a system of three linear equations:

$$\begin{cases} 9 \times 10^6 = \dfrac{N_x}{2.88 \times 10^{-2}} + \dfrac{M_y \cdot 3 \times 10^{-2}}{1.3824 \times 10^{-4}} + \dfrac{M_z \cdot 3 \times 10^{-2}}{0.3456 \times 10^{-4}}; \\ 6 \times 10^6 = \dfrac{N_x}{2.88 \times 10^{-2}} + \dfrac{M_y \cdot (-3 \times 10^{-2})}{1.3824 \times 10^{-4}} + \dfrac{M_z \cdot 3 \times 10^{-2}}{0.3456 \times 10^{-4}}; \\ 12 \times 10^6 = \dfrac{N_x}{2.88 \times 10^{-2}} + \dfrac{M_y \cdot 3 \times 10^{-2}}{1.3824 \times 10^{-4}} + \dfrac{M_z \cdot (-3 \times 10^{-2})}{0.3456 \times 10^{-4}}. \end{cases}$$

Solving the system, we obtain the magnitudes of the internal forces and indicate their direction (Fig. 4.19):

$$N_x = 0.2592\ kN;$$

$$M_y = 0.6912 \times 10^{-2}\ kN{\cdot}m;$$

$$M_z = -0.1728 \times 10^{-2}\ kN{\cdot}m.$$

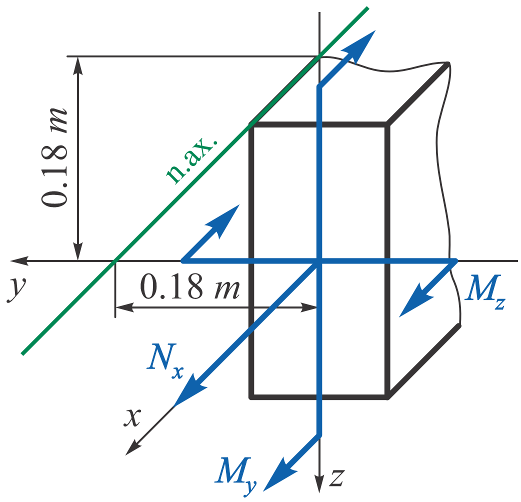


Fig. 4.19

We write the neutral axis equation:

$$\frac{0.2592}{2.88 \times 10^{-2}} + \frac{0.6912 \times 10^{-2}}{1.3824 \times 10^{-4}} z + \frac{-0.1728 \times 10^{-2}}{0.3456 \times 10^{-4}} y = 0.$$

After simplifications we get:

$$9 + 50 \cdot z - 50 \cdot y = 0;$$

$$50 \cdot z = 50 \cdot y - 9;$$

$$z = y - 0.18.$$

The neutral axis position is shown in Fig. 4.19.

Example 4.13

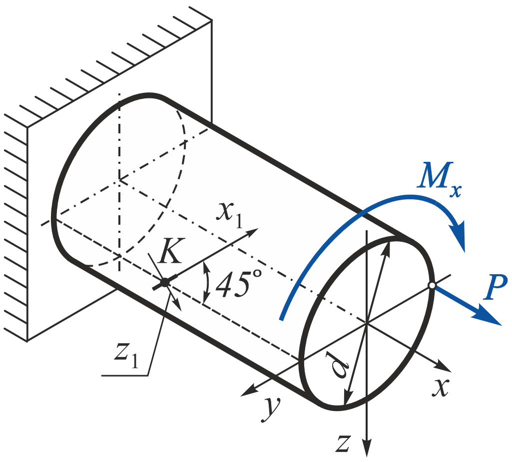


Fig. 4.20

It is known that at point K, the linear strain at a 45° angle to the generator of a circular cross-section bar is zero ($\varepsilon_{x_1}^K = 0$) (Fig. 4.20). The diameter of the bar is $d = 200\ mm$, the applied torque is $M_x = 5\ kN{\cdot}m$, and the Poisson's ratio of the bar material is $\mu = 0.25$.

Determine the maximum normal (principal) stress σ_{max}.

Solution

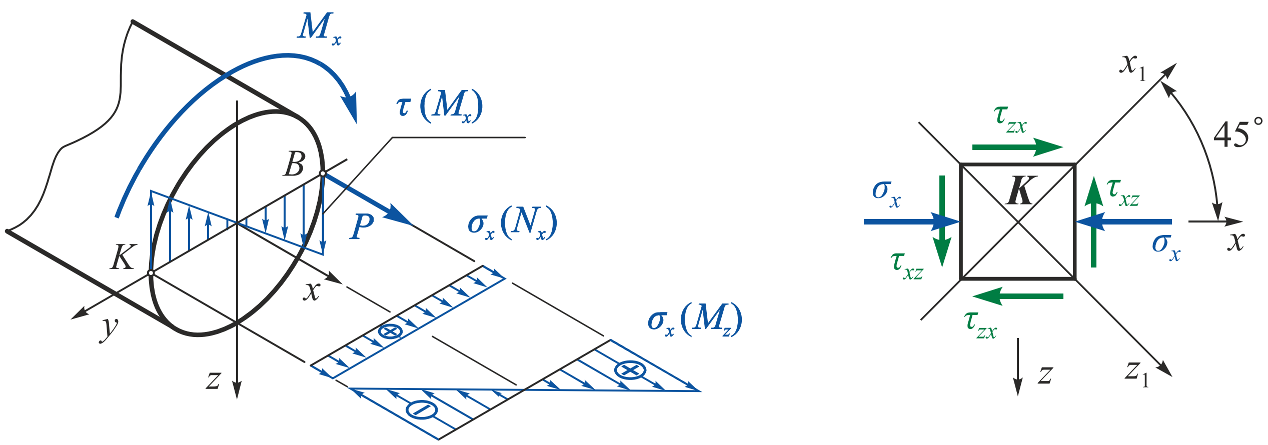
In any cross-section of the bar, a pure bending with tension and torsion is realized (Fig. 4.21, a).

The stress state at point K on the x, y, and z planes is

$$\tau = |-\tau_{xz}| = \tau_{zx} = \frac{M_x}{W_\rho} = \frac{16M_x}{\pi d^3}; \tag{4.3}$$

$$\sigma = |\sigma_x| = \left|\frac{P}{F} - \frac{P}{W_z}\cdot\frac{d}{2}\right| = \left|\frac{4P}{\pi d^2} - \frac{32P}{\pi d^3}\cdot\frac{d}{2}\right| = \frac{12P}{\pi d^2}; \tag{4.4}$$

$$\sigma_y = \sigma_z = 0.$$



a b

Fig. 4.21

Let us determine the normal stresses on mutually perpendicular planes x_1 and z_1, rotated relative to the y and z-axes by the angle $\alpha = 45°$ with respect to the x-axis, as shown in Fig. 4.21, b:

$$\begin{cases} \sigma_{x_1} = \sigma_x \cos^2\alpha + \sigma_z \sin^2\alpha + \tau \sin 2\alpha = \dfrac{\sigma}{2} + \tau; \\ \sigma_{y_1} = \sigma_y = 0; \\ \sigma_{z_1} = \sigma_x \sin^2\alpha + \sigma_z \cos^2\alpha - \tau \sin 2\alpha = \dfrac{\sigma}{2} - \tau. \end{cases}$$

According to Hooke's law, the linear strain at point *K* in the x_1-axis direction is

$$\varepsilon_{x_1}^{(K)} = \frac{1}{E}\left[\sigma_{x_1} - \mu\left(\sigma_{y_1} + \sigma_{z_1}\right)\right] = \frac{1}{E}\left[(1-\mu)\frac{\sigma}{2} - (1+\mu)\tau\right].$$

By the problem statement, $\varepsilon_{x_1}^{(K)} = 0$. Hence

$$\tau = \frac{1-\mu}{2(1+\mu)}\sigma.$$

Taking into account (4.3) and (4.4):

$$\frac{16M_x}{\pi d^3} = \frac{1-\mu}{2(1+\mu)} \cdot \frac{12P}{\pi d^2}.$$

From this, we get:

$$P = \frac{8}{3} \cdot \frac{(1+\mu)}{(1-\mu)} \cdot \frac{M_x}{d} = \frac{8}{3} \cdot \frac{(1+0.25)}{(1-0.25)} \cdot \frac{5 \times 10^3}{0.2} = \frac{1000 \times 10^3}{9} = 111.11\ kN.$$

Now we determine the greatest normal stress, i.e. the maximum principal stress, which will occur at point *B* of the bar (see the stress diagrams in Fig. 4.21, a). On the x and z planes the following stresses act:

$$\begin{cases} \sigma_x = \dfrac{P}{F} + \dfrac{P}{W_y} \cdot \dfrac{d}{2} = \dfrac{4P}{\pi d^2} + \dfrac{32P}{\pi d^3} \cdot \dfrac{d}{2} = \dfrac{20P}{\pi d^2}; \\ \sigma_z = \sigma_y = 0; \\ \tau_{xz} = \dfrac{M_x}{W_\rho} = \dfrac{16M_x}{\pi d^3}. \end{cases}$$

The maximum normal (principal) stress is:

$$\sigma_{max} = \frac{\sigma_x + \sigma_z}{2} + \frac{1}{2}\sqrt{(\sigma_x + \sigma_z)^2 + 4\tau_{xz}^2} = \frac{10P}{\pi d^2} + \frac{1}{2}\sqrt{\left(\frac{20P}{\pi d^2}\right)^2 + 4\left(\frac{16M_x}{\pi d^3}\right)^2} =$$

$$= \frac{10 \cdot 111.11 \times 10^3}{\pi \cdot 0.2^2} + \frac{1}{2}\sqrt{\left(\frac{20 \cdot 111.11 \times 10^3}{\pi \cdot 0.2^2}\right)^2 + 4\left(\frac{16 \cdot 5 \times 10^3}{\pi \cdot 0.2^3}\right)^2} = 18.239\ MPa.$$

Example 4.14

The beam material (Fig. 4.22) is concrete, characterized by different ultimate tensile $\left(\sigma_{ult_t} = \sigma_0\right)$ and compressive $\left(\sigma_{ult_c} = 5\sigma_0\right)$ strengths. The beam span length is l, the cross-section is an isosceles triangle with height h and base b, and the safety factor for concrete is n_{ult}. It is required to determine the optimal initial prestress of the reinforcement located along the centroidal axis of the cross-section, i.e. such prestress N_0 that the bending capacity of the beam is maximized ($P = P_{max}$). What is the value of P_{max}?

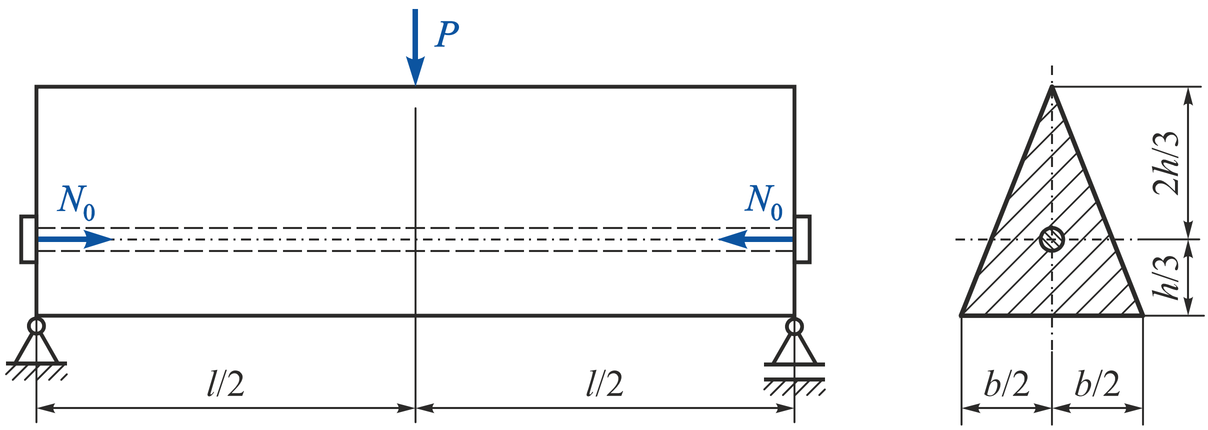


Fig. 4.22

Solution

The beam is subjected to bending with compression.

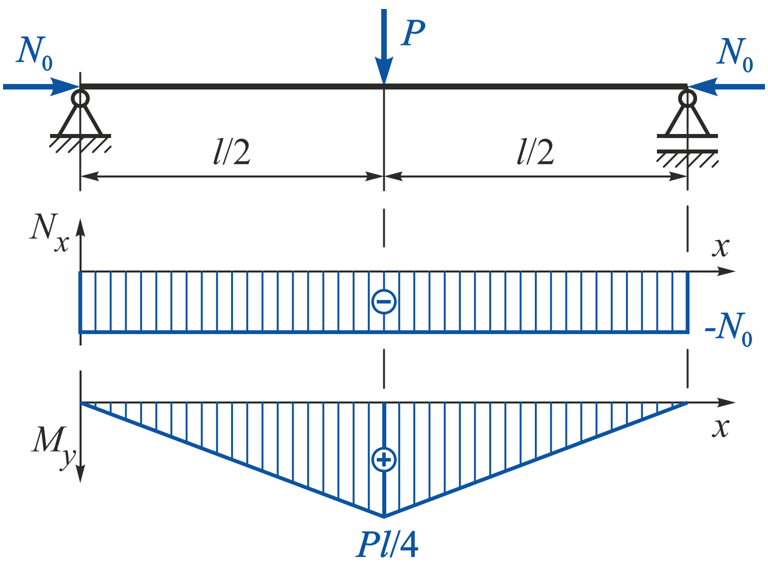


Fig. 4.23

In the beam critical section (Fig. 4.23) at midspan, two internal loads will act:

– initial prestress of the reinforcement

$$N_x = -N_0;$$

– and bending moment

$$M_y = \frac{P}{2} \cdot \frac{l}{2} = \frac{Pl}{4}.$$

We determine the optimal value of the initial reinforcement prestress N_0 by ensuring the beam's most tensioned (A) and most compressed (B) points are ***equally strong (the equal strength condition)***:

$$\begin{cases} \sigma_{t\,max} = [\sigma]_t = \dfrac{\sigma_{ult_t}}{n_{ult}} = \dfrac{\sigma_0}{n_{ult}}; \\ \left|\sigma_{c\,max}\right| = [\sigma]_c = \dfrac{\sigma_{ult_c}}{n_{ult}} = \dfrac{5\sigma_0}{n_{ult}}, \end{cases}$$

where n_{ult} is the concrete safety factor.

The maximum tensile stresses act at point *A*, and the maximum compressive stresses act at point *B* (Fig. 4.24):

$$\sigma_A = \sigma_{t\,max} = \frac{M_y}{I_y} \cdot \frac{1}{3}h - \frac{|N_x|}{F} = \frac{Plh}{12I_y} - \frac{N_0}{F};$$

$$|\sigma_B| = \left|\sigma_{c\,max}\right| = \left|\frac{M_y}{I_y} \cdot \left(-\frac{2}{3}h\right) + \frac{N_x}{F}\right| = \left|-\frac{Pl}{4I_y} \cdot \frac{2}{3}h - \frac{N_0}{F}\right| = \frac{Plh}{6I_y} + \frac{N_0}{F},$$

where $I_y = \dfrac{bh^3}{36}$; $F = \dfrac{bh}{2}$.

Then, we can rewrite the equal strength condition as:

$$\begin{cases} \dfrac{P_{max} l}{4I_y} \cdot \dfrac{1}{3}h - \dfrac{N_0}{F} = \dfrac{\sigma_0}{n_{ult}}; \\ \dfrac{P_{max} l}{4I_y} \cdot \dfrac{2}{3}h + \dfrac{N_0}{F} = \dfrac{5\sigma_0}{n_{ult}}. \end{cases}$$

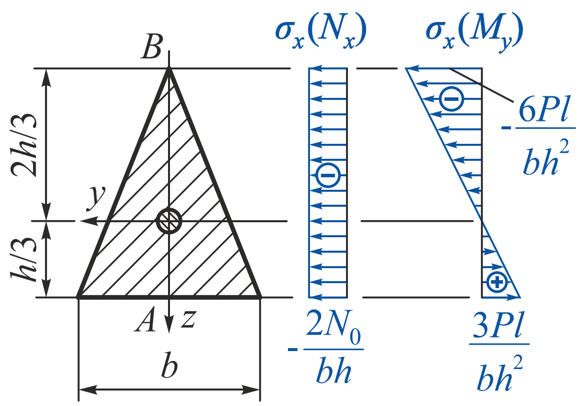


Fig. 4.24

After substituting I_y and F, we obtain:

$$\begin{cases} \dfrac{3l}{bh^2} P_{max} - \dfrac{2}{F} N_0 = \dfrac{\sigma_0}{n_{ult}}; \\ \dfrac{6l}{bh^2} P_{max} + \dfrac{2}{F} N_0 = \dfrac{5\sigma_0}{n_{ult}}. \end{cases}$$

As a result of solving the linear equations system, we get:

$$N_0 = \frac{\sigma_0}{n_{ult}} \cdot \frac{bh}{2};$$

$$P_{max} = \frac{\sigma_0}{n_{ult}} \cdot \frac{2bh^2}{3l}.$$

Example 4.15

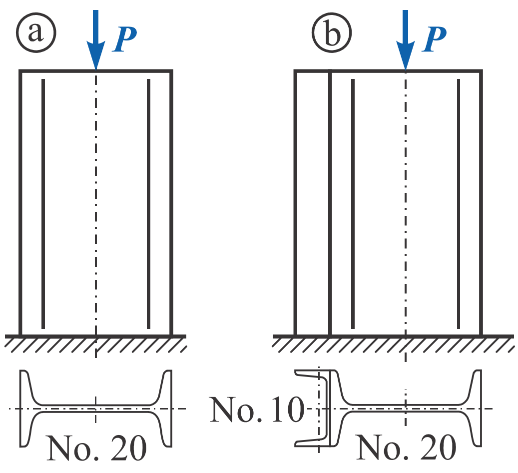


Fig. 4.25

A short column made of an I-beam No. 20, centrally loaded with a force $P = 200\ kN$ (Fig. 4.25, a), is "reinforced" by a channel No. 10, welded to the column along its entire length (see Fig. 4.25, b). What was the maximum compressive stress in the I-beam column and in the column "reinforced" by the channel?

Solution

From the steel section tables, we write down the necessary geometric characteristics of the rolled sections for the subsequent calculation:

I-beam No. 20: $h_1 = 20 \times 10^{-2}\ m$; $F_1 = 26.8 \times 10^{-4}\ m^2$;
$I_{z_1} = 1840 \times 10^{-8}\ m^4$;

Channel No. 10: $y_0 = 1.44 \times 10^{-2}\ m$; $F_2 = 10.9 \times 10^{-4}\ m^2$;
$I_{z_2} = 20.4 \times 10^{-8}\ m^4$.

In the first case, pure axial compression is realized, where the normal stress is:

$$\sigma = \frac{P}{F_1} = \frac{200 \times 10^3}{26.8 \times 10^{-4}} = 74.63\ MPa.$$

In the second case, it is eccentric compression. Then, the maximum normal stresses occur at the points on the right face of the column:

$$\sigma_{max\ (A)} = \frac{P}{F} + \frac{M_z}{I_z} \cdot y_{(A)}.$$

In this expression,

F is the area of the ***compound section:***

$$F = F_1 + F_2 = 26.8 \times 10^{-4} + 10.9 \times 10^{-4} = 37.7 \times 10^{-4}\ m^4;$$

M_z is the bending moment relative to the z-axis, caused by the eccentricity of the applied force P:

$$M_z = P \cdot y_c;$$

I_z is the inertia moment of the compound section relative to the z-axis;

$y_{(A)}$ is the coordinate of point A in the in the system of centroidal axes yOz of compound cross-section.

We find the coordinates of the compound section centroid (Fig. 4.26). Since the y-axis is the axis of symmetry of the compound section, the centroid lies on this axis ($z_c = 0$). We find the second coordinate of the centroid:

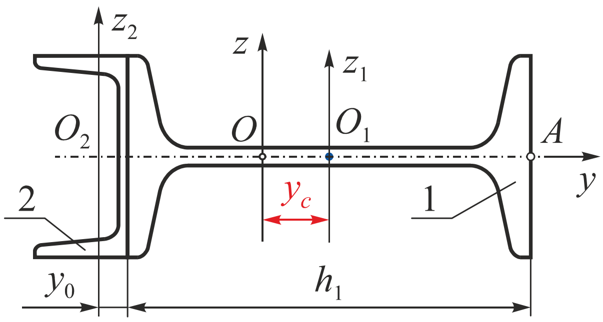


Fig. 4.26

$$y_c = \frac{\sum_{i=1}^{n} S_z^{(i)}}{\sum_{i=1}^{n} F_i} = \frac{0 \cdot F_1 + \left(-\left(\frac{h_1}{2} + y_0\right)\right) F_2}{F_1 + F_2} =$$

$$= \frac{-\left(\frac{20 \times 10^{-2}}{2} + 1.44 \times 10^{-2}\right) \cdot 10.9 \times 10^{-4}}{26.8 \times 10^{-4} + 10.9 \times 10^{-4}} =$$

$$= \frac{-1.247 \times 10^{-4}}{37.7 \times 10^{-4}} = -3.308 \cdot 10^{-2}\ m;$$

Let us determine the compound section inertia moment relative to the central z-axis:

$$I_z = I_z^{(1)} + I_z^{(2)} = I_{z_1} + y_c^2 \cdot F_1 + I_{z_2} + \left(\frac{h_1}{2} + y_0 - y_c\right)^2 F_2 =$$

$$= 1840 \times 10^{-8} + (3.308 \times 10^{-2})^2 \cdot 26.8 \times 10^{-4} +$$

$$+20.4 \times 10^{-8} + \left(\frac{20 \times 10^{-2}}{2} + 1.44 \times 10^{-2} - 3.308 \times 10^{-2}\right)^2 \cdot 10.9 \times 10^{-4} =$$

$$= 2133.269 \times 10^{-8} + 741.211 \times 10^{-8} = 2874.48 \times 10^{-8}\ m^4.$$

Hence, the expression for determining the maximum normal stresses can be rewritten:

$$\sigma_{\max\ (A)} = \frac{P}{F} + \frac{P \cdot y_c}{I_z}\left(\frac{h_1}{2} + y_c\right) =$$

$$= \frac{200 \times 10^3}{37.7 \times 10^{-4}} + \frac{200 \times 10^3 \cdot 3.308 \times 10^{-2}}{2874.48 \times 10^{-8}}\left(\frac{20 \times 10^{-2}}{2} + 3.308 \times 10^{-2}\right) =$$

$$= 53.05 \times 10^6 + 30.63 \times 10^6 = 83.68\ MPa.$$

Thus, the "reinforcement" of the column led to a 12.13% increase in the maximum normal stresses.

Example 4.16

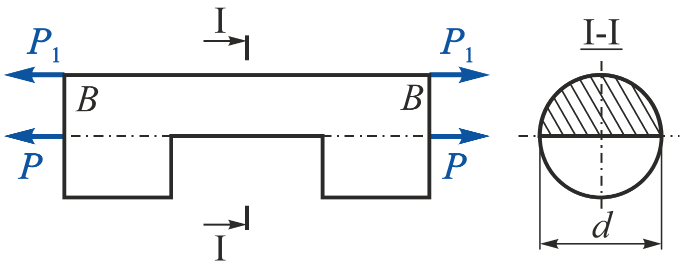


Fig. 4.27

A variable cross-section bar is loaded with tensile forces P (Fig. 4.27). Determine the magnitude of forces P_1, applied at points B, so that the normal stress distribution is uniform in the bar critical section.

Will the load-bearing capacity of the bar increase in this case, and by how much? Neglect the stress concentration.

Solution

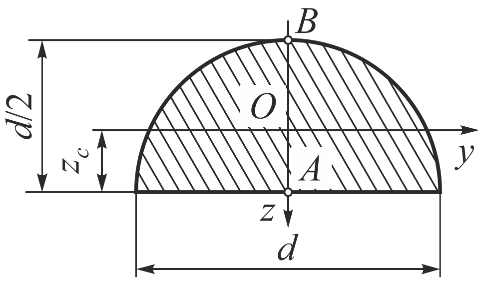


Fig. 4.28

We find the force P_1 from the equality condition of normal stresses at the lower point (A) of the weakened cross-section under the moments caused by the forces P and P_1 relative to the section centroid (Fig. 4.28):

$$\sigma_x\left(M_y(P)\right) = \sigma_x\left(M_y(P_1)\right).$$

Then

$$\frac{M_y(P)}{W_A} = \frac{M_y(P_1)}{W_A},$$

where $M_y(P) = Pz_A$; $M_y(P_1) = P_1 z_B$ are the bending moments caused by forces P and P_1;

$z_A = z_c = 2d/3\pi = 0.2122d$; $z_B = d/2 - z_c = 0.2878d$ are the coordinates of the forces P and P_1 application points relative to the section centroid;

F is the cross-sectional area;

$W_A = 0.0239d^3$; $W_B = 0.0324d^3$ are the section moduli of the semicircular cross-section relative to the y-axis for the upper (point B) and lower (point A) fibers.

Let us substitute into the normal stress equality condition:

$$\frac{P \cdot 0.2122d}{0.0324d^3} = \frac{P_1 \cdot 0.2878d}{0.0324d^3}$$

and after transformations, we get:

$$P_1 = 0.7373 \cdot P.$$

According to the calculation scheme (Fig. 4.29), ***in the first loading case***, the maximum stress in the critical cross-section will be at the bottom fibers:

$$\sigma_{x\,max}^{(1)} = \sigma_x(N_x) + \sigma_{x\,max}(M_y) = \frac{P}{F} + \frac{Pz_A}{W_A} =$$

$$= \frac{P \cdot 8}{\pi d^2} + \frac{P \cdot 0.2122d}{0.0324d^3} = \frac{P}{d^2}(2.547 + 6.549) = 9.096\frac{P}{d^2}.$$

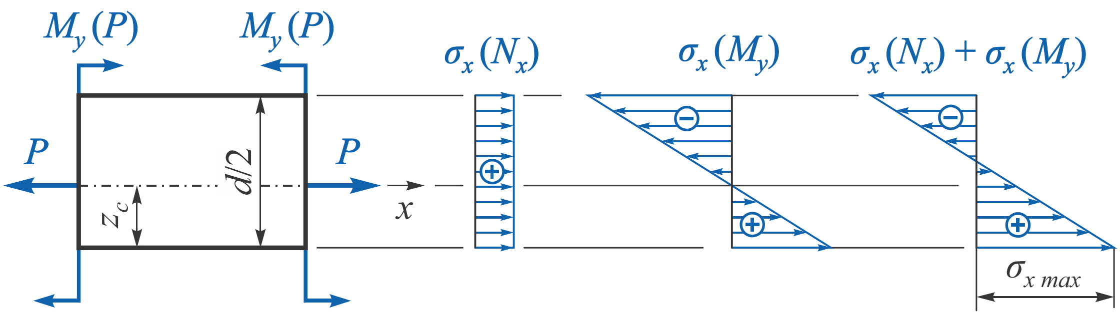


Fig. 4.29

In the second loading variant (Fig. 4.30), with $P_1 = 0.7373 \cdot P$, the maximum stresses will be at any point of the weakened cross-section, since the normal stresses due to the bending moments will compensate each other (the bending moments are not shown in Fig. 4.30):

$$\sigma_{x\,max}^{(2)} = \frac{P + P_1}{F} = \frac{(P + 0.7373 \cdot P) \cdot 8}{\pi d^2} = \frac{P}{d^2}4.424.$$

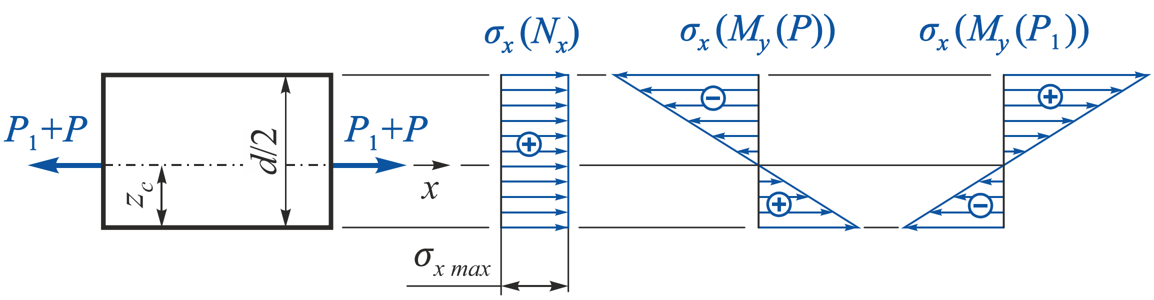


Fig. 4.30

In the second loading variant, the maximum normal stress in the critical cross-section decreases in 2.056 times compared to the first case: При втором варианте нагружения наибольшее нормальное напряжение в опасном сечении уменьшается в 2,056 раза по сравнению с первым вариантом.

$$\frac{\sigma_{x\,max}^{(1)}}{\sigma_{x\,max}^{(2)}} = \frac{9.096}{4.424} = 2.056.$$

QUESTIONS FOR SELF-CHECK

1. What is a limiting stress state?
2. What are the strength theories, and for what purpose are they applied?
3. Which stress states at a point are called equally critical?
4. What is the equivalent stress?
5. What is the purpose of strength hypotheses (theories of limiting stress states)?
6. How is the theory of maximum normal stresses (the first strength theory) formulated, and what is its strength condition?
7. How is the theory of maximum linear strains (the second strength theory) formulated, and what is its strength condition?
8. How is the theory of maximum shear stresses (the third strength theory) formulated, and what is its strength condition?
9. How is the energy theory of strength (the fourth strength theory) formulated, and what is its strength condition?
10. What is Mohr's strength theory?
11. Write down the strength conditions according to the third and fourth strength theories for a special case of a plane stress state.
12. What are diagrams? Present the basic rules of their construction and the sign conventions when constructing diagrams.
13. What is a planar cranked bar with out-of-plane loading?
14. What is combined loading of a bar?
15. Formulate the principles upon which the analysis of bars under combined loading is based.
16. Which points of a rectangular cross-section will be potentially critical, and what stress state arises at these points under the combined action of bending with torsion and tension?
17. Write the strength conditions for the critical points of a bar with a rectangular cross-section under bending with torsion and tension.
18. What is the procedure for selecting the dimensions of a bar with a rectangular cross-section under combined loading?
19. Which points of a circular cross-section are critical, and what stress state arises at these points under the combined action of bending with torsion?
20. Write the strength conditions for the critical points of a bar with a circular cross-section under the combined action of bending and torsion.
21. What is the procedure for selecting the diameter of a bar with a circular cross-section under the combined action of bending with torsion and tension (compression)?

22. How do you find the value of the design (equivalent) moment according to the third and fourth strength theories for bending with torsion of a bar with a circular cross-section?
23. Which point of a circular cross-section will be critical under the combined action of bending with torsion and tension (compression)? Write the strength conditions for this point.
24. Why are the shear stresses from the action of transverse (shear) forces usually not taken into account in the analysis of bars for bending with torsion?
25. What is oblique bending?
26. What is called pure oblique bending and transverse oblique bending?
27. For which cross-sectional shapes of bars is oblique bending not possible?
28. To what resultants do the internal loads lead under oblique bending
29. How do you determine the neutral axis position under oblique bending?
30. Does the neutral axis pass through the centroid of the section under oblique bending?
31. How do you determine the critical points in a cross-section under oblique bending?
32. What is the procedure of bar analysing under oblique bending?
33. What kind of combined loading is called eccentric tension-compression?
34. What types of stresses arise in a bar subjected to eccentric tensile or compressive loading?
35. What formulas are used to determine the normal stresses in the cross-sections of a bar under eccentric tension and compression? What kind of diagram do these stresses have?
36. How do you determine the neutral axis position under eccentric tension-compression?
37. How does the neutral axis move when the coordinates of the external force application point change under eccentric tension-compression?
38. How do you determine the position of the most critical point of a bar cross-section under eccentric tension-compression?
39. What is the core of a cross-section?
40. How is the core of a cross-section constructed?
41. What is the neutral axis position when the pressure center lies on the contour of the cross-section core?
42. What will the stresses be at all points of the cross-section if it is known that the tensile force lies inside the core of a cross-section?
43. Can compressive stresses arise at points of the cross-section under eccentric tension?

PROCEDURE AND EXAMPLE OF PERFORMING THE HOME ASSIGNMENT

Objective. To perform a strength analysis of a cranked bar subjected to combined action of bending and torsion.

1. Draw to scale the analytical scheme of the cranked bar with the given loads.
2. Divide the cranked bar into segments.
3. In arbitrary cross-sections of each segment at a distance x from its beginning, place a coordinate system so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downward, and the horizontal y-axis together with the first two axes forms a right-handed orthogonal basis.

Remark	To obtain a formally ordered sign system for the internal forces and moments on all segments, it is recommended to construct the coordinate system of segment **II** by a simple translation, i.e. by rotating the coordinate system of segment **I** by 90° about the z-axis, and so on.

4. Using the method of sections, write expressions for all internal forces and moments in arbitrary cross-sections within each segment, following the adopted sign conventions.
5. Construct (draw) the internal forces and moments diagrams.

Remarks	When constructing the internal forces and moments diagrams for a cranked bar, keep in mind: а) The N_x and M_x diagrams can be drawn in any plane.; б) The diagrams of Q_z, Q_y, M_y, and M_z must be drawn only in their ***respective planes of action***; в) The diagrams of ***bending moments*** M_y and M_z must be constructed ***on the tensioned fibers***.

6. Verify the correctness of the constructed diagrams (equilibrium at the nodal points).
7. Determine the critical cross-section.

Remark	If the position of the critical cross-section is not evident from the diagrams, all potentially critical cross-sections must be considered in the strength analysis.

8. Select a rectangular cross-section ($k = h/b$ is prescribed by the instructor). Perform the analysis for all potentially critical points of the critical cross-section. Apply the maximum shear stress hypothesis (the third strength theory). Construct diagrams of the normal and shear stresses distributions in the critical cross-section of the considered bar element due to the action of the axial force N_x, torsional moment M_x, and bending moments M_y and M_z. Identify the truly critical point. At all potentially critical points determine and show the stress state. Construct the combined diagram of normal stresses and indicate the position of the cross-section neutral axis.

Remark	If $\boldsymbol{M_z > M_y}$, the cross-section should be oriented ***horizontally***; if; $\boldsymbol{M_y > M_z}$, it should be oriented ***vertically***, so that the greater bending moment is opposed by the greater bending stiffness of the cross-section (ensuring strength with smaller cross-section dimensions and, accordingly, reduced bar weight).

9. Select a circular cross-section. Apply the maximum shear stress hypothesis (the third strength theory) and the strain energy hypothesis (the fourth strength theory). Show the stress state at the critical point.

Remarks	1. The strength analysis is performed taking into account normal stresses from bending and axial force (if present in the considered cross-section), and shear stresses from torsion. Shear stresses due to transverse (shear) forces are neglected. 2. If there is an axial force, first-approximation selection of section dimensions is performed neglecting this axial force. After calculating section dimensions, determine the actual design stresses in the critical cross-section. If this stress exceeds allowable stress by more than 5%, increase section dimensions so that the overload does not exceed 5%; 3. For material, assume $[\sigma] = 160 \dots 240\ MPa$, coefficients α and γ for ratio $k = h/b$ are given in the appendix.

10. Compare the weight of the rectangular and circular cross-sections.

Example

Construct diagrams of internal forces and moments for the given cranked bar. For the critical cross-section, determine the dimensions h and b of the rectangular cross-section and the diameter d of the circular cross-section (Fig. 1).

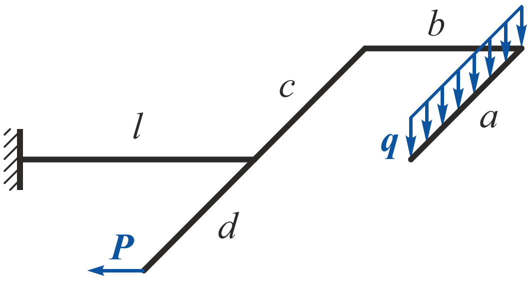


Fig. 1

Given: $P = 2\ kN$; $q = 5\ kN/m$; $a = 2\ m$;
$b = 0.5\ m$; $c = 2\ m$; $d = 0.5\ m$;
$l = 1\ m$; $[\sigma] = 180\ MPa$;
$k = h/b = 1.5$; $\alpha = 0.231$; $\gamma = 0.859$.

It is necessary to construct the N_x, Q_z, Q_y, M_x, M_y, and M_z diagrams, and to determine the dimensions of the rectangular and the diameter of the circular cross-sections.

Solution

1. We ***draw to scale the analytical scheme*** of the cranked bar with the given loads. In arbitrary cross-sections of each segment at a distance x from its beginning, place the coordinate system xyz so that the x-axis coincides with the longitudinal axis of the bar, the z-axis is directed downward, and the horizontal y-axis together with the first two forms a right-handed orthogonal basis (Fig. 2).

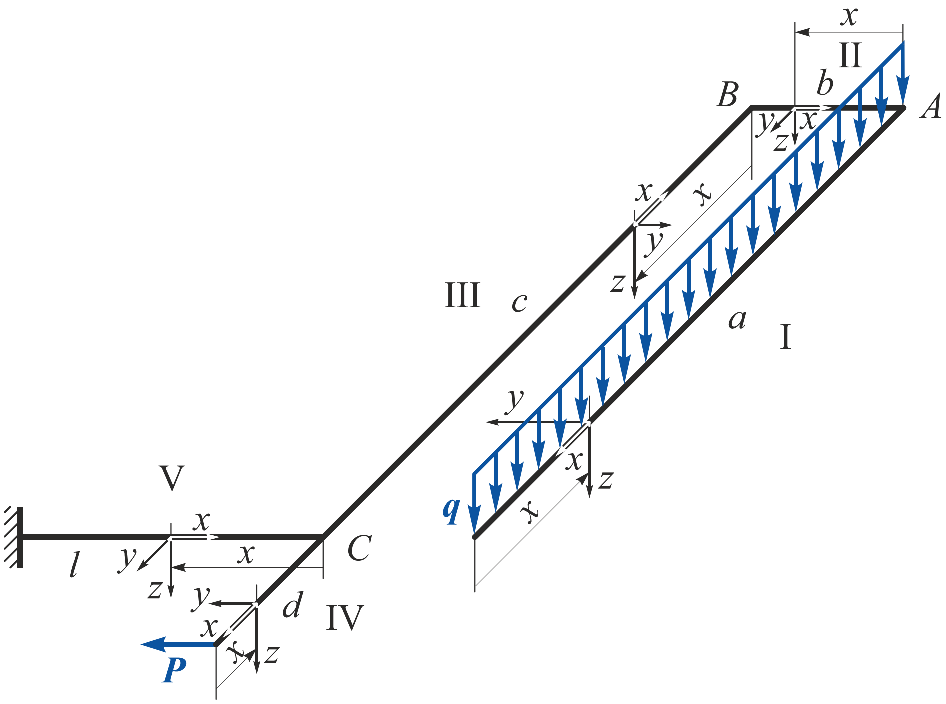


Fig. 2

2. Using the method of sections, ***we write expressions for all internal forces and moments*** in arbitrary cross-sections within each segment, following the adopted sign conventions.

Segment I $(0 \leq x \leq a,\ \ a = 2\ m)$.

$N_x^I = 0;$

$Q_z^I = qx = 5x \ \Big|_{x=0} = 0 \ \Big|_{x=a=2\ m} = 10\ kN;$

$Q_y^I = 0;$

$M_x^I = 0;$

$M_y^I = -\dfrac{qx^2}{2} = -\dfrac{5x^2}{2} \ \Big|_{x=0} = 0 \ \Big|_{x=a=2\ m} = -10\ kN{\cdot}m;$

$M_z^I = 0.$

Segment II $(0 \leq x \leq b,\ \ b = 0.5\ m)$.

We will construct a separate analytical scheme, replacing the distributed load acting within the first section with its resultant force (Fig. 3).

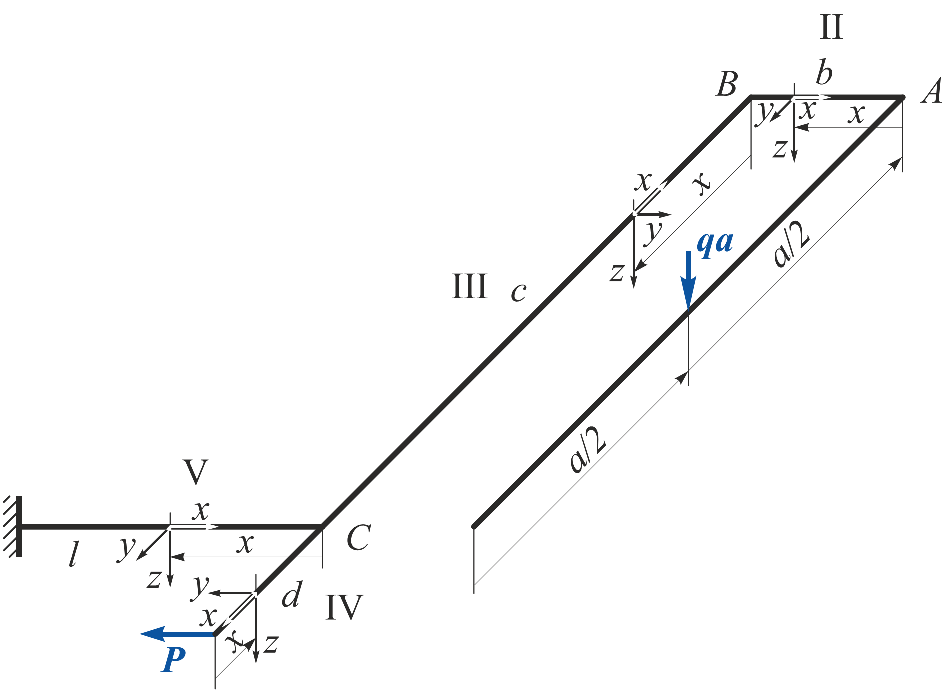


Fig. 3

$N_x^{II} = 0;$

$Q_z^{II} = qa = 5 \cdot 2 = 10\ kN;$

$Q_y^{II} = 0;$

$$M_x^{II} = qa\frac{a}{2} = 5 \cdot 2 \cdot \frac{2}{2} = 10\ kN{\cdot}m;$$

$$M_y^{II} = -qax = -5 \cdot 2 \cdot x \quad \Big|_{x=0} = 0 \quad \Big|_{x=b=0.5\ m} = -5\ kN{\cdot}m;$$

$$M_z^{II} = 0.$$

Segment III $(0 \le x \le c,\ \ c = 2\ m)$.

$$N_x^{III} = 0;$$

$$Q_z^{III} = qa = 5 \cdot 2 = 10\ kN;$$

$$Q_y^{III} = 0;$$

$$M_x^{III} = qab = 5 \cdot 2 \cdot 0.5 = 5\ kN{\cdot}m;$$

$$M_y^{III} = qa\frac{a}{2} - qax = 5 \cdot 2 \cdot \frac{2}{2} - 5 \cdot 2 \cdot x \quad \Big|_{x=0} = 10\ kN{\cdot}m \quad \Big|_{x=c=2\ m} = -10\ kN{\cdot}m;$$

$$M_z^{III} = 0.$$

Segment IV $(0 \le x \le d,\ \ d = 0.5\ m)$.

$$N_x^{IV} = 0;$$

$$Q_z^{IV} = 0;$$

$$Q_y^{IV} = -P = -2\ kN;$$

$$M_x^{IV} = 0;$$

$$M_y^{IV} = 0;$$

$$M_z^{IV} = -Px = -2 \cdot x \quad \Big|_{x=0} = 0 \quad \Big|_{x=d=0.5\ m} = -1\ kN{\cdot}m.$$

Segment V $(0 \le x \le l,\ \ l = 1\ m)$.

$$N_x^{V} = -P = -2\ kN;$$

$$Q_z^{V} = qa = 5 \cdot 2 = 10\ kN;$$

$$Q_y^{V} = 0;$$

$$M_x^{V} = -qa\left(c - \frac{a}{2}\right) = -5 \cdot 2 \cdot \left(2 - \frac{2}{2}\right) = -10\ kN{\cdot}m;$$

$$M_y^{V} = -qa(b + x) = 5 \cdot 2 \cdot (0.5 + x) \quad \Big|_{x=0} = -5\ kN{\cdot}m \quad \Big|_{x=l=1\ m} = -15\ kN{\cdot}m;$$

$$M_z^{V} = -Pd = -2 \cdot 0.5 = -1\ kN{\cdot}m.$$

3. ***Let us construct the internal forces and moments diagrams*** (Fig. 4).

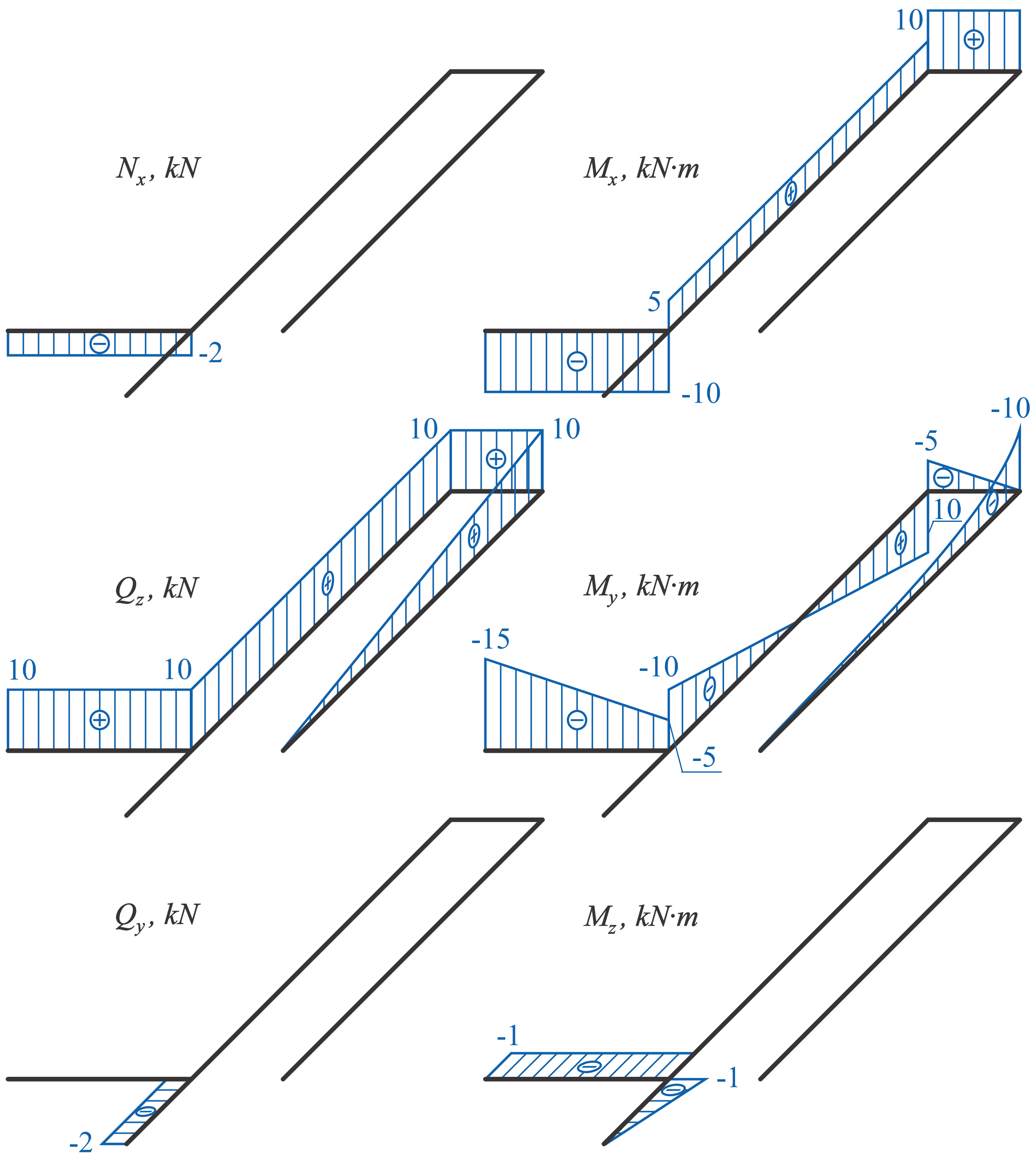


Fig. 4

The diagrams of N_x, Q_y and M_z are drawn to a larger scale.

4. ***We check the correctness of the constructed diagrams***.

To do this, we isolate infinitesimal elements of the cranked bar at the joints of its parts (nodes A, B, and C) and examine their equilibrium under the action of internal and external loads applied within these nodes (Fig. 5).

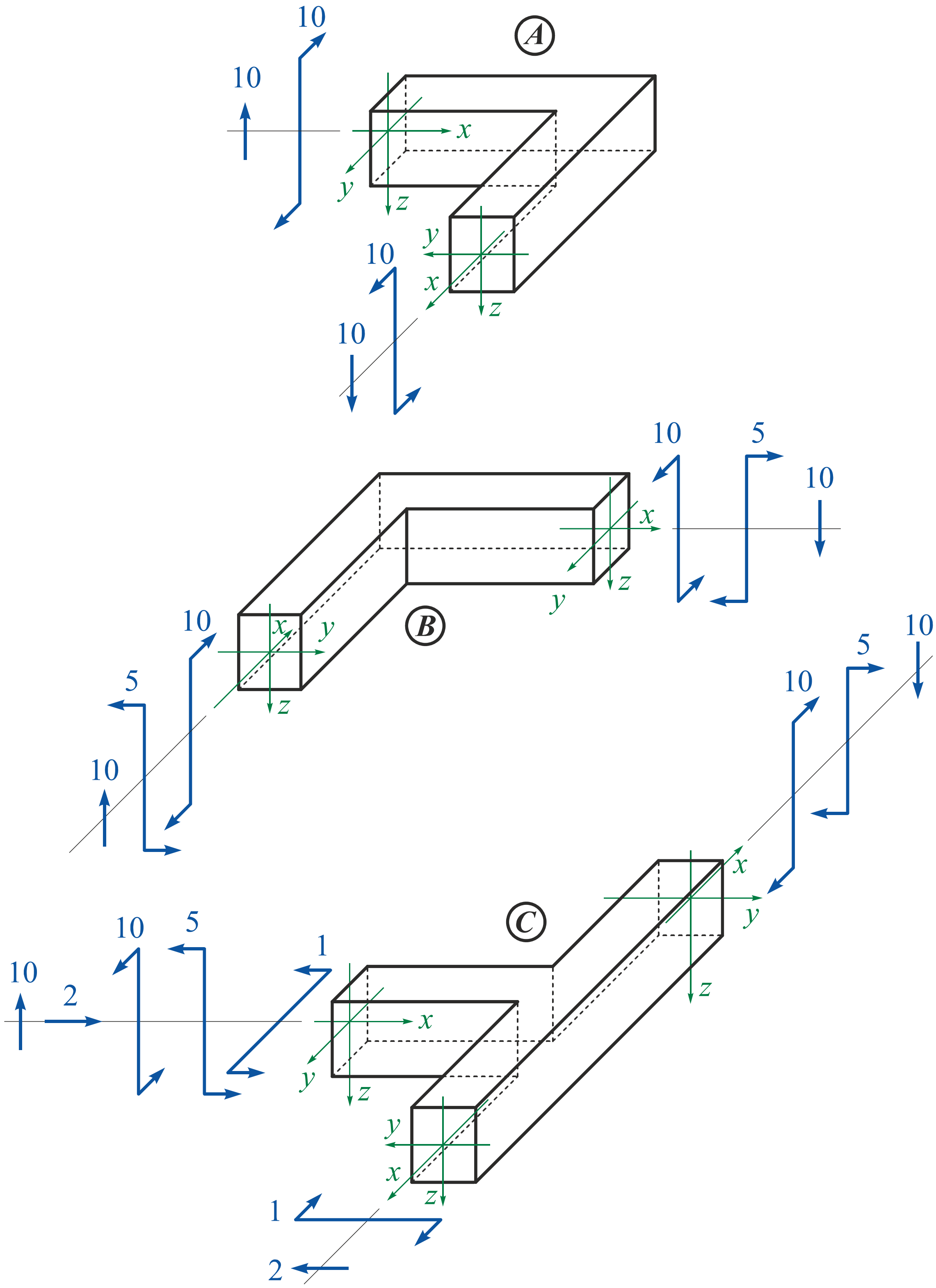


Fig. 5

Equilibrium equations for node *A*:

$\sum P_x = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 10 - 10 = 0;$ $\sum M_y = 0;$ $\sum M_z = 0.$

Equilibrium equations for node B:

$\sum P_x = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 10 - 10 = 0;$ $\sum M_y = 5 - 5 = 0;$ $\sum M_z = 0.$

Equilibrium equations for node C:

$\sum P_x = 2 - 2 = 0;$ $\sum P_y = 0;$ $\sum P_z = 10 - 10 = 0;$

$\sum M_x = 10 - 10 = 0;$ $\sum M_y = 5 - 5 = 0;$ $\sum M_z = 1 - 1 = 0.$

5. *Let us identify the critical cross-section*.

From the analysis of the diagrams (Fig. 4) it follows that the most critical cross-section is at the fixed support, where the following internal forces and moments act:

$$N_x = -2\ kN; \quad M_x = -10\ kN{\cdot}m; \quad M_y = -15\ kN{\cdot}m; \quad M_z = -1\ kN{\cdot}m.$$

6. *We determine the dimensions of the rectangular cross-section.*

Since $M_y > M_z$, we orient the section vertically to ensure the section's strength with smaller dimensions.

The section with the applied internal forces and moments is shown in Fig. 6.

The internal forces and moments are applied in accordance with the adopted sign conventions:

- a negative axial force N_x signifies compression;
- negative torsional moment M_x signifies clockwise rotation;
- a negative bending moment M_y signifies tension of the upper fibers and compression of the lower fibers;
- a negative bending moment M_z signifies tension of the right fibers and compression of the left fibers.

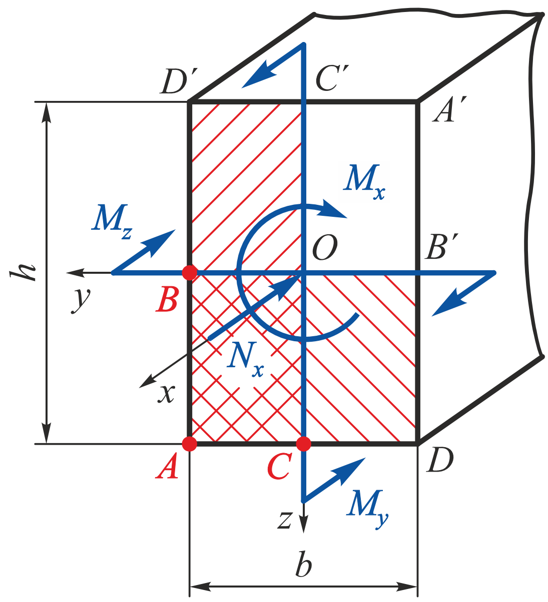


Fig. 6

Let us determine the potentially critical points of the cross-section. Select the triaxially compressed quarter (since $N_x < 0$) of the cross-section (the shaded area in Fig. 6) and mark its three corner points A, B, and C. These will be the potentially critical points.

We construct the normal and shear stress distribution diagrams across the section (Fig. 7–10).

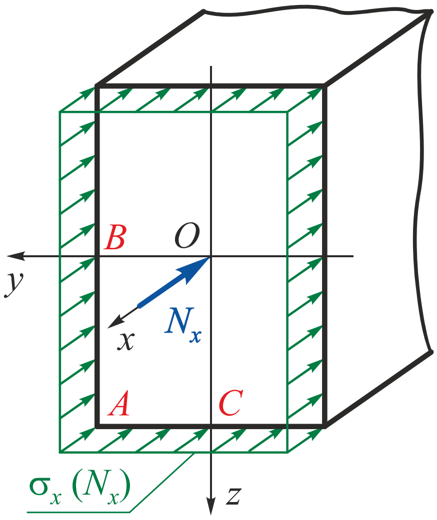


Fig. 7

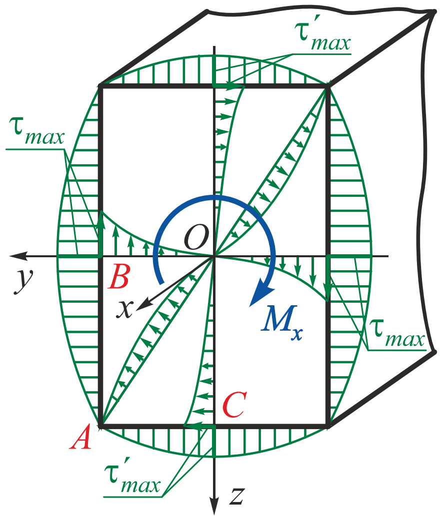


Fig. 8

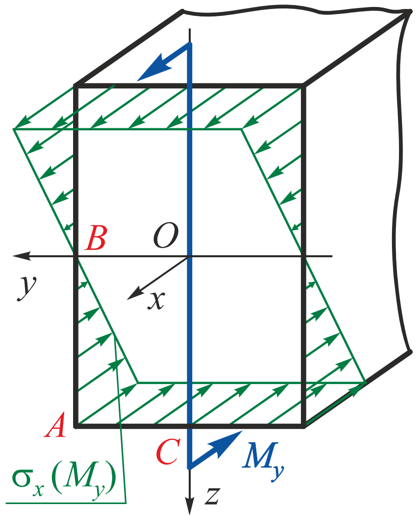


Fig. 9

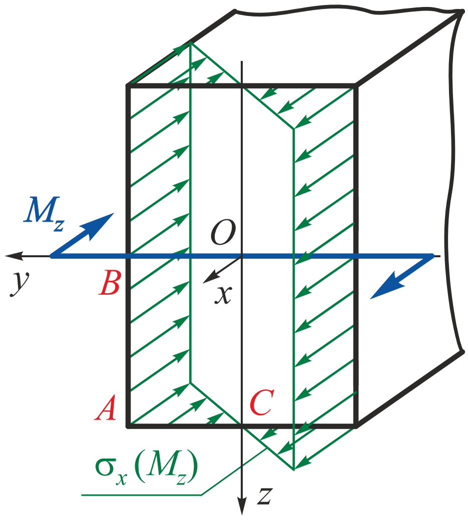


Fig. 10

For each of the three potentially critical points (*A*, *B*, and *C*) of the cross-section, we show the stress state type and write the strength conditions (neglecting the influence of the axial force N_x).

Point *A*

At point *A* of the cross-section, a uniaxial stress state is realized (Fig. 11).

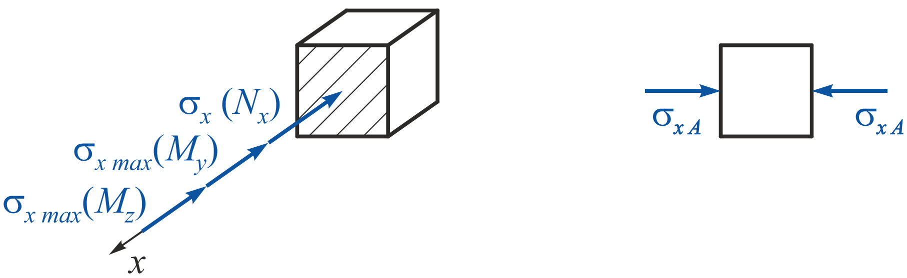


Fig. 11

The strength condition at this point is written as:

$$\sigma_{x\,A} = \sigma_{x_{max}} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \le [\sigma],$$

where $W_y = \dfrac{bh^2}{6} = \left\{ \text{since } k = \dfrac{h}{b} \implies h = kb \right\} = \dfrac{k^2 b^3}{6};$

$W_z = \dfrac{hb^2}{6} = \left\{ \text{since } k = \dfrac{h}{b} \implies h = kb \right\} = \dfrac{kb^3}{6}.$

After substituting the values of W_y and W_z into the strength condition and performing transformations, we obtain:

$$b \ge \sqrt[3]{\frac{6|M_y| + 6k|M_z|}{k^2[\sigma]}} = \sqrt[3]{\frac{6 \cdot |-15 \times 10^3| + 6 \cdot 1.5 \cdot |-1 \times 10^3|}{1.5^2 \cdot 180 \times 10^6}} = 0.0625\ m.$$

Point *B*

At point B of the cross-section, a plane stress state is realized (Fig. 12).

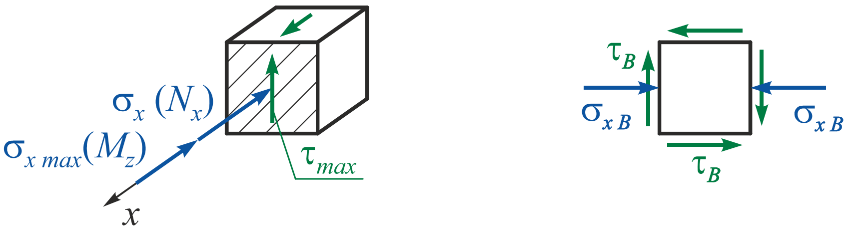


Fig. 12

Using the third strength theory, write the strength condition:

$$\sigma_{eq\,B}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{W_{torsional}}\right)^2} \le [\sigma],$$

where $W_z = \dfrac{hb^2}{6} = \left\{ \text{since } k = \dfrac{h}{b} \implies h = kb \right\} = \dfrac{kb^3}{6};$

$W_{torsional} = \alpha h b^2 = \left\{ \text{since } k = \dfrac{h}{b} \implies h = kb \right\} = \alpha k b^3;$

$\alpha = 0.231$ is a coefficient that depends on the ratio $h/b = 1.5$ (see Appendix).

After substituting the values of W_z and $W_{torsional}$ into the strength condition, we obtain:

$$b^{\text{III}} \ge \sqrt[6]{\frac{\left(\frac{6M_z}{k}\right)^2 + 4\left(\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{\left(\frac{6 \cdot (-1 \times 10^3)}{1.5}\right)^2 + 4\left(\frac{-10 \times 10^3}{0.231 \cdot 1.5}\right)^2}{(180 \times 10^6)^2}} = 0.0685\ m.$$

Point *C*

At point *C* of the cross-section, a plane stress state is realized (Fig. 13).

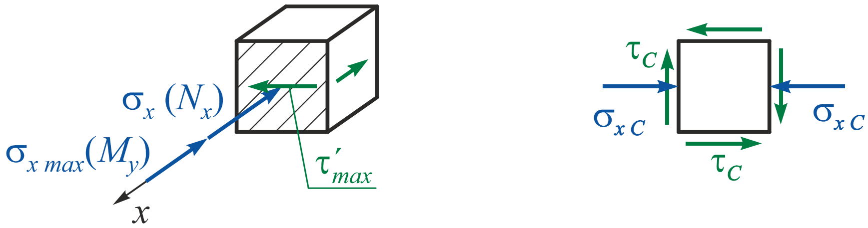


Fig. 13

Using the third strength theory, we write the strength condition:

$$\sigma_{eq\,C}^{III} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{M_y}{W_y}\right)^2 + 4\left(\gamma\frac{M_x}{W_{torsional}}\right)^2} \le [\sigma],$$

where $W_y = \dfrac{bh^2}{6} = \left\{\text{since } k = \dfrac{h}{b} \;\Rightarrow\; h = kb\right\} = \dfrac{k^2b^3}{6}$;

$W_{torsional} = \alpha h b^2 = \left\{\text{since } k = \dfrac{h}{b} \;\Rightarrow\; h = kb\right\} = \alpha k b^3$;

$\alpha = 0.231$, $\gamma = 0.859$ are coefficients that depend on the ratio $h/b = 1.5$ (see Appendix).

After substituting the values of W_y and $W_{torsional}$ into the strength condition, we obtain

$$b^{III} \ge \sqrt[6]{\frac{\left(\frac{6M_y}{k^2}\right)^2 + 4\left(\gamma\frac{M_x}{\alpha k}\right)^2}{[\sigma]^2}} = \sqrt[6]{\frac{\left(\frac{6\cdot(-15\times10^3)}{1.5^2}\right)^2 + 4\left(0.859\cdot\frac{-10\times10^3}{0.231\times1.5}\right)^2}{(180\cdot10^6)^2}} =$$

$$= 0.0707\ m.$$

We choose the largest of the three dimensions for b.

The calculated dimensions of the rectangular cross-section and its geometric characteristics are:

$b = 0.0707\ m,\qquad h = kb = 1.5\cdot0.0707 = 0.1061\ m;$

$F = bh = 0.0707\cdot0.1061 = 0.0075\ m^2;$

$$W_y = \frac{bh^2}{6} = \frac{0.0707\cdot0.1061^2}{6} = 1.3265\times10^{-4}\ m^3;$$

$$W_z = \frac{hb^2}{6} = \frac{0.1061\cdot0.0707^2}{6} = 0.8839\times10^{-4}\ m^3;$$

$W_{torsional} = \alpha h b^2 = 0.231\cdot0.1061\cdot0.0707^2 = 1.2251\cdot10^{-4}\ m^3.$

Point C is identified as the most critical point of the cross-section.

We determine the actual design stresses in the critical point taking into account the action of the axial force N_x

$$\sigma_{eq\,C}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_y|}{W_y}\right)^2 + 4\left(\gamma \frac{M_x}{W_{torsional}}\right)^2} =$$

$$= \sqrt{\left(\frac{|-2 \times 10^3|}{0.0075} + \frac{|-15 \times 10^3|}{1.3265 \times 10^{-4}}\right)^2 + 4\left(0.859 \cdot \frac{-10 \times 10^3}{1.2251 \times 10^{-4}}\right)^2} = 180.31\ MPa.$$

The overstress is

$$\Delta\sigma\ \% = \frac{\sigma_{eq\,C}^{\text{III}} - [\sigma]}{[\sigma]} \cdot 100\ \% = \frac{180.31 - 180}{180} \cdot 100\ \% = 0.17\ \% < 5\ \%.$$

Thus, the strength of the cranked bar is ensured.

For the critical cross-section, we construct the diagram of normal stress distribution and show the position of the cross-section neutral axis.

Let us determine the normal stresses at the corner points of the cross-section, having calculated in advance the stress contributions from each of the internal loads N_x, M_y, and M_z:

$$|\sigma_x(N_x)| = \frac{N_x}{F} = \frac{2 \times 10^3}{0.0075} = 0.27\ MPa;$$

$$|\sigma_{x\,max}(M_y)| = \frac{M_y}{W_y} = \frac{15 \times 10^3}{1.3265 \times 10^{-4}} = 113.08\ MPa;$$

$$|\sigma_{x\,max}(M_z)| = \frac{M_z}{W_z} = \frac{1 \times 10^3}{0.8839 \times 10^{-4}} = 11.31\ MPa.$$

Then

$$\sigma_{x\,A} = -\sigma_x(N_x) - \sigma_{x\,max}(M_y) - \sigma_{x\,max}(M_z) = -0.27 - 113.08 - 11.31 =$$
$$= -124.66\ MPa;$$

$$\sigma_{x\,D} = -\sigma_x(N_x) - \sigma_{x\,max}(M_y) + \sigma_{x\,max}(M_z) = -0.27 - 113.08 + 11.31 =$$
$$= -102.04\ MPa;$$

$$\sigma_{x\,A'} = -\sigma_x(N_x) + \sigma_{x\,max}(M_y) + \sigma_{x\,max}(M_z) = -0.27 + 113.08 + 11.,31 =$$
$$= 124.12\ MPa;$$

$$\sigma_{x\,D'} = -\sigma_x(N_x) + \sigma_{x\,max}(M_y) - \sigma_{x\,max}(M_z) = -0.27 + 113.08 - 11.31 =$$
$$= 101.50\ MPa.$$

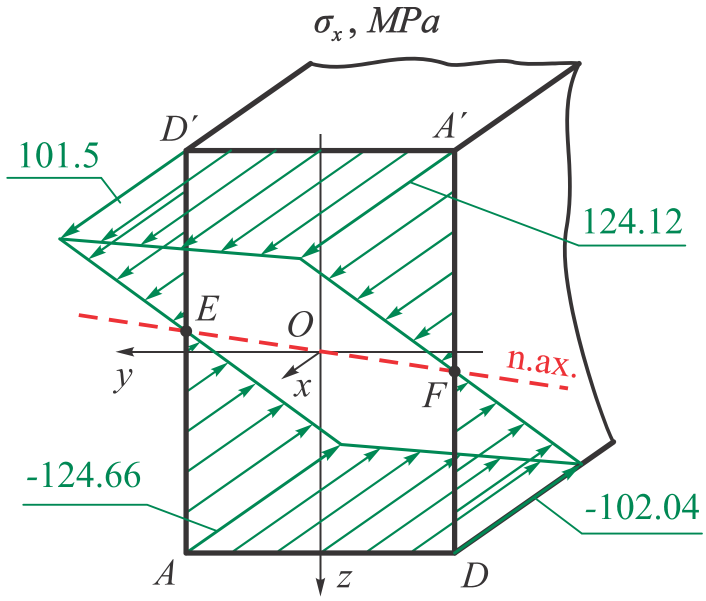


Fig. 14

Let us draw the rectangular cross-section to scale according to the obtained dimensions ($b = 0.0707\ m$, $h = 0.1061\ m$) and construct the normal stress diagram (Fig. 14).

To determine the neutral axis position, we connect the intersection points of the diagram with the cross-sectional plane (E and F) with a straight line.

7. We determine the diameter of the circular cross-section.

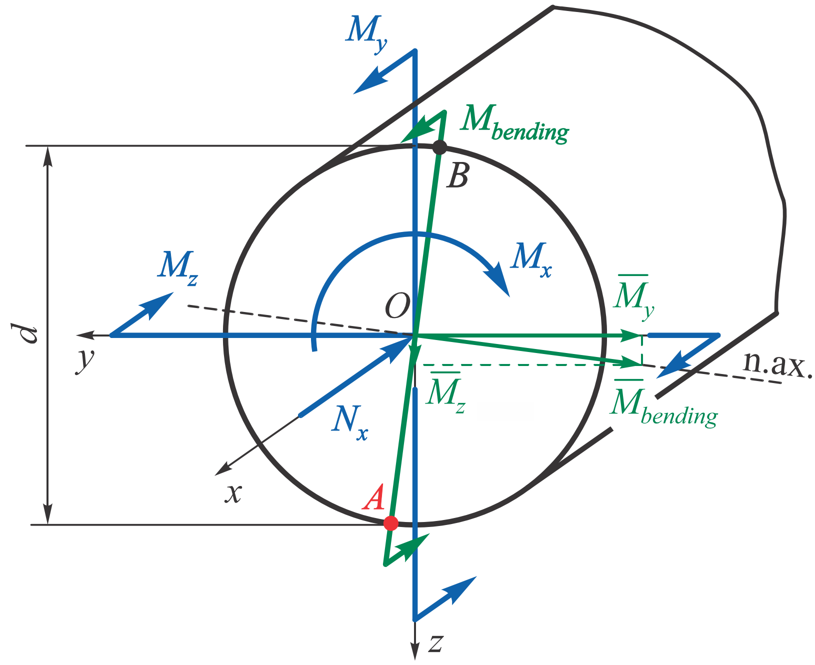


Fig. 15

Since all axes passing through the circular cross-section's centroid are its principal central axes of inertia, the bending should be considered in the plane of the total bending moment $M_{bending} = \sqrt{M_y^2 + M_z^2}$, which determines the position of the points with the maximum bending normal stresses (points A and B) (Fig. 15).

Due to the compressive normal stresses caused by the axial force N_x, the maximum value of the normal stress occurs at point A, where the stresses from the axial force and the bending moment are additive. At the same time, this point is also the location of maximum shear stresses due to torsion, since it lies on the circumference of the cross-section.

Thus, point A is the only critical point of the circular cross-section.

Since the stress state at the considered point is plane, the strength analysis must be performed according to one of the strength theories (the third or the fourth) (Fig. 16).

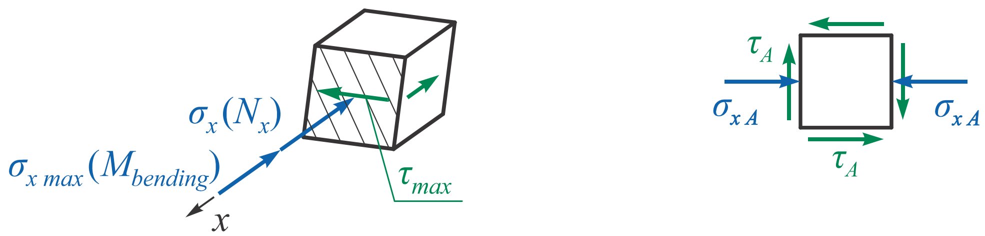


Fig. 16

Using the ***third strength theory***, we write the strength condition:

$$\sigma_{eq\,A}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{\text{n.ax.}}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \le [\sigma],$$

where $M_{bending} = \sqrt{M_y^2 + M_z^2}$ is the total bending moment;

$F = \dfrac{\pi d^2}{4}$ is the cross-sectional area;

$W_{\text{n.ax.}} = \dfrac{\pi d^3}{32}$ is the ***section modulus*** relative to the neutral axis;

$W_\rho = \dfrac{\pi d^3}{16} = 2W_{\text{n.ax.}}$ is the ***polar moment of inertia***.

In the first approximation, to determine the diameter d, we write the strength condition without taking into account the effect of the axial force N_x

$$\sigma_{eq_A}^{\text{III}} = \sqrt{\left(\frac{M_{bending}}{W_{\text{n.ax.}}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} = \sqrt{\frac{M_{bending}^2}{W_{\text{n.ax.}}^2} + \frac{M_x^2}{W_{\text{n.ax.}}^2}} = \frac{\sqrt{M_{bending}^2 + M_x^2}}{W_{\text{n.ax.}}} \le [\sigma].$$

We denote:

$$M_{design}^{\text{III}} = \sqrt{M_{bending}^2 + M_x^2} = \sqrt{M_y^2 + M_z^2 + M_x^2} = \sqrt{15^2 + 1^2 + 10^2} = 18.055\ kN{\cdot}m,$$

where M_{design}^{III} is the design moment determined according to the third strength theory.

Then the condition of strength takes the form:

$$\sigma_{eq_A}^{\text{III}} = \frac{M_{design}^{\text{III}}}{W_{\text{n.ax.}}} = \frac{32M_{design}^{\text{III}}}{\pi d^3} \le [\sigma],$$

from which

$$d^{\text{III}} \ge \sqrt[3]{\frac{32M_{design}^{\text{III}}}{\pi[\sigma]}} = \sqrt[3]{\frac{32 \cdot 18.055 \times 10^3}{\pi \cdot 180 \times 10^6}} = 0.1007\ m.$$

Let us determine the actual design stresses in the critical point taking into account the action of the axial force N_x:

$$M_{bending} = \sqrt{M_y^2 + M_z^2} = \sqrt{15^2 + 1^2} = 15.033\ kN{\cdot}m;$$

$$F = \frac{\pi d^2}{4} = \frac{\pi \cdot 0.1007^2}{4} = 7.964 \times 10^{-3}\ m^2;$$

$$W_{\text{n.ax.}} = \frac{\pi d^3}{32} = \frac{\pi \cdot 0.1007^3}{32} = 1.0025 \times 10^{-4}\ m^3;$$

$$W_\rho = \frac{\pi d^3}{16} = 2W_{\text{n.ax.}} = \frac{\pi \cdot 0.1007^3}{16} = 2.005 \times 10^{-4}\ m^3;$$

$$\sigma_{eq\,A}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{\left(\frac{|N_x|}{F} + \frac{|M_{bending}|}{W_{\text{n.ax.}}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} =$$

$$= \sqrt{\left(\frac{|-2 \times 10^3|}{7.964 \times 10^{-3}} + \frac{|-15.033 \times 10^3|}{1.0025 \times 10^{-4}}\right)^2 + 4\left(\frac{-10 \times 10^3}{2.005 \times 10^{-4}}\right)^2} = 180.31\ MPa.$$

The overstress is:

$$\Delta\sigma\ \% = \frac{\sigma_{eq\,A}^{\text{III}} - [\sigma]}{[\sigma]} \cdot 100\ \% = \frac{180.31 - 180}{180} \cdot 100\ \% = 0.17\ \% < 5\ \%.$$

Thus, the strength of the cranked bar is ensured.

Let us determine the diameter of the section using ***the fourth strength theory***:

$$M_{design}^{\text{IV}} = \sqrt{M_{bending}^2 + 0.75 \cdot M_x^2} = \sqrt{M_y^2 + M_z^2 + 0.75 \cdot M_x^2} =$$

$$= \sqrt{15^2 + 1^2 + 0.75 \cdot 10^2} = 17.349\ kN{\cdot}m;$$

$$d^{\text{IV}} \geq \sqrt[3]{\frac{32 M_{design}^{\text{IV}}}{\pi[\sigma]}} = \sqrt[3]{\frac{32 \cdot 17.349 \times 10^3}{\pi \cdot 180 \times 10^6}} = 0.0994\ m.$$

8. Let us ***compare the weight*** of bars with rectangular and circular cross-sections:

$$\frac{G^{\bigcirc}}{G^{\square}} = \frac{F^{\bigcirc}}{F^{\square}} = \frac{7.964 \cdot 10^{-3}}{0.0075} = 1.062.$$

Thus, for the given combination of internal forces and moments and the rectangle aspect ratio $k = h/b = 1.5$, it is more beneficial to use a rectangular cross-section to reduce the weight of the structure.

At the same time, the largest overall dimension of the rectangular cross-section exceeds the diameter of the circular one: $h = 0.1061\ m > d = 0.1007\ m$.

PROBLEM VARIANTS

Figure 1 (Table 1)

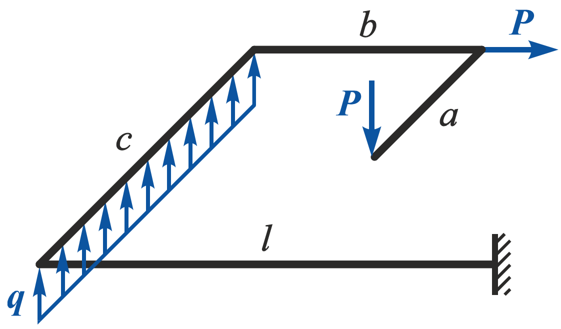
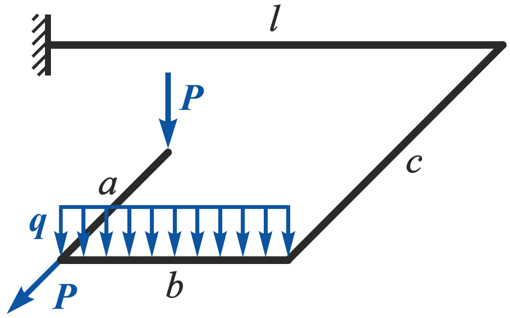


Figure 2 (Table 1)



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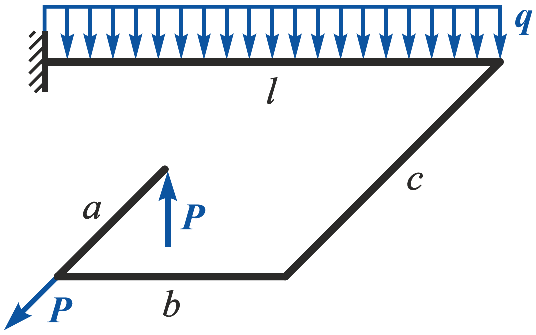

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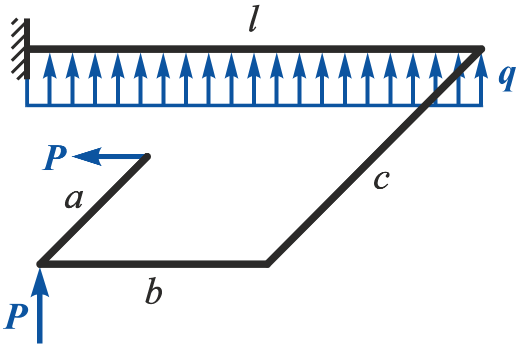

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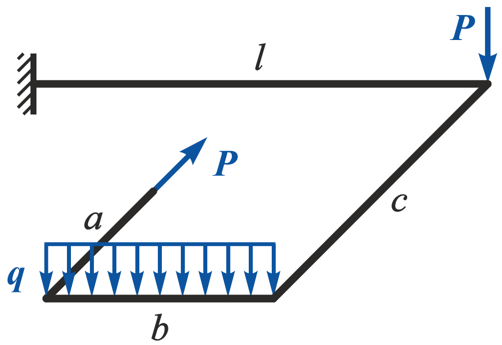

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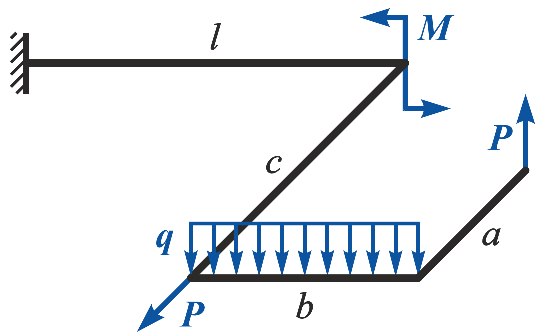

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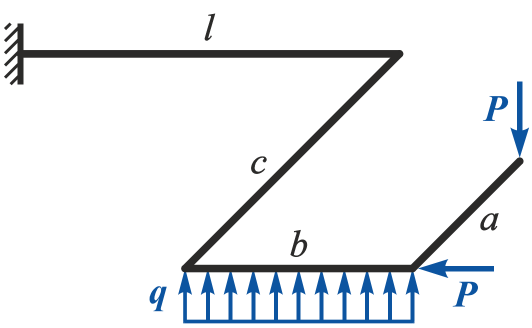

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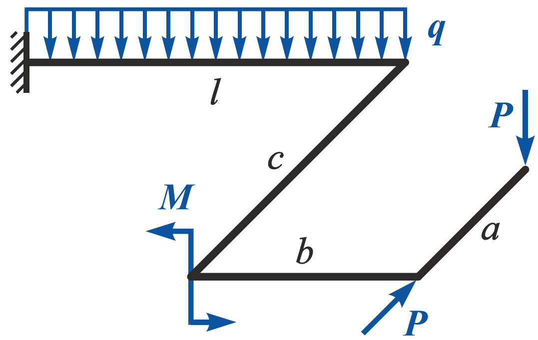

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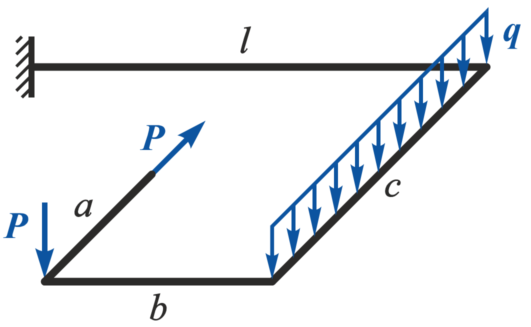


Figure 10 (Table 1)

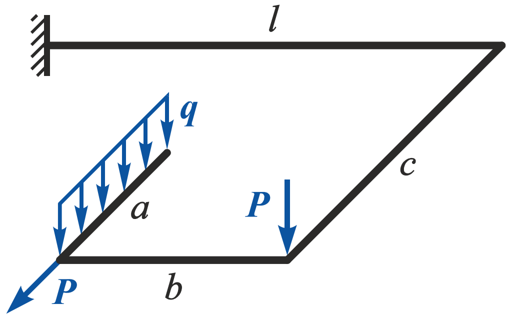
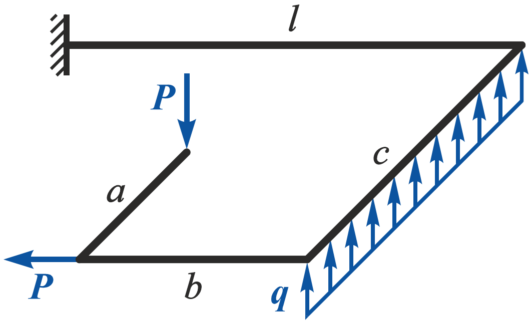


Figure 11 (Table 1)



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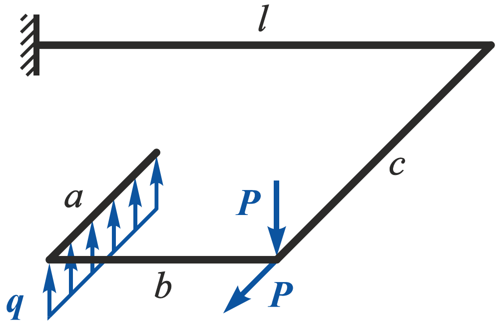
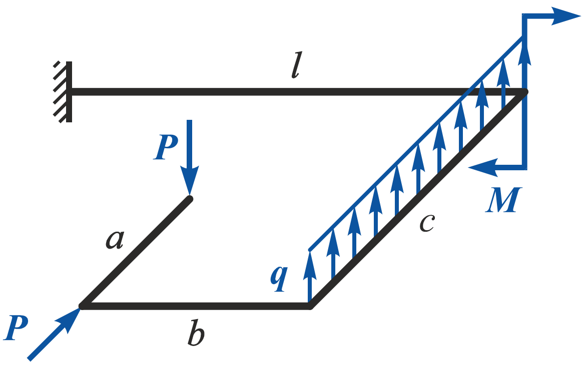


Figure 13 (Table 2)



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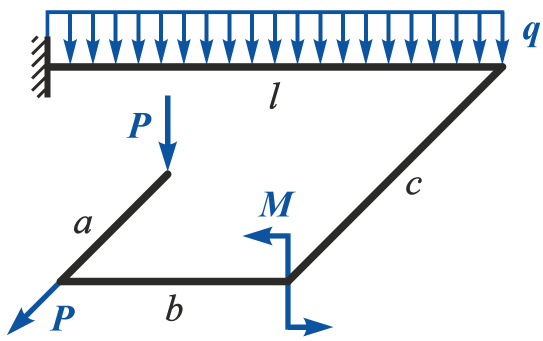

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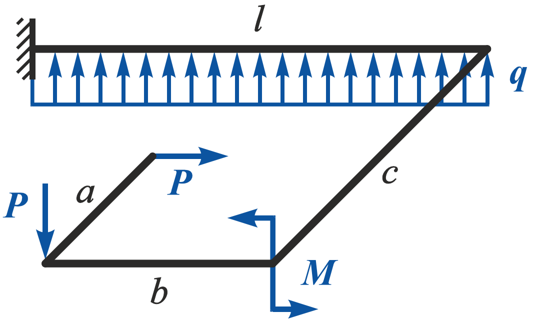

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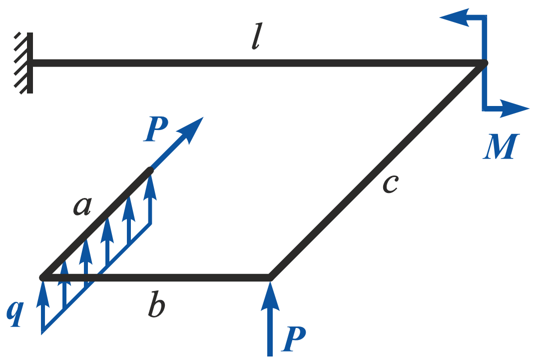

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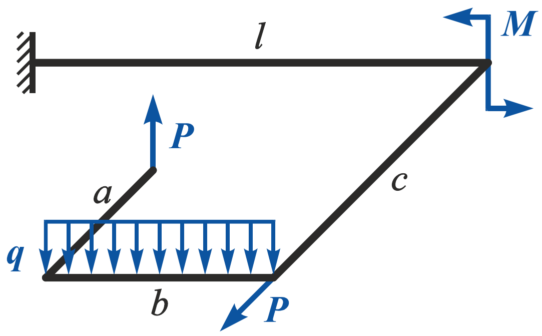

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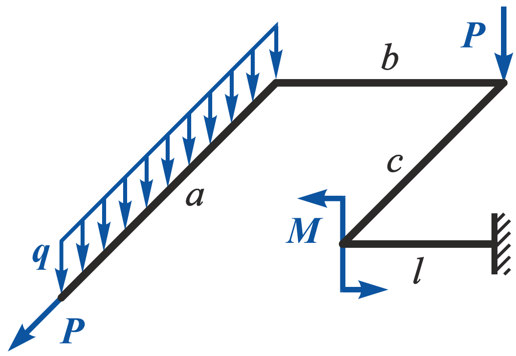

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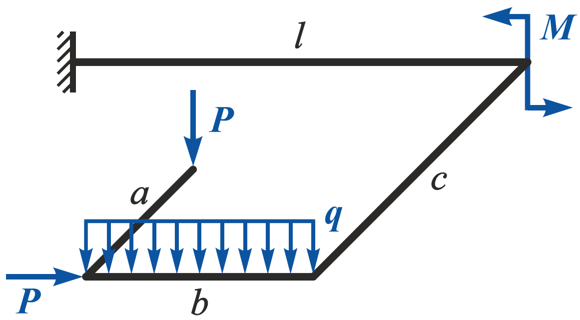

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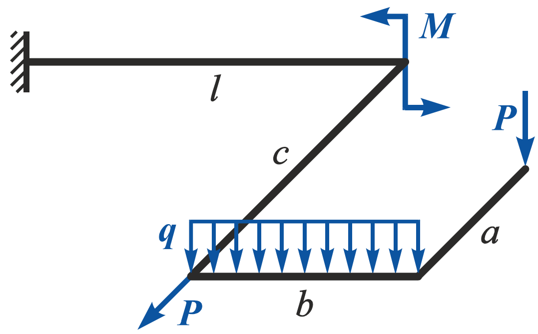
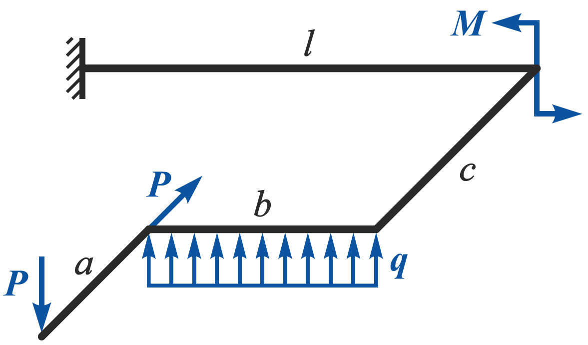


Figure 21 (Table 3)



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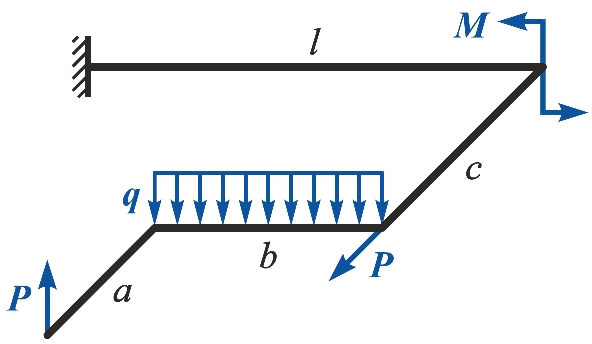

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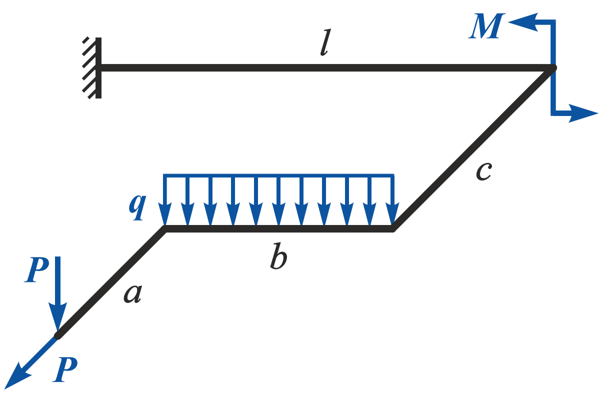

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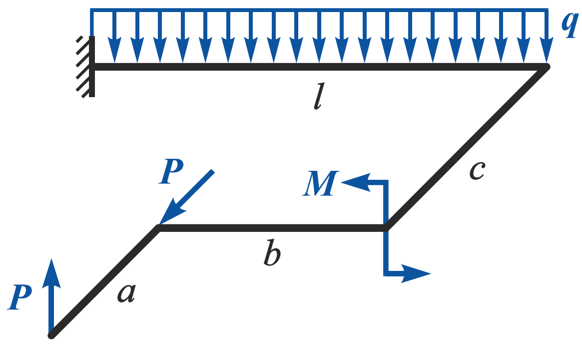

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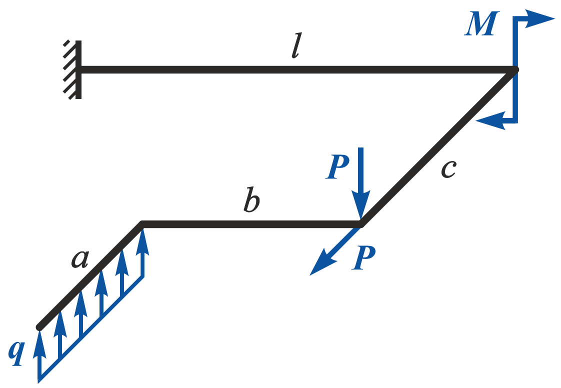

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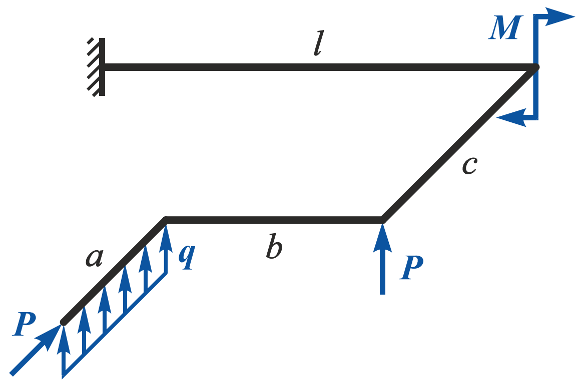

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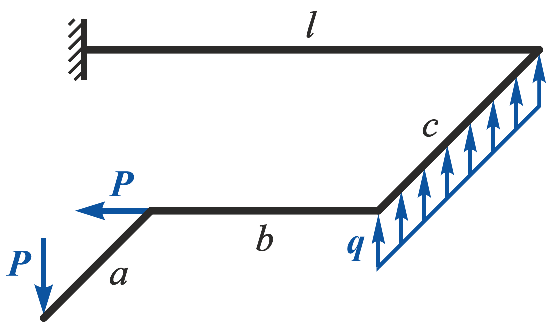

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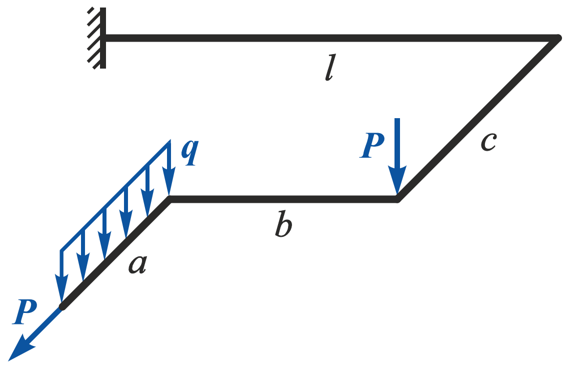

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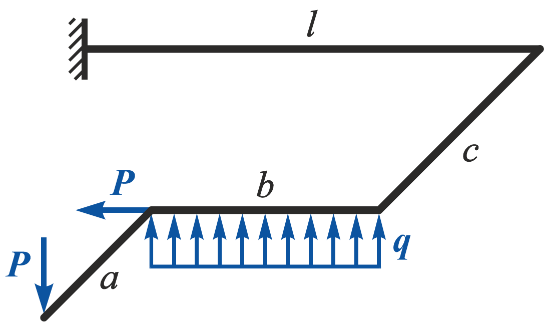

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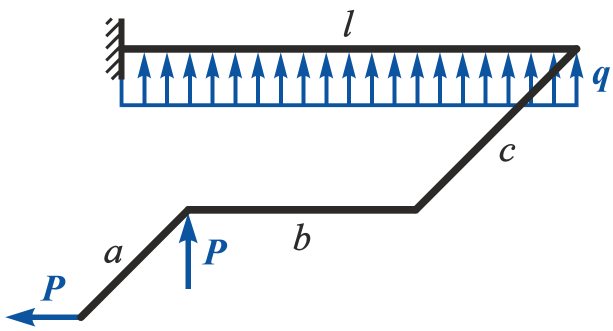

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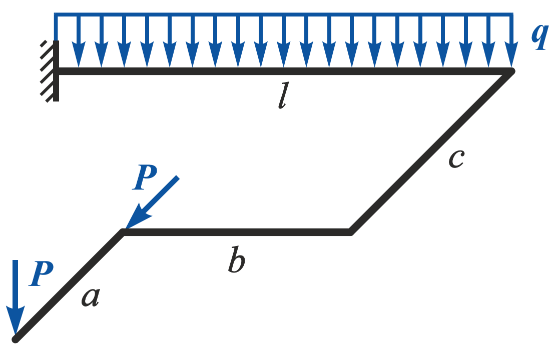

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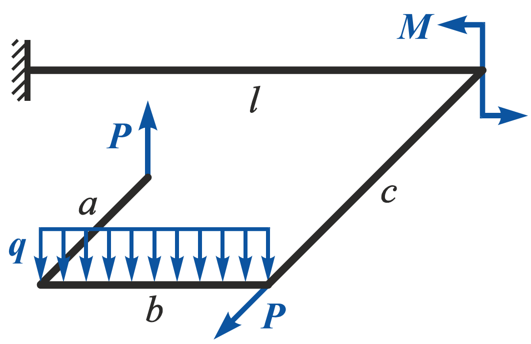

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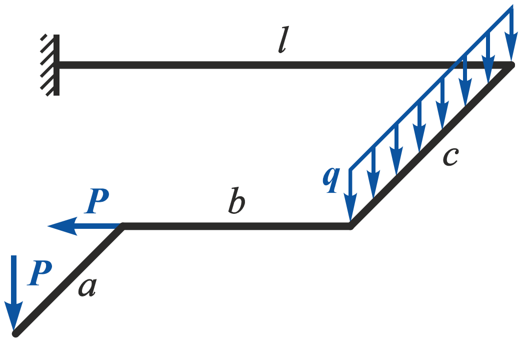

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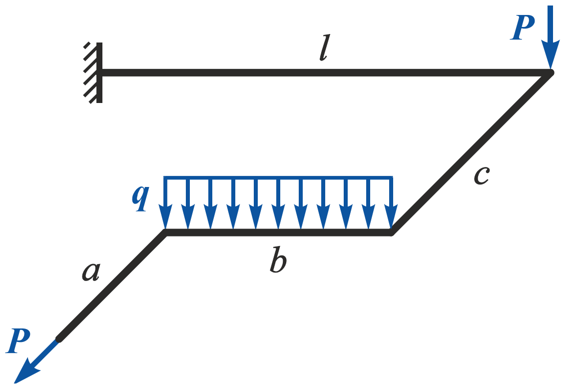

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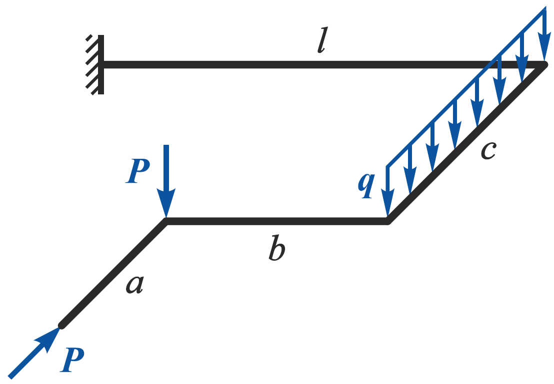

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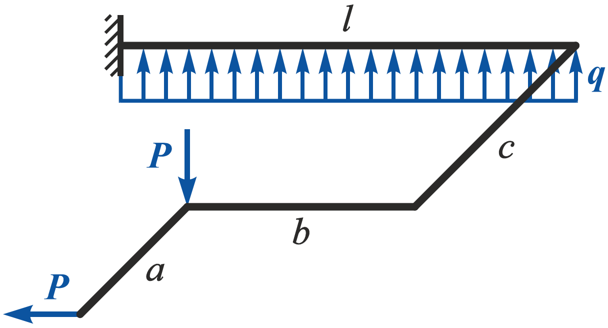

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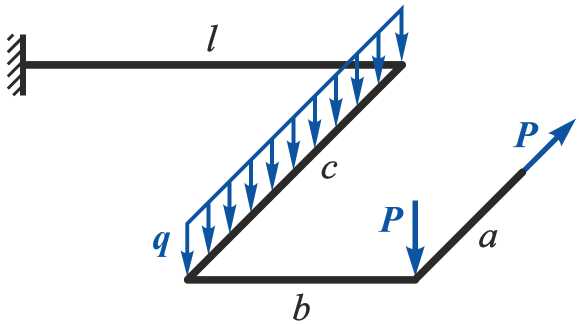

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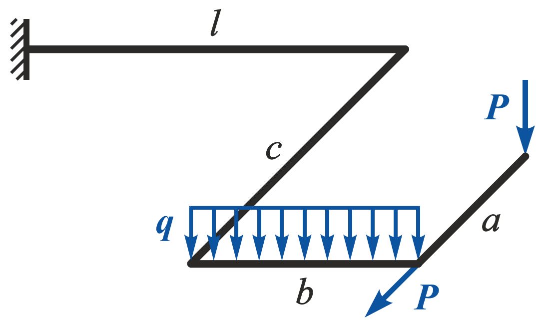

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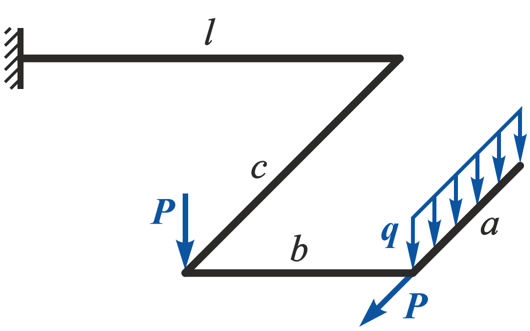

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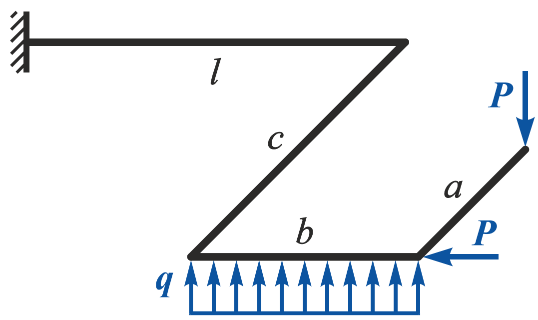

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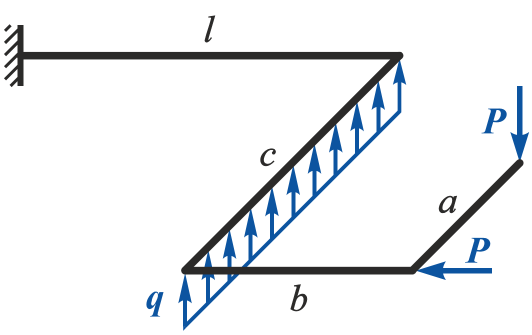

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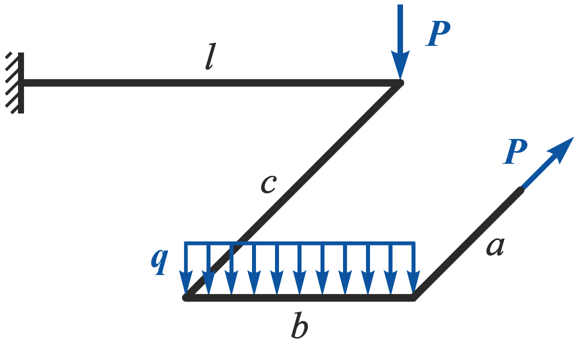

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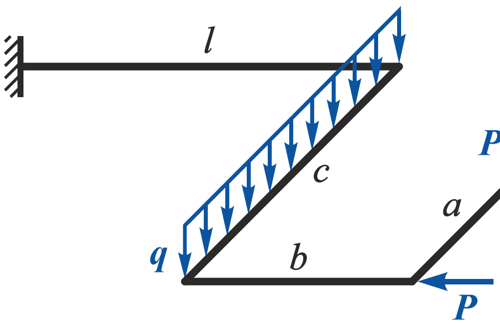

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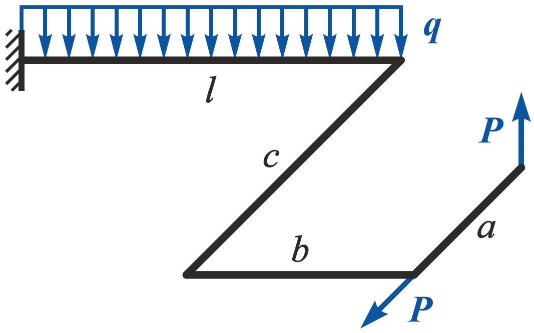

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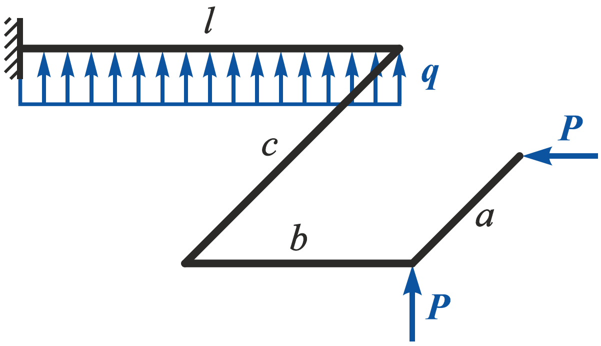

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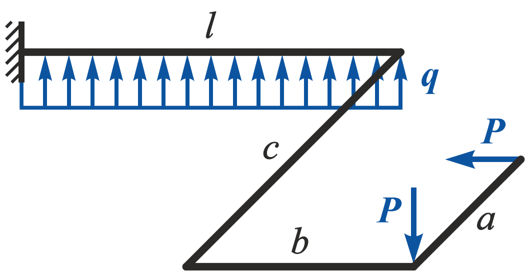

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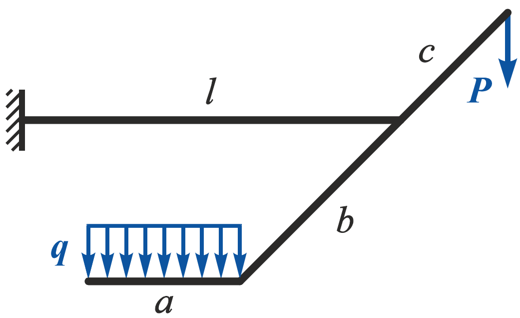

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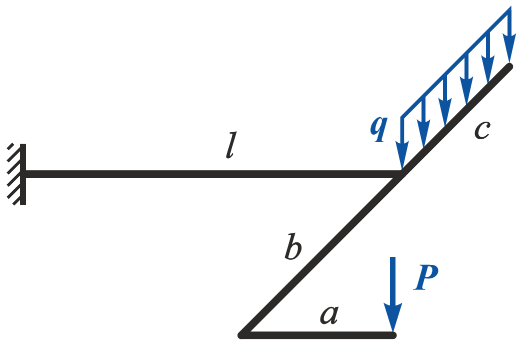
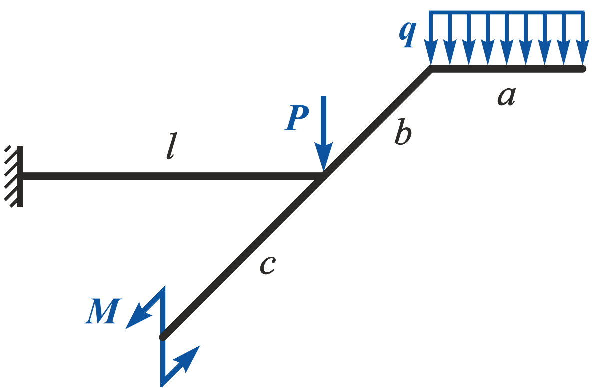


Figure 49 (Table 8)



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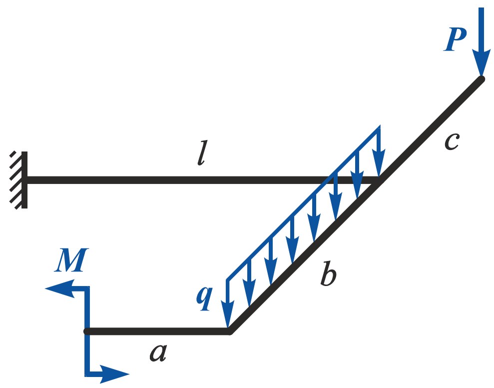

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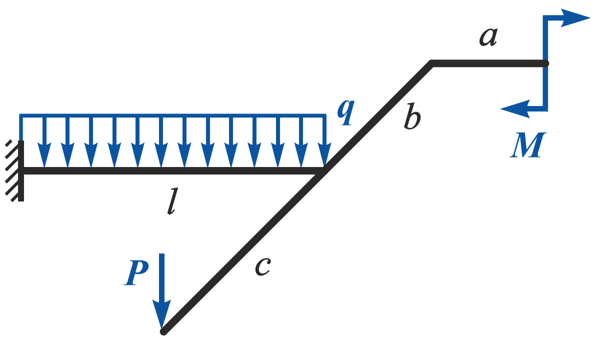

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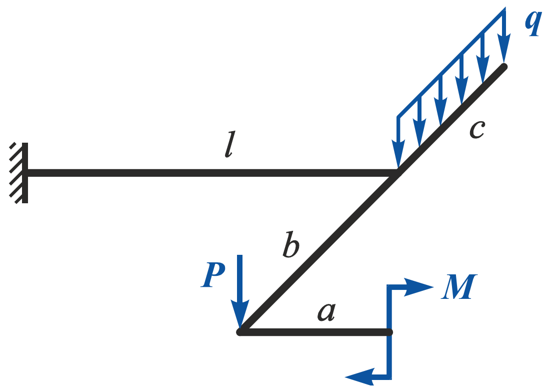

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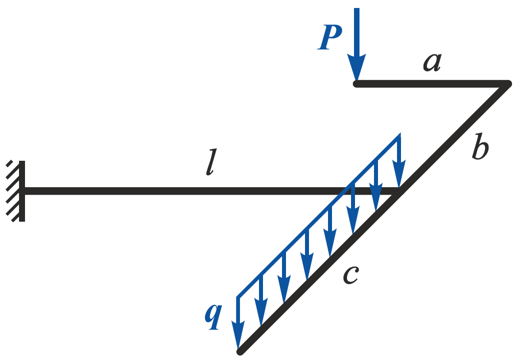

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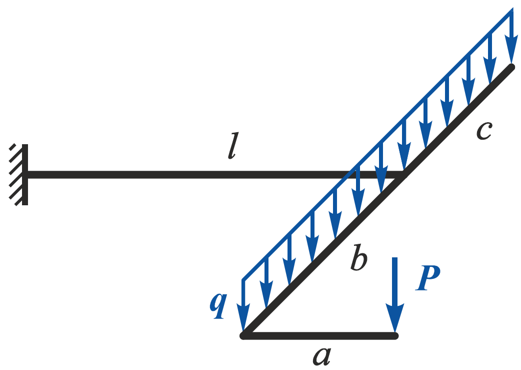

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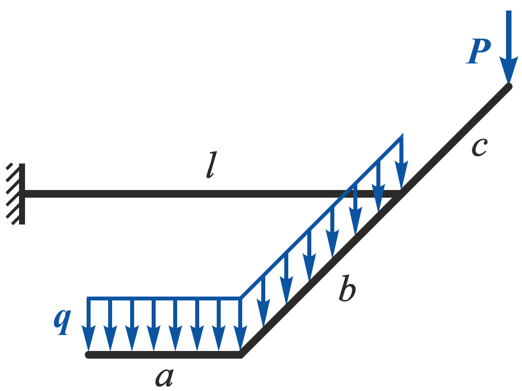

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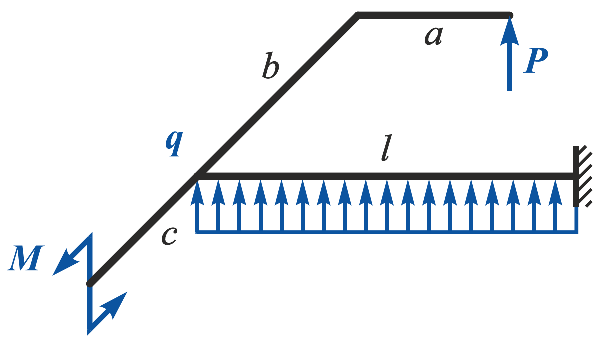

Table 1

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	0.5	0.8	0.5	0.8	0.5	0.2	1	1.2	0.5	0.8	0.8	1	1	1.1	0.4	0.6	1.5	0.4	1	1.2
b, *m*	0.5	0.4	1	0.8	0.5	1.2	1.5	1.2	1.5	1.5	1.5	1.2	1.5	1.2	1	0.8	1	1.2	1.5	1.2
c, *m*	1.5	1.2	1.5	1.2	1	0.3	0.5	0.4	1	0.9	1	1.2	1.5	1.2	0.5	0.4	0.5	0.4	0.5	0.4
l, *m*	1.5	1.2	1.5	1.2	1	0.8	1	0.8	1	0.8	1	0.8	1	0.8	0.5	0.4	0.5	0.4	0.5	0.4
P, *kN*	10	5	20	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	12	8
q, *kN/m*	5	10	5	4	10	20	8	4	10	25	6	8	5	4	10	15	10	10	8	12

Table 2

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	0.5	0.5	0.5	1	0.5	0.5	1	1.5	0.5	1.5	0.5	1	1.5	0.5	1	1.5	0.5	0.5	0.5	0.5
b, *m*	0.5	0.5	1	1	0.5	1.5	1.5	1.5	1.5	1.2	1.5	1.5	1.5	1	1	1.5	0.5	0.5	0.5	0.5
c, *m*	1.5	1.5	1.5	1.5	1	1	1.5	1.5	1	1.5	1.5	1.5	1.5	0.7	1.2	2	1	1.5	1.5	1.5
l, *m*	1.5	1.5	2	1.5	1	1	1	0.5	1.5	1	1	1	1	0.5	0.5	2	1.5	0.5	1	1.5
P, *kN*	10	5	10	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	10	8
M, *kN·m*	10	15	20	30	10	15	20	30	10	15	20	30	10	15	20	30	10	15	20	30
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12

Table 3

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1	1	0.5	0.5	0.5	0.5	0.5	1	1.5	1.5
b, *m*	1.5	1.5	1.5	1.5	1	1	1	1	1	0.5	1.5	1	1.5	1	0.5	0.5	1	1.6	0.5	0.5
c, *m*	0.5	0.5	1	1	0.5	0.5	1	1	1	1	1	0.5	1	1	1	1.5	1.5	1.5	0.5	1
l, *m*	0.5	1	0.5	1	0.5	0.8	0.5	0.8	0.5	0.4	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	1
P, *kN*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12
M, *kN·m*	10	15	20	30	10	15	20	30	10	15	20	25	30	10	15	20	25	30	10	15
q, *kN/m*	10	5	20	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	10	8

Table 4

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1	1	0.5	0.5	0.5	0.5	0.5	1	1.5	1.5
b, *m*	1.8	1.5	1.5	1.5	1	1	1	1	0.5	0.5	1.5	1	1.5	1	0.5	0.5	1	1	0.5	0.5
c, *m*	0.5	0.5	1	1	0.5	0.5	1	1	1	1	1	0.5	1	1	1	1.5	1.5	1.5	0.5	1
l, *m*	0.5	1	0.5	1	0.5	0.8	0.5	0.8	0.5	0.4	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	1	1
P, *kN*	10	5	10	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	10	8
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	8	8	5	4	10	10	10	10	8	12

Table 5

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	1	1	1	0.5	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	1	1.5	1.5	1.5	1.5	0.5
b, *m*	1.5	1	1.5	1	0.5	0.5	1	1	1	1.5	1	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5
c, *m*	0.5	0.5	1	1	1	1.5	1.5	1.5	0.6	1	1	1	1.5	1.5	1.5	0.5	1	1	0.5	1
l, *m*	1	1	1	1	1	1	1	1	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1	1	0.5
P, *kN*	10	5	20	10	10	5	10	12	8	5	4	5	5	5	20	10	5	8	10	8
M, *kN·m*	10	15	20	25	30	10	15	20	25	30	10	15	20	25	30	10	15	20	25	30
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12

Table 6

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	1	1	1	0.5	0.5	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	1	1.5	1.5	1.5	1.5	0.5
b, *m*	1.5	1	1.5	1	0.5	0.5	1	1	1	1.5	1	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5
c, *m*	0.5	0.5	1	1	1	1.5	1.5	1.5	0.5	1	1	1	1.5	1.5	1.5	0.5	1	1	0.5	1
l, *m*	1	1	1	1	1	1	1	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1	1	0.5
P, *kN*	10	5	12	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	10	8
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12

Table 7

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	0.5	0.5	0.5	1.5	0.5	0.5	1	1	1.5	1	1	0.5	0.5	1	1.5	1	1	1.5	0.5	0.5
b, *m*	0.5	0.5	1	1	1.5	1	0.5	0.5	1	0.5	1	1.5	1	1	0.5	1.5	1	0.5	0.5	1
c, *m*	1	0.5	1	1	0.5	0.5	0.5	1	0.5	0.5	0.5	1	1.5	1.5	0.5	0.5	0.5	0.5	1	1.5
l, *m*	1	1	2	2	1.8	1	2	2	2	1.5	1.5	1	1.5	1.5	2	1.5	1.5	2	1.5	1.5
P, *kN*	10	5	12	10	10	5	10	12	8	5	6	5	20	10	5	8	5	8	10	8
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12

Table 8

Variant number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a, *m*	0.5	0.5	0.5	1.5	0.5	0.5	1	1	1.5	1	1	0.5	0.5	1	1.5	1	1	1.5	0.5	0.5
b, *m*	0.5	0.5	1	1	1.5	1	0.5	0.5	1	0.5	1	1.5	1	1	0.5	1.5	1	0.5	0.5	1
c, *m*	1	0.5	1	1	0.5	0.5	0.5	1	0.5	0.5	0.5	1	1.5	1.5	0.5	0.5	0.5	0.5	1	1.5
l, *m*	1	1	2	2	1.8	1	2	2	2	1.5	1.5	1	1.5	1.5	2	1.5	1.5	2	1.5	1.5
P, *kN*	10	5	12	10	10	5	10	12	8	5	4	5	6	5	20	10	5	8	10	8
M, *kN·m*	10	15	20	25	30	10	15	20	25	30	10	15	20	25	30	10	15	20	25	30
q, *kN/m*	5	10	5	4	10	10	8	4	10	10	6	8	5	4	10	10	10	10	8	12

TORSION OF A BAR WITH A RECTANGULAR CROSS-SECTION

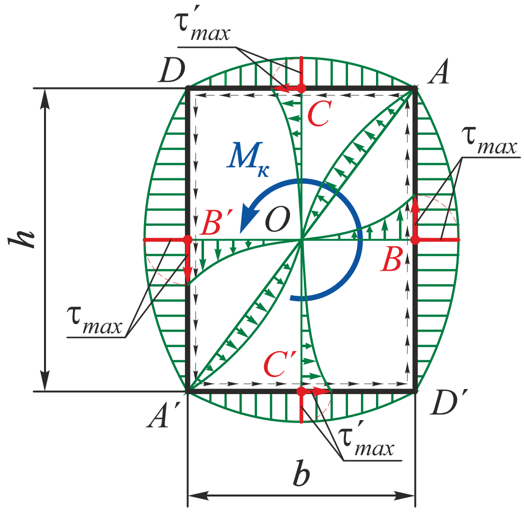

Fig. A.1

At the middle of the long sides:

$$\tau_{max} = \tau_B = \tau_{B'} = \frac{M_{torsional}}{W_{torsional}}.$$

At the middle of the short sides:

$$\tau'_{max} = \tau_C = \tau_{C'} = \gamma \tau_{max}.$$

At the corner points and at the center:

$$\tau_A = \tau_{A'} = \tau_D = \tau_{D'} = \tau_O = 0.$$

Twisting angle:

$$\varphi = \frac{M_{torsional} l}{G I_{torsional}}.$$

$$W_{torsional} = \alpha h b^2; \qquad I_{torsional} = \beta h b^3,$$

where $M_{torsional}$ is a torsional moment acting in the cross-section;
$W_{torsional}$ is torsional section modulus of the rectangular cross-section;
$I_{torsional}$ is torsional moment of inertia of the rectangular cross-section;
$GI_{torsional}$ is torsional stiffness of the rectangular cross-section;
α, β, γ are coefficients depending on the rectangle aspect ratio $k = h/b$.

$k = \frac{h}{b}$	α	β	γ
1,0	0,208	0,141	1,000
1,2	0,219	0,166	0,935
1,25	0,221	0,172	0,910
1,5	0,231	0,196	0,859
1,75	0,239	0,214	0,820
2,0	0,246	0,229	0,795
2,5	0,258	0,249	0,766
3,0	0,267	0,263	0,753
4,0	0,282	0,281	0,745
5,0	0,291	0,291	0,744
6,0	0,299	0,299	0,743
8,0	0,307	0,307	0,742
10,0	0,313	0,313	0,742
> 10,0	0,333	0,333	0,742

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