МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ
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ЛІНІЙНА АЛГЕБРА

Навчальний посібник

LINEAR ALGEBRA

Tutorial

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Розглянуто теоретичні та практичні питання, пов’язані з такими важливими розділами вищої математики, як векторна алгебра, аналітична геометрія і лінійна алгебра. Матеріал цих розділів подано згідно з програмою курсу "Вища математика", який викладається у вищих технічних навчальних закладах. Усі приклади наведено з розв’язанням. Останній розділ містить практичні задачі для самостійної роботи.

Теоретичний матеріал методично узгоджено з практичною частиною, до якої віднесено деякі найважливіші питання практичного характеру.

Для студентів першого курсу вищих технічних навчальних закладів.

Іл. 24. Бібліогр.: 15 назв.

Theoretical and practical questions connected with such important parts of the Higher Mathematics as Vector Algebra, and Analytic Geometry and Linear Algebra are considered. The material of these parts corresponds with the program of the "Higher Mathematics" course, which is taught in Technical Institutes. The last chapter contains problems and exercises to be solved by students.

Theoretical material methodologically corresponds with the practical part, which contains most important practical problems.

For the first year students of High Technical Educational Establishments.

Fig. 24. Bibliogr.: 15 items.

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Contents

DETERMINANTS

- **Definition and basic properties.**

Let us consider the system of two linear algebraic equations in two variables

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

Using the method of substitution we can write the solution of this system

$$x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \text{ and } y = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.$$

The numbers that are in numerator and denominator are often written as

$$a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ b_1a_{22} - b_2a_{12} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},\ b_2a_{11} - b_1a_{21} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

The number $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ is called the ***determinant of the second order***.

Analogously, let us consider the system of three linear algebraic equations in three variables

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1, \\ a_{21}x + a_{22}y + a_{23}z = b_2,. \\ a_{31}x + a_{32}y + a_{33}z = b_3. \end{cases}$$

Using the method of substitution we can write the solution of this system

$$x = \frac{b_1a_{22}a_{33} + a_{12}a_{23}b_3 + a_{13}b_2a_{32} - a_{13}a_{22}b_3 - a_{12}b_2a_{33} - b_1a_{23}a_{32}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}};$$

$$y = \frac{a_{11}b_2a_{33} + b_1a_{23}a_{31} + a_{13}a_{21}b_3 - a_{13}b_2a_{31} - b_1a_{21}a_{33} - a_{11}a_{23}b_3}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}};$$

$$z=\frac{a_{11}a_{22}b_3+a_{12}b_2a_{31}+b_1a_{21}a_{32}-b_1a_{22}a_{31}-a_{12}a_{21}b_3-a_{11}b_2a_{32}}{a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}}.$$

The numbers that are in numerator and denominator are often written as

$$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-$$
$$-a_{11}a_{23}a_{32}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$b_1a_{22}a_{33}+a_{12}a_{23}b_3+a_{13}b_2a_{32}-a_{13}a_{22}b_3-a_{12}b_2a_{33}-b_1a_{23}a_{32}=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$a_{11}b_2a_{33}+b_1a_{23}a_{31}+a_{13}a_{21}b_3-a_{13}b_2a_{31}-b_1a_{21}a_{33}-a_{11}a_{23}b_3=\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$a_{11}a_{22}b_3+a_{12}b_2a_{31}+b_1a_{21}a_{32}-b_1a_{22}a_{31}-a_{12}a_{21}b_3-a_{11}b_2a_{32}=\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

The number

$$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-$$
$$-a_{11}a_{23}a_{32}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is called the ***determinant of the third order***.

It easily can be proved that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

which is called the decomposition by the first row. Note that the signs in this decomposition are alternating. Now we can give the inductive definition to the determinant of any order.

1. $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

2.

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} -$$

$$- a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + \ldots + (-1)^n a_{1n} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2\ n-1} \\ a_{31} & a_{32} & \cdots & a_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\ n-1} \end{vmatrix}.$$

Thus, the determinants of the n^{th} order are expressed in the terms of the determinants of the $(n-1)^{th}$ order.

The properties of the determinants are as follows.

1. If all rows are substituted for all columns the determinant stays the same. This operation is called ***transposition***:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}.$$

   It follows from this property that every statement that is true for rows is also true for columns.
2. If one of the rows consists only of zeros then the determinant is equal to zero.
3. If a determinant is obtained from another by interchanging two rows then the determinants are distinguished by the sign.

$$\begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix} = -\begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

4. If a determinant contains two equal rows then it is equal to zero.
5. If all entries are multiplied by a number k then the determinant also multiplies by k:

$$\begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & \cdots & \cdots & \cdots & ka_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix} = k\begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \cdots & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

6. If two rows of a determinant are proportional then the determinant is equal to zero.
7. If all entries of a row of a determinant can be represented as $a_{ij} = b_j + c_j, j = \overline{1,n}$ then the determinant can be represented as a sum of two determinants:

$$\begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 + c_1 & \cdots & \cdots & b_n + c_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 & \cdots & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & \cdots & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

8. If a row in a determinant is a linear combination of other rows then the determinant is equal to zero.
9. A determinant stays the same if we multiply all entries of a row by the same number and add them to the entries of another row.

All of these properties can be proved using the principle of mathematical induction. For instance let us prove the second property.

Let us consider the determinant of the second order $\begin{vmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{a} & \boldsymbol{b} \end{vmatrix}$. By the definition it is equal to zero in the same way as $\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \mathbf{0} & \mathbf{0} \end{vmatrix}$. Let us assume that this property is true for all determinants of the order less than $\boldsymbol{n}$. Then, we consider the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

By the definition

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} +$$

$$+ \ldots + (-1)^{n-1} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2\ n-1} \\ a_{31} & a_{32} & \cdots & a_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\ n-1} \end{vmatrix}.$$

There are two possible variants:

1) if the entries of the first row are equal to zero, then in the right-hand side all coefficients before the determinants are equal to zero. Consequently, the determinant in the left-hand side is also equal to zero.
2) if the entries of another row are equal to zero, then all determinants in right-hand side contain the row which entries are equal to zero. Finally, since by the induction hypothesis, all of them are equal to zero. Consequently, the determinant in the left-hand side is also equal to zero.

Thus, the property is proved.

Note that it follows from the properties of determinants that the definition of it can be given as a decomposition by any row or column.

Example 1: find $\begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix}$.

Solution: $\begin{vmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{vmatrix} = \cos^2\varphi - \left(-\sin^2\varphi\right) = \cos^2\varphi + \sin^2\varphi = 1$.

Example 2: find $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 1\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - a\begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} + a^2\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} = bc^2 - b^2c - ac^2 + ab^2 + a^2c - a^2b =$$

$$= bc(c-b) - a(c-b)(c+b) + a^2(c-b) = (c-b)\left(bc - ac - ab + a^2\right) =$$
$$= (c-b)\left(c(b-a) - a(b-a)\right) = (b-a)(c-a)(c-b).$$

- **Systems of *n* linear algebraic equations in *n* variables. Cramer's rule.**

In the first paragraphs it was shown that the solution of the systems of two and three equations in two and three variables is represented as the quotient of determinants. Now we will generalize this to the general case.

Let us consider the system of n linear algebraic equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \qquad (*)$$

The system is called ***consistent***, if there exists a set of values of variables, which satisfies every equation in the system.

The system is called ***determined***, if there exists a unique set of values of variables, which satisfies every equation in the system.

The determinant formed with the coefficients of this system is called the ***determinant of the system***

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

The determinant of the system, in which the i^{th} column is substituted by the column of right-hand values, is called the ***determinant of the i^{th} variable***:

$$\Delta_i = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}.$$

Now we can formulate the ***Cramer's rule***.

1. If $\Delta = 0, \exists i : \Delta_i \neq 0$ then the system (*) is inconsistent.
2. If $\Delta = 0, \Delta_i = 0, i = \overline{1,n}$ then the system (*) is consistent, but undefined.
3. If $\Delta \neq 0$ then the system (*) is defined and $x_i = \dfrac{\Delta_i}{\Delta}, i = \overline{1,n}$.

Example 3: solve the system

$$\begin{cases} x + y - z = 36, \\ x + z - y = 13, \\ y + z - x = 7. \end{cases}$$

Solution: let us find the determinant of the system. Note that in the first column formed by the coefficients before x, second by the coefficients before y, third by the coefficients before z:

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -2 - 2 + 0 = -4.$$

Then, let us find the determinants of the variables:

$$\Delta_x = \begin{vmatrix} 36 & 1 & -1 \\ 13 & -1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 36\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 13 & 1 \\ 7 & 1 \end{vmatrix} - 1\begin{vmatrix} 13 & -1 \\ 7 & 1 \end{vmatrix} = -72 - 6 - 20 = -98,$$

$$\Delta_y = \begin{vmatrix} 1 & 36 & -1 \\ 1 & 13 & 1 \\ -1 & 7 & 1 \end{vmatrix} = 1\begin{vmatrix} 13 & 1 \\ 7 & 1 \end{vmatrix} - 36\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 13 & -1 \\ 7 & 1 \end{vmatrix} = 6 - 72 - 20 = -86,$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 36 \\ 1 & -1 & 13 \\ -1 & 1 & 7 \end{vmatrix} = 1\begin{vmatrix} -1 & 13 \\ 1 & 7 \end{vmatrix} - 1\begin{vmatrix} 1 & 13 \\ -1 & 7 \end{vmatrix} + 36\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -20 - 20 + 0 = -40.$$

Thus, the system is determined and

$$x = \frac{\Delta_x}{\Delta} = \frac{-98}{-4} = \frac{49}{2}, \; y = \frac{\Delta_y}{\Delta} = \frac{-86}{-4} = \frac{43}{2}, \; z = \frac{\Delta_z}{\Delta} = \frac{-40}{-4} = 10.$$

Example 4: solve the system

$$\begin{cases} 2x - y + z = -2, \\ x + 2y + 3z = -1, \\ x - 3y - 2z = 3. \end{cases}$$

Solution: at first let us find the determinant of the system:

$$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & -3 & -2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = 10 - 5 - 5 = 0.$$

Then, let us find the determinants of the variables.

$$\Delta_x = \begin{vmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ 3 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 3 & -3 \end{vmatrix} = -10 - 4 - 9 = -23,$$

$$\Delta_y = \begin{vmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 22 - 10 + 4 = 26,$$

$$\Delta_z = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 2 & -1 \\ 1 & -3 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -3 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = 18 - 4 - 10 = 4.$$

Since, $\Delta = 0, \Delta_x, \Delta_y, \Delta_z \neq 0$ the system is inconsistent.

VECTOR ALGEBRA

- **Vectors and vector operations.**

Many quantities in geometry and physics, such as area, time, and temperature, can be represented by a single real number. Other quantities, such as force, velocity, involve both magnitude and direction and cannot be completely characterized by a single real number. To represent such a quantity, we use a ***directed line segment***, as shown in Fig. 1. The directed line segment $\overrightarrow{PQ}$ has ***initial point P*** and ***terminal point Q***, and we denote its ***length*** by $\left|\overrightarrow{PQ}\right|$.

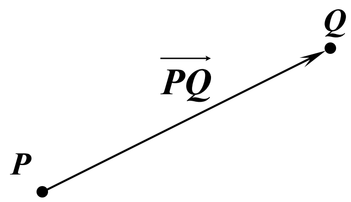

Fig. 1

Two directed line segments that have the same length (or magnitude) and direction are called ***equivalent***. For example, the directed line segments in Fig. 2 are all equivalent. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a ***vector*** $\vec{a}$. And we write $\vec{a} = \overrightarrow{PQ}$. Two vectors that have the same or opposite direction are called ***collinear***. A vector with zero length is called ***zero vector*** $\vec{0}$. Zero vector has not got direction.

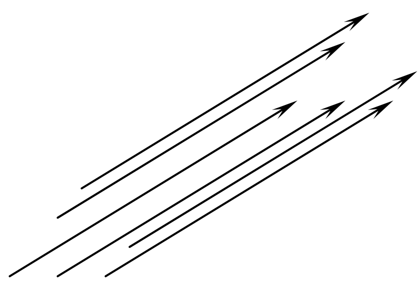

Fig. 2

The two basic vector operations are ***scalar multiplication*** and ***vector addition***. We will usually use the term scalar to mean a real number. Geometrically, the product of a vector $\vec{a}$ and a scalar k is the collinear vector that is $|\boldsymbol{k}|$ times as long as $\vec{a}$. If k is positive, then $k\vec{a}$ has the same direction as $\vec{a}$, and if k is negative, then $k\vec{a}$ has the opposite direction of $\vec{a}$, as shown in Fig. 3.

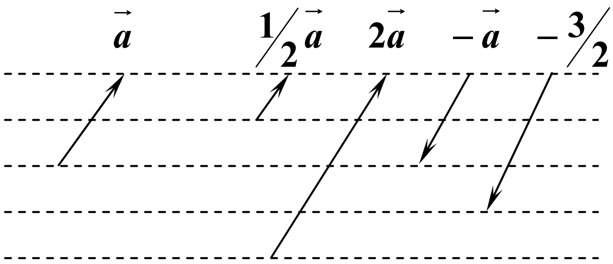

Fig. 3

To add vectors geometrically, place them (without changing length or direction) so that initial point of one coincides with the terminal point of the other. The sum $\vec{a}+\vec{b}$ is formed by joining the initial point of the first vector $\vec{a}$ with the terminal point of the second vector $\vec{b}$, as shown in Fig 4. Because the vector $\vec{a}+\vec{b}$ is the diagonal of a parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides, we call this the ***parallelogram law*** for vector addition.

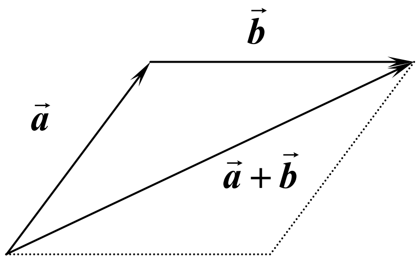

Fig. 4

The ***negative*** of vector $\vec{a}$ is vector $-\vec{a}=(-1)\vec{a}$ and the ***difference*** of $\vec{a}$ and $\vec{b}$ is $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$. To represent graphically, we use directed line segments with the same initial points. The difference $\vec{a}-\vec{b}$ is the vector from the terminal point of $\vec{b}$ to the terminal point of $\vec{a}$, as shown in Fig. 5.

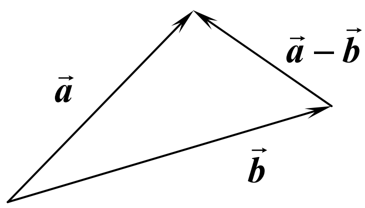

Fig. 5

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic. Here are some of them. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors and let α and β be scalars. Then the following properties are true:

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.
2. $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$.
3. $\vec{a}+\vec{0}=\vec{a}$.
4. $\vec{a}+(-\vec{a})=\vec{0}$.
5. $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$.
6. $(\alpha+\beta)\vec{a}=\alpha\vec{a}+\beta\vec{a}$.
7. $\alpha(\vec{a}+\vec{b})=\alpha\vec{a}+\alpha\vec{b}$.
8. $(1)\vec{a}=\vec{a}$, $(0)\vec{a}=\vec{0}$.
9. $|\alpha\vec{a}|=|\alpha||\vec{a}|$.

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector $\vec{a}$ and its length is equal to 1: $\vec{e}=\dfrac{\vec{a}}{|\vec{a}|}$.

We call $\vec{e}$ a ***unit vector in the direction of*** $\vec{a}$.

Example 5: in a triangle ABC $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{AC}=\vec{b}$. Find the vector $\overrightarrow{AD}$ if it is a bisector of the angle A.

Solution: from the property of the bisector we know that $\dfrac{CD}{BD}=\dfrac{AC}{AB}$. But $BD+CD=AC$, or in the vector form $\overrightarrow{CD}+\overrightarrow{DB}=\overrightarrow{AB}-\overrightarrow{AC}=\vec{a}-\vec{b}$ and $\overrightarrow{CD}=\dfrac{AC}{AB}\left(\overrightarrow{CB}-\overrightarrow{CD}\right)$. Thus $\overrightarrow{CD}=\dfrac{AC}{AB+AC}\overrightarrow{CB}=\dfrac{AC}{AB+AC}\left(\vec{a}-\vec{b}\right)$. Finally the vector $\overrightarrow{AD}$ can be expressed as a sum of vectors $\overrightarrow{AC}$ and $\overrightarrow{CD}$:

$$\overrightarrow{AD}=\vec{b}+\frac{AC}{AB+AC}\left(\vec{a}-\vec{b}\right)=\vec{b}+\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\left(\vec{a}-\vec{b}\right)=\frac{|\vec{b}|}{|\vec{a}|+|\vec{b}|}\vec{a}+\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}\vec{b}$$ (Fig. 6).

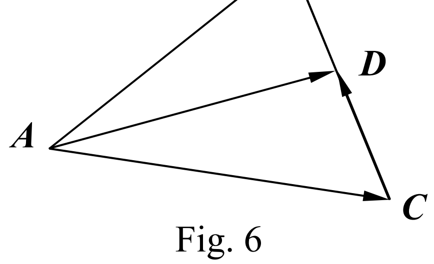

Fig. 6

- **Linear independence of vectors.**

The ***linear combination*** of vectors $\{\vec{a}_1,...,\vec{a}_n\}$ with coefficients $\{\alpha_1,...,\alpha_n\}$ is called a vector of the form

$$\vec{b} = \sum_{i=1}^{n} \alpha_i \vec{a}_i \, .$$

A linear combination is called ***trivial***, if all coefficients are equal to zero, and is called ***non trivial*** in the opposite case. Trivial linear combination of vectors is obviously equal to zero.

The system of vectors $\{\vec{a}_1,...,\vec{a}_n\}$ is called ***linear independent*** if the linear combination of them is equal to zero only if it is trivial. The system is called ***linear dependent*** in the opposite case.

- **The Cartesian coordinate system.**

Just as real numbers are represented by points on the real number line; ordered triple of real numbers is represented by points in space. This space is called a ***rectangular coordinate system*** or the ***Cartesian space***.

The Cartesian space is formed by three real number lines intersecting at right angles in one point called ***origin***. The lines are called ***axes***.

Each point in the plane corresponds to an ordered triple of real numbers $\langle x, y, z \rangle$, which are called ***coordinates*** of a point. Each coordinate tells how far the point is from appropriate axis, as shown in Fig. 7.

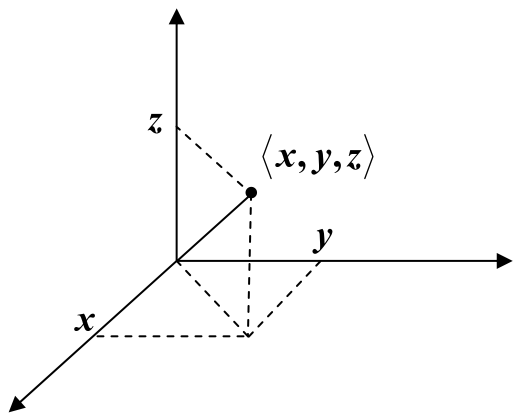

Fig. 7

Since a directed line segment is defined by its initial and terminal points, and these points are defined by coordinates, a vector can be defined by coordinates as well, by the following rule. Let the coordinates of the initial point ***P*** be $\langle x_1, y_1, z_1 \rangle$, and let the coordinates of the terminal point ***Q*** be $\langle x_2, y_2, z_2 \rangle$. Then if we subtract the coordinates of initial point from the coordinates of the terminal point we obtain the coordinates of the vector defined by this directed line segment. This representation of a vector is called ***component form***:

$$\vec{a} = \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

The unit vectors $\langle 1,0,0 \rangle$, $\langle 0,1,0 \rangle$, and $\langle 0,0,1 \rangle$ are called ***standard unit vectors*** and are denoted by $\vec{i} = \langle 1,0,0 \rangle$, $\vec{j} = \langle 0,1,0 \rangle$, and $\vec{k} = \langle 0,0,1 \rangle$ as shown in Fig. 8.

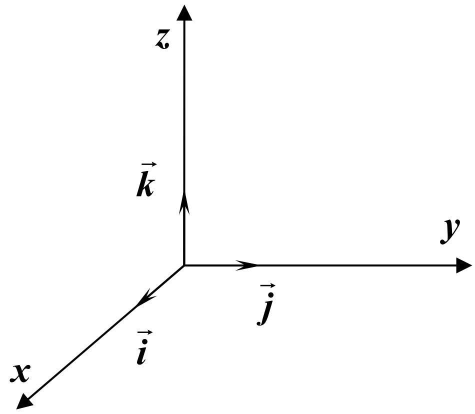

Fig. 8

The operations of addition and scalar multiplication of vectors represented in component form are introduced as follows. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle ,\ k\vec{a} = \langle ka_1, ka_2, ka_3 \rangle .$$

Thus, every vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be represented as $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, or as a linear combination of standard unit vectors.

The length or magnitude of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be found using the Pythagorean Theorem

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} .$$

- **Scalar product of two vectors.**

The ***scalar*** or ***dot product*** of two vectors denoted by $(\vec{a}, \vec{b})$ or $\vec{a}\vec{b}$ is a number, which is equal to the product of their lengths multiplied by the cosine of the angle between them:

$$(\vec{a}, \vec{b}) = |\vec{a}||\vec{b}|\cos(\vec{a} \wedge \vec{b}).$$

The properties of scalar product are as follows:

1. If $(\vec{a},\vec{b})=0$ and $\vec{a}\neq\vec{0},\vec{b}\neq\vec{0}$ then the vectors are perpendicular or ***orthogonal***.
2. $(\vec{a},\vec{b})=(\vec{b},\vec{a})$.
3. $(\vec{a}_1+\vec{a}_2,\vec{b})=(\vec{a}_1,\vec{b})+(\vec{a}_2,\vec{b})$.
4. $\lambda(\vec{a},\vec{b})=(\lambda\vec{a},\vec{b})$.
5. $(\vec{a},\vec{a})=|\vec{a}|^2\geq 0$, $(\vec{a},\vec{a})=0$ if and only if $\vec{a}=\vec{0}$.

From the definition it follows that $(\vec{i},\vec{i})=(\vec{j},\vec{j})=(\vec{k},\vec{k})=1$, $(\vec{i},\vec{j})=(\vec{j},\vec{k})=(\vec{i},\vec{k})=0$. So we can find the expression of scalar product in component form.

Let $\vec{a}=\langle a_1,a_2,a_3\rangle$, and $\vec{b}=\langle b_1,b_2,b_3\rangle$, then

$$(\vec{a},\vec{b})=(a_1\vec{i}+a_2\vec{j}+a_3\vec{k},b_1\vec{i}+b_2\vec{j}+b_3\vec{k}),$$

removing brackets by the third property and using the orthogonality of standard unit vectors, we obtain

$$(\vec{a},\vec{b})=a_1b_1+a_2b_2+a_3b_3.$$

Example 6: find the cosine of the angle between vectors $\vec{a}=2\vec{p}-\vec{q}$ and $\vec{b}=\vec{p}+2\vec{q}$, if $|\vec{p}|=|\vec{q}|=1$ and $(\vec{p}\wedge\vec{q})=\dfrac{\pi}{3}$.

Solution: from the definition of scalar product $\cos(\vec{a}\wedge\vec{b})=\dfrac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}$;

from the 3rd and 4th properties

$$(\vec{a},\vec{b})=(2\vec{p}-\vec{q},\vec{p}+2\vec{q})=2(\vec{p},\vec{p})-(\vec{q},\vec{p})+4(\vec{p},\vec{q})-2(\vec{q},\vec{q}),$$

and from the 2nd and 5th properties $(\vec{a},\vec{b})=2|\vec{p}|^2+3(\vec{p},\vec{q})-2|\vec{q}|^2$, $(\vec{p},\vec{q})=|\vec{p}||\vec{q}|\cos(\vec{p}\wedge\vec{q})$.

Thus, $(\vec{a},\vec{b})=2|\vec{p}|^2+3(\vec{p},\vec{q})-2|\vec{q}|^2=2+3\dfrac{1}{2}-2=\dfrac{3}{2}$. Analogously

$$|\vec{a}|=\sqrt{(\vec{a},\vec{a})}=\sqrt{(2\vec{p}-\vec{q},2\vec{p}-\vec{q})}=\sqrt{4|\vec{p}|^2-4(\vec{p},\vec{q})+|\vec{q}|^2}=\sqrt{4-4\frac{1}{2}+1}=\sqrt{3},$$

$$|\vec{b}|=\sqrt{(\vec{b},\vec{b})}=\sqrt{(\vec{p}+2\vec{q},\vec{p}+2\vec{q})}=\sqrt{|\vec{p}|^2+4(\vec{p},\vec{q})+4|\vec{q}|^2}=\sqrt{1+4\frac{1}{2}+4}=\sqrt{7}.$$

Hence, $\cos(\vec{a}\wedge\vec{b})=\frac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}=\frac{\frac{3}{2}}{\sqrt{3}\sqrt{7}}=\frac{3}{2\sqrt{21}}$.

Example 7: find the cosine of the angle between vectors $\vec{a}$ and $\vec{b}$ if $\vec{a}=\langle 3,2,-6\rangle$ and $\vec{b}=\langle 2,-2,1\rangle$.

Solution: from the definition of scalar product $\cos(\vec{a}\wedge\vec{b})=\frac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}$.

$(\vec{a},\vec{b})=a_1b_1+a_2b_2+a_3b_3=3\cdot 2-2\cdot 2-6\cdot 1=-4$;

$|\vec{a}|=\sqrt{(\vec{a},\vec{a})}=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{9+4+36}=\sqrt{49}=7$;

$|\vec{b}|=\sqrt{(\vec{b},\vec{b})}=\sqrt{b_1^2+b_2^2+b_3^2}=\sqrt{4+4+1}=\sqrt{9}=3$.

Hence, $\cos(\vec{a}\wedge\vec{b})=\frac{(\vec{a},\vec{b})}{|\vec{a}||\vec{b}|}=\frac{-4}{7\cdot 3}=-\frac{4}{21}$.

Example 8: find the components of vector $\vec{p}$ if $\vec{p}\perp\vec{a}$, $\vec{p}\perp\vec{b}$, $|\vec{p}|=4$, $\vec{a}=\langle 1,1,1\rangle$, $\vec{b}=\langle 1,2,1\rangle$.

Solution: let $\vec{p}=\langle x,y,z\rangle$, then since $\vec{p}\perp\vec{a}$, $x+y+z=0$. And since $\vec{p}\perp\vec{b}$, $x+2y+z=0$. And finally if $|\vec{p}|=4$, then $x^2+y^2+z^2=16$. We obtained the system:

$$\begin{cases} x+y+z=0, \\ x+2y+z=0, \\ x^2+y^2+z^2=16. \end{cases}$$

Subtracting the first equation from the second we have $y=0$. Thus

$$\begin{cases} x+z=0, \\ x^2+z^2=16. \end{cases}$$

And finally $x=\pm 2\sqrt{2}, z=\mp 2\sqrt{2}$, hence $\vec{p}=\langle \pm 2\sqrt{2},0,\mp 2\sqrt{2}\rangle$.

- **Projection of a vector on an axis.**

The expression $\frac{\vec{a}\vec{l}}{|\vec{l}|}$ is called the ***projection of the vector $\vec{a}$ on the axis which direction is given by the vector $\vec{l}$***. By the definition of the scalar product we see that the projection can be written as $|\vec{a}|\cos(\vec{a}\wedge\vec{l})$ (Fig. 9).

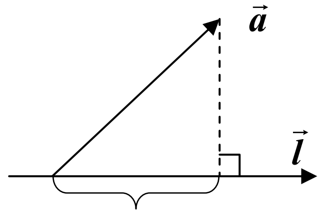

Fig. 9

Properties of the projection follow from the properties of the scalar product. The main property is that vector sum projection is equal to the sum of projections.

For instance the components of a vector in the Cartesian coordinate system are nothing else than projections of a vector on the coordinate axes.

Example 9: find the projection of the vector $\vec{a} = \langle 1,2,-3\rangle$ on the vector $\vec{l} = \langle -1,-2,2\rangle$.

Solution: $\mathbf{proj}_{\vec{j}}\,\vec{a} = \frac{\vec{a}\vec{l}}{|\vec{l}|} = \frac{1\cdot(-1)+2\cdot(-2)-3\cdot 2}{\sqrt{(-1)^2+(-2)^2+2^2}} = \frac{-11}{3} = -\frac{11}{3}$.

- **Vector product of two vectors.**

A vector $\vec{c}$ is said to be a ***vector product*** or ***cross product*** of two vectors $\vec{a}$ and $\vec{b}$ denoted by $[\vec{a},\vec{b}]$ or $\vec{a}\times\vec{b}$ if it satisfies three conditions:

1. Vector $\vec{c}$ is orthogonal to vectors $\vec{a}$ and $\vec{b}$ each.
2. The length of vector $\vec{c}$ is equal to the product of vectors $\vec{a}$ and $\vec{b}$ lengths and the sine of the angle between vectors $\vec{a}$ and $\vec{b}$:
$$|\vec{c}| = |\vec{a}||\vec{b}|\sin(\vec{a}\wedge\vec{b}).$$
3. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the so-called ***right triple*** that is if we superpose the initial points of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and look down from the terminal point of vector $\vec{c}$ to the plane of vectors $\vec{a}$ and $\vec{b}$, we will see the shortest angle

between vectors $\vec{a}$ and $\vec{b}$ in anticlockwise direction (Fig. 10).

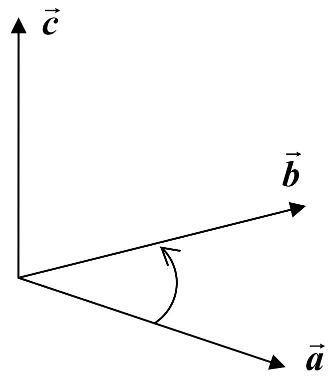

Fig. 10

Note that vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ form the right triple (see Fig. 8).

The vector product of two vectors has the following properties:

1. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$, hence $\vec{a}\times\vec{a}=\vec{0}$.
2. $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.
3. $(\alpha\cdot\vec{a})\times\vec{b}=\alpha(\vec{a}\times\vec{b})$.
4. $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$.

Using the properties of vector product of two vectors we can prove the following statement.

If two vectors $\vec{a}$ and $\vec{b}$ are represented in component form $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$ then the vector product of them can be written as:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

Example 10: find the area of the triangle ABC if $A\langle 1,1,0\rangle$, $B\langle 1,0,1\rangle$, $C\langle 2,1,1\rangle$.

Solution: the area of a triangle ABC can be found by the formula:

$$S_{\Delta ABC}=\frac{1}{2}AB\cdot AC\cdot\sin(AB\wedge AC),$$

but the right-hand part of it is a half of the $\overrightarrow{AB}$ and $\overrightarrow{AC}$ vector product length.

Hence $S_{\Delta ABC}=\frac{1}{2}AB\cdot AC\cdot\sin(AB\wedge AC)=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$. Let us find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. $\overrightarrow{AB}=\langle 0,-1,1\rangle$, $\overrightarrow{AC}=\langle 1,0,1\rangle$.

Then $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & -1 & 1\\ 1 & 0 & 1\end{vmatrix}=\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix}\vec{i}-\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}\vec{j}+\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}\vec{k}=-\vec{i}+\vec{j}+\vec{k}$. Hence the area of the triangle is equal to $\frac{1}{2}\sqrt{(-1)^2+1^2+1^2}=\frac{\sqrt{3}}{2}$.

Example 11: find $\left|(\vec{a}-\vec{b})\times(2\vec{a}+3\vec{b})\right|$ if $|\vec{a}|=|\vec{b}|=2$ and $(\vec{a}\wedge\vec{b})=\frac{5\pi}{6}$.

Solution: let us simplify the given expression using the properties of vector product:

$$\left|(\vec{a}-\vec{b})\times(2\vec{a}+3\vec{b})\right|=\left|\vec{a}\times(2\vec{a})-\vec{b}\times(2\vec{a})+\vec{a}\times(3\vec{b})-\vec{b}\times(3\vec{b})\right|=$$

$$=\left|2(\vec{a}\times\vec{a})-2(\vec{b}\times\vec{a})+3(\vec{a}\times\vec{b})-3(\vec{b}\times\vec{b})\right|=\left|2(\vec{a}\times\vec{b})+3(\vec{a}\times\vec{b})\right|=$$

$$=5\left|(\vec{a}\times\vec{b})\right|=5|\vec{a}||\vec{b}|\sin(\vec{a}\wedge\vec{b})=5\cdot 2\cdot 2\cdot\frac{1}{2}=10.$$

- **Scalar triple product.**

The expression $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the ***scalar triple product*** of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, and denoted as $\vec{a}\vec{b}\vec{c}$.

The properties of scalar product are as follows.

1. $\vec{a}\vec{b}\vec{c}=\vec{b}\vec{c}\vec{a}=\vec{c}\vec{a}\vec{b}=-\vec{b}\vec{a}\vec{c}=-\vec{a}\vec{c}\vec{b}=-\vec{c}\vec{b}\vec{a}$.
2. If at least two vectors in the triple $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear, then the scalar triple product is equal to zero (e.g. $\vec{a}\vec{b}\vec{a}=\vec{b}\vec{c}\vec{b}=\vec{c}\vec{a}\vec{c}=0$).
3. $\vec{a}\vec{b}(\vec{c}+\vec{d})=\vec{a}\vec{b}\vec{c}+\vec{a}\vec{b}\vec{d}$.
4. $(\alpha\vec{a})\vec{b}\vec{c}=\alpha(\vec{a}\vec{b}\vec{c})$.
5. If $\vec{a}\vec{b}\vec{c}>0$ then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the right triple, if $\vec{a}\vec{b}\vec{c}<0$ then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the left triple, if $\vec{a}\vec{b}\vec{c}=0$ then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie in one plane or called ***coplanar***.
6. If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are given in a component form $\vec{a}=\langle a_1,a_2,a_3\rangle$, $\vec{b}=\langle b_1,b_2,b_3\rangle$ and $\vec{c}=\langle c_1,c_2,c_3\rangle$ then

$$\vec{a}\vec{b}\vec{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

7. The volume of the parallelepiped, which adjacent edges are vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is equal to the scalar triple product absolute value. $V = |\vec{a}\vec{b}\vec{c}|$.

Example 12: find $\vec{a}\vec{b}(3\vec{a} - 2\vec{b} + 3\vec{c})$ if $\vec{a}\vec{b}\vec{c} = 2$.

Solution: using the properties of scalar triple product let us simplify the given expression:

$$\vec{a}\vec{b}(3\vec{a} - 2\vec{b} + 3\vec{c}) = \vec{a}\vec{b}(3\vec{a}) + \vec{a}\vec{b}(-2\vec{b}) + \vec{a}\vec{b}(3\vec{c}) = 3\vec{a}\vec{b}\vec{a} - 2\vec{a}\vec{b}\vec{b} + 3\vec{a}\vec{b}\vec{c} = 3\vec{a}\vec{b}\vec{c} = 6.$$

Example 13: find the volume of the pyramid $ABCD$ if $A\langle 2,-1,1\rangle$, $B\langle 5,5,4\rangle$, $C\langle 3,1,-1\rangle$, $D\langle 4,1,3\rangle$.

Solution: vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ form the adjacent edges of the pyramid. Volume of the pyramid is equal to $\frac{1}{6}$ of the volume of the parallelepiped. Hence $V_{ABCD} = \frac{1}{6}\left|\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right|$. Let us find the components of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$. $\overrightarrow{AB} = \langle 3,6,3\rangle$, $\overrightarrow{AC} = \langle 1,3,-2\rangle$, $\overrightarrow{AD} = \langle 2,2,2\rangle$.

Then $\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD} =$

$$\begin{vmatrix} 3 & 6 & 3 \\ 1 & 3 & -2 \\ 2 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & -2 \\ 2 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 3\cdot 10 - 6\cdot 6 - 3\cdot 4 = -18.$$

$$V_{ABCD} = \frac{1}{6}\left|\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right| = \frac{1}{6}|-18| = 3.$$

Example 14: find λ such that vectors $\langle 1,2,-3\rangle$, $\langle 4,-2,1\rangle$, $\langle -1,0,\lambda\rangle$ are coplanar.

Solution: vectors are coplanar if their scalar triple product is equal to zero. Let us find the scalar triple product of given vectors:

$$\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ -1 & 0 & \lambda \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & \lambda \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ -1 & \lambda \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ -1 & 0 \end{vmatrix} = -\lambda - 2(4\lambda + 1) - 3(-1) = -9\lambda + 1.$$

Solving the equation $-9\lambda + 1 = 0$ we obtain $\lambda = \frac{1}{9}$.

ANALYTIC GEOMETRY

- **Equation of a plane.**

An equation of a plane passing through a point $M\langle x_0, y_0, z_0\rangle$, perpendicular to a vector $\vec{n}\langle A, B, C\rangle$ using scalar product can be written as $\vec{n}(\vec{r}-\vec{r}_0)=0$, where $\vec{r}-\vec{r}_0$ is a vector which initial point is a point with components $\langle x, y, z\rangle$ and terminal point is M_0 (Fig. 11). $\vec{n}\langle A, B, C\rangle$ is called ***normal vector*** to a plane.

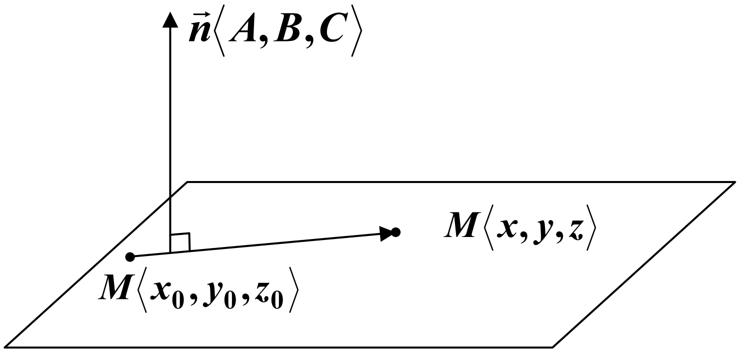

Fig. 11

By the equation of a plane we understand the equation where if variables x, y, z are substituted for components of a point the equation is satisfied if and only if the point lies on the plane. Writing the expression $\vec{n}(\vec{r}-\vec{r}_0)=0$ in scalar form we obtain an equation of the plane passing through the point $M\langle x_0, y_0, z_0\rangle$, perpendicular to vector $\vec{n}\langle A, B, C\rangle$:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

If we remove brackets we obtain the so-called ***general equation of a plane***:

$$Ax+By+Cz+D=0.$$

Example 15: find equation of the plane passing through the point $\langle 1,1,1\rangle$, perpendicular to the vector $\langle 2,-1,0\rangle$.

Solution: $A(x-x_0)+B(y-y_0)+C(z-z_0)=2(x-1)-(y-1)+0(z-1)=$ $=2x-2-y+1=2x-y-1$. So the equation is $2x-y-1=0$.

Example 16: find equation of the plane passing through the point $\langle 2,0,1\rangle$, parallel to vectors $\vec{a}=\langle 1,-1,3\rangle$ and $\vec{b}=\langle 0,2,-1\rangle$.

Solution: to write the equation of the plane we need the components of the normal vector. It is a vector which is perpendicular to the plane, but therefore it is perpendicular to any vector parallel to the plane, and hence it is perpendicular to given vectors. To find normal vector we find the vector product of $\vec{a}$ and $\vec{b}$.

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 0 & 2 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -5\vec{i} + \vec{j} + 2\vec{k}.$$

Putting into the equation of a plane we obtain:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = -5(x - 2) + (y - 0) + 2(z - 1) = -5x + y + 2z + 8$$

So the equation is $-5x + y + 2z + 8 = 0$.

Example 17: find the equation of the plane passing through the points $A\langle -1,0,1\rangle$, $B\langle 0,3,1\rangle$, $C\langle 4,-2,0\rangle$.

Solution: we can reduce the problem to the previous one by finding two vectors lying on the plane. For instance $\overrightarrow{AB} = \langle 1,3,0\rangle$, $\overrightarrow{AC} = \langle 5,-2,-1\rangle$. Then $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} =$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 0 \\ 5 & -2 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 0 \\ -2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ 5 & -2 \end{vmatrix} = -3\vec{i} + \vec{j} - 17\vec{k}.$$

Taking point A we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = -3(x + 1) + (y - 0) - 17(z - 1) = -3x + y - 17z + 14$$

So the equation is $-3x + y - 17z + 14 = 0$.

- **Angle between planes.**

It is obvious that the angle between two planes is equal to the angle between their normal vectors.

Example 18: find the angle between planes $x - y + \sqrt{2}z + 2 = 0$ and $x + y + \sqrt{2}z - 3 = 0$.

Solution: normal vector to the first plane is $\vec{n}_1 = \langle 1,-1,\sqrt{2}\rangle$, and to the second one is $\vec{n}_2 = \langle 1,1,\sqrt{2}\rangle$. Using the scalar product we can write that $\cos(\vec{n}_1 \wedge \vec{n}_2) = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \dfrac{1 \cdot 1 - 1 \cdot 1 + \sqrt{2} \cdot \sqrt{2}}{\sqrt{1 + 1 + 2}\sqrt{1 + 1 + 2}} = \dfrac{2}{4} = \dfrac{1}{2}$. Hence the angle between the planes is equal to $\dfrac{\pi}{3}$.

Moreover two planes are parallel if and only if their normal vectors are collinear.

Example 19: find λ and μ such that planes $x - y + \lambda z + 2 = 0$ and $\mu x + y + 3z - 3 = 0$ are parallel.

Solution: $\vec{n}_1 = \langle 1, -1, \lambda \rangle$, $\vec{n}_2 = \langle \mu, 1, 3 \rangle$. Since these vectors are collinear their components are proportional: $\frac{1}{\mu} = \frac{-1}{1} = \frac{\lambda}{3}$, hence $\mu = -1$ and $\lambda = -3$.

- **Distance from a point to a plane.**

Let us consider Fig. 12. We are to find the distance between the point $M\langle x_0, y_0, z_0 \rangle$ and the plane L. Let us take any point $N\langle x, y, z \rangle$ on the plane, then the distance is equal to the absolute value of the projection of the vector $\overrightarrow{NM}$ to the vector $\vec{n}\langle A, B, C \rangle$.

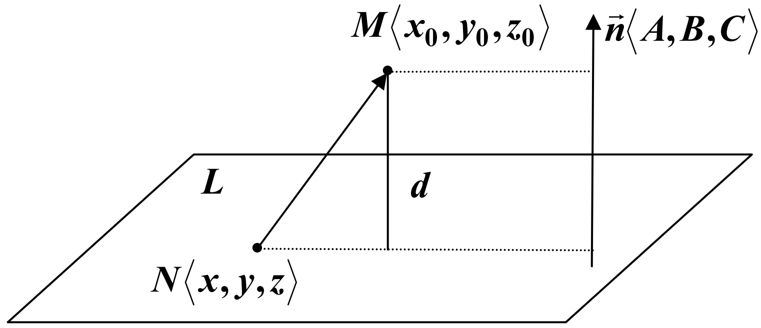

Fig. 12

Using the formula $\mathbf{proj}_{\vec{n}} \overrightarrow{NM} = \frac{\vec{n} \cdot \overrightarrow{NM}}{|\vec{n}|}$ we obtain $d = \left| \frac{\langle A, B, C \rangle \cdot \langle x_0 - x, y_0 - y, z_0 - z \rangle}{\sqrt{A^2 + B^2 + C^2}} \right| = \frac{|Ax_0 + By_0 + Cz_0 - Ax - By - Cz|}{\sqrt{A^2 + B^2 + C^2}}$. But the point $N\langle x, y, z \rangle$ lies on the plane, hence it satisfies the equation of the plane $Ax + By + Cz + D = 0$. Finally $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Example 20: find the bisector of planes $x - 2y + 2z + 2 = 0$ and $4y + 3z - 3 = 0$. The bisector of two planes is a plane which divides the dihedral angle between them by two.

Solution: the bisector consists of points equidistant to the given planes. Let us take a point $N\langle x, y, z\rangle$ and find distances to the given planes: $d_1 = \frac{|x - 2y + 2z + 2|}{\sqrt{1+4+4}}$ and $d_2 = \frac{|4y + 3z - 3|}{\sqrt{16+9}}$. Since $d_1 = d_2$, we obtain $5|x - 2y + 2z + 2| = 3|4y + 3z - 3|$, or $5(x - 2y + 2z + 2) = \pm 3(4y + 3z - 3)$. Taking '+' we get $-7x - 10y + z + 19 = 0$, and taking '–' we get $17x - 10y + 19z + 1 = 0$.

- **Equation of a straight line.**

If two planes intersect the intersection is a straight line. Thus the equation of line can be written as

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

which is called a ***general equation of a straight line***. From the other hand a straight line is defined by a point on it and a vector codirectional to the straight line (Fig. 13).

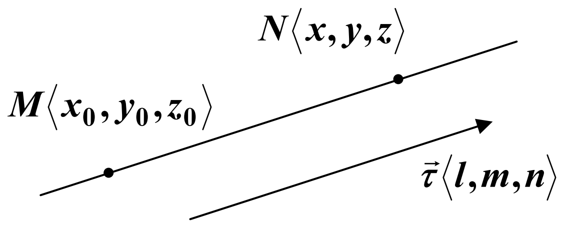

Fig. 13

If we take a point $M\langle x_0, y_0, z_0\rangle$ on the line and any point $N\langle x, y, z\rangle$ on the same line, the obtained vector is collinear to the vector $\vec{\tau}\langle l, m, n\rangle$. Thus the equation of the straight line can be written as

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

which is called ***the canonical equation of a straight line***. And the vector $\vec{\tau}\langle l, m, n\rangle$ is called **the direction vector** of the straight line. If in the canonical equation we define the coefficient of proportionality by t and express x, y, z in terms of it we obtain ***the parametrical equation of the straight line***:

$$\begin{cases} x = x_0 + lt, \\ y = y_0 + mt, \quad t \in \Re, \\ z = z_0 + nt. \end{cases}$$

Note that there are a lot of different general as well as canonical equations of a straight line.

Example 21: find the equation of the straight line passing through two points $A\langle 1,2,-3\rangle$ and $B\langle 0,-1,2\rangle$.

Solution: the direction vector for the straight line is the vector $\overrightarrow{AB}=\langle -1,-3,5\rangle$. Let us write the canonical equation of the straight line $\frac{x-1}{-1}=\frac{y-2}{-3}=\frac{z+3}{5}$.

Example 22: find the canonical equation of a straight line $\begin{cases} 3x-2y+z=0, \\ x-2z+1=0. \end{cases}$

Solution: To write the canonical equation we need to know the direction vector of the straight line and a point on it. As far as the direction vector is parallel to both planes it is perpendicular to both normal vectors and can be expressed as their vector product:

$$\langle 3,-2,1\rangle\times\langle 1,0,-2\rangle=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 1 & 0 & -2 \end{vmatrix}=\vec{i}\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix}-\vec{j}\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}+\vec{k}\begin{vmatrix} 3 & -2 \\ 1 & -2 \end{vmatrix}=4\vec{i}+\vec{j}-4\vec{k}.$$

To find a point on the plane we have to find any solution of the system $\begin{cases} 3x-2y+z=0, \\ x-2z+1=0. \end{cases}$ Let us add two times the first equation to the second one. We have: $7x-4y+1=0$, or $y=\frac{1}{4}(7x+1)$. If we put $x=1$, then $y=2$ and $z=1$. So the canonical equation has the form $\frac{x-1}{4}=\frac{y-2}{1}=\frac{z-1}{-4}$.

The parametrical equation of the straight line is convenient to use when finding the intersection of a straight line and a plane.

Example 23: find the point of intersection between the straight line $\frac{x+1}{2}=\frac{y-1}{0}=\frac{z-4}{-3}$ and a plane $2x-3y+4z-3=0$.

Solution: let us write the equation of the straight line in the parametrical form

$$\begin{cases} x=-1+2t, \\ y=1, \\ z=4-3t \end{cases}$$

and put these in the equation of the plane $2(-1+2t)-3(1)+4(4-3t)-3=0$, or $-8t+8=0$, hence $t=1$ and $\langle x,y,z\rangle=\langle 1,1,1\rangle$.

- **Positional relationship between straight lines.**

Two lines in a space can be parallel (their direction vectors are collinear, and they do not have common points), coincide (every point of one straight line is a point of another), intersect (have only one common point), be skew (not parallel and having no common points).

Two straight lines defined by their direction vectors $\vec{\tau}_1$, $\vec{\tau}_2$ and points M_1, M_2 are parallel if $\vec{\tau}_1\times\vec{\tau}_2=0$ and $\vec{\tau}_1\times\overrightarrow{M_1M_2}\neq 0$.

Two straight lines defined by their direction vectors $\vec{\tau}_1$, $\vec{\tau}_2$ and points M_1, M_2 are skew if $\vec{\tau}_1\vec{\tau}_2\,\overrightarrow{M_1M_2}\neq 0$.

Two straight lines defined by their direction vectors $\vec{\tau}_1$, $\vec{\tau}_2$ and points M_1, M_2 intersect if $\vec{\tau}_1\vec{\tau}_2\,\overrightarrow{M_1M_2}=0$ and $\vec{\tau}_1\times\vec{\tau}_2\neq 0$.

Two straight lines defined by their direction vectors $\vec{\tau}_1$, $\vec{\tau}_2$ and points M_1, M_2 coincide if $\vec{\tau}_1\times\vec{\tau}_2\neq 0$ and $\vec{\tau}_1\times\overrightarrow{M_1M_2}=0$.

Example 24: determine the positional relationship between the straight lines $\frac{x+2}{2}=\frac{y}{-3}=\frac{z-1}{4}$ and $\frac{x-3}{\alpha}=\frac{y-1}{4}=\frac{z-7}{2}$.

Solution: we have $\vec{\tau}_1=\langle 2,-3,4\rangle$ and $\vec{\tau}_2=\langle \alpha,4,2\rangle$, $M_1\langle -2,0,1\rangle$ and $M_1\langle 3,1,7\rangle$. Then $\overrightarrow{M_1M_2}=\langle 5,1,6\rangle$. Let us find $\vec{\tau}_1\vec{\tau}_2\,\overrightarrow{M_1M_2}$.

$\vec{\tau}_1\vec{\tau}_2\,\overrightarrow{M_1M_2}=\begin{vmatrix} 2 & -3 & 4 \\ \alpha & 4 & 2 \\ 5 & 1 & 6 \end{vmatrix}=22\alpha-66$. It is equal to zero when $\alpha=3$. Thus these straight lines are skew if $\alpha\neq 3$.

Let us find $\vec{\tau}_1\times\vec{\tau}_2=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ \alpha & 4 & 2 \end{vmatrix}=-22\vec{i}+(4\alpha-4)\vec{j}+(3\alpha+8)\vec{k}$. This vector is never equal to zero, so the straight lines cannot be parallel or coincide. Hence if $\alpha=3$ these straight lines are intersected.

- **Other problems on a plane and a straight line.**

Example 25: find the projection of the point $M\langle 1,-2,2\rangle$ on the straight line $\frac{x}{1}=\frac{y+3}{0}=\frac{z}{2}$ and the equation of the perpendicular from the point M on the straight line.

Solution: let us write the equation of the plane which is perpendicular to the straight line and passes through the point M. Then the intersection of the plane and the straight line is the projection of the point M on the straight line (Fig. 14).

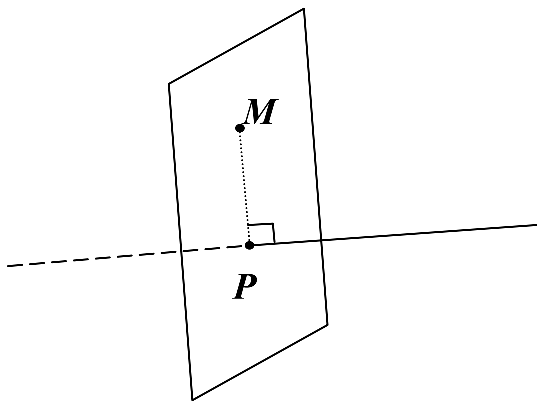

Fig. 14

Since the direction vector of the straight line is perpendicular to the plane, we can write the equation of the plane $1(x-1)+0(y+2)+2(z-2)=0$ or $x+2z-5=0$.

Let us write the equation of the straight line in the parametric form $\begin{cases} x=t, \\ y=-3, \\ z=2t \end{cases}$ and put it into the equation of the plane $t+4t-5=0$ or $t=1$. Hence the point $P\langle 1,-3,2\rangle$. To find the equation of the perpendicular we are to find the direction vector. It will be vector $\overrightarrow{MP}=\langle 0,-1,0\rangle$. Finally the equation of the perpendicular is $\frac{x-1}{0}=\frac{y+2}{-1}=\frac{z-2}{0}$.

Example 26: find the angle between the plane $x-2y+2z-6=0$ and the straight line $\frac{x-1}{1}=\frac{y+2}{-1}=\frac{z}{0}$.

Solution: the angle between a plane and a straight line is equal to $\frac{\pi}{2}$ minus the angle between the direction vector of the straight line and the normal vector to the plane (Fig. 15).

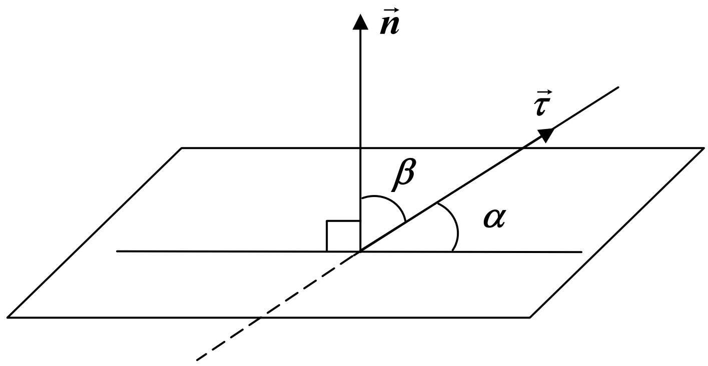

Fig. 15

Thus $\sin\alpha = \cos\beta = \dfrac{\vec{n}\cdot\vec{\tau}}{|\vec{n}|\cdot|\vec{\tau}|}$. We have $\vec{n} = \langle 1,-2,2\rangle$ and $\vec{\tau} = \langle 1,-1,0\rangle$.

Hence $\sin\alpha = \dfrac{\vec{n}\cdot\vec{\tau}}{|\vec{n}|\cdot|\vec{\tau}|} = \dfrac{1\cdot 1 - 2\cdot(-2) + 2\cdot 0}{\sqrt{1^2 + (-2)\ \ + 2^2}\cdot\sqrt{1^2 + (-1)^2 + 0^2}} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$. Finally the angle between the plane and the straight line is equal to $\arcsin\dfrac{1}{\sqrt{2}} = \dfrac{\pi}{4}$.

Example 27: find the projection of a point $M\langle 1,0,-2\rangle$ to the plane $x - 2y + 3z - 9 = 0$.

Solution: to find the projection we are to find the equation of the perpendicular from the point M to the plane and then the intersection P of the perpendicular and the plane. The direction vector of the perpendicular is the normal vector of the plane (see Fig. 16).

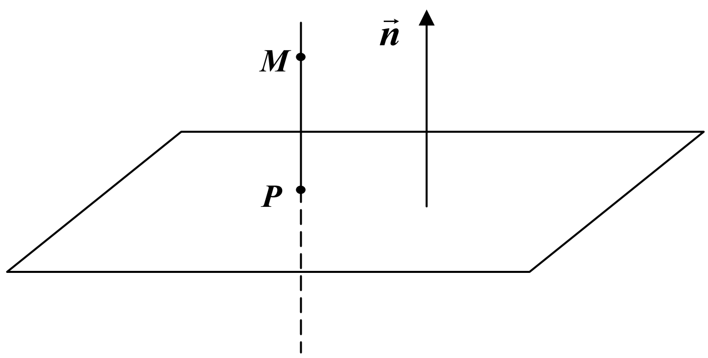

Fig. 16

Thus the equation of the perpendicular is $\dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z+2}{3}$. To find the intersection let us write this equation in parametric form: $\begin{cases} x = 1 + t \\ y = -2t \\ z = -2 + 3t \end{cases}$ and put these in to the equation of the plane. $(1 + t) - 2 \cdot (-2t) + 3 \cdot (-2 + 3t) - 9 = 0$ or $t = 1$. Finally $P\langle 2,-2,1 \rangle$.

Example 28: find the projection of the straight line $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-1}{3}$ on the plane $3x - 2y + z - 2 = 0$.

Solution: the projection of a straight line is another straight line which lies on the plane. Its direction vector ($\vec{\tau}_1$) then is perpendicular to the normal vector to the plane. From the other hand it is perpendicular to a vector product of the normal vector to the plane and the direction vector of the straight line (Fig. 17).

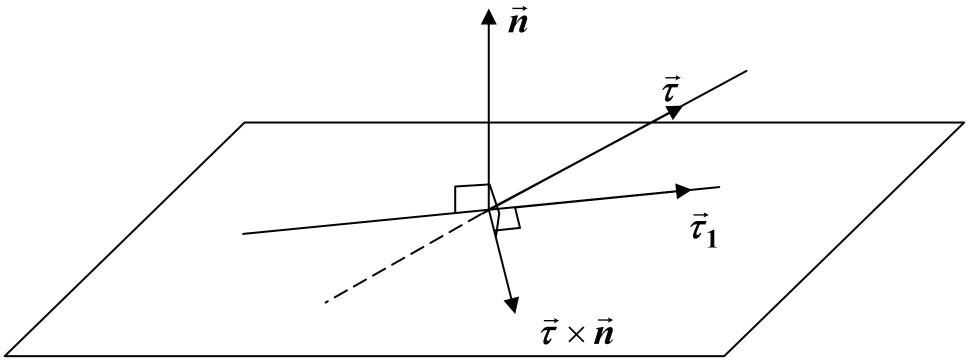

Fig. 17

We have $\vec{n} = \langle 3,-2,1 \rangle$ and $\vec{\tau} = \langle 1,-2,3 \rangle$. Let us find $\vec{\tau} \times \vec{n}$:

$$\vec{\tau} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 4\vec{i} + 8\vec{j} + 4\vec{k} = \langle 4,8,4 \rangle.$$

Then $\vec{\tau}_1 = \vec{n} \times (\vec{\tau} \times \vec{n}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 4 & 8 & 4 \end{vmatrix} = -16\vec{i} - 8\vec{j} + 32\vec{k}$.

To write the equation of the projection we need to find a point on it. But it can be the point of the straight line and the plane intersection. Let us write the equation of the straight line in the parametrical form: $\begin{cases} x = 1 + t, \\ y = 1 - 2t, \\ z = 1 + 3t \end{cases}$ and put these into the equation of the plane. We have $3(1+t)-2(1-2t)+(1+3t)-2=0$ or $t=0$. Thus the point of intersection is $\langle 1,1,1\rangle$. Finally we can write the equation of the projection:

$$\frac{x-1}{-16}=\frac{y-1}{-8}=\frac{z-1}{32} \text{ or } \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-1}{-4}.$$

Example 29: find the distance between two skew straight lines $\frac{x+7}{3}=\frac{y-3}{4}=\frac{z+3}{-2}$ and $\frac{x-21}{6}=\frac{y+1}{-4}=\frac{z-2}{-1}$.

Solution: the distance between two skew straight lines is the length of the common perpendicular to these straight lines. Since this perpendicular is common to both straight lines its direction vector is equal to the vector product of the two given direction vectors. And the distance is the absolute value of the projection of any vector connecting two straight lines on the common perpendicular (Fig. 18).

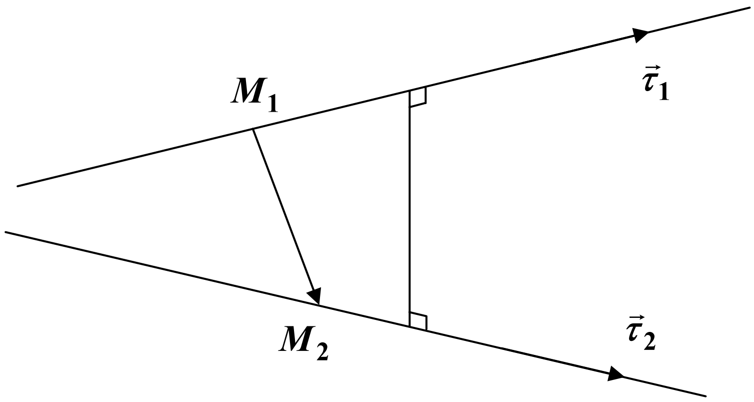

Fig. 18

Thus the distance $d=\left|\mathbf{proj}_{\vec{\tau}_1\times\vec{\tau}_2}\ \overrightarrow{M_1M_2}\right|=\left|\frac{\overrightarrow{M_1M_2}(\vec{\tau}_1\times\vec{\tau}_2)}{|\vec{\tau}_1\times\vec{\tau}_2|}\right|$.

Let us find $\vec{\tau}_1\times\vec{\tau}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 4 & -2\\ 6 & -4 & -1\end{vmatrix}=-12\vec{i}-9\vec{j}-36\vec{k}$. Then $\overrightarrow{M_1M_2}=\langle 28,-1,5\rangle$.

Putting into the formula we

obtain: $d=\left|\frac{28\cdot(-12)-1\cdot(-9)+5\cdot(-36)}{\sqrt{(-12)^2+(-9)^2+(-36)^2}}\right|=\frac{507}{39}=13$.

MATRICES

- **Basic definitions and operations on matrices.**

If n and m are natural numbers, then $n \times m$ ***matrix*** (read "n by m") is a rectangular array

$$[a_{ij}]_{i=1}^{n}{}_{j=1}^{m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix},$$

in which each ***entry***, a_{ij}, is a real (sometimes complex) number. $n \times m$ matrix has m rows and n columns.

If $n = m$, the matrix is ***square of order n***. For a square matrix, the entries $a_{11}, a_{22}, ..., a_{nn}$ are the ***main diagonal*** entries.

A matrix that has only one row is a ***row matrix***, and a matrix that has only one column is a ***column matrix***.

The matrix having all entries equal to zero is called ***zero matrix***, usually denoted as O. The square matrix that consists of zeros anywhere, except for the main diagonal, is called the ***diagonal matrix***. The square $n \times n$ matrix that consists of ones on its main diagonal and zeros elsewhere is called the ***identity matrix of order n*** and is denoted by I_n.

Two matrices are ***equal*** if their corresponding entries are equal.

The following operations are defined on matrices:

1. Matrix addition.
2. Scalar multiplication.
3. Matrix multiplication.

If $A = \left[a_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ and $B = \left[b_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ are the matrices of order $n \times m$, then their sum is the $n \times m$ matrix given by $A + B = \left[a_{ij} + b_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$. The sum of two matrices of the different order is undefined.

Example 30 : find $A + B$ if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$.

Solution: $A + B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$.

When working on matrices, we usually refer to numbers as scalars.

If $A=\left[a_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ is the matrix of order $n\times m$ and the c is a scalar, then the ***scalar multiple*** of A by c is $n\times m$ matrix given by $cA=\left[c\cdot a_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$. We use $-A$ to represent a scalar product $(-1)A$. Moreover, if $A=\left[a_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ and $B=\left[b_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ are the matrices of order $n\times m$, the difference $A-B$ represents the sum of A and $(-1)B$.

Example 31: for the matrices $A=\begin{bmatrix}1 & 2 & 4\\ -3 & 0 & -1\\ 2 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0 & 0\\ 1 & -4 & 3\\ -1 & 3 & 2\end{bmatrix}$ find $3A-B$.

Solution: $3A=\begin{bmatrix}3\cdot 1 & 3\cdot 2 & 3\cdot 4\\ 3\cdot(-3) & 3\cdot 0 & 3\cdot(-1)\\ 3\cdot 2 & 3\cdot 1 & 3\cdot 2\end{bmatrix}=\begin{bmatrix}3 & 6 & 12\\ -9 & 0 & -3\\ 6 & 3 & 6\end{bmatrix}$; $3A-B=$

$$=\begin{bmatrix}3 & 6 & 12\\ -9 & 0 & -3\\ 6 & 3 & 6\end{bmatrix}-\begin{bmatrix}2 & 0 & 0\\ 1 & -4 & 3\\ -1 & 3 & 2\end{bmatrix}=\begin{bmatrix}3-2 & 6-0 & 12-0\\ -9-1 & 0-(-4) & -3-3\\ 6-(-1) & 3-3 & 6-2\end{bmatrix}=\begin{bmatrix}1 & 6 & 12\\ -10 & 4 & -6\\ 7 & 0 & 4\end{bmatrix}.$$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix: $\begin{bmatrix}1/2 & -3/2\\ 5/2 & 1/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -3\\ 5 & 1\end{bmatrix}$.

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers, and we summarize them in the following list. If A, B, and C are $n\times m$ matrices and c and d are scalars, then the following properties are true:

1. $A+B=B+A$.
2. $A+(B+C)=(A+B)+C$.
3. $(cd)A=c(dA)$.
4. $1A=A$.
5. $c(A+B)=cA+cB$.
6. $(c+d)A=cA+dA$.
7. $A+O=A$.
8. $0\cdot A=O$.

Note that the second property of matrix addition allows us to write expressions such as $A+B+C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. The same reasoning applies to sums of four or more matrices.

The third basic matrix operation is ***matrix multiplication***. If $A=\left[a_{ij}\right]_{i=1}^{n}{}_{j=1}^{m}$ is the matrix of order $n\times m$ and $B=\left[b_{ij}\right]_{i=1}^{m}{}_{j=1}^{p}$ is the matrix of order $m\times p$, then their ***product*** AB is $n\times p$ matrix $AB=\left[c_{ij}\right]_{i=1}^{n}{}_{j=1}^{p}$, where $c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$. This definition indicates a ***row-by-column*** multiplication, where the entry c_{ij} in the i^{th} row and j^{th} column of the product AB is obtained by multiplying the entries in the i^{th} row of A by the corresponding entries in the j^{th} column of B and then adding the results.

Example 32: find the product AB, if $A=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$.

Solution: first note that the product AB is defined because the number of columns of A is equal to the number of rows of B. Moreover, the product AB has order 3×2 and will take the form

$$\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix}\cdot\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\\ c_{31} & c_{32}\end{bmatrix}.$$

To find c_{11} (the entry in the first row and first column of the product), multiply corresponding entries in the first row of A and the first column of B. That is, $c_{11}=(-1)(-3)+3(-4)=-9$. Similarly, to find c_{12}, multiply corresponding entries in the first row of A and the second column of B to obtain $c_{12}=(-1)2+3\cdot 1=1$. Continuing the pattern produces the following results:

$$c_{21}=4(-3)+(-2)(-4)=-4,$$
$$c_{22}=4\cdot 2+(-2)1=6,$$
$$c_{31}=5(-3)+0(-4)=-15,$$
$$c_{32}=5\cdot 2+0\cdot 1=10.$$

Thus, the product is

$$AB=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix}\cdot\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}=\begin{bmatrix}-9 & 1\\ -4 & 6\\ -15 & 10\end{bmatrix}.$$

Example 33: find AB and BA where $A = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution: since the order of A is 1×3 and the order of B is 3×1, the order of product AB is 1×1. Multiplying the row of A and the column of B we obtain

$$AB = \left[1 \cdot 2 + (-2)(-1) + (-3)1\right] = [1].$$

The product BA has the order 3×3 and

$$BA = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2(-2) & 2(-3) \\ (-1)1 & (-1)(-2) & (-1)(-3) \\ 1 \cdot 1 & 1(-2) & 1(-3) \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}.$$

Note that the two products are different. Matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$.

Main properties of matrix multiplication are as follows. If A, B, and C are matrices and c is a scalar, then the following properties are true:

1. $A(BC) = (AB)C$.
2. $A(B + C) = AB + AC$.
3. $(A + B)C = AC + BC$.
4. $c(AB) = (cA)B = A(cB)$.
5. If A is a square matrix then $IA = AI = A$.

- **Elementary row operations. Gauss-Jordan elimination method.**

Two matrices are called ***row-equivalent*** if one can be obtained from the other by a sequence of ***elementary row operations***. The elementary row operations are as follows:

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Example 34:

a) Interchange the first and the second rows:

Original matrix	*New row-equivalent matrix*
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$

b) Multiply the first row by $\frac{1}{2}$.

Original matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New row-equivalent matrix

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c) Add **–2** times the first row to the third row.

Original matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New row-equivalent matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

It is said that the matrix is in ***row-echelon form*** if it has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is **1** (a ***leading*** **1**).
3. For two successive (nonzero) rows, the leading **1** in the higher row is farther to the left than the leading **1** in the lower row.

A matrix in row-echelon form is in ***reduced row-echelon form*** if every column that has leading **1** has zeros in every position above and below its leading **1**. Every matrix is row-equivalent to a matrix in row-echelon form.

Example 35: the following matrices are in row-echelon form. The matrices ***B*** and ***D*** also happen to be in reduced row-echelon form.

$$A.\ \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad B.\ \begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C.\ \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad D.\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elementary row operations are applied in solving systems of linear equations. The method is called ***Gauss-Jordan elimination***. This method works well for solving systems with a computer. Before we consider the method, we must give the following definitions.

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is called the ***augmented matrix*** of the system. Moreover, the matrix derived from the coefficients of the system (but which does not include the constant terms) is called the ***coefficient matrix*** of the system. We use zeros for the missing variables. Here is an example.

System	Augmented matrix	Coefficient matrix
$\begin{cases} x-4y+3z=5, \\ -x+3y-z=-3, \\ 2x-4z=6. \end{cases}$	$\left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right]$	$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$

Guidelines for using Gauss-Jordan elimination to solve a system of linear equation are summarized as follows:

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form and then in reduced row-echelon form.
3. If a coefficient matrix is the identity matrix then the solution of the system is in the column separated by vertical dots.

For this algorithm, the order in which the elementary row operations are performed is important. We suggest operating from ***left-to-right by columns***, using elementary row operations to obtain zeros in all entries directly below the leading ones, and then above the leading ones.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in elimination process, we obtain a row with zeros except for the last entry, it is unnecessary to continue the elimination process. We can simply conclude that the system is inconsistent.

Example 36: solve the following system:

$$\begin{cases} y+z-2w=-3, \\ x+2y-z=2, \\ 2x+4y+z-3w=-2, \\ x-4y-7z-w=-19. \end{cases}$$

Solution: the augmented matrix for this system is

$$\left[\begin{array}{cccc:c} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right].$$

Let us start with obtaining a leading one in the upper left corner by interchanging the first and second rows, and then proceed to obtain zeros elsewhere in the first column.

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{-3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{-4} & \mathbf{-7} & \mathbf{-1} & \mathbf{-19} \end{array}\right] \sim \left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{-3} & \mathbf{-6} \\ \mathbf{0} & \mathbf{-6} & \mathbf{-6} & \mathbf{-1} & \mathbf{-21} \end{array}\right].$$

We added –**2** times the first row to the third row, and –**1** times the first row to the fourth row. After that the first column has zeroes below its leading one. Now after the first column is already in the desired form, we can change the second, third, and fourth columns as follows.

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{-3} & \mathbf{-6} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-13} & \mathbf{-39} \end{array}\right]$$
. We added **6** times the second row to the forth row.

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-13} & \mathbf{-39} \end{array}\right]$$
. We divided the third row by **3**.

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \end{array}\right]$$
. We divided the fourth row by –**13**.

The matrix is now in row-echelon form, and now we continue elimination, starting with the last column:

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} \end{array}\right].$$

We added the fourth row to the third row and **2** times the forth row to the second row. Then we added –**1** times the third row to the second row and the third row to the first row. And finally we added –**2** times the second row to the first row. We obtained the identity matrix, hence in the last column we have the solution of the system:

$$\begin{cases} x = 1, \\ y = 2, \\ z = 1, \\ w = 3. \end{cases}$$

- **The rank of a matrix. Theorem of Kronecker-Capelli.**

A determinant composed of entries of a matrix by crossing out some rows and columns is called a minor. The size of the maximum nonzero minor is called the ***rank*** of a matrix. For example let us consider the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}.$$

The different minors that can be composed of entries of this matrix are as follows:

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}, |1|, |2|, |3|.$$

It can be easily seen that the determinants of the third and second order are all equal to zero. Thus, the rank of a matrix is equal to **1**.

If $A = [a_{ij}]_{i=1}^{n}{}_{j=1}^{m}$ is the matrix of order $n \times m$, then the rank of the matrix *A*, denoted by $r(A)$, satisfies the following inequality $1 \le r(A) \le \min(n, m)$, except for zero matrix, which rank is equal to zero.

If a matrix has a big size then it is difficult to write and especially to calculate all minors. But the following theorem gives us an opportunity to do this easily.

Theorem 1. Two row-equivalent matrices have the same rank.

Thus, to find the rank of a matrix, we rewrite it in the row-echelon form. The number of nonzero rows will be the rank of the matrix.

Example 37: find the rank of the following matrix:

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 2 \\ 1 & -3 & -1 \\ 4 & 6 & 14 \end{bmatrix}.$$

Solution:

Step 1. Interchange the first and the third row:

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 2 \\ 1 & -3 & -1 \\ 4 & 6 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & -4 \\ 4 & 6 & 14 \end{bmatrix}.$$

Step 2. Add **2** times the first row to the third row and –**4** times the first row to the fourth row:

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 2 & 2 \\ -2 & 0 & -4 \\ 4 & 6 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 2 & 2 \\ 0 & -6 & -6 \\ 0 & 18 & 18 \end{bmatrix}.$$

Step 3. Divide the second row by **2**, the third row by –**6**, the fourth row by **18**:

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 2 & 2 \\ 0 & -6 & -6 \\ 0 & 18 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Step 4. Add –**1** times the second row to the third and fourth rows:

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We obtained the row-echelon matrix, with **2** nonzero rows. Thus, the rank of the matrix is equal to **2**.

Theorem 2. (Kronecker-Capelli). The system of ***n*** linear equations in ***m*** variables is consistent if and only if the rank of the augmented matrix is equal to the rank of the coefficient matrix.

It is not necessary to find the ranks of augmented and coefficient matrices separately. We can find both ranks just considering the augmented matrix.

Example 38: check if the following system is consistent.

$$\begin{cases} \mathbf{x_1 + x_2 = 1,} \\ \mathbf{x_1 + x_2 + x_3 = 4,} \\ \mathbf{x_2 + x_3 + x_4 = -3,} \\ \mathbf{x_3 + x_4 + x_5 = 2,} \\ \mathbf{x_4 + x_5 = -1.} \end{cases}$$

Solution: the augmented matrix for this system is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

Step 1. Add –**1** times the first row to the second row:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

Step 2. Interchange the second and the third rows:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

Step 3. Add –**1** times the third row to the fourth row:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

Step 4. Add –**1** times the fourth row to the fifth row:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, we have the row-echelon form of the matrix. It is seen that the ranks of the augmented and coefficient matrices are equal to **4**. Hence, the system is consistent.

Example 39: check if the following system is consistent:

$$\begin{cases} 2x_1 + x_2 + 3x_3 - x_4 + x_5 = 2, \\ x_1 - 2x_2 + x_3 + 3x_4 - x_5 = 1, \\ 3x_1 + 4x_2 + 5x_3 - 5x_4 + 3x_5 = 3, \\ x_1 + 3x_2 + 2x_3 - 4x_4 + 2x_5 = 1, \\ 5x_1 - 5x_2 + 6x_3 + 8x_4 - 2x_5 = 4. \end{cases}$$

Solution: the augmented matrix for this system is

$$\left[\begin{array}{ccccc|c} 2 & 1 & 3 & -1 & 1 & 2 \\ 1 & -2 & 1 & 3 & -1 & 1 \\ 3 & 4 & 5 & -5 & 3 & 3 \\ 1 & 3 & 2 & -4 & 2 & 1 \\ 5 & -5 & 6 & 8 & -2 & 4 \end{array}\right].$$

Step 1. Interchange the first and the second rows:

$$\left[\begin{array}{ccccc|c} 2 & 1 & 3 & -1 & 1 & 2 \\ 1 & -2 & 1 & 3 & -1 & 1 \\ 3 & 4 & 5 & -5 & 3 & 3 \\ 1 & 3 & 2 & -4 & 2 & 1 \\ 5 & -5 & 6 & 8 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 2 & 1 & 3 & -1 & 1 & 2 \\ 3 & 4 & 5 & -5 & 3 & 3 \\ 1 & 3 & 2 & -4 & 2 & 1 \\ 5 & -5 & 6 & 8 & -2 & 4 \end{array}\right].$$

Step 2. Add –**2** times the first row to the second row, –**3** times to the third, –**1** times to the fourth, and –**5** times to the fifth:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 2 & 1 & 3 & -1 & 1 & 2 \\ 3 & 4 & 5 & -5 & 3 & 3 \\ 1 & 3 & 2 & -4 & 2 & 1 \\ 5 & -5 & 6 & 8 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & -3 & 1 & -7 & -1 & 0 \\ 0 & -2 & 2 & -14 & 6 & 0 \\ 0 & 5 & 1 & -7 & 3 & 1 \\ 0 & 5 & 1 & -7 & 3 & -1 \end{array}\right].$$

Step 3. Add –**2** times the third row to the second row:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & -3 & 1 & -7 & -1 & 0 \\ 0 & -2 & 2 & -14 & 6 & 0 \\ 0 & 5 & 1 & -7 & 3 & 1 \\ 0 & 5 & 1 & -7 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & -2 & 2 & -14 & 6 & 0 \\ 0 & 5 & 1 & -7 & 3 & 1 \\ 0 & 5 & 1 & -7 & 3 & -1 \end{array}\right].$$

Step 4. Add **2** times the second row to the third row, –**5** times to the fourth and fifth:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & -2 & 2 & -14 & 6 & 0 \\ 0 & 5 & 1 & -7 & 3 & 1 \\ 0 & 5 & 1 & -7 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & 0 & -4 & 28 & -20 & 0 \\ 0 & 0 & -14 & -112 & 68 & 1 \\ 0 & 0 & -14 & -112 & 68 & -1 \end{array}\right].$$

Step 5. Divide the third row by –**4**:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & 0 & -4 & 28 & -20 & 0 \\ 0 & 0 & -14 & -112 & 68 & 1 \\ 0 & 0 & -14 & -112 & 68 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & 0 & 1 & -7 & 5 & 0 \\ 0 & 0 & -14 & -112 & 68 & 1 \\ 0 & 0 & -14 & -112 & 68 & -1 \end{array}\right].$$

Step 6. Add **14** times the third row to the fourth and fifth rows:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & 0 & 1 & -7 & 5 & 0 \\ 0 & 0 & -14 & -112 & 68 & 1 \\ 0 & 0 & -14 & -112 & 68 & -1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & -1 & 1 \\ 0 & 1 & -3 & 21 & -13 & 0 \\ 0 & 0 & 1 & -7 & 5 & 0 \\ 0 & 0 & 0 & -210 & 138 & 1 \\ 0 & 0 & 0 & -210 & 138 & -1 \end{array}\right].$$

Step 7. Add –**1** times the fourth row to the fifth row:

$$\begin{bmatrix} 1 & -2 & 1 & 3 & -1 & \vdots & 1 \\ 0 & 1 & -3 & 21 & -13 & \vdots & 0 \\ 0 & 0 & 1 & -7 & 5 & \vdots & 0 \\ 0 & 0 & 0 & -210 & 138 & \vdots & 1 \\ 0 & 0 & 0 & -210 & 138 & \vdots & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 3 & -1 & \vdots & 1 \\ 0 & 1 & -3 & 21 & -13 & \vdots & 0 \\ 0 & 0 & 1 & -7 & 5 & \vdots & 0 \\ 0 & 0 & 0 & -210 & 138 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 2 \end{bmatrix}.$$

Thus, we have the row-echelon form of the matrices. The rank of the augmented matrix is equal to **5**, but the rank of the coefficient matrix is equal to **4**. Thus, the system is inconsistent.

- **Homogeneous systems of linear algebraic equations.**

The following system of linear algebraic equations is called ***homogeneous***.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{nm}x_n = 0. \end{cases}$$

From the theorem of Kronecker-Capelli's it follows that this system is always consistent, since adding zero column does not change the rank. Moreover, it is seen that this system has the zero solution.

Let the rank of the coefficient matrix of this system equal r. If $r = n$ then the zero solution will be unique. From the other hand, if the number of equations is less than the number of variables or $r < n$, then the system always has nonzero solutions.

The solutions of homogeneous systems of linear algebraic equations have the following properties:

1. If $(c_1,\dots,c_n)$ is a solution then $(kc_1,\dots,kc_n)$ is also a solution for every k.
2. If $(c_1,\dots,c_n)$ and $(d_1,\dots,d_n)$ are solutions then $(c_1 + d_1,\dots,c_n + d_n)$ is also a solution.

In other words, a linear combination of solutions of the homogeneous system of linear algebraic equations is also a solution of the system.

Theorem 3. If the rank r of the coefficient matrix of the homogeneous system of linear algebraic equations in n variables is less than the number of variables, then there exist $n - r$ linear independent solutions, which are called the fundamental system of solutions.

To find the fundamental system of solutions we take $\boldsymbol{r}$ linear independent equations and solve it for $\boldsymbol{r}$ variables. Then, we take $\boldsymbol{n-r}$ linear independent vectors which are, for instance, columns of the identity matrix of the order $\boldsymbol{n-r}$. Substituting their components in the found solution for $\boldsymbol{r}$ variables, we obtain the fundamental system of solutions.

Example 40: find the fundamental system of solutions of the following system:

$$\begin{cases} 3x_1 + x_2 - 8x_3 + 2x_4 + x_5 = 0, \\ 2x_1 - 2x_2 - 3x_3 - 7x_4 + 2x_5 = 0, \\ x_1 + 11x_2 - 12x_3 + 34x_4 - 5x_5 = 0, \\ x_1 - 5x_2 + 2x_3 - 16x_4 + 3x_5 = 0. \end{cases}$$

Solution: first of all let us derive the coefficient matrix to the reduced row-echelon form.

Step 1. Interchange the first and the third row:

$$\begin{bmatrix} 3 & 1 & -8 & 2 & 1 \\ 2 & -2 & -3 & -7 & 2 \\ 1 & 11 & -12 & 34 & -5 \\ 1 & -5 & 2 & -16 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 2 & -2 & -3 & -7 & 2 \\ 3 & 1 & -8 & 2 & 1 \\ 1 & -5 & 2 & -16 & 3 \end{bmatrix}.$$

Step 2. Add –**2** times the first row to the second row, –**3** times to the third, –**1** times to the fourth:

$$\begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 2 & -2 & -3 & -7 & 2 \\ 3 & 1 & -8 & 2 & 1 \\ 1 & -5 & 2 & -16 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & -24 & 21 & -75 & 12 \\ 0 & -32 & 28 & -100 & 16 \\ 0 & -16 & 14 & -50 & 8 \end{bmatrix}.$$

Step 3. Divide the second row by –**3**, the third by **4**, the fourth by **2**:

$$\begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & -24 & 21 & -75 & 12 \\ 0 & -32 & 28 & -100 & 16 \\ 0 & -16 & 14 & -50 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & -8 & 7 & -25 & 4 \\ 0 & -8 & 7 & -25 & 4 \end{bmatrix}.$$

Step 4. Add the second row to the third and fourth:

$$\begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & -8 & 7 & -25 & 4 \\ 0 & -8 & 7 & -25 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 5. Divide the second row by **8**:

$$\begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 8 & -7 & 25 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 1 & -{}^{7}\!/_{8} & {}^{25}\!/_{8} & -{}^{1}\!/_{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 6. Add –**11** times the second row to the first row:

$$\begin{bmatrix} 1 & 11 & -12 & 34 & -5 \\ 0 & 1 & -{}^{7}\!/_{8} & {}^{25}\!/_{8} & -{}^{1}\!/_{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -{}^{19}\!/_{8} & -{}^{3}\!/_{8} & {}^{1}\!/_{2} \\ 0 & 1 & -{}^{7}\!/_{8} & {}^{25}\!/_{8} & -{}^{1}\!/_{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have $r = 2$ and $\begin{cases} x_1 = \frac{19}{8}x_3 + \frac{3}{8}x_4 - \frac{1}{2}x_5, \\ x_2 = \frac{7}{8}x_3 - \frac{25}{8}x_4 + \frac{1}{2}x_5. \end{cases}$ Then we take three independent vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and after substituting the components of these vectors for x_3, x_4, x_5 we obtain the fundamental system of solutions:

$$\begin{bmatrix} {}^{19}\!/_{8} \\ {}^{7}\!/_{8} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} {}^{3}\!/_{8} \\ -{}^{25}\!/_{8} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -{}^{1}\!/_{2} \\ {}^{1}\!/_{0} \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- **The inverse of a matrix. Matrix equations.**

Let A be a square matrix of order n. If there exists a matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$, then A^{-1} is the ***inverse*** of A. A^{-1} is read "A inverse". Recall that it is not always true that $AB=BA$, even if both products are defined. However, if A and B are both square matrices and $AB=I$, then it can be shown that $BA=I$.

If a matrix A has an inverse, then A is ***invertible*** (or ***non-singular***); otherwise, A is ***singular***. A nonsquare matrix cannot have an inverse. Not all square matrices possess an inverse.

Theorem 4. The matrix A is invertible if and only if the determinant of the matrix A is not equal to zero. If the matrix does possess an inverse, then that inverse is unique.

Example 41: show that B is inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Solution:

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are several methods of finding the inverse of a matrix. Let us consider two of them. The first method is the determinant method. Let A be a square matrix of order n, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix},$$

where A_{ij} is a minor obtained from A by crossing out i^{th} row and j^{th} column, taken with "+" if $i+j$ is even, and with "–" if $i+j$ is odd.

Example 42: find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & -5 \\ 0 & -2 & 6 \end{bmatrix}.$$

Solution: let us find the determinant of A:

$$\det(A)=\begin{vmatrix}2 & 1 & 3\\ 1 & 4 & -5\\ 0 & -2 & 6\end{vmatrix}=2\begin{vmatrix}4 & -5\\ -2 & 6\end{vmatrix}-1\begin{vmatrix}1 & -5\\ 0 & 6\end{vmatrix}+3\begin{vmatrix}1 & 4\\ 0 & -2\end{vmatrix}=2\cdot 14-1\cdot 6+3(-2)=16.$$

Since, the determinant of A is not equal to zero the inverse matrix exists, and

$$A^{-1}=\frac{1}{16}\begin{vmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{vmatrix}.$$

Let us find A_{ij}:

$$A_{11}=\begin{vmatrix}4 & -5\\ -2 & 6\end{vmatrix}=24-10=14,\ A_{12}=-\begin{vmatrix}1 & -5\\ 0 & 6\end{vmatrix}=-(6-0)=-6,$$

$$A_{13}=\begin{vmatrix}1 & 4\\ 0 & -2\end{vmatrix}=-2-0=-2,\ A_{21}=-\begin{vmatrix}1 & 3\\ -2 & 6\end{vmatrix}=-(6+6)=-12,$$

$$A_{22}=\begin{vmatrix}2 & 3\\ 0 & 6\end{vmatrix}=12-0=12,\ A_{23}=-\begin{vmatrix}2 & 1\\ 0 & -2\end{vmatrix}=-(-4-0)=4,$$

$$A_{31}=\begin{vmatrix}1 & 3\\ 4 & -5\end{vmatrix}=-5-12=-17,\ A_{32}=-\begin{vmatrix}2 & 3\\ 1 & -5\end{vmatrix}=-(-10-3)=13,$$

$$A_{33}=\begin{vmatrix}2 & 1\\ 1 & 4\end{vmatrix}=8-1=7.$$

Thus, we can write the inverse of A:

$$A^{-1}=\frac{1}{16}\begin{bmatrix}14 & -12 & -17\\ -6 & 12 & 13\\ -2 & 4 & 7\end{bmatrix}=\begin{bmatrix}7/8 & -3/4 & -17/16\\ -3/8 & 3/4 & 13/16\\ -1/8 & 1/4 & 7/16\end{bmatrix}.$$

The second method is as follows. If we by a sequence of elementary row operations reduce a square matrix to the identity matrix, then with the same sequence of elementary row operations the identity matrix is reduced to the inverse of the given matrix.

Example 43: find the inverse of the matrix

$$A=\begin{bmatrix} 1 & 3 & -1 & -2 \\ 2 & 7 & 0 & -1 \\ -1 & -4 & 0 & -1 \\ -3 & 10 & -1 & -2 \end{bmatrix}.$$

Solution: at first we form the matrix, which consists of two parts. The first is the matrix A, the second is the identity matrix. Then with row operations we transform this matrix, such that the first part reduces to the identity matrix. Moreover, the second part will be the inverse A^{-1}:

$$\left[\begin{array}{cccc:cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & -4 & 0 & -1 & 0 & 0 & 1 & 0 \\ 3 & 10 & -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right].$$

Step 1. Add –**2** times the first row to the second row, add to the third, –**3** times to the fourth:

$$\left[\begin{array}{cccc:cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & -4 & 0 & -1 & 0 & 0 & 1 & 0 \\ 3 & 10 & -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc:cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right].$$

Step 2. Add the second row to the third, –**1** times to the fourth:

$$\left[\begin{array}{cccc:cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc:cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right].$$

Step 3. Add –**3** times the fourth row to the second row, and **2** times to the first row:

$$\left[\begin{array}{cccc|cccc} 1 & 3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right].$$

Step 4. Add –**2** times the third row to the second row, and to the first row:

$$\left[\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & -2 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 & -2 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right].$$

Step 5. And finally add –3 times the second row to the first row:

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & -2 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 & -2 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -11 & -7 & 7 & 11 \\ 0 & 1 & 0 & 0 & 3 & 2 & -2 & -3 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right].$$

$$\text{Thus, } A^{-1} = \begin{bmatrix} -11 & -7 & 7 & 11 \\ 3 & 2 & -2 & -3 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}.$$

The system of n linear algebraic equations in n variables can be rewritten in the following matrix form.

<u>*System*</u>   <u>*Matrix form*</u>

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Or the system can be rewritten as $\boldsymbol{AX} = \boldsymbol{B}$. If we multiply this equation from the left by $\boldsymbol{A}^{-1}$, we obtain $\boldsymbol{A}^{-1}\boldsymbol{AX} = \boldsymbol{A}^{-1}\boldsymbol{B}$, then $\boldsymbol{IX} = \boldsymbol{A}^{-1}\boldsymbol{B}$, and finally $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

Example 44: solve the following system using the inverse matrix:

$$\begin{cases} 2x_1 - x_2 + x_3 = 6, \\ -x_1 + x_2 + 2x_3 = 4, \\ 3x_1 + 2x_2 - 3x_3 = -8. \end{cases}$$

Solution: let us rewrite this system as $\boldsymbol{AX} = \boldsymbol{B}$, where

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 2 \\ 3 & 2 & -3 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 6 \\ 4 \\ -8 \end{bmatrix}, \ \boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Using one of the mentioned methods we obtain

$$\boldsymbol{A}^{-1} = \frac{1}{22}\begin{bmatrix} 7 & 1 & 3 \\ -3 & 9 & 5 \\ 5 & 7 & -1 \end{bmatrix}. \ \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \frac{1}{22}\begin{bmatrix} 7 & 1 & 3 \\ -3 & 9 & 5 \\ 5 & 7 & -1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \\ -8 \end{bmatrix} =$$

$$= \frac{1}{22}\begin{bmatrix} 7\cdot 6 + 1\cdot 4 - 3\cdot 8 \\ -3\cdot 6 + 9\cdot 4 - 5\cdot 8 \\ 5\cdot 6 + 7\cdot 4 + 1\cdot 8 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 22 \\ -22 \\ 66 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 1, \\ x_2 = -1, \\ x_3 = 3. \end{cases}$$

VECTOR SPACES

- **Definition and examples.**

Let a set V be given; we will denote its elements by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, etc., and let two operations be defined on this set – ***adding***, i.e. for every two elements $\boldsymbol{a}$ and $\boldsymbol{b}$ in V there exists the uniquely determined element $\boldsymbol{a}+\boldsymbol{b}$ also in V, which is called the sum of $\boldsymbol{a}$ and $\boldsymbol{b}$; and ***scalar multiplication***, i.e. for every element $\boldsymbol{a}$ in V and every real number α there exists the uniquely determined element $\alpha\boldsymbol{a}$ also in V. The elements of the set V we will call ***vectors***.

This space is called a ***vector space*** if the mentioned operations have the following properties:

I. $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$.

II. $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$.

III. In V there exists ***zero element*** $\mathbf{0}$, such that $\boldsymbol{a}+\mathbf{0}=\boldsymbol{a}$ for all $\boldsymbol{a}\in V$; it is easy to prove that zero element is unique.

IV. For every $\boldsymbol{a}\in V$ there exists the ***opposite element*** $-\boldsymbol{a}$, such that $\boldsymbol{a}+(-\boldsymbol{a})=\mathbf{0}$; it is easy to prove that the opposite element is unique.

V. $\alpha(\boldsymbol{a}+\boldsymbol{b})=\alpha\boldsymbol{a}+\alpha\boldsymbol{b}$.

VI. $(\alpha+\beta)\boldsymbol{a}=\alpha\boldsymbol{a}+\beta\boldsymbol{a}$.

VII. $(\alpha\beta)\boldsymbol{a}=\alpha(\beta\boldsymbol{a})$.

VIII. $\mathbf{1}\cdot\boldsymbol{a}=\boldsymbol{a}$.

From axioms I–IV we can infer the existence and uniqueness of the difference $\boldsymbol{a}-\boldsymbol{b}$, that is such an element, which satisfies the equation $\boldsymbol{b}+\boldsymbol{x}=\boldsymbol{a}$.

Let us state some properties following from these axioms:

1. $\alpha\cdot\mathbf{0}=\mathbf{0}$.
2. $\mathbf{0}\cdot\alpha=\mathbf{0}$.
3. If $\alpha\boldsymbol{a}=\mathbf{0}$ then $\boldsymbol{a}=\mathbf{0}$ or $\alpha=\mathbf{0}$.
4. $\alpha(-\boldsymbol{a})=-\alpha\boldsymbol{a}$.
5. $(-\alpha)\boldsymbol{a}=-\alpha\boldsymbol{a}$.
6. $\alpha(\boldsymbol{a}-\boldsymbol{b})=\alpha\boldsymbol{a}-\alpha\boldsymbol{b}$.
7. $(\alpha-\beta)\boldsymbol{a}=\alpha\boldsymbol{a}-\beta\boldsymbol{a}$.

The definition given above is the definition of a real vector space, but if we use complex numbers instead of real numbers in the axioms, we obtain the definition of a complex vector space.

Here are some examples of vector spaces.

1. Space of vectors on a plane or in space. Recall that a vector in a plane is an ordered pair of real numbers, and a vector in space is an ordered triple of real numbers. Thus, all axioms are true.

2. The set of all sequences. A sequence is a function defined on the set of natural numbers, usually denoted by $(\alpha_1,\alpha_2,...,\alpha_n,...)$, where α_1 is the image of **1**, α_2 is the image of **2**, etc. Adding and scalar multiplication are defined as follows.

$$(\alpha_1,\alpha_2,...,\alpha_n,...)+(\beta_1,\beta_2,...,\beta_n,...)=(\alpha_1+\beta_1,\alpha_2+\beta_2,...,\alpha_n+\beta_n,...),$$
$$\gamma(\alpha_1,\alpha_2,...,\alpha_n,...)=(\gamma\alpha_1,\gamma\alpha_2,...,\gamma\alpha_n,...).$$

It is easy to notice that all axioms are true.

3. The set of all polynomials of order less or equal than n, denoted by P^n. All axioms follow from the properties of polynomials.
4. The set of infinitely continuously differentiable functions on the segment $[a,b]$, denoted by $C^{\infty}_{[a,b]}$. All axioms follow from the properties of continuous functions.

- **Linear independence and basis.**

The linear independence of vectors in linear space is defined exactly as in a plane or a space.

The ***linear combination*** of vectors $\{a_1,...,a_n,...\}$ with coefficients $\{\alpha_1,...,\alpha_n,...\}$ is called a vector of the form

$$b=\sum_{i=1}^{\infty}\alpha_i a_i,$$

if this series converges.

A linear combination is called ***trivial***, if all coefficients are equal to zero, and is called ***non trivial*** in the opposite case. Trivial linear combination of vectors is obviously equal to zero.

The system of vectors $\{a_1,...,a_n,...\}$ is called ***linear independent*** if their linear combination is equal to zero only if it is trivial. The system is called ***linear dependent*** in the opposite case.

The set of all linear combinations of vectors in the system is called the ***linear span*** of the system of vectors Γ, and denoted as $L(\Gamma)$. Let a linear space L and a system Γ be given, if $L=L(\Gamma)$, then the system Γ is called ***full***. A full linear independent system in the linear space L is called the ***basis*** of the linear space. A basis is usually denoted by $\{e_1,e_2,...,e_n,...\}$. If a vector can be represented as $a=\sum_{i=1}^{\infty}\alpha_i e_i$ then this representation is unique and $\{\alpha_i\}_{i=1}^{\infty}$ are called the components of the vector a in the basis $\{e_1,e_2,...,e_n,...\}$. All bases of a linear space consist of equal number of vectors. The number of vectors in the basis is called the ***dimension*** of the linear space $\boldsymbol{L}$, denoted by $\mathbf{dim}(\boldsymbol{L})$. If it is a finite number the vector space is called ***finite-dimensional*** linear space, and ***infinite-dimensional*** in the opposite case.

Now, let us consider examples mentioned above.

1. Vector spaces in a plane or space. Since every vector in space is an ordered triple of real numbers $\langle x, y, z\rangle$, then it can be represented as $\langle x, y, z\rangle = x\vec{i} + y\vec{j} + z\vec{k}$. Thus, $\{\vec{i}, \vec{j}, \vec{k}\}$ is a basis in this linear space, and its dimension equals **3**, and this space is finite-dimensional.
2. The set of sequences. Every sequence can be represented as a linear combinations of the following sequences. $(1,0,0,...),(0,1,0,...)$ etc. For example: $(\alpha_1, \alpha_2, ..., \alpha_n, ...) = \alpha_1(1,0,0,...) + \alpha_2(0,1,0,...) + ...$. Thus, this system is a basis and the linear space is infinite-dimensional, because the number of such sequences is infinite.
3. The set of polynomials of order less or equal than $\boldsymbol{n}$. Every polynomial can be represented as $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, and the system $\{1, x, x^2, ..., x^n\}$ is a basis of this linear space and $\mathbf{dim}(P^n) = n + 1$.

Let us consider a finite-dimensional linear space $\boldsymbol{L}$, and let $\{e_1, e_2, ..., e_n\}$ and $\{e'_1, e'_2, ..., e'_n\}$ be two bases of $\boldsymbol{L}$. Every vector in the second basis, like every vector in $\boldsymbol{L}$, can be represented as a linear combination of the first basis $e'_i = \sum_{j=1}^{n} \tau_{ij} e_j, i = \overline{1,n}$. The matrix

$$T = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \cdots & \tau_{nn} \end{bmatrix},$$

the rows of which are the components of $\{e'_1, e'_2, ..., e'_n\}$ in the basis $\{e_1, e_2, ..., e_n\}$, is called the ***transition matrix*** from basis $\{e_1, e_2, ..., e_n\}$ to the basis $\{e'_1, e'_2, ..., e'_n\}$. This can be written as

$$\begin{bmatrix} e'_1 \\ e'_2 \\ \vdots \\ e'_n \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1n} \\ \tau_{21} & \tau_{22} & \cdots & \tau_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n1} & \tau_{n2} & \cdots & \tau_{nn} \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix},$$

or if we denote the two bases written as a column by e and e', then $e' = Te$. From the other hand, if T' is the transition matrix from e' to e, then $e = T'e'$. Hence,

$$e = T'Te \text{ and } e' = TT'e'.$$

Thus, on account of linear independence of $\boldsymbol{e}$ and $\boldsymbol{e}'$ $\boldsymbol{T}'\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}' = \boldsymbol{I}$, and $\boldsymbol{T}' = \boldsymbol{T}^{-1}$. This states that the transition matrix is an invertible matrix. And the converse, every invertible matrix is a transition matrix from a basis to another basis.

- **Linear transformations.**

Let a finite-dimensional linear space $\boldsymbol{L}$ be given. The representation $\varphi : \boldsymbol{L} \to \boldsymbol{L}$ is called a transformation of the linear space.

The transformation φ is called a ***linear transformation*** if $\varphi(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha\varphi(\boldsymbol{a}) + \beta\varphi(\boldsymbol{b})$. This means that a linear transformation transform any linear combination of vectors in a linear combination of the images of these vectors, moreover, with the same coefficients.

The following properties of linear transformations are true. $\varphi(\boldsymbol{0}) = \boldsymbol{0}$, $\varphi(-\boldsymbol{a}) = -\varphi(\boldsymbol{a})$.

Let us show some examples:

1. identity transformation, that is $\varphi(\boldsymbol{a}) = \boldsymbol{a}$;
2. zero transformation, that is $\varphi(\boldsymbol{a}) = \boldsymbol{0}$.

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$ be a basis in the linear space $\boldsymbol{L}$.

Theorem 1. For an ordered system of vectors $\{\boldsymbol{c}_1, \boldsymbol{c}_2, ..., \boldsymbol{c}_n\}$ there exists only one transformation $\boldsymbol{\varphi}$, such that $\boldsymbol{\varphi}(\boldsymbol{e}_i) = \boldsymbol{c}_i, \boldsymbol{i} = \overline{\boldsymbol{1}, \boldsymbol{n}}$.

Thus, we have one-to-one correspondence between all linear transformations and all ordered systems of vectors $\{\boldsymbol{c}_1, \boldsymbol{c}_2, ..., \boldsymbol{c}_n\}$. However every vector from this system can be represented as a linear combination of the basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$:

$$\boldsymbol{c}_i = \sum_{j=1}^{n} \alpha_{ij}\boldsymbol{e}_j, \boldsymbol{i} = \overline{\boldsymbol{1}, \boldsymbol{n}}.$$

From the components of the vectors $\{\boldsymbol{c}_1, \boldsymbol{c}_2, ..., \boldsymbol{c}_n\}$ in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$ we can form the square matrix $\boldsymbol{A} = [\alpha_{ij}]_{i,j=1}^{n}$. Thus, we have one-to-one correspondence between all linear transformations and all square matrices of the order $\boldsymbol{n}$, this correspondence, certainly, depends on the choice of the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$.

We will say that the matrix $\boldsymbol{A}$ defines the linear transformation φ or $\boldsymbol{A}$ is the ***matrix of the linear transformation*** φ in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_n\}$. If we define $\varphi(\boldsymbol{e})$ as the images column of the basis, then $\boldsymbol{\varphi}(\boldsymbol{e}) = \boldsymbol{A}\boldsymbol{e}$ and if $\boldsymbol{b} = \sum_{i=1}^{n} \beta_i \boldsymbol{e}_i$, then $\varphi(\boldsymbol{b}) = [\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n] \cdot \boldsymbol{A} \cdot \boldsymbol{e}$. The linear transformation is called ***non-singular*** if it is a surjection. The matrix of a non-singular linear transformation is non-singular (i.e. its determinant is not equal to zero).

Example 45: let $\{e_1, e_2, e_3\}$ be the basis of linear space, matrix A be the matrix of linear transformation φ. Find the image of the element $a = 5e_1 + e_2 - 2e_3$, if

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & -4 & 1 \end{bmatrix}.$$

Solution:

$$\varphi(a) = [5 \quad 1 \quad -2] \cdot \begin{bmatrix} -2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = [-9 \quad 16 \quad 0] \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = -9e_1 + 16e_2.$$

Let two bases e and e', with the transition matrix T ($e' = Te$) be given. And let the linear transformation φ in these bases be defined by matrices A and A' ($\varphi(e) = Ae$ and $\varphi(e') = A'e'$. We have $\varphi(Te) = A'Te$, but $\varphi(Te) = T\varphi(e)$, because φ is the linear transformation. Thus, $(TA)e = (A'T)e$, and because e is the linear independent system $TA = A'T$. Finally, because T is invertible,

$$A' = TAT^{-1} \text{ and } A = T^{-1}A'T.$$

If two matrices are connected with such a correlation they are called ***similar***. The determinants of similar matrices are equal.

- **Characteristic numbers and characteristic vectors of a linear transformation.**

Let a linear transformation φ in a real linear space be given. If a non-zero vector b transforms under this linear transformation into a proportional vector, then such vector is called a ***characteristic vector*** or ***eigenvector*** of the linear transformation and a proportional coefficient is called a ***characteristic number*** or ***eigenvalue*** of the linear transformation. If $\varphi(b) = \lambda b$ then λ is a characteristic number and b is a characteristic vector of the linear transformation φ. We will usually say that the characteristic vector b corresponds to the characteristic number λ.

If $A = [a_{ij}]_{i,j=1}^{n}$ is a matrix of the linear transformation φ, then $Ab = \lambda b$ or if $b = \langle b_1, b_2, ..., b_n \rangle$ then

$$\begin{cases} a_{11}b_1 + a_{12}b_2 + ... + a_{1n}b_n = \lambda b_1, \\ a_{21}b_1 + a_{22}b_2 + ... + a_{2n}b_n = \lambda b_2, \\ ...., \\ a_{n1}b_1 + a_{n2}b_2 + ... + a_{nn}b_n = \lambda b_n. \end{cases}$$

Since $b \neq 0$, not all $b_1, b_2, ..., b_n$ are equal to zero. This means that the system

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0, \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + ... + a_{2n}x_n = 0, \\ ...., \\ a_{n1}x_1 + a_{n2}x_2 + ... + (a_{nn} - \lambda)x_n = 0. \end{cases} \qquad (*)$$

has got non-zero solution, which means that the matrix of the system is singular (i.e. $\det(A - \lambda I) = 0$). This is called the ***characteristic equation***. The roots of this equation are the characteristic numbers of the linear transformation. Every non-zero solution of the system (*) will satisfy the corresponding matrix equation $Ax = \lambda x$ and will be components of characteristic vector corresponding to the characteristic number λ in the same basis as A.

Example 46: find characteristic numbers and characteristic vectors of the linear transformation φ represented in a basis by a matrix $A = \begin{bmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix}$.

Solution: let us write the characteristic equation $\det(A - \lambda I) = 0$, or

$$\begin{vmatrix} 7-\lambda & -12 & -2 \\ 3 & -4-\lambda & 0 \\ -2 & 0 & -2-\lambda \end{vmatrix} = (7-\lambda)(-4-\lambda)(-2-\lambda) - 4(-4-\lambda) + 36(-2-\lambda) =$$

$= (7-\lambda)(-4-\lambda)(-2-\lambda) - 32\lambda - 56 = -\lambda^3 + \lambda^2 + 2\lambda = -\lambda(\lambda - 2)(\lambda + 1) = 0$, i.e. $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = -1$.

Firstly, let us consider the case $\lambda_1 = 0$. Then the system (*) will be:

$$\begin{cases} 7x_1 - 12x_2 - 2x_3 = 0, \\ 3x_1 - 4x_2 = 0, \\ -2x_1 - 2x_3 = 0. \end{cases}$$

It has got non-zero solution because the determinant of the system is equal to zero. If we put $x_1 = 4t$, where t is any real number, then $x_2 = 3t$ from the second equation and $x_3 = -4t$ from the third one. Thus the first characteristic vector will be $e_1 = \langle 4t, 3t, -4t \rangle$, where t is any real number.

Then, let we have $\lambda_2 = 2$. Then the system (*) will be:

$$\begin{cases} 5x_1 - 12x_2 - 2x_3 = 0, \\ 3x_1 - 6x_2 = 0, \\ -2x_1 - 4x_3 = 0. \end{cases}$$

If we put $x_2 = t$, then $x_1 = 2t$ and $x_3 = -2t$. And the second characteristic vector will be $e_2 = \langle 2t, t, -2t \rangle$, where t is any real number.

And finally, let we have $\lambda_2 = -1$. Then the system (*) will be:

$$\begin{cases} 8x_1 - 12x_2 - 2x_3 = 0, \\ 3x_1 - 3x_2 = 0, \\ -2x_1 - x_3 = 0. \end{cases}$$

If we put $x_2 = t$, then $x_1 = t$ and $x_3 = -2t$. And the third characteristic vector will be $e_3 = \langle t, t, -2t \rangle$, where t is any real number.

- **Euclid spaces.**

We will say that in finite-dimensional real linear space the ***scalar product*** is defined, if every two elements a and b corresponds with a real number (a, b), which satisfies the following properties. (a, b, c are arbitrary vectors, α – a real number):

I. $(a, b) = (b, a)$.
II. $(a + b, c) = (a, c) + (b, c)$.
III. $\alpha(a, b) = (\alpha a, b)$.
IV. If $a \neq 0$, then $(a, a) > 0$.

From II and III it follows that $\sum_{i=1}^{n} \alpha_i a_i \sum_{j=1}^{n} \beta_j b_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \beta_j (a_i, b_j)$. If in a finite-dimensional linear space the scalar product is defined, then such space is called ***Euclid space***. In every finite-dimensional linear space the scalar product can be defined. For example, let $\{e_1, e_2, ..., e_n\}$ be a basis in the linear space L.

Then, if $a = \sum_{i=1}^{n} \alpha_i e_i$ and $b = \sum_{i=1}^{n} \beta_i e_i$, we can state that

$$(a, b) = \sum_{i=1}^{n} \alpha_i \beta_i$$

satisfies all the properties of scalar product.

Two vectors a and b in a Euclid space are called ***orthogonal*** if $(a, b) = 0$. A system of vectors $\{a_1, ..., a_n\}$ is called orthogonal system if $(a_i, a_j) = 0, i \neq j$.

Theorem 2. Every orthogonal system is linear independent.

A vector $\boldsymbol{a}$ is called ***normalized*** if $(\boldsymbol{a},\boldsymbol{a})=\mathbf{1}$. If a vector $\boldsymbol{a}\neq\mathbf{0}$ then normalization is the following transformation of the vector:

$$\boldsymbol{b}=\frac{\mathbf{1}}{\sqrt{(\boldsymbol{a},\boldsymbol{a})}}\boldsymbol{a}.$$

This vector will be normalized. The basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_n\}$ is called ***orthonormal***, if it is orthogonal and $(\boldsymbol{e}_i,\boldsymbol{e}_i)=\mathbf{1}, \boldsymbol{i}=\overline{\mathbf{1},\boldsymbol{n}}$. Every Euclid space has orthonormal basis.

- **Orthogonal and symmetric transformations.**

A linear transformation φ of a Euclid space is called ***orthogonal***, if for every vector $\boldsymbol{a}$ from the space $(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{a}))=(\boldsymbol{a},\boldsymbol{a})$. It has the following properties:

1. For every vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ it is true that $(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{b}))=(\boldsymbol{a},\boldsymbol{b})$. From the definition $(\varphi(\boldsymbol{a}+\boldsymbol{b}),\varphi(\boldsymbol{a}+\boldsymbol{b}))=(\boldsymbol{a}+\boldsymbol{b},\boldsymbol{a}+\boldsymbol{b})=(\boldsymbol{a},\boldsymbol{a})+\mathbf{2}(\boldsymbol{a},\boldsymbol{b})+(\boldsymbol{b},\boldsymbol{b})$, but on the other hand $(\varphi(\boldsymbol{a}+\boldsymbol{b}),\varphi(\boldsymbol{a}+\boldsymbol{b}))=(\varphi(\boldsymbol{a})+\varphi(\boldsymbol{b}),\varphi(\boldsymbol{a})+\varphi(\boldsymbol{b}))=(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{a}))+$
   $+\mathbf{2}(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{b}))+(\varphi(\boldsymbol{b}),\varphi(\boldsymbol{b}))=(\boldsymbol{a},\boldsymbol{a})+\mathbf{2}(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{b}))+(\boldsymbol{b},\boldsymbol{b})$.
   Hence $(\varphi(\boldsymbol{a}),\varphi(\boldsymbol{b}))=(\boldsymbol{a},\boldsymbol{b})$.
2. The images of an orthonormal system under an orthogonal transformation form another orthonormal system. Let $\{\boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_n\}$ be a orthonormal system in the space. Let us consider $(\varphi(\boldsymbol{e}_i),\varphi(\boldsymbol{e}_j))=(\boldsymbol{e}_i,\boldsymbol{e}_j)=\mathbf{0}$, if $\boldsymbol{i}\neq\boldsymbol{j}$. And $(\varphi(\boldsymbol{e}_i),\varphi(\boldsymbol{e}_i))=(\boldsymbol{e}_i,\boldsymbol{e}_i)=\mathbf{1}$. Hence $\{\varphi(\boldsymbol{e}_1),\varphi(\boldsymbol{e}_2),...,\varphi(\boldsymbol{e}_n)\}$ is orthonormal.
3. In every orthonormal basis the matrix of an orthogonal transformation is an orthogonal matrix (i.e. $\boldsymbol{A}^{-1}=\boldsymbol{A}^{\boldsymbol{T}}$).
4. Every orthogonal transformation is non-singular.

A linear transformation φ of a Euclid space is called ***symmetrical***, if for every vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ from the space $(\varphi(\boldsymbol{a}),\boldsymbol{b})=(\boldsymbol{a},\varphi(\boldsymbol{b}))$. It has the following properties:

1. In every orthonormal basis the matrix of a symmetrical transformation is also a symmetric matrix (i.e. $\boldsymbol{A}=\boldsymbol{A}^{\boldsymbol{T}}$). Let $\{\boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_n\}$ be an orthonormal system in the space. Let us consider $(\varphi(\boldsymbol{e}_i),\boldsymbol{e}_j)=\left(\sum_{k=1}^{n}\boldsymbol{a}_{ik}\boldsymbol{e}_k,\boldsymbol{e}_j\right)=\sum_{k=1}^{n}\boldsymbol{a}_{ik}(\boldsymbol{e}_k,\boldsymbol{e}_j)=\boldsymbol{a}_{ij}$, and on the other hand

$$(\varphi(\boldsymbol{e}_i),\boldsymbol{e}_j)=(\boldsymbol{e}_i,\varphi(\boldsymbol{e}_j))=\left(\boldsymbol{e}_i,\sum_{k=1}^{n}\boldsymbol{a}_{jk}\boldsymbol{e}_k\right)=\sum_{k=1}^{n}\boldsymbol{a}_{jk}(\boldsymbol{e}_i,\boldsymbol{e}_k)=\boldsymbol{a}_{ji}.$$ Hence $\boldsymbol{a}_{ij}=\boldsymbol{a}_{ji}$.

2. All characteristic numbers of a symmetrical transformation are real. Let us have λ that is a characteristic number (maybe complex) of the linear transformation with the matrix $A = [a_{ij}]_{i,j=1}^{n}$. Let us consider the system

$$\begin{cases} (a_{11} - \lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \ldots + a_{2n}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn} - \lambda)x_n = 0. \end{cases}$$

This system has got a non-zero solution because λ is a characteristic number. Let it be $\langle \beta_1, \ldots, \beta_n \rangle$ (maybe complex). Then $\sum_{j=1}^{n} a_{ij}\beta_j = \lambda\beta_i$, $i = \overline{1,n}$. Let us multiply the i^{th} equation by $\overline{\beta_i}$ (complex conjugate to β_i) and sum up all the equations. We obtain $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\beta_j\overline{\beta_i} = \lambda\sum_{i=1}^{n}\beta_i\overline{\beta_i}$. If we prove that the left-hand part of the equality is a real number it will imply that λ is a real number as well, because $\sum_{i=1}^{n}\beta_i\overline{\beta_i}$ is always a real number. Let us consider

$$\overline{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\beta_j\overline{\beta_i}} = \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a_{ij}\beta_j\overline{\beta_i}} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\overline{\beta_j}\beta_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}\overline{\beta_i}\beta_j =$$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\beta_j\overline{\beta_i}$. Thus this is a real number and hence λ is a real number as well.

3. Characteristic vectors of a symmetrical transformation form an orthogonal system. Let λ_1 and λ_2 be two different characteristic numbers. And let ***a*** and ***b*** be corresponding characteristic vectors. Let us consider an expression $0 = (\varphi(a), b) - (a, \varphi(b)) = \lambda_1(a,b) - \lambda_2(b,a) = \lambda_1(a,b) - \lambda_2(a,b) = (\lambda_1 - \lambda_2)(a,b)$. Then, because $\lambda_1 \neq \lambda_2$ $(a,b) = 0$.

4. For every symmetrical transformation there exists an orthonormal basis in which the matrix of this transformation is diagonal. This basis consists of normalized characteristics vectors. On the diagonal in the matrix there are characteristic numbers.

Example 47: find the basis in which the matrix $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ of a symmetric transformation is diagonal.

Solution: let us find characteristic numbers of this transformation: $\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}=(2-\lambda)^2-1=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0$. Thus $\lambda_1=1$ and $\lambda_2=3$.

Let us find the first normalized characteristic vector: $\begin{cases}x_1+x_2=0,\\ x_1+x_2=0\end{cases}$ or $\begin{cases}x_1=t,\\ x_2=-t,\end{cases}$ where t is a real number. Then, $e_1=\dfrac{1}{\sqrt{t^2+(-t)^2}}\begin{bmatrix}t\\ -t\end{bmatrix}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$.

Let us find the second normalized characteristic vector: $\begin{cases}-x_1+x_2=0,\\ x_1-x_2=0\end{cases}$ or $\begin{cases}x_1=t,\\ x_2=t,\end{cases}$ where t is a real number. Then, $e_2=\dfrac{1}{\sqrt{t^2+t^2}}\begin{bmatrix}t\\ t\end{bmatrix}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$. Hence the matrix A is diagonal in the basis $\{e_1,e_2\}$, and it is $A'=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}$. Let us check it. $A'=TAT^{-1}$, where T is a matrix composed of characteristic vectors rows. In our case $T=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ and $T^{-1}=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$. Then,

$A'=TAT^{-1}=$

$$=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=$$

$$=\frac{1}{2}\begin{bmatrix}1 & -1\\ 3 & 3\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}.$$

IDENTIFYING SECOND DEGREE EQUATIONS

- **Classification of second degree curves.**

Thus far we have considered the first degree equations, where the variables in the equations were in the first degree. Next step is the second degree equations. The general equation of the second degree has the following form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Later you will see that there are only three different second order curves: ellipse, parabola, and hyperbola. Let us define them.

An ***ellipse*** is the set of all points $\langle x, y \rangle$ the sum of whose distances from two distinct fixed points, called ***foci***, is constant. The line through the foci intersects the ellipse at two points, called ***vertices***. The chord joining the vertices is the ***major axis***, and its midpoint is the ***center*** of the ellipse. The chord perpendicular to the major axis at the center is called the ***minor axis*** (Fig. 19). F_1 and F_2 are foci, A and C are vertices, AC is major axis and BD is minor axis, O is the center and $F_1K + F_2K = const$ for any point K on the ellipse.

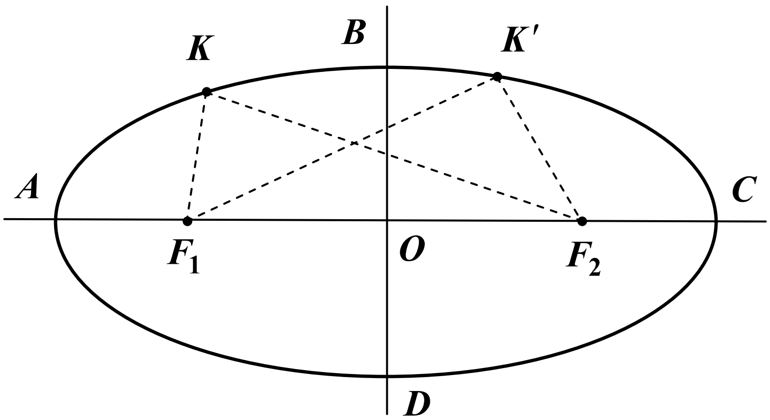

Fig. 19

Let us deduce the standard equation of an ellipse. Let us put the center in the origin and foci at points $\langle c,0 \rangle$ and $\langle -c,0 \rangle$. Let us put major axis as $2a$. Then the point will lie on the ellipse if $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$. Raising to the second power and simplifying we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b=\sqrt{a^2-c^2}$ is the half of the minor axis. This is called the ***standard equation of an ellipse***.

A ***parabola*** is the set of all points $\langle x,y\rangle$ that are equidistant from a fixed straight line called directrix and a fixed point called the ***focus*** (not on the straight line). The midpoint between the focus and the directrix is called the ***vertex***, and the straight line passing through the focus and the vertex is called the ***axis*** of the parabola (Fig. 20).

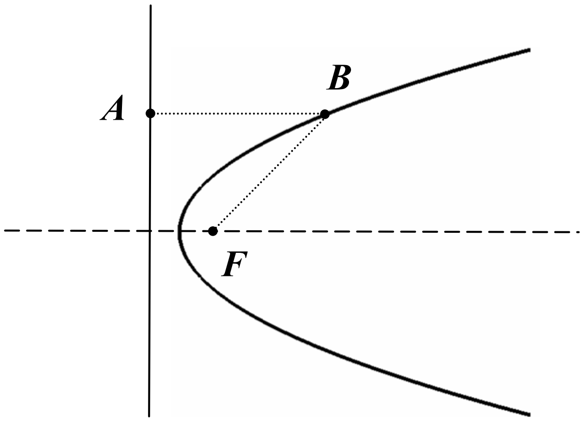

Fig. 20

Using the definition we can easily derive the following standard form of the equation of a parabola. Let us suppose that the equation of the directrix is $x=-\frac{p}{2}$ and the focus is in the point $\left\langle \frac{p}{2},0\right\rangle$. Because the point $\langle x,y\rangle$ is equidistant from $x=-\frac{p}{2}$ and $\left\langle \frac{p}{2},0\right\rangle$ we can write $\sqrt{\left(x-\frac{p}{2}\right)^2+(y-0)^2}=x+\frac{p}{2}$. Raising to the square both sides and simplifying the expression we obtain:

$$y^2=2px,$$

which is called the ***standard equation of a parabola***.

A ***hyperbola*** is the set of all points $\langle x,y\rangle$ the difference of whose distances from two distinct fixed points, called ***foci***, is constant. The graph of a hyperbola has two disconnected parts, called ***branches***. The straight line through the two foci intersects the hyperbola at two points, called ***vertices***. The straight line segment connecting the vertices is the ***transverse axis***, and the midpoint of the transverse axis is the ***center*** of the hyperbola (Fig. 21).

An important aid in graphing a hyperbola is the determination of its asymptotes. Each hyperbola has two asymptotes that intersect in the center of the hyperbola.

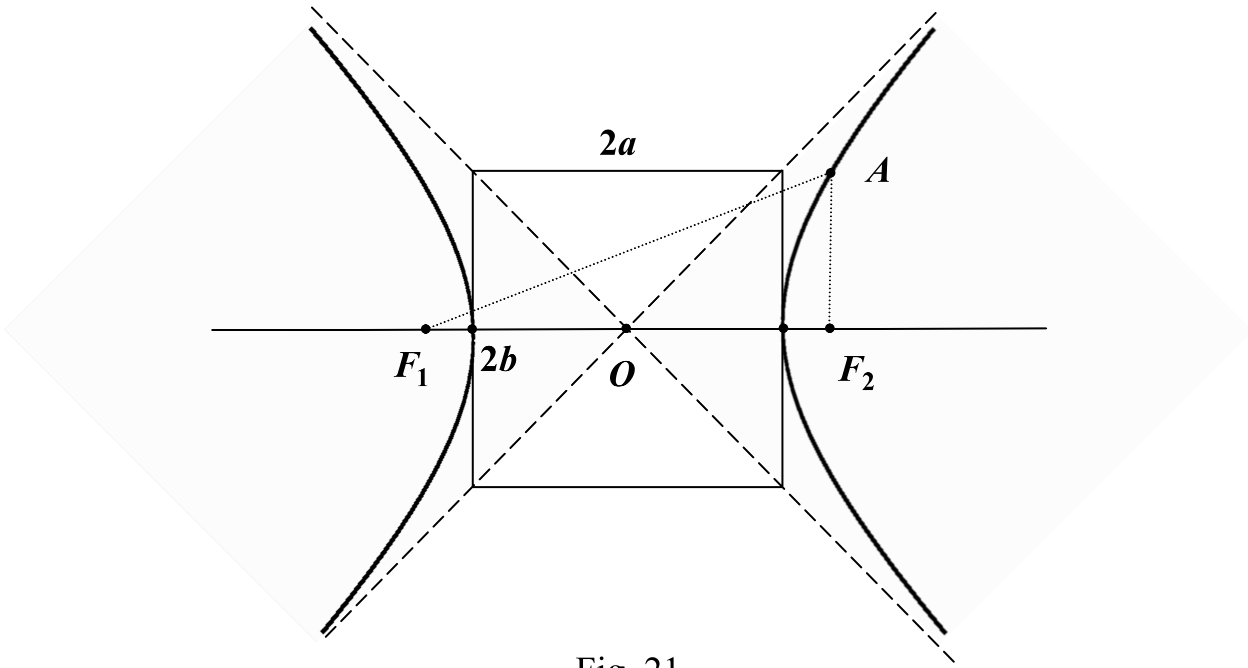

Fig. 21

Let us deduce the standard equation of a hyperbola. Let us put the center in the origin and foci at points $\langle c,0\rangle$ and $\langle -c,0\rangle$. Let us put the transverse axis as $2a$. Then the point will lie on the hyperbola if $\left|\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}\right|=2a$. Raising to the second power and simplifying we get

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where $b=\sqrt{a^2-c^2}$ is referred to as the ***conjugate axis*** of the hyperbola. Note that asymptotes of a hyperbola pass through the corners of a rectangle of dimensions $2a$ and $2b$.

Example 48: find the standard equation of $x^2+4y^2+6x-8y+9=0$.

Solution: let us at first group terms: $(x^2+6x)+4(y^2-2y)+9=0$, then complete the squares: $(x+3)^2+4(y-1)^2-4=0$ or $\frac{(x+3)^2}{4}+\frac{(y-1)^2}{1}=1$.

Finally making the substitution $\begin{cases} \bar{x} = x + 3 \\ \bar{y} = y - 1 \end{cases}$, we obtain the standard equation of an ellipse with major axis 2 and minor axis 1:

$$\frac{\bar{x}^2}{2^2} + \frac{\bar{y}^2}{1^2} = 1.$$

Example 49: find the standard equation of $-4x^2 + y^2 + 24x + 4y - 41 = 0$.

Solution: as in previous example let us complete the squares. We obtain: $-4(x-3)^2 + (y+2)^2 = 9$. Making the substitution $\begin{cases} \bar{x} = y + 2 \\ \bar{y} = x - 3 \end{cases}$, we obtain the standard equation of a hyperbola:

$$\frac{\bar{x}^2}{3^2} - \frac{\bar{y}^2}{\left(3/2\right)^2} = 1.$$

Example 50: find the standard equation of $x^2 - 2x + 4y - 3 = 0$.

Solution: we can rewrite the equation as $-4(y-1) = (x-1)^2$. Making the substitution $\begin{cases} \bar{x} = y - 1 \\ \bar{y} = x - 1 \end{cases}$, we obtain the standard equation of a parabola:

$$\bar{y}^2 = -4\bar{x}.$$

From the previous examples we can see, that if the general equation of the second degree $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ does not contain the term Bxy, completing the square and making the linear substitution, we can easily identify the curve and write its equation in the standard form. The following method helps us in case $B \neq 0$.

- **The eigenvalues method.**

Let us consider the first part of the general equation of the second degree $Ax^2 + Bxy + Cy^2$, which is called the quadratic form. It can be written in the form $\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$. The matrix $\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$ is called the matrix of the quadratic form. It is a symmetric matrix, and we know that there exists a basis in which this matrix is diagonal. Furthermore, this basis is the basis of eigenvectors. Thus, making the substitution $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$, where T is the transition matrix, we make the quadratic form not to contain the term Bxy. Moreover, the quadratic form becomes $\lambda_1\tilde{x}^2 + \lambda_2\tilde{y}^2$.

Example 51: find the standard equation of

$$3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0.$$

Solution: let us write the matrix of the quadratic form $A = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$. Then, let us find its eigenvalues. The characteristic equation is

$$\begin{vmatrix} 3-\lambda & 5 \\ 5 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 16$$

Solving it we get $\lambda_1 = 8$ and $\lambda_2 = -2$. Let us now find the corresponding eigenvectors.

Let us find the first normalized eigenvector: $\begin{cases} -5x_1 + 5x_2 = 0, \\ 5x_1 - 5x_2 = 0 \end{cases}$ or $\begin{cases} x_1 = t, \\ x_2 = t, \end{cases}$ where t is a real number. Then, $e_1 = \frac{1}{\sqrt{t^2 + t^2}}\begin{bmatrix} t \\ t \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us find the second normalized eigenvector: $\begin{cases} 5x_1 + 5x_2 = 0, \\ 5x_1 + 5x_2 = 0 \end{cases}$ or $\begin{cases} x_1 = -t, \\ x_2 = t, \end{cases}$ where t is a real number. Then, $e_2 = \frac{1}{\sqrt{t^2 + (-t)^2}}\begin{bmatrix} -t \\ t \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence the matrix A is diagonal in the basis $\{e_1, e_2\}$, and the transition matrix $T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then, let us make the substitution $\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{x} - \tilde{y} \\ \tilde{x} + \tilde{y} \end{bmatrix}$. Putting this into the equation we obtain $8\tilde{x}^2 - 2\tilde{y}^2 - 8\sqrt{2}\tilde{x} + 6\sqrt{2}\tilde{y} - 13 = 0$. After completing the squares, we get $8\left(\tilde{x} - \frac{\sqrt{2}}{2}\right)^2 - 2\left(\tilde{y} + \frac{3\sqrt{2}}{2}\right)^2 - 8 = 0$. Making the substitution

$\begin{cases} \bar{x} = \tilde{x} - \frac{\sqrt{2}}{2}, \\ \bar{y} = \tilde{y} + \frac{3\sqrt{2}}{2}, \end{cases}$ we obtain the standard form of the equation of a hyperbola

$$\frac{\bar{x}^2}{1^2} - \frac{\bar{y}^2}{2^2} = 1.$$

Example 52: find the standard equation of

$$9x^2-24xy+16y^2-20x+110y-50=0.$$

Solution: let us write the matrix of the quadratic form $A=\begin{bmatrix}9 & -12\\ -12 & 16\end{bmatrix}$. Then, let us find its eigenvalues. The characteristic equation is $\begin{vmatrix}9-\lambda & -12\\ -12 & 16-\lambda\end{vmatrix}=\lambda^2-25\lambda=0$. Solving it we get $\lambda_1=0$ and $\lambda_2=25$. Let us now find the corresponding eigenvectors.

Let us find the first normalized eigenvector: $\begin{cases}9x_1-12x_2=0,\\ -12x_1+16x_2=0\end{cases}$ or $\begin{cases}x_1=4t,\\ x_2=3t,\end{cases}$ where t is a real number. Then, $e_1=\frac{1}{\sqrt{16t^2+9t^2}}\begin{bmatrix}4t\\ 3t\end{bmatrix}=\frac{1}{5}\begin{bmatrix}4\\ 3\end{bmatrix}$. Let us find the second normalized eigenvector: $\begin{cases}-16x_1-12x_2=0,\\ -12x_1-9x_2=0\end{cases}$ or $\begin{cases}x_1=-3t,\\ x_2=4t,\end{cases}$ where t is a real number. Then, $e_2=\frac{1}{\sqrt{16t^2+9(-t)^2}}\begin{bmatrix}-3t\\ 4t\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-3\\ 4\end{bmatrix}$. Hence the matrix A is diagonal in the basis $\{e_1,e_2\}$, and the transition matrix $T=\frac{1}{5}\begin{bmatrix}4 & -3\\ 3 & 4\end{bmatrix}$. Then, let us make the substitution $\begin{bmatrix}x\\ y\end{bmatrix}=T\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}=\frac{1}{5}\begin{bmatrix}4 & -3\\ 3 & 4\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}=\frac{1}{5}\begin{bmatrix}4\tilde{x}-3\tilde{y}\\ 3\tilde{x}+4\tilde{y}\end{bmatrix}$. Putting this into the equation we obtain $25\tilde{y}^2+50\tilde{x}+100\tilde{y}-50=0$. After completing the squares and dividing by 25, we get $(\tilde{y}+2)^2+2\tilde{x}-6=0$. Making the substitution $\begin{cases}\bar{x}=\tilde{x}-3,\\ \bar{y}=\tilde{y}+2,\end{cases}$ we obtain the standard form of the equation of a parabola

$$\bar{y}^2=-2\bar{x}.$$

Example 53: find the standard equation of

$$25x^2-14xy+25y^2+64x-64y-224=0.$$

Solution: let us write the matrix of the quadratic form $A=\begin{bmatrix}25 & -7\\ -7 & 25\end{bmatrix}$. Then, let us find its eigenvalues. The characteristic equation is $\begin{vmatrix}25-\lambda & -7\\ -7 & 25-\lambda\end{vmatrix}=\lambda^2-50\lambda-674=0$. Solving it we get $\lambda_1=18$ and $\lambda_2=32$. Let us now find the corresponding eigenvectors.

Let us find the first normalized eigenvector: $\begin{cases} 7x_1 - 7x_2 = 0, \\ -7x_1 + 7x_2 = 0 \end{cases}$ or $\begin{cases} x_1 = t, \\ x_2 = t, \end{cases}$ where t is a real number. Then, $e_1 = \frac{1}{\sqrt{t^2 + t^2}} \begin{bmatrix} t \\ t \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let us find the second normalized eigenvector: $\begin{cases} -7x_1 - 7x_2 = 0, \\ -7x_1 - 7x_2 = 0 \end{cases}$ or $\begin{cases} x_1 = -t, \\ x_2 = t, \end{cases}$ where t is a real number. Then, $e_2 = \frac{1}{\sqrt{t^2 + (-t)^2}} \begin{bmatrix} -t \\ t \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence the matrix A is diagonal in the basis $\{e_1, e_2\}$, and the transition matrix $T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then, let us make the substitution $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{x} - \tilde{y} \\ \tilde{x} + \tilde{y} \end{bmatrix}$. Putting this into the equation we obtain $18\tilde{x}^2 + 32\tilde{y}^2 - 64\sqrt{2}\tilde{y} - 224 = 0$. After completing the squares and dividing by **288**, we get $\frac{\tilde{x}^2}{16} + \frac{(\tilde{y} - \sqrt{2})^2}{9} = 1$. Making the substitution $\begin{cases} \bar{x} = \tilde{x}, \\ \bar{y} = \tilde{y} - \sqrt{2}, \end{cases}$ we obtain the standard form of the equation of an ellipse

$$\frac{\bar{x}^2}{4^2} + \frac{\bar{y}^2}{3^2} = 1.$$

- **Sketching second degree curves.**

It is easy to sketch a second degree curve if its equation is given in the standard form. However, it is possible in general case. Each substitution we made in examples 51-53 is a transformation of the coordinate axes. Since eigenvectors of a symmetric matrix are orthogonal their directions could be coordinate axis direction. After the first substitution the coordinate axes “rotate” with respect to the origin. The second substitution is linear, thus the coordinate axes shift with respect to the origin. And in the obtained coordinate system $\bar{x}, \bar{y}$ we sketch the curve given in the standard form.

Example 54: sketch the curve $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$.

Solution: as it is shown in the example 51 the standard form of this equation $\frac{\bar{x}^2}{1^2} - \frac{\bar{y}^2}{2^2} = 1$, where $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{x} - \tilde{y} \\ \tilde{x} + \tilde{y} \end{bmatrix}$ and $\begin{cases} \bar{x} = \tilde{x} - \frac{\sqrt{2}}{2} \\ \bar{y} = \tilde{y} + \frac{3\sqrt{2}}{2} \end{cases}$. Firstly, $\tilde{x}$-axis has the direction of $e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\tilde{y}$-axis has the direction of $e_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Secondly, $\bar{x}$-axis has the same direction as $\tilde{x}$-axis and has a shift $\frac{-3\sqrt{2}}{2}$. $\bar{y}$-axis has the same direction as $\tilde{y}$-axis and has a shift $\frac{\sqrt{2}}{2}$. Finally, in the new coordinate system $\bar{x}, \bar{y}$ we sketch a hyperbola with the equation $\frac{\bar{x}^2}{1^2} - \frac{\bar{y}^2}{2^2} = 1$ (Fig. 22).

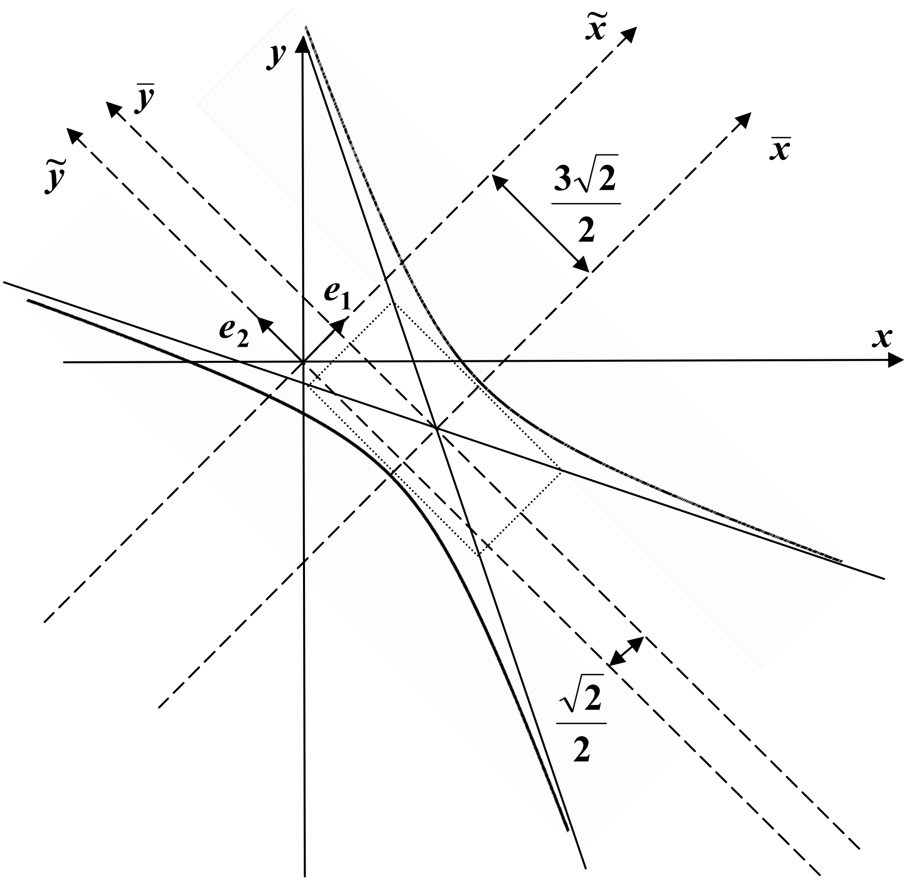

Fig. 22

Example 55: sketch the curve $\boldsymbol{9x^2 - 24xy + 16y^2 - 20x + 110y - 50 = 0}$.

Solution: as it is shown in the example 52 the standard form of this equation $\bar{y}^2 = -2\bar{x}$, where $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4\tilde{x} - 3\tilde{y} \\ 3\tilde{x} + 4\tilde{y} \end{bmatrix}$ and $\begin{cases} \bar{x} = \tilde{x} - 3, \\ \bar{y} = \tilde{y} + 2. \end{cases}$ Firstly, $\tilde{x}$-axis has the direction of $e_1 = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and $\tilde{y}$-axis has the direction of $e_2 = \frac{1}{5}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$. Secondly, $\bar{x}$-axis has the same direction as $\tilde{x}$-axis and has a shift 2. $\bar{y}$-axis has the same direction as $\tilde{y}$-axis and has a shift -1. Finally, in the new coordinate system $\bar{x}, \bar{y}$ we sketch a parabola with the equation $\bar{y}^2 = -2\bar{x}$ (Fig. 23).

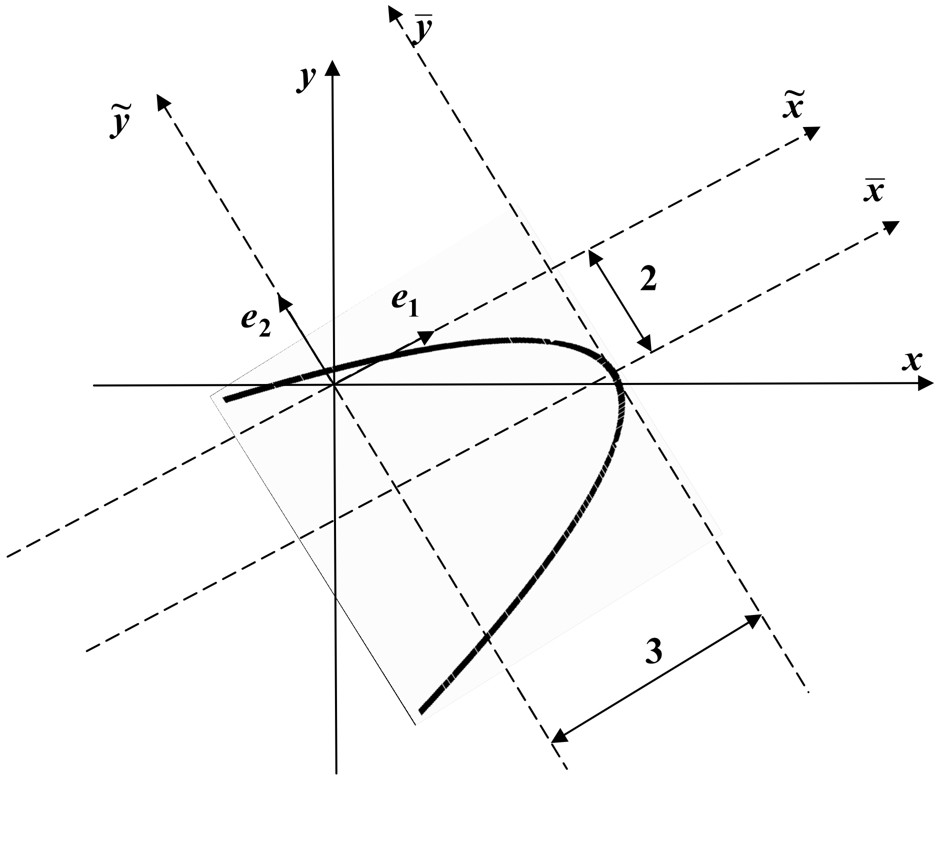

Fig. 23

Example 56: sketch the curve $\boldsymbol{25x^2 - 14xy + 25y^2 + 64x - 64y - 224 = 0}$.

Solution: as it is shown in the example 53 the standard form of this equation $\frac{\bar{x}^2}{4^2} + \frac{\bar{y}^2}{3^2} = 1$, where $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \tilde{x} - \tilde{y} \\ \tilde{x} + \tilde{y} \end{bmatrix}$ and $\begin{cases} \bar{x} = \tilde{x}, \\ \bar{y} = \tilde{y} - \sqrt{2}. \end{cases}$. Firstly, $\tilde{x}$-

axis has the direction of $\boldsymbol{e_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\tilde{y}$-axis has the direction of $\boldsymbol{e_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Secondly, $\bar{x}$-axis has the same direction as $\tilde{x}$-axis and has a shift $\sqrt{2}$. $\bar{y}$-axis coincides with $\tilde{y}$-axis. Finally, in the new coordinate system $\bar{x}, \bar{y}$ we sketch a hyperbola with the equation $\frac{\bar{x}^2}{4^2} + \frac{\bar{y}^2}{3^2} = 1$ (Fig. 24).

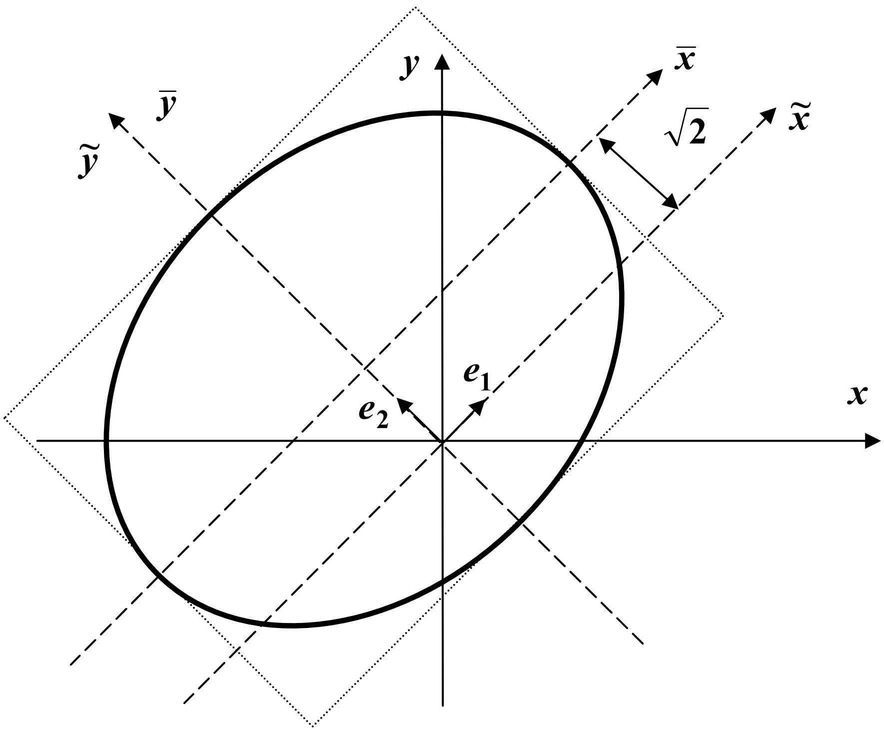

Fig. 24

PROBLEMS

- **Determinants.**

Find the determinants:

1. $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ 3. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$ 4. $\begin{vmatrix} 1 & 2 & 3 & 5 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \end{vmatrix}$ 6. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$ 7. $\begin{vmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{vmatrix}$

8*. $\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix}$ 9*. $\begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix}$ 10*. $\begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{vmatrix}$

Solve the systems by the Cramer's rule:

11. $\begin{cases} x - y + 2z = 3, \\ -2x + 3y - z = 2, \\ 3x - y + 12z = 0. \end{cases}$ 12. $\begin{cases} x + 2y - z = 2, \\ -2x + y + 2z = 6, \\ 3x - 2y + z = -6. \end{cases}$ 13. $\begin{cases} x - 2y + 4z = 3, \\ 2x - 4y + 3z = 1, \\ 3x - y + 5z = 2. \end{cases}$

14. $\begin{cases} 2x + 3y + 5z = 10, \\ x + 2y + 2z = 3, \\ 3x + 7y + 4z = 3. \end{cases}$ 15. $\begin{cases} -3x + y - z = -3, \\ x + y + z = 3, \\ 2x - y - z = 0. \end{cases}$ 16. $\begin{cases} x_1 + x_2 + x_3 = 0, \\ 2x_1 + x_2 - 3x_3 = -6, \\ -x_1 + 3x_2 - 2x_3 = 3. \end{cases}$

17. $\begin{cases} x + 2y - z = 3, \\ 2x + y - 3z = -2, \\ 3x + 12y - z = 0. \end{cases}$ 18. $\begin{cases} 2x_1 + x_2 + x_3 = 3, \\ 3x_1 - x_2 + 2x_3 = 3, \\ 5x_1 + 2x_2 = 2. \end{cases}$ 19. $\begin{cases} x_1 + x_2 + x_3 = 36, \\ 2x_1 - 3x_3 = -17, \\ 6x_1 - 5x_3 = 7. \end{cases}$

20. $\begin{cases} x_1 + x_2 - x_3 = 36, \\ x_1 - x_2 + x_3 = 13, \\ -x_1 + x_2 + x_3 = 7. \end{cases}$ 21. $\begin{cases} x_1 + 2x_2 + x_3 = 4, \\ 3x_1 - 5x_2 + 3x_3 = 1, \\ 2x_1 + 7x_2 - x_3 = 8. \end{cases}$ 22. $\begin{cases} 2x_1 - 4x_2 + 9x_3 = 28, \\ 7x_1 + 3x_2 - 6x_3 = -1, \\ 7x_1 + 9x_2 - 9x_3 = 5. \end{cases}$

23. $\begin{cases} 2x_1 + x_2 = 5, \\ x_1 + 3x_3 = 16, \\ 5x_2 - x_3 = 10. \end{cases}$ 24. $\begin{cases} 7x_1 + 2x_2 + 3x_3 = 15, \\ 5x_1 - 3x_2 + 2x_3 = 15, \\ 10x_1 - 11x_2 + 5x_3 = 36. \end{cases}$ 25. $\begin{cases} x_1 + x_2 + x_3 = 17, \\ x_1 - x_2 + x_3 = 13, \\ -x_1 + x_2 + x_3 = -7. \end{cases}$

26. $\begin{cases} x_1 + x_2 + x_3 = 2, \\ x_1 - x_2 + x_3 = 10, \\ -x_1 + x_2 + x_3 = -4. \end{cases}$ 27. $\begin{cases} x_1 + x_2 + x_3 = 21, \\ x_1 - x_2 + x_3 = -13, \\ -x_1 + x_2 + x_3 = 17. \end{cases}$ 28. $\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1, \\ 3x_1 - x_2 - x_3 - 2x_4 = -4, \\ 2x_1 + 3x_2 - x_3 - x_4 = -6, \\ x_1 + 2x_2 + 3x_3 - x_4 = -4. \end{cases}$

29. $\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ 2x_1 - x_2 - 2x_3 - 3x_4 = 8, \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 4, \\ 2x_1 - 3x_2 + 2x_3 + x_4 = -8. \end{cases}$ 30. $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 5, \\ 2x_1 + x_2 + 2x_3 + 3x_4 = 1, \\ 3x_1 + 2x_2 + x_3 + 2x_4 = 1, \\ 4x_1 + 3x_2 + 2x_3 + x_4 = -5. \end{cases}$

- **Vector algebra.**

1. Find the angle between vectors $\vec{a} = \vec{q} + \vec{p}$ and $\vec{b} = \vec{p} - \vec{q}$, if $|\vec{p}| = \sqrt{2},\ |\vec{q}| = 1$, $\left(\vec{p} \wedge \vec{q}\right) = 135^0$.
2. Find the angle between vectors $\vec{a} = 2\vec{p} - \vec{q}$ and $\vec{b} = \vec{p} + 2\vec{q}$, if $|\vec{p}| = |\vec{q}| = 1$ and $(\vec{p} \wedge \vec{q}) = \frac{\pi}{3}$.
3. Find the angle between vectors $\vec{a} = \vec{m} + 2\vec{n}$ and $\vec{b} = 3\vec{m} - \vec{n}$, if $|\vec{m}| = 1$, $|\vec{n}| = 2$, $\left(\vec{m} \wedge \vec{n}\right) = 120^0$.
4. Prove that in a parallelogram the sum of diagonal squares is equal to the sum of all sides' squares.
5. Find $|\vec{m} + 2\vec{n} - \vec{p}|$, if $|\vec{n}| = |\vec{m}| = 2$, $|\vec{p}| = 1$, $\left(\vec{m} \wedge \vec{n}\right) = 60^0$, $\left(\vec{m} \wedge \vec{p}\right) = \left(\vec{n} \wedge \vec{p}\right) = 90^0$. Find the angle between vectors $\vec{a}$ and $\vec{b}$ if $\vec{a} = \langle 3,2,-6 \rangle$ and $\vec{b} = \langle 2,-2,1 \rangle$.
6. Find $\mathbf{proj}_{\vec{b}+\vec{c}}\, \vec{a}$, if $\vec{a}\langle 1,-3,4 \rangle$, $\vec{b}\langle 3,-4,2 \rangle$, $\vec{c}\langle -1,1,4 \rangle$.
7. Find $\mathbf{proj}_{\vec{b}}(2\vec{a} + 3\vec{c})$, if $\vec{a}\langle -2,-1,1 \rangle$, $\vec{b}\langle 2,-1,0 \rangle$, $\vec{c}\langle -1,3,-7 \rangle$.
8. Find the angles of the triangle ABC, if $A\langle 1,2,1 \rangle$, $B\langle 3,-1,7 \rangle$, $C\langle 7,4,-2 \rangle$.

9. Find the components of the vector $\vec{x}$, if $\vec{x} \perp \vec{i} - 2\vec{j} + \vec{k}$, $\vec{x} \perp 2\vec{i} - 3\vec{j} + \vec{k}$, and modulus of the vector $\vec{x}$ equals $6\sqrt{3}$.
10. Find the components of the vector $\vec{x}$, if $\vec{x} \perp \vec{i} - \vec{j} - \vec{k}$, $\vec{x}(\vec{i} - \vec{j}) = 2$, and $|\vec{x}| = \sqrt{6}$.
11. Find the components of the vector $\vec{x}$, if $\vec{x} \perp \Delta ABC$: $A\langle 1,3,4\rangle$, $B\langle -1,0,9\rangle$, $C\langle 3,2,3\rangle$, and $|\vec{x}| = 5\sqrt{3}$.
12. Find the area of the parallelogram $ABCD$ if $A\langle 1,0,3\rangle$, $B\langle -2,1,5\rangle$ and $C\langle -1,5,-4\rangle$. Find the area of the triangle ABC if $A\langle 1,1,0\rangle$, $B\langle 1,0,1\rangle$, $C\langle 2,1,1\rangle$.
13. Find $|\vec{a} \times \vec{b}|$ if $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = -1$.
14. Find $\vec{a} \cdot \vec{b}$ if $|\vec{a}| = 5$, $|\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 9$.
15. Find $|(\vec{a} - \vec{b}) \times (2\vec{a} + 3\vec{b})|$ if $|\vec{a}| = |\vec{b}| = 2$ and $(\vec{a} \wedge \vec{b}) = \dfrac{5\pi}{6}$.
16. Find $(2\vec{a} - 3\vec{b}) \times (2\vec{b} - \vec{a})$, if $\vec{a} \times \vec{b} = 2$.
17. Prove that $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}$.
18. Find $\vec{a}\vec{b}(3\vec{a} - 2\vec{b} + 3\vec{c})$ if $\vec{a}\vec{b}\vec{c} = 2$.
19. Find $(2\vec{a} - 3\vec{b} + 2\vec{c}) \times (-\vec{a} + \vec{b} + 3\vec{c}) \cdot (3\vec{a} + \vec{b} - 5\vec{c})$, if $\vec{a}\vec{b}\vec{c} = 2$.
20. Find the volume of the pyramid $ABCD$ if $A\langle 1,2,-3\rangle$, $B\langle 0,-1,2\rangle$, $C\langle -1,5,-4\rangle$ and $D\langle 2,7,5\rangle$.
21. Prove that vectors $\vec{a} + 4\vec{b} + 7\vec{c}$, $2\vec{a} + 5\vec{b} + 8\vec{c}$ and $3\vec{a} + 6\vec{b} + 9\vec{c}$ are always coplanar.
22. Find λ such that vectors $\langle 1,2,-3\rangle$, $\langle 4,-2,1\rangle$, $\langle -1,0,\lambda\rangle$ are coplanar.

- **Analytic geometry.**

1. Find the equation of the plane passing through the point **A(0; 1; 3)** and the line $\dfrac{x}{2} = \dfrac{y+1}{-1} = \dfrac{z}{2}$.
2. Find the equation of the plane passing through the points ***A*(-1; 1; 2), *B*(0; 1; 3)** and ***C*(2; 0; -1**.
3. Find the equation of the plane passing through two parallel lines: $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z}{4}$, $\dfrac{x}{3} = \dfrac{y}{-1} = \dfrac{z+5}{4}$.

4. Find the equation of the plane passing through the point **A(0; 1; 3)** and the line $\frac{x}{2}=\frac{y+1}{-1}=\frac{z}{2}$.
5. Find the equation of the plane passing through the point $\langle 2,0,1\rangle$, parallel to vectors $\vec{a}=\langle 1,-1,3\rangle$ and $\vec{b}=\langle 0,2,-1\rangle$.
6. Find the equation of the plane passing through the points **A(-1; 1; 2), B(0; 1; 3)** and $C\langle 2,0,-1\rangle$.
7. Find the equation of the plane passing through the points $A\langle -1,0,1\rangle$, $B\langle 0,3,1\rangle$, $C\langle 4,-2,0\rangle$.
8. Find the bisector to planes $x-2y+2z+2=0$ and $4y+3z-3=0$.
9. Find the angle between planes $2x-6y+3z+1=0$ and $-4x+12y-6z+1=0$.
10. Find the angle between planes $x-y+\sqrt{2}z+2=0$ and $x+y+\sqrt{2}z-3=0$.
11. Find λ and μ such that planes $x-y+\lambda z+2=0$ and $\mu x+y+3z-3=0$ are parallel.
12. Determine the positional relationship between the straight lines $\frac{x+2}{2}=\frac{y}{-3}=\frac{z-1}{4}$ and $\frac{x-3}{\alpha}=\frac{y-1}{4}=\frac{z-7}{2}$.
13. Find the canonical equation of a straight line $\begin{cases} 3x-2y+z=0 \\ x-2z+1=0 \end{cases}$.
14. Find the distance between the point **M(1; 2; -3)** and the plane $x-6y+18z-1=0$.
15. Find the intersection of the line $\frac{x+1}{2}=\frac{y+1}{-3}=\frac{z+4}{-1}$ and the plane $x+2y+z-6=0$.
16. Find the plane passing through two parallel lines: $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z}{4}$, $\frac{x}{3}=\frac{y}{-1}=\frac{z+5}{4}$.
17. Find the point of intersection between the straight line $\frac{x+1}{2}=\frac{y-1}{0}=\frac{z-4}{-3}$ and a plane $2x-3y+4z-3=0$.
18. Find the projection of the point $M\langle 1,0,-2\rangle$ to the plane $x-2y+3z-9=0$.
19. Find the projection of the point $M\langle 1,-2,2\rangle$ on the straight line $\frac{x}{1}=\frac{y+3}{0}=\frac{z}{2}$ and the equation of the perpendicular from the point M on the straight line.

20. Find the volume of the pyramid $ABCD$ if $A\langle 1,2,-3\rangle$, $B\langle 0,-1,2\rangle$, $C\langle -1,5,-4\rangle$ and $D\langle 2,7,5\rangle$.

21. Find the volume of the pyramid $ABCD$ if $A\langle 2,-1,1\rangle$, $B\langle 5,5,4\rangle$, $C\langle 3,1,-1\rangle$, $D\langle 4,1,3\rangle$.

- **Matrices.**

Find products

1. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. 2. $\begin{bmatrix} 3 & 5 \\ 6 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$. 3. $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$. 5. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. 6. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. 7. $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$.

8*. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n$ 9*. $\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}^n$

Find $f(A)$

10. $f(x) = x^2 - x - 1$, $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. 11. $f(x) = x^2 - 5x + 3$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

Solve the systems using Gauss-Jordan elimination method

12. $\begin{cases} x + 2y - z = 2, \\ -2x + y + 2z = 6, \\ 3x - 2y + z = -6. \end{cases}$ 13. $\begin{cases} x - 2y + 4z = 3, \\ 2x - 4y + 3z = 1, \\ 3x - y + 5z = 2. \end{cases}$ 14. $\begin{cases} 2x + 3y + 5z = 10, \\ x + 2y + 2z = 3, \\ 3x + 7y + 4z = 3. \end{cases}$

15. $\begin{cases} -3x + y - z = -3, \\ x + y + z = 3, \\ 2x - y - z = 0. \end{cases}$ 16. $\begin{cases} x_1 + x_2 + x_3 = 0, \\ 2x_1 + x_2 - 3x_3 = -6, \\ -x_1 + 3x_2 - 2x_3 = 3. \end{cases}$

17. $\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ 2x_1 - x_2 - 2x_3 - 3x_4 = 8, \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 4, \\ 2x_1 - 3x_2 + 2x_3 + x_4 = -8. \end{cases}$ 18. $\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ 2x_1 - x_2 - 2x_3 - 3x_4 = 4, \\ 3x_1 + 2x_2 - x_3 + 2x_4 = 8, \\ 2x_1 - 3x_2 + 2x_3 + x_4 = -8. \end{cases}$

Find the inverse of the matrices

19. $A=\begin{bmatrix} 7 & -1 & 0 \\ -5 & 0 & 3 \\ 0 & 1 & -5 \end{bmatrix}$. 20. $A=\begin{bmatrix} 0 & 2 & 5 \\ 2 & -1 & 0 \\ -2 & 0 & -3 \end{bmatrix}$. 21. $A=\begin{bmatrix} 0 & -3 & 2 \\ 1 & 3 & -2 \\ 1 & -1 & 0 \end{bmatrix}$.

22. $A=\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. 23. $A=\begin{bmatrix} -1 & 5 & 5 \\ 3 & -1 & 0 \\ 1 & 8 & 0 \end{bmatrix}$. 24. $A=\begin{bmatrix} 1 & 3 & -1 & -2 \\ 2 & 7 & 0 & -1 \\ -1 & -4 & 0 & -1 \\ -3 & 10 & -1 & -2 \end{bmatrix}$.

Solve the systems using an inverse of the matrix

25. $\begin{cases} x_1+x_2+2x_3=31, \\ 5x_1+x_2+2x_3=29, \\ 3x_1-x_2+x_3=10. \end{cases}$ 26. $\begin{cases} 2x_1+4x_2-x_3=52, \\ -x_1+5x_2+3x_3=72, \\ 3x_1-7x_2+2x_3=10. \end{cases}$ 27. $\begin{cases} 5x_1-4x_2=10, \\ 4x_2-5x_3=-5, \\ 3x_1-2x_3=20. \end{cases}$

28. $\begin{cases} x+2y-z=2, \\ -2x+y+2z=6, \\ 3x-2y+z=-6. \end{cases}$ 29. $\begin{cases} x-2y+4z=3, \\ 2x-4y+3z=1, \\ 3x-y+5z=2. \end{cases}$ 30. $\begin{cases} 2x+3y+5z=10, \\ x+2y+2z=3, \\ 3x+7y+4z=3. \end{cases}$

Find the rank of the matrices

31. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 3 & -1 & 3 & 2 \\ 7 & -2 & 8 & 5 \end{bmatrix}$. 32. $\begin{bmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{bmatrix}$. 33. $\begin{bmatrix} 1 & 4 & -1 & 1 & 4 \\ 2 & 9 & 3 & -1 & -2 \\ 3 & 13 & 2 & 0 & 2 \\ 4 & 17 & 1 & 1 & 6 \\ 2 & 8 & -2 & 1 & 8 \end{bmatrix}$.

34. $\begin{bmatrix} 1 & -2 & 3 & 6 & -3 \\ 3 & -7 & 2 & -4 & 3 \\ 4 & -9 & 5 & 2 & 0 \\ 7 & -16 & 7 & -2 & -3 \\ 11 & -25 & 12 & 0 & -3 \end{bmatrix}$. 35. $\begin{bmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & 8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{bmatrix}$.

Find the fundamental system of solutions of the following systems

36. $\begin{cases} 3x_1+x_2-8x_3+2x_4+x_5=0, \\ 2x_1-2x_2-3x_3-7x_4+2x_5=0, \\ x_1+11x_2-12x_3+34x_4-5x_5=0, \\ x_1-5x_2+2x_3-16x_4+3x_5=0. \end{cases}$ 37. $\begin{cases} x_1+x_2+x_3+2x_4+x_5=0, \\ 2x_1-x_2-3x_3-x_4+3x_5=0, \\ 3x_1+2x_3+x_4+4x_5=0, \\ x_1-2x_2-4x_3-3x_4+2x_5=0. \end{cases}$

38. $\begin{cases} 2x_1 + x_2 - 2x_3 + x_4 - x_5 = 0, \\ 3x_1 - 5x_2 + x_3 - 3x_4 + 2x_5 = 0, \\ x_1 - 6x_2 + 3x_3 - 4x_4 + 3x_5 = 0, \\ 7x_1 - 3x_2 - 3x_3 - x_4 = 0. \end{cases}$

39. $\begin{cases} x_1 - x_2 - 4x_3 + 3x_4 + 2x_5 = 0, \\ 5x_1 - 2x_2 - 3x_3 + 5x_4 + 2x_5 = 0, \\ 6x_1 - 3x_2 - 7x_3 + 8x_4 + 4x_5 = 0, \\ 4x_1 - x_2 + x_3 + 2x_4 = 0. \end{cases}$

- **Linear transformations and second degree curves.**

1. Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix}$.
2. Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.
3. Sketch the curve $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$.
4. Sketch the curve $9x^2 - 24xy + 16y^2 - 20x + 110y - 50 = 0$.
5. Sketch the curve $25x^2 - 14xy + 25y^2 + 64x - 64y - 224 = 0$.
6. Sketch the curve $4xy + 3y^2 + 16x - 12y - 36 = 0$.
7. Sketch the curve $7x^2 + 6xy - y^2 + 28x + 12y + 28 = 0$.
8. Sketch the curve $19x^2 + 6xy + 11y^2 + 38x + 6y + 29 = 0$.
9. Sketch the curve $5x^2 - 2xy + 5y^2 - 4x + 20y + 20 = 0$.
10. Sketch the curve $14x^2 + 24xy + 21y^2 - 4x + 18y - 139 = 0$.
11. Sketch the curve $11x^2 - 20xy - 4y^2 - 20x - 8y + 1 = 0$.
12. Sketch the curve $7x^2 + 60xy + 32y^2 - 14x - 60y + 7 = 0$.
13. Sketch the curve $50x^2 - 8xy + 35y^2 + 100x - 8y + 67 = 0$.
14. Sketch the curve $41x^2 + 24xy + 34y^2 + 34x - 112y + 129 = 0$.
15. Sketch the curve $29x^2 - 24xy + 36y^2 + 82x - 96y - 91 = 0$.
16. Sketch the curve $4x^2 + 24xy + 11y^2 + 64x + 42y + 51 = 0$.

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