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AIRCRAFT AS THE CONTROLLED OBJECT

Educational supply

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Initial data on aircraft as controlled objects are resulted. Stages of obtaining of analytical models of aircrafts for problems of projection of control systems, and also - examples of models are considered. For students of trades "avionics", "air navigation", "system engeneering" at study of special disciplines and accomplishment of outlet baccalaureate activities, course and degree designs and activities.

Представлены первоначальные сведения о летательных аппаратах как объектах управления. Сведение используются на этапах получения аналитических моделей летательных аппаратов в задачах проектирования систем управления. Приведены также примеры моделей в указанных задачах. Для студентов специальностей «Авионика», «Аэронавигация», «Системная инженерия» при изучении специальных дисциплин и выполнении выпускных бакалаврских работ, курсовых и дипломных проектов и работ.

Illustration 1. Bibliographies: 28 names

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1. THE CLASSIFICATION OF AIRCRAFT AS THE CONTROLLED OBJECTS

Diagram, criteria and principles of the aircraft classification. Features of aircraft separate classes and their account in elaboration models process. The principles of the flight and creation of control forces and moments

Aircraft is a complex technical hierarchic system used for target guide control moving in space without a direct Earth contact on the usage of one or several lift creation methods [1 – 8].

There is a big amount of aircraft types (or kinds) that differ from each other by design dimensions, flying speed, kind of taking off, functional mission etc. at the present time. All these factors define the requirements (relatively) to control system, its kind, the structure and mission. During the solution of control system design tasks, analyzing and synthesis systems, which contain aircraft as control objects (links) for the mathematical aircraft description. It is necessary for mathematical dynamics model obtaining to perform it as the definite mechanical model, taking into account specific character, structural peculiarities, kinds of control elements and other factors mentioned above. The more pieces of aircraft information as a control object, the better adequate description of its dynamics using obtained Mathematical dynamics model

It is convenient to take the analysis of distinctive special features and peculiarities of aircraft as control objects using their classification. The definite criteria are chosen for the aircraft classification. They definite peculiarities of each of them with a certain completeness degree depending on the chosen criteria (attributes) all aircraft can be divided into separate groups. Each this group (class) has a typical feature, which is the most important for the concrete comparison of different aircraft. Any aircraft classification can't be absolute and the requirements at once.

The different principles of the aircraft classification are possible. They are defined by its missions: aerodynamic, exploitation, strength, controllability etc.

The aircraft classification in accordance with the way of mission in the most useful for searching new versions and for comparing present ones.

The Fig.1.1 represents a diagram of the aircraft full classification, which shows their peculiarities and special features as possible as control objects.

On the basic of such classification the following criteria are included.

1. Flight principle – aerodynamic, jet, aerostatic and combinatorial. According to this criterion aircraft are divided into classes: heavier and lighter than air. In the first or second cases lift is created by special aircraft equipment, in the third case the lift creation has on aerostatic origin. Concerning control heavier than air aircraft are the most interesting ones. They are the most numerous groups and classes of the different aircraft subjected to be considered as further classification.

2. Methods of performing mission. According to this criterion lighter-than-air aircraft are divided into separate kinds: balloons, exploratory air balloons and air-ships dirigible. Each of this kind is intended for performing concrete tasks.

3. Person's presence in the control system (outline). This criterion causes two aircraft classes: piloted and pilotless. It's important for control system is designing. In the first case aircraft contains either manual or automatical control systems or both systems at once. Several aircraft types contain an automatic control system as well. In the second case all aircraft control tasks are just performed with the automatic system (without person's participation).

It's convenient to classify pilotless aircraft on the basic of the criterion "method of performing mission":

a) airplanes, controlled automatically (reconnaissance, drones, remote controlled fighters and so on);

b) spacecrafts (artificial Earth satellites of different assigning, interplanetary spacecraft, orbiter, orbital transport vehicle and so on);

c) rockets (missiles) are automatically controlled flight vehicles intended for transporting payload to the given point of space or a surface of the Earth (as a rule, flight time of rockets ≤ 10 minutes);

4. Lift creation method (way). It can be distinguished three methods: usage of an airfoil carrier, which is fixed concerning fuselages-wing; usage of surfaces, rotated relatively to a fuselage, these surfaces are propeller plates of helicopter; usage of jet force of the efflux engine gases. According to that such aircraft classes are inserted: airplanes (the first method of the lift creation); helicopters (the second method of the lift creation); hypersonic flight vehicles (the combination of the first and third methods), exotic craft (experimented, exploration, original craft; the combination of three lift creation methods), missiles (the first and (or) third methods of the lift creation). Spacecraft take a special place among piloted and pilotless ones. They are: Earth satellites, transport spacecrafts, manned orbital stations. Such craft motion is described using

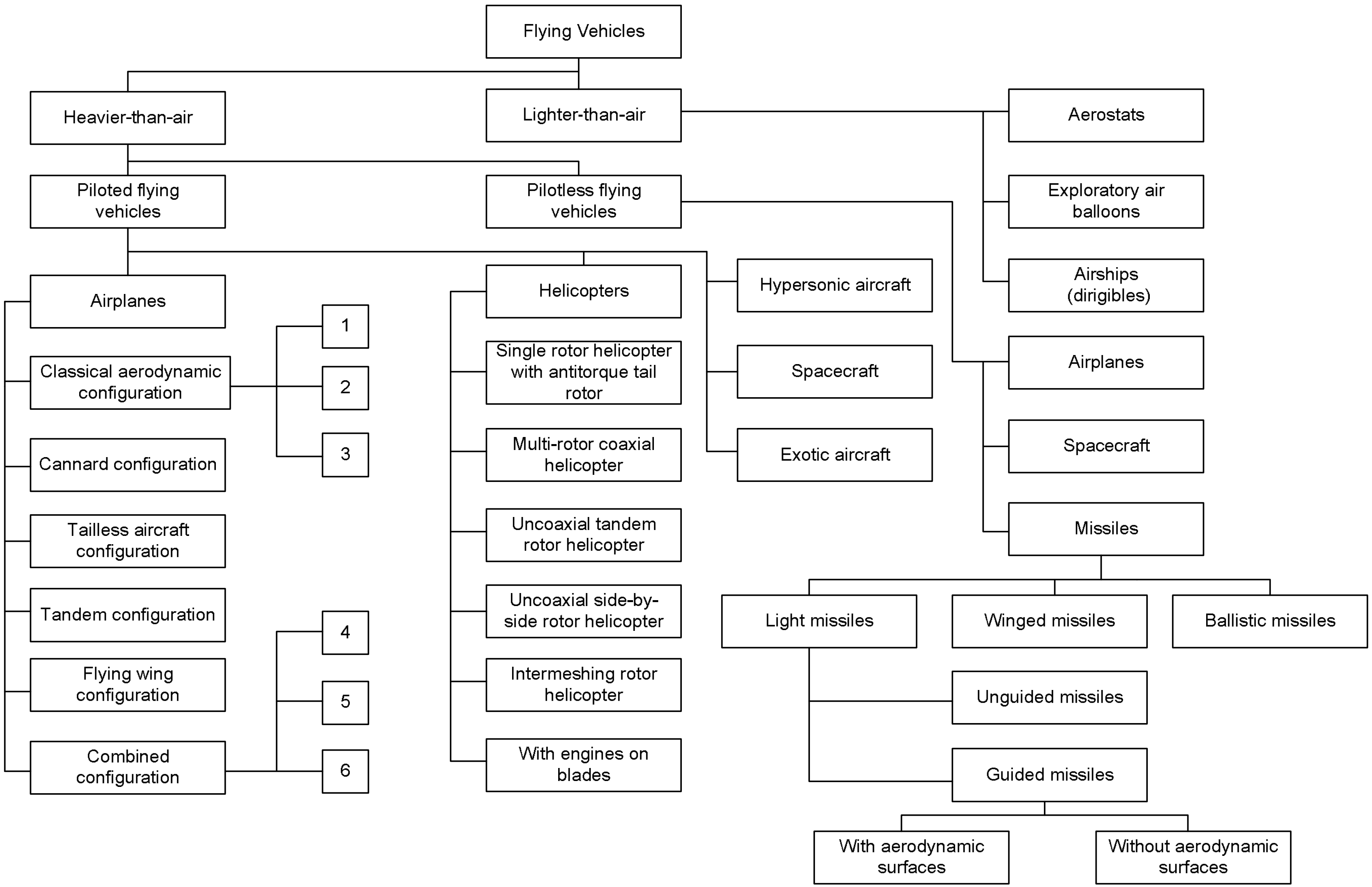

Fig. 1.1. Classification diagram of flying vehicles as control objects:
1 – Propelled airplanes; 2 – Prop-fan airplanes; 3 – Jet airplanes; 4 – Horizontal taking off and landing airplanes;
5 – Short taking off and landing airplanes; 6 – Vertical taking off and landing airplanes

space dynamics laws (laws of body's motion in a central gravitation field) and it can't be said about a lift creation. Artificial satellites don't "fall down" on the Earth due to gravitation force to be compensated by centrifugal force taking place in process of body's motion along the curved trajectory.

It should be mentioned that aircraft dividing into classes, mentioned in item 4 can be made on the basis of the other classification criterion-value of a cruising flight speed. It's a typical and a important attribute of any aircraft.

Airplanes are characterized by flight speeds conforming $M < 3.5$, helicopters – $M \leq 0.26$; hypersonic craft - $M \leq 27.5$; spacecrafts – $M \approx \approx 20...33$; exotic crafts – $M \leq 0.8$.

Criterion "lift creation principle" inconvenient to use for pilotless aircraft. Both in manned and pilotless airplanes are used the first method of the lift creation, rockets (missiles) are used the first, third or a combination of these two methods. Pilotless spacecraft as far as manned ones move in a central gravitational field according to the laws of space dynamics.

5. The kind of an airplane aerodynamic configuration. It is one of the basic criteria of airplane classification – aircraft, moving in the atmosphere and using an aerodynamic flight principle of criterion the lift and controlling forces and moments. Aerodynamic configuration of an airplane define different construction airplane's attribute and corresponds to an effective range of speed of their motion [9 – 12].

Aerodynamic configuration of an airplane is usually understood as the system of main and auxiliary airfoil surfaces.

This system can be characterized both by mutual location of airfoil surfaces and relative sizes and shapes. In the system of airfoil surfaces there main surfaces (wings), creating the main part of an aerodynamic lift creation. There also auxiliary (horizontal and vertical tail and control equipment) using for an airplane stabilization and motion control.

Depending on location and a kind of the auxiliary surfaces in accordance with wings systems there are such aerodynamic configurations:

- normal (classical) configuration, if horizontal and vertical tail is located behind;
- "tailless aircraft" if horizontal tail is absent;
- "canard scheme" if horizontal tail or control surfaces are in front of the wing;
- "flying wing" if all the aggregates and units are included into the airfoil surface – the wing;
- "tandem" if lifting surfaces consist of two pairs of wings;

- "closed wing" if the lifting surfaces consist of two pairs of wings united by their tip edges;

- combined aerodynamic scheme, which is characterized by separate attributes of the previous configurations.

All enumerated aerodynamic configurations must have common properties: to be trimmed in different values of the lift creation and to keep the static motion in case of the definite lift creation's value. Control (trim) properties are main in the creation of any aerodynamic airplanes configuration. In connection with this indicated aerodynamic configurations are called the trimmed ones. They can be created by flat lifting surfaces. And "a tailless aircraft" configuration can be trimmed just on the longitudinal static stability (neutral stability). The usage of "twisted" the airfoil bearing surfaces allow to execute the trim of all configurations rationally (with a minimum losses of lift-to-drag ratio) under condition of their stability.

6. Kind of aircraft propulsive device (the instrument directly creating a thrust force capacity). At present there are famous three airplane classes, defined by propeller type: propeller, prop-fan and jet. Two first propulsive devices are based on the aerodynamic force taking place during the interaction between turning blade and space to create thrust. The third type of propulsive device is based on the usage of reaction jet's force of engines.

A propeller is a blade propulsive device, turned in motion by aircraft engine and used for thrust obtaining in airspace, which is necessary to aircraft motion.

Prop-fan is on multiblade big diameter propeller (up to 8 blades). It has the complicated configuration (form) blades with a sabrely fold (and bend) of a thin profile (roll-formed section). Prop-fan in the combination with the modern gas turbine creates a power plant, which allows to reduce a fuel consumption by 20...30 percent as compared with the present turbojet engines. It allows to fly at speeds of 800...900 km/h. Prop-fan noise and vibration are considerable lower than noise and vibration from usual propeller. For a high-power plant so called coaxial scheme is accepted. In that case there are two aircrews with antispin direction on one shaft. Such a solution gets except diameter reduction and jet moment's elimination on the wing also a supplementary fuel economy. Because of a big amount of the complex form blades and its thin sections it is impossible to make prop-fans from the traditional aluminium alloy. They are made of composite materials such as reinforced plastic on the basis of glass, coal and organic fibres. Prop-fan as a propulsive is supposed to be used on passenger and transport aircraft. The special blade form allows to maintain high useful action ratio (approximately 0.8) during high subsonic flight speed (M ≈ 0.8). When the axial forces (thrust) are equal prop-fan diameter is essentially less than an usual propeller diameter. The propulsive device's type of an airplane is connected with a type of its engine (engine installation on power unit). Propeller is included into the power unit structure of a piston or a gas turbine type. The prop-fan is included into the power unit of gas-turbine type. The third type of a propulsive device (jet type) corresponds to a turbojet or ramjet engines of different classes. On the classification scheme (mind Fig.1.1) there are conditionally inserted airplane types (on the basis of showed criterion) of a normal aerodynamic configuration, which is characterized by a wide range of flight speeds (M=0.25...3.5). Rotor or prop-fan, propeller are used on low-speed airplanes (M≤0.5) and jet propellers are used on high-speed ones (M=0.5...3.5). As a rule, airplanes, which are made on the basis of other aerodynamical configuration, are high-speed. That's why they are equipped by jet (turbojet) power units.

7. Take-off and landing type. This criterion defines three classes of airplane such as: horizontal take-off and landing, short take-off and landing and vertical take-off and landing.

Short take-off and landing (shortening take-off and landing run) and a vertical take-off and landing are obtained on the basis of the following special equipment's usage: thrust-to-weight ratio increasing more power engine's installation, using the afterburning of power unit or a start booster motor, creation of the thrust vertical component, trailing-edge three section flap's deflection and other devices of aerodynamic lift force and drag force control; flap inflow by the escaping gas stream's engine. Airplane division in accordance with the considered criterion defines their constructural features, the features of the control devices and the control system. For example, with a control system it is necessary to solve special tasks of control devices which changes lift force and drag force. It is also necessary to solve tasks of control the swiveling engines or a nozzle, control surfaces providing a centre of mass motion along the trajectory and the angle stabilization of the aircraft body.

Classification diagram (mind Fig.1.1*)* represents conditional division according to the aircraft's form of taking off and landing only of airplanes made concerning the combined aerodynamic configuration. However, only airplanes made on any of the shown diagram can be classified, especially - on a normal aerodynamic configuration can be.

Pilotless aircraft contain an automatic control system [4, 13]. They use an aerodynamic, jet and combined principle of flight and creation of control forces.

To pilotless aircraft attribute ballistic missiles and launch vehicles of space plants [4, 14 -18]. They use only a jet principle of flight and creation of control actions and always are statically unstable plants.

Helicopters as control plants classify basically by such criterion as a carrier system kind [5, 6, 19, 20]. Helicopters use only an aerodynamic principle of flight and creation of control forces and the moments.

Space aircraft operate in a space [21, 22]. Their principle of flight is based on use of laws of motion of a particle in a gravitational field of planets, and trajectories represent circles, parabolas and hyperbolas.

Test questions

1. Name criteria of classification of aircraft.
2. What basic classification criteria of planes?
3. List kinds of aerodynamic schemes of planes.
4. What basic classification criteria of helicopters?
5. Give definitions the aircraft of the basic classes.
6. What basic classification criteria of ballistic missiles?
7. Present settlement-kinematic schemes of planes of various aerodynamic schemes.

2. STAGES AND PRINCIPLES OF OBTAINING MODELS OF AIRCRAFT, AGGREGATES AND SYSTEMS

The stages of obtaining models of aircraft, aggregates and systems. The methods of models development. The system of conditions, assumptions and requirements in the process of models development. The theoretical basis for the process of the models formation

Formation of aircraft mathematical models is a complex and complicated process which requires great attention and carefully selected approach. By convention this process may be divided into the following stages [1, 9, 14, 16, 19, 23, 24].

1. Systematization and determination of primary (physical) characteristics of the object discussed, all its peculiarities, modes of operations, laws of parameters variations being taken into consideration.

Systematizing and formalizing apriori information concerning the aircraft it is necessary to obtain and describe:

1) aircraft purpose;

2) structural and arrangement configuration;
3) structural and functional scheme of control system;
4) flight operational models;
5) principle of control;
6) type of control elements;
7) type of aerodynamic forces control facilities;
8) type of power unit (type of engines, propulsors);
9) specific features of the system for the object of control;
10) primary (fundamental) characteristics.

By convention the object primary characteristics may be divided into the following groups:

1) flight and technical characteristics;
2) geometrical (dimensional) characteristics;
3) mass and inertial characteristics;
4) thrust performance;
5) aerodynamic characteristics.

Information on primary (fundamental) characteristics may be found in technical documentation for the object of control or in control system design specification. Some parameters: mass and inertial characteristics are calculated in the process of development of the objects of control models.

2. Systematization of parameters and formation of environment and contacts with it directly. Environment directly affects the aircraft motion and its flight and technical characteristics. That's why the demand arises for creation and application of environment fundamental parameters necessary in the process of formation of aerodynamic forces and moments applied to the aircraft.

The air transport speed is different at different levels of the atmosphere. The atmosphere is thermally different both in vertical and horizontal direction. All these factors result in the conditions favorable for turbulence occurrence. In some cases the atmospheric model should include the description of turbulence phenomenon.

Turbulence is defined as phenomenon of significant variations of environment parameters in the adjacent points with considerable non-stationariety of air velocity field.

When analyzing the aircraft longitudinal motion in turbulent atmosphere, the special attention is paid to the determination of pitch angle variations and vertical overfloating acting on the aircraft. These parameters of longitudinal motion are of great interests in the majority of practical tasks relating to the aircraft flight in turbulence atmosphere conditions.

Atmosphere turbulence is characterized by wind velocity variable component. The turbulent air motion whose speed at any points of the area where this motion occurs represents a random function of coordinates of this point and time.

Immediate causes of turbulence occurrence in the atmosphere are vertical and horizontal gradients of the temperature and wind velocity.

In order to describe aircraft dynamic in turbulent atmosphere it is necessary to use analytical methods of wind velocity field representation. Two methods are used: method of dynamics the frequency-response methods are widely used as well as spectral densities describing turbulence.

3. Formation of the system involving conditions, requirements and assumptions used in the process of the object dynamic description.

Any object can be described by a great number of models. For the purpose of control systems design and dynamic the different models of the same object may be used. That's why the process of models making suggest the use of the following information:

1) type of the task the model is made for;
2) type of the planet figure model;
3) type of the system of the planet motion parameters in absolute space;
4) type of the model of the planet gravitational field as the basic body;
5) type of the atmospheric (environment) model;
6) type and characteristics of basic (reference) coordinate system;
7) type of projection coordinate system where flight conditions, type of measuring system elements and combination of object state parameters are taken into account;
8) combinations of object of control state parameters (system of variables characterizing the object state);
9) type of the object of control mechanical model (these mechanical models are: solid body, system of interrelated solid bodies, elastic body, solid body with cavities partially filled with liquid, combined models corresponding to the type combinations mentioned above);
10) types of the object aggregates (devices) and control system models (power unit engines, propulsors, fuel lines etc.);
11) types of controlling actions system (the actions which are applied to the object);
12) system of amplifying factors in the process of formation of object state original equations.

4. Selection of the fundamental theories and basic analytical relationships corresponding the object state equations.

Depending on the complexity and type of mathematical model the equations used for mechanical objects are the following ones:

1) equation of kinetostatics (Dalamber principle), which includes forces applied to the body, taking into account the forces of "inertia";

2) equation of solid state angular motion

$$I\varepsilon = M,$$

where M is a moment applied to the body; I is moment of inertia relative to the axis the body is rotating around;

$$\varepsilon = \frac{d\omega}{dt}$$

is angular acceleration; ω is the body angular velocity relative to the axis in question;

3) Newton's equation

$$m\frac{dV}{dt} = F,$$

where m is the mass of the body; $\frac{dV}{dt}$ is the absolute acceleration of the body ; F is the force applied to the body;

4) the laws of change of the momentum and angular momentum (moment of momentum) of the solid body

$$\frac{d\overline{K}}{dt} = \sum_{i=1}^{n_1} \overline{F_i}; \qquad \frac{d\overline{L}}{dt} = \sum_{i=1}^{n_2} \overline{M_i},$$

where $\overline{K}$ *is* the momentum of the body specified point; $\overline{L}$ is moment of momentum of the solid body; $\sum_{i=1}^{n_1} \overline{F_i}$ is the sum (total) of the forces discussed; $\sum_{i=1}^{n_2} \overline{M_i}$ is the sum of the moments applied to the body; n_1 is the number of forces into account; n_2 is the number of moments taken into account;

5) Lagrange's equation of the second kind

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i - \frac{\partial \Pi}{\partial q_i} \qquad (i = 1, 2, ..., n);$$

$$T_i = \frac{1}{2}[m_i V_{oi}^2 + 2m_i(\overline{V}_{oi} \times \overline{\omega}_i) \cdot \overline{r}'_{ci} + \overline{\omega}_i \cdot Q_i^0 \cdot \overline{\omega}_i];$$

$$T = \sum_{i=0}^{n} T_i,$$

where T is kinetic energy of the system; Π is the potential energy of the system; q_i is the i-th independent generalized coordinate; Q_i is i-th generalized force assigned to the generalized coordinate q_i; n is the number of generalized coordinate characteristing the system state; T_i is the kinetic energy of i-th element; $\bar{V}_{oi}$ is the pole 0 velocity of i-th element; $\overline{r'}_{ci}$ is the vector - radius $\overline{OC}$ of the body internal center in the system of axes having their origin in the pole 0 for i-th element; m_i is the mass of i-th element; θ_i^0 is the tensor inertia of i-th element in this point; $\bar{\omega}_i$ is the vector of absolute angular velocity of i-th element;

6) equations of center of mass kinematics and of solid body angular motion (such equations can be presented in different forms);

7) Kirchoff's laws are used for electrical and electromechanical objects:

- Algebraic sum of currents in the electric circuit unit equals zero if all currents incoming to the unit are considered to be positive and all outgoing currents are considered negative;
- Algebraic sum of voltage drops in any closed loop is equal to algebraic sum of electromotive force acting in this loop (current and electromotive forces coinciding with arbitrary choosing direction of the loop by-pass are considered positive, while those acting in the direction opposite to by-pass are considered negative).

Aggregation method is used for the description of complex systems. Aggregation is defined as conventional separational of the general structure of the object or system into some isolated substructures in accordance with the feature of physical character, functional purpose or complexity, with isolation of incoming actions and outgoing parameters.

5. Formation of original equations of the state of the object defined as solid body when conditions, requirements and assumptions are taken into consideration.

The purpose of this stage may be defined as analytical description of the object state without taking into account any additional factors on the basis of fundamental theories and (or) original equations assumptions, prerequisites.

For example, speaking about aircraft, this description may be defined as the system of scalar equations describing it's motion as the motion of solid body.

6. Formation of the additional models of the complete original systems of equations.

The additional models are:

1) planet figure models;
2) gravitational field models;
3) atmospheric models;
4) control system aggregates or object models;
5) aerodynamic forces and moments models ;
6) differential and (or) integral equations describing elastic oscillations of object structure elements;
7) differential and (or) integral equations describing the fluid sloshing in the cavities partially filled with this fluid.

7. Formation of the complete original system of object state equations in accordance with the accepted assumption and requirement.

The complete original system of object state equations is formed by using the subsystems of kinematic and dynamic equations of solid body specific point, equations of body angular motion, additional models, accepted conditions and assumptions. Such system should include:

1) equations of object pole kinematic ;
2) equations of pole dynamics;
3) equations of solid body motion with regard to the pole (angular motion equations);
4) kinematic equations of aircraft angular motion;
5) additional models (equations).

As a rule, the obtained system has a smaller number of equations than a number of variables contained in these equations. In order to make the number of equations equal to the number of variables, the additional equations should be used. In this case it is convenient to use the system of analytical relationship which determine the intersection between the state parameters.

For example, the complete system of aircraft state parameters can include the additional models of:

1) power unit engines;
2) servo elements and control elements;
3) planet figure;

4) atmosphere etc.

The complete original system of equations is referred to as the combination of nonlinear ordinary differential equations, equations in partial derivatives, algebraic and transcendental equations. The original equations establish the analytical interaction between the variables which characterize the object state and input actions can be defined as the non simplified mathematical model of simplified models of different type.

8. Simplification of original equations system and creation of the object phenomenon models.

In order to obtain the object phenomenon model in the general case it is necessary to use its complete original system and the methods of simplification: reduction, decomposition and linearization. For obtaining the models of such objects as the aircraft is necessary to use the special types of motion, that is the separation of the total three-dimensional motion into some independent types. For example, in case of linear and non-linear systems such partial types of motion are separated as longitudinal, lateral and phugoid (long-period) motion.

Linearization is:

1) approximation of complex dependence by linear function;

2) reducing of non-linear tasks to linear ones.

Method of linearization: static linearization, tangent approximation, Taylor series expansion, secant method, statistical linearization, linearization by the describing function method, perturbation method.

Reduction method is defined as simplification, transformation of something complicated into simple one, understandable, limited more suitable for analysis or solution.

Decomposition is the division (separation) of the equation system or structure into special, independent subsystem or structure each of which describes the selected conventionally simplified type of the object motion.

9. Analysis of accuracy and adequacy of the obtained models.

In order to solve certain tasks of control systems design it is necessary to make an analysis of accuracy and adequacy of the objects and units of models. At this stage it is necessary to analyze the quality to the obtained model and to make a conclusion concerning the possibility of its application in the tasks of the system synthesis and analysis. There are some methods for analyzing the accuracy and adequacy of the obtained models.

One of them is the comparative analysis of the model with different degree of simplification. The comparison of results of equation system solution is made for each pair of the model versions obtained as the result of simplification. In order to simplify the models such methods as reduction and decomposition are used.

At each stage the model should satisfy the certain type of requirements, for example trade-off of accuracy against complexity.

Another method which may be used is the experimental research of the object whose model is being formed (developed). This method is based on the comparison and analysis of the results of full-scale tests and those obtained analytically. This method is called verification.

Generally recognized definition of this notion is: check, empirical corroboration of theoretically obtained results by means of their correlation with observed processes of the object functioning, with experiment. Verification is possible only in the case the really acting object is available.

If discrepancies exceed the acceptable limits the model is said not to satisfy the requirements.

10. Formation of special models.

Special models are obtained on the basis of the system of complete original equations of the object state in accordance with the requirements of the task being solved.

Special mathematical model is the combination of the analytical and (or) structural relationships formed on the basis of additional requirements, conditions or factors.

Special models can be classified in the following way:

1) static models, that is, models correlating the dependence between input and output parameters without considering their time dependence (when $t \to \infty$);

2) accuracy models, that is, models where such factors as deviations from nominal or design values of the object or supporting motion parameters are taken into account; accuracy models contain the maximum body of information determined by conditions, requirements and assumptions accepted at the process of the original model formation;

3) statical models, that is, models containing probabilistic descriptions of individual elements of equations, primary characteristics, supporting motion parameters, describing actions;

4) structural models, that is, schemes making graphical representation of correlations between input and output parameters of the object;

5) computer models, that is, formalized representation of analytical and (or) structural models by means of characters (symbols) and programmed packages aids;

6) operating models, that is the analytical relationships obtained by Laplace integral transformation;

7) frequency and impulse characteristics are analytical or graphical dependences of output parameter value from input action frequency;

8) combined models are various types of combinations of the versions

mentioned above.

Test questions

1. Name the basic stages of the development process of models of system elements, units and the object of control.
2. What is the essence of each stage of the model development process?
3. Name the types of aircraft mechanical models development process.
4. What conditions, prerequisites and assumptions are to be taken into account when the models are being obtained?
5. What laws and theories are used in the development process of models of units objects and systems?
6. Write down the formulae for the laws of the solid body momentum change and moment of momentum (angular momentum) change.
7. Write down Lagrange equation of the second kind and make components .
8. Obtain the mathematical model for simple mathematical pendulum whose parameters are given.
9. What models can be referred as additional and special?
10. Name the simplification methods of units, objects and system models. Make comments.
11. Is it possible to obtain the aircraft models using Newton's equation?

3. ADDITIONAL MODELS IN THE TASKS OF AIRCRAFT DYNAMICS DESCRIPTION

Elements of solid (rigid) body turn (rotation) theory. Formation and application of directional cosines matrices. Angular parameters of aircraft state (condition). Atmosphere models. Models of planet's figure and gravitational field

3.1. Elements of theory of solid body end turn (rotation)

The position of two orthogonal coordinate systems relative to each other is determined by radius-vector $\bar{r}$ which connects their origin points and body rotation angles of one of these systems with respect to the other. As a rule, one of the two coordinate systems considered is taken as original one and the other is referred to as the moving coordinate system. In this case radius-vector $\bar{r}$ may be predetermined by its projections on the axis of the original coordinate system.

Of great interest is the description of the angular position of the coordinate systems relative to each other. In this case real or imaginary coincidence of origins of both coordinate systems in one point is taken into consideration.

In order to the positions (motions) of solid body or coordinate system tightly connected with this body relative to the original coordinate system with one fixed point in which the origins of both system coincide the different systems of kinematic parameters and methods are used:

a) Eulerian angles and Eulerian-Krylov angles;
b) directional cosines matrices;
c) Rodrigo-Hamilton and Kelley-Klein parameters;
d) quaternions.

3.1.1. Classical Eulerian angles and Eulerian-Krylov angles

Let's consider two positions of the solid body or coordinate system Oxyz tightly connected with this body: the initial position in which the Oxyz system axes connected with the body coincide with the corresponding axes of original ('fixed') coordinate system OXYZ and the end position when (in the general case) the axes of systems OXYZ and Oxyz do not coincide (the origins of both coordinate systems are located in the point 0).

Trihedral Oxyz may be transferred from initial position to the end position by means of three rotations made in certain sequence round the properly choosen axes. The angles of these turns (rotations) are called Eulerian angles (classical Eulerian angles) and they represent three independent values [2,3].

As an example of Eulerian-Krylov angles application let's consider the transfer from normal coordinate system $OX_gY_gZ_g$ to the body-axis system OXYZ by three successive end turns of the moving coordinate system OXYZ with respect to its axes. In the initial position the corresponding axes of coordinate systems $OX_gY_gZ_g$ and OXYZ coincide.

The first turn of the moving coordinate system is performed relative to the axis OZ (OZ_g) at angle υ:

$$OX_gY_gZ_g \xrightarrow[\upsilon]{Z_g, Z'} OX'Y'Z'.$$

As the result of this, the orthogonal trihedral of the moving system will occupy the position OX'Y'Z' and axes OX' and OY' will change their position in plane X_gOY_g.

The second turn of the system OX'Y'Z' at angle Ψ is made round its axis OY' which coincides with axis OY":

$$OX'Y'Z' \xrightarrow[\psi]{Y_g, Y'} OX''Y''Z''.$$

The position of the trihedral of moving coordinate system after its second turn is designated as 0X"Y"Z". Axes OX" and OZ" are moving in the plane X'OZ'. The third turn of moving system 0X"Y"Z" is performed around the axis OX" at angle γ:

$$OX''Y''Z'' \xrightarrow[\gamma]{X'', X} OXYZ.$$

Axes OY and OZ are changing their position in plane Y"0Z". As the result of three turn the position of axes of moving coordinate system OXYZ is referred to as the end position.

The choice of turns making sequence (that is the order in which the angles each other) is determined in the following way: the angles whose value changes most and (or) the most important (the most critical) angle is taken as the first turn angle, the angle whose value changes least of all is taken as the last one.

Eulerian-Krylov angles υ, ψ and γ change in the range of ±90°. Unlike the classical Eulerian angles, the Eulerian-Krylov angles are always small and only in the case if the end turn angle which directly transfers the moving system trihedral from initial to the end position is small.

Eulerian-Krylov angles are not rearranged, that is in case the moving system turns order is changed, the values of corresponding angles are not equal to each other. But when these angles are of small values (<<1) the corresponding angles are approximately equal.

3.1.2. Directional cosines and directional cosines matrices

Let's consider axis l in space when the direction of this axis is determined by unit vector $\bar{l}$. This axis is crossed by unit vector $\bar{V}$ in some point at angle α.

The projection of unit vector $\bar{V}$ to axis l may be found in the following way:

$$Pr_l \overline{V} = \left|\overline{V}\right| \cos\alpha.$$

When $|\bar{V}|$ =1, $Pr_l\bar{V}$ = cosα, that is relative position of unit vector and axis or of two unit vectors is determined by means of cosine of the angle between them (directional cosine).

Vector directional cosines are the cosines of angles that the vector forms with the coordinate axes. If vector $\bar{V}$ is predetermined by its projection in Oxyz system, then:

$$cos(\bar{V},\bar{\mathrm{x}}) = \frac{V\mathrm{x}}{|\bar{V}|}; \;\; cos(\bar{V},\bar{\mathrm{y}}) = \frac{V\mathrm{y}}{|\bar{V}|}; \;\; cos(\bar{V},\bar{\mathrm{z}}) = \frac{V\mathrm{z}}{|\bar{V}|}. \qquad \text{Thus:} \qquad (3.1)$$

$$cos^2(\bar{V},\bar{\mathrm{x}}) + cos^2(\bar{V},\bar{\mathrm{y}}) + cos^2(\bar{V},\bar{\mathrm{z}}) = \frac{V_{\mathrm{x}}^2}{V^2} + \frac{V_{\mathrm{y}}^2}{V^2} + \frac{V_{\mathrm{z}}^2}{V^2} = 1. \qquad (3.2)$$

It should also be remembered that the scalar product of two vectors which may belong to different basis sets is equal to the cosine of the angle between these vectors.

The vector's directional cosines are enough to determine the vector's direction, but they do not contain any information about its length. The directional cosines are defined as the basis of one of the methods for description of relative position of orthogonal coordinate systems-application of directional cosines matrices (cosine tables). The directional cosines matrix is the table of dimensions 3x3. The elements of this table are the cosines of angles between the axes of original and moving coordinate systems.

In the general case the directional cosines matrix can be given as:

$$M=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad (3.3)$$

a_{ij} (i,j = 1,2,3) are the cosines of the angles between the axes of original and moving coordinate systems.

Each element a_{ij} (i,j = 1,2,3) of the directional cosines matrix (3.3) is defined as the scalar product of the unit vectors of corresponding axes of original (e.g. OXYZ) and moving (e.g. Oxyz) coordinate systems, that is:

$$\begin{gathered} a_{11}=\bar{X}\cdot\bar{\mathrm{x}} = cos(X,\mathrm{x}), \\ a_{12}=\bar{Y}\cdot\bar{\mathrm{y}} = cos(Y,\mathrm{y}), \\ \ldots \quad \ldots \quad \ldots \\ a_{31}=\bar{X}\cdot\bar{\mathrm{z}} = cos(X,\mathrm{z}), \\ \ldots \quad \ldots \quad \ldots \; ; \end{gathered}$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} X & Y & Z \end{array} \\
M= \begin{array}{c} \mathrm{x} \\ y \\ \mathrm{z} \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.
\end{array}
$$

For the sake of convenience of directional cosines application we may introduce the designation of axes of original and moving coordinate systems.

Each unit vector of moving coordinate system Oxyz may be represented by the following decomposition by the unit vectors of original system OXYZ:

$$
\begin{aligned}
\bar{\mathrm{x}} &= a_{11}\bar{X} + a_{12}\bar{Y} + a_{13}\bar{Z}; \\
\bar{\mathrm{y}} &= a_{21}\bar{X} + a_{22}\bar{Y} + a_{23}\bar{Z}; \\
\bar{\mathrm{z}} &= a_{31}\bar{X} + a_{32}\bar{Y} + a_{33}\bar{Z}.
\end{aligned} \tag{3.4}
$$

Vectors $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ of the original coordinate system axes and $\bar{\mathrm{x}}$, $\bar{\mathrm{y}}$ and $\bar{\mathrm{z}}$ of the moving coordinate system are defined as unit vectors which are orthogonal relative to each other. They form the orthonormalized vector systems. Hence:

$$
\begin{aligned}
&\bar{X}\cdot\bar{Y} = \bar{X}\cdot\bar{Z} = \bar{X}\cdot\bar{Z} = 0; \qquad && \bar{X}\cdot\bar{X} = \bar{Y}\cdot\bar{Y} = \bar{Z}\cdot\bar{Z} = 1; \\
&\bar{\mathrm{x}}\cdot\bar{\mathrm{y}} = \bar{\mathrm{x}}\cdot\bar{\mathrm{z}} = \bar{\mathrm{x}}\cdot\bar{\mathrm{z}} = 0; && \bar{\mathrm{x}}\cdot\bar{\mathrm{x}} = \bar{\mathrm{y}}\cdot\bar{\mathrm{y}} = \bar{\mathrm{z}}\cdot\bar{\mathrm{z}} = 1.
\end{aligned} \tag{3.5}
$$

On the basis of these equations and using the dependence (3.4) we can write down six relationships which connect nine directional cosines:

$$
\begin{aligned}
&a_{11}^2 + a_{12}^2 + a_{13}^2 = 1; \qquad && a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{23} = 0; \\
&a_{21}^2 + a_{22}^2 + a_{23}^2 = 1; && a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0; \\
&a_{31}^2 + a_{32}^2 + a_{33}^2 = 1; && a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} = 0.
\end{aligned} \tag{3.6}
$$

In the determinant

$$
\mathrm{detM} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{3.7}
$$

of directional cosines matrix M each element is equal to its own algebraic adjunct, that is:

$$a_{11} = (a_{22}a_{33} - a_{23}a_{32});$$
$$a_{12} = -\,(a_{21}a_{33} - a_{23}a_{31}) \quad \text{etc.} \tag{3.8}$$

The directional cosines matrix is referred to as nine-parametric system. The use of directional cosines matrices in the tasks of dynamics and navigation suggests the availability of specific expressions of their elements. These expressions are formed with consideration for the type of coordinate systems considered and Eulerian-Krylov angles which determine their relative position.

3.1.3. Directional cosines matrices applications

Properties and nature of directional cosine matrices make it possible to use them effectively for the tasks of coordinates transformation and in geometry of orthogonal systems.

If the elements a_{ij}(i,j =1,2,3) of the directional cosine matrix are given then using the expressions (3.3) it is possible to determine the angles between the axes of original and the moving coordinate systems:

$$(\bar{X}, \hat{}\ \bar{\mathrm{x}}) = arccos\, a_{11},$$
$$(\bar{Y}, \hat{}\ \bar{\mathrm{x}}) = arccos\, a_{12},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(\bar{X}, \hat{}\ \bar{\mathrm{z}}) = arccos\, a_{13} \quad \text{etc.} \tag{3.9}$$

Thus, for example, the angle between the axes OY_g and OZ_a may be found by using the matrix that describes the position of wind coordinate system relative to the normal one:

$$(\bar{Y}_g, \hat{}\ \bar{Z}_a) = arccos\ [-sin\gamma_a cos v_a + sin v_a sin\psi_a cos\gamma_a].$$

If the considered systems are orthonormalized it is possible to find the axes projections of one coordinate system to the axes of the other by using directional cosine matrices. In accordance with expression (3.3):

$$Pr_{\mathrm{x}}X = Pr_{\mathrm{X}}\mathrm{x} = cos(\bar{X}, \hat{}\ \bar{\mathrm{x}}) = a_{11},$$
$$Pr_{\mathrm{x}}Y = Pr_{Y}\mathrm{x} = cos(\bar{Y}, \hat{}\ \bar{\mathrm{x}}) = a_{12},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$Pr_{\mathrm{z}}X = Pr_{\mathrm{X}}z = cos(\bar{X}, \hat{}\ \bar{\mathrm{x}}) = a_{31} \quad \text{etc.} \tag{3.10}$$

For example, on the basis of matrix M_{gk} we may find:

$$Pr_{X_k}X_g = Pr_{X_g}X_k = cos\theta\, cos\Psi,$$
$$Pr_{X_k}Y_g = Pr_{Y_g}X_k = sin\theta\, cos\Psi, \quad \text{etc.}$$

In order to obtain the values of the angles between the axes of the original and the moving coordinate systems and their relative projection it is possible to use the directional cosine matrices of both direct and inverse coordinates transformation.

The directional cosine matrices obtained above may be used for the description of the relative position of any given coordinate system not related to each other directly by these of those angles. For example, the position the trajectory coordinated system with respect to body-axis and wind body coordinate systems is determined by means of the following matrix relations:

$$M_{lk} = M_{gk}(\Psi,\theta)\cdot M_{lg}(\gamma,\Psi,\upsilon) = M_{gk}(\Psi,\theta)\cdot M_{g1}^{T}(\gamma,\Psi,\upsilon),$$
$$M_{ak} = M_{gk}(\Psi,\theta)\cdot M_{ag}(\gamma_a,\Psi_a,\upsilon_a) = M_{gk}(\Psi,\theta)\cdot M_{ga}^{T}(\gamma_a,\Psi_a,\upsilon_a).$$

Directional cosine matrices are very convenient for making coordinates orthogonal transformations, that is for determining the projections of force vectors, moments and kinematic parameters given in the original coordinate system to the axes of any other system. Such transformations are described by vector-matrix relationships of the type given in expressions:

$$\bar{P}^{a} = M_{1a}(\alpha,\beta)\bar{P}^{1} = M_{a1}^{T}(\alpha,\beta)\bar{P}^{1};$$
$$\bar{V}^{1} = M_{a1}(\alpha,\beta)\bar{V}^{a};$$
$$\bar{G}^{a} = M_{ca}(\gamma_a,\psi_a,\upsilon_a)\bar{G}^{c} = M_{ca}(\gamma_a,\psi_a,\upsilon_a)\begin{bmatrix}0\\-mg\\0\end{bmatrix};$$
$$\bar{W}^{и} = M_{си}(\varphi_{Ц},\lambda_{Ц},A,t)M_{c1}^{T}(\gamma,\psi,\upsilon)\bar{W}^{1};$$
$$\bar{W}^{1} = [W_X \quad W_Y \quad W_Z]^{T}.$$

The definition of the angles describing the relative position of the normal, body-axis and wind-body coordinate system are given above. Only five of eight $\gamma, \Psi, \upsilon, \gamma_a, \Psi_a, \upsilon_a$, α and β angles are independent. The rest three angles may be expressed through the other angles. The task of determination the geometrical relationships between the angles may be solved by means of matrix identities:

$$M_{g1}(\gamma,\psi,\upsilon) \equiv M_{a1}(\alpha,\beta)\cdot M^{T}_{ga}(\gamma_a,\psi_a,\upsilon_a), \tag{3.11}$$
$$M_{ga}(\gamma_a,\psi_a,\upsilon_a) \equiv M_{1a}(\alpha,\beta)\cdot M_{g1}(\gamma,\psi,\upsilon) \equiv M^{T}_{a1}(\alpha,\beta)\cdot M_{g1}(\gamma,\psi,\upsilon).$$

Using the first of these identities and one of the matrices equality properties (two matrices are equal if the elements of corresponding to them are equal to each other) and equating the elements a_{13}, a_{12} and a_{23} of both identity parts we obtain:

$$\begin{aligned}
&-\sin\psi \quad -\cos\alpha\cos\beta\sin\psi_a + \sin\alpha\cos\beta\sin\gamma_a\cos\psi_a - \sin\beta\cos\gamma_a\cos\psi_a;\\
&\sin\upsilon\cos\psi = \cos\alpha\cos\beta\sin\upsilon_a\cos\psi_a + \sin\alpha\cos\beta(\cos\upsilon_a\cos\gamma_a +\\
&+\sin\gamma_a\sin\upsilon_a\sin\psi_a) - \sin\beta(\sin\upsilon_a\sin\psi_a\cos\gamma_a - \sin\gamma_a\cos\upsilon_a);\\
&\sin\gamma\cos\psi = \sin\alpha\sin\psi_a + \cos\alpha\sin\gamma_a\cos\psi_a.
\end{aligned} \tag{3.12}$$

These equations describe geometrical relations between groups of angles (γ, γ_a), ($\upsilon, \upsilon_a, \alpha$), ($\psi, \psi_a$, β). In case the angles are small, that is $\gamma, \psi, \upsilon, ..., \beta \ll 1$ and assuming that sinq≈q, cosq≈1, $\sin q_i \sin q_j \approx 0$ (q= =γ, ψ, υ, ..., β), we obtain simple evident relationships between the angles (on the basis of (3.12)):

$$\upsilon \approx \upsilon_a + \alpha; \qquad \psi \approx \psi_a + \beta; \qquad \gamma = \gamma_a. \tag{3.13}$$

3.2. Atmosphere models

The atmosphere conditions is very changeable. It depends on season and time of the day, geographical coordinates of the locality, meteorological conditions and the flight altitude.

The atmosphere is the environment where the aircraft is operating. The aircraft interaction with the atmosphere determines the movement character. The atmosphere parameters are arguments of aerodynamic forces applied to the aircraft. They influence the operation of power plant engines and determine the intensity of disturbance effects. That's why the condition of the atmosphere as environment should be described analytically. The received relationship should be used as additional mathematical model in the formation process of general original system of the aircraft movement equations. Such interconnections are called atmosphere models and may be divided into the following types.

1. The simplest model which is the whole complex of constant values of density *(*ρ*)*, pressure (p),air temperature (T) and sound velocity (a) which correspond to the given flight altitude. Numerical values of these parameters may be found in the special tables or calculated analytically. The disadvantage of this model is that it can't be used in the case of changing aircraft flight.

2. Simplified exponential model. The model consist of the following relationship for atmospheric density:

$$\rho(H)=\rho_* exp\left[-\frac{g_0}{RT}H\right], \tag{3.14}$$

where H is the flight altitude; g_0 is mean value of gravitational field acceleration at the sea level; ρ_*=1,225 $\frac{kg}{m^3}$ is the air dencity at the sea level; R=287,053 $\frac{J}{kg\cdot deg}$ is specific gas constant for air.

The model accuracy is valid for altitudes range H = 0... 11000m.

3. Simplified power model. It consists of one empirical relationship for atmosphere density:

$$\rho(H)=\rho_* \left[1-\frac{H}{44300}\right]^{4,2561}. \tag{3.15}$$

The model accuracy is valid for altitudes range H = 0...11000m.

4. Standard atmosphere. This standard sets numerical values of basic atmosphere parameters for altitudes from minus 2000 to 120000 meters [25].

Analytical relationships describe the ideal gas condition. The equation for ideal gas condition is:

$$\rho=\frac{\rho R^* T}{M_B}, \tag{3.16}$$

where R^* = 8314,32 $\frac{J}{deg\cdot kmole}$ is universal gas constant of the air; M_B is molar mass of the air.

The air molar mass is constant till the altitude of 94000m and equal to molar mass M_c at the sea level,

$$M_c = 28{,}96442\,\frac{kg}{kmole}. \tag{3.17}$$

For altitudes to 94000m, where $M_B = M_c = const$,

$$\frac{R^*}{M_B} = \frac{R^*}{M_C} = const = R\ , \tag{3.18}$$

then the equation for ideal gas condition coincides with the formula given above:

$$p = \rho RT\ .$$

Then, with the altitude increase the molar mass decreases to 28,850$\frac{\text{kg}}{\text{kmole}}$ at the altitude 97500m according to the formula:

$$M_B = 28{,}82 + 0{,}158\left[1 - 7{,}5\cdot 10^{-8}(H-94000)^2\right]^{1/2} - \\ -2{,}479\cdot 10^{-4}(97000-H)^{1/2}, \tag{3.19}$$

where H is geometrical altitude.

The further decrease of molar mass to h = 120000m occurs in accordance with linear law with gradient $\frac{\mathrm{dM_B}}{\mathrm{dH}}$ = 0,0001511$\frac{\text{kg}}{\text{m}\cdot\text{kmole}}$. The relationship between geometrical and geoidal potentiality rise are set by means of the following formulae:

$$H = \frac{rh}{r-h}, \qquad h = \frac{rH}{r+H}\ , \tag{3.20}$$

where h is geoidal potentiality rise; r = 6356767m - earth standard radius at which the earth gravitational field and acceleration vertical gradient at the sea level are the closest to the true values at latitude 45°32'33".

For atmosphere parameters calculations at the altitudes to h = 94000 meters, the temperature is approximated by linear function of geoidal potentiality rise

$$T = T_* + \beta(h - h_*)\ , \tag{3.21}$$

$\beta = \frac{dT}{dh}$ is temperature gradient at geoidal potentiality rise (here in after the values of parameters with index at the bottom refer to the lowest level of air layer which is considered).

For altitudes 94000 ≤ H ≤ 120000m we use the correlation of molar temperature

$$T_M = \frac{T \cdot M_C}{M_B} \ , \tag{3.22}$$

which undergoes linear changes at geoidal potentiality rise. The equation looks as follows:

$$T_M = T_{M*} + \beta_M(h - h_*) \ , \tag{3.23}$$

where $\beta_M = \frac{\partial T_M}{\partial h}$ is gradient of molar temperature at geoidal potentiality rise. So, β_M values are given as constant numbers for certain altitude ranges (for example 0 ≤ h ≤ 11000m, β_M = - 0,0065).

It case of linear change of molar temperature at geoidal potentiality rise we have the following equation for atmosphere statics:

$$-dp = \rho g_0 dH \tag{3.24}$$

and the following equation for ideal gas condition if the relationships for temperature are taken into account:

$$\begin{aligned} lgp &= lgp_* - \frac{g_0}{\beta_M R} lg \frac{T_M + \beta_M(h - h_*)}{T_{M*}} \quad \text{for} \quad \beta_M \neq 0; \\ lgp &= lgp_* - \frac{0{,}434294 g_0}{RT}(h - h_*) \quad \text{for} \quad \beta_M \neq 0. \end{aligned} \tag{3.25}$$

The air density is calculated by means of the equation of state:

$$\rho = \frac{pM_B}{R^* T} \ . \tag{3.26}$$

The sound velocity is described by means of the following formula

$$a = 20{,}046796\sqrt{T} \ . \tag{3.27}$$

Knowing the standard atmosphere it is also possible to calculate specific gravity, particles concentration, particles mean velocity, mean length of particles free run, dynamic and kinematic viscosity heat conduction of the air.

5. The simplified combined model includes the equation for air density such as (3.14) or (3.15).

These equations for simplified combined model are taken from the above simplified exponential and simplified power models. Besides this the simplified combined model includes relationship of temperature and sound velocity of standard atmosphere.

6. Upper Earth atmosphere. The standard determines the density model, methods of calculation and mean density value of upper Earth atmosphere at altitude range from 120 to 1500km for different levels of solar activity. Such model is used for the tasks of spacecraft dynamics and control.

7. Dynamic atmosphere model. All the above mentioned types of atmosphere models didn't take into account changes of air parameters during the day or the year. The dynamical atmosphere model is free of this disadvantage. The given model can be made on the base of such models as "2", "3", "4" and "5" for particular space.

The given model takes into account the changes of the atmosphere condition which occur (take place) during the day, the season, the year.

3.3. Models of planet figure

In the general case the reference surface or datum plane of seas and oceans extended along the narrow channels inward the mainland without the mass distribution change is taken as the surface of the planet (Earth) figure. The reference surface is determined by the combined actions of gravitational forces and centripetal acceleration due to the planet rotation. The plane tangent to the reference surface in any point is perpendicular to the vector of planet gravitational acceleration:

$$\bar{g} = \bar{g}_{\Gamma} + \bar{g}_{Ц} , \qquad (3.28)$$

where $\bar{g}_{\Gamma}$ is gravitational acceleration vector; $\bar{g}_{Ц}$ is centrifugal acceleration vector.

Due to the non-uniform distribution of mass inside the planet the gravity acceleration vector deflects from its ideal position that could occur in case the uniform distribution of mass. The deflection of the vector $\bar{g}$ value and direction from their ideal values results in the fact that the reference surface differs from the ideal surface that could be easily described mathematically. The reference Earth's surface is very complex and can't be exactly represented by this or that geometrical figure. In order to determine the reference surface the special term suggested by L. Listing in 1873 was accepted. This term is geoid. In the tasks of dynamics, control and navigation the application of geoid as the planet figure is not possible because of the difficult mathematic description. That's why different approximations are used (figures that can be easily described mathematically). These are such as:

- spheres;
- Spheroids and ellipsoids;
- reference-ellipsoids.

The first two mentioned types are used for describing the planet figure as a whole. In some cases it is required that the local surface of the planet figure be described with a high degree of accuracy.

In case of the reference- ellipsoid application the error in the angular position gravity acceleration vector that determines the geoid surface and the reference ellipsoid surface is equal to 2...3 seconds of arc. The error of the point to the altitude is equal to 100...150 meters. The aircraft attitude relative to the planet figure is determined by the following coordinates: orthogonal – X_0,Y_0,Z_0; spherical – radius-vector ($\bar{\rho}_{Ц}$), geocentric latitude ($\varphi_{Ц}$), geocentric longitude ($\lambda_{Ц}$).

Analytically, the expressions are to relate the point orthogonal coordinates to the spherical coordinates and parameters of the planet figure model. Such expressions are used when forming the complete original equation system of the aircraft motion.

In the tasks of dynamics, navigation and control the use can be made of the following types of the planet figure model:

1. Plane determined by the basic coordinate system.
2. Sphere with the constant radius R = constant.
3. Biaxial ellipsoids of different types.
4. Reference-ellipsoids.

The model of the planet figure is described if the following parameters are known:

a) radius-vector that determines the relative position of the origin of the planetocentric coordinate system and the system related to the approximating figure $\bar{r} = [x_0\ y_0\ z_o]^T$;

b) directional cosine matrix that determines the position of the coordinate system related to the planet and related to the approximating figure;

c) analytical description of the approximating figure;

d) parameters of approximating figure control: semi-axes – a, b, c, eccentricities of the first and second kind e_1, e_2, compressions – α_1, α_2.

3.4. Models of the planet gravitational field

Models of the gravitational field are defined as the mathematical description of the gravitational acceleration. In order to find the relationships between the gravitational acceleration and the coordinates of the aircraft specific point the use is made of the expression of the planet gravitational field potential $П_Т$.

Let's use the accompanying trihedral of the geocentric coordinate system O_0 $X_Ц$ $Y_Ц$ $Z_Ц$. In this case the different of directions and elements of the gravitational acceleration vector looks like:

$$\partial S_1 = \partial\rho_Ц\,; \qquad g_{\Gamma\rho} = \frac{\partial П_Т}{\partial\rho_Ц}\,;$$
$$\partial S_2 = \rho_Ц\partial\varphi_Ц\,; \qquad g_{\Gamma\varphi} = \frac{1}{\rho_Ц}\frac{\partial П_Т}{\partial\varphi_Ц}\,; \qquad (3.29)$$
$$\partial S_3 = \rho_Ц\partial\lambda_Ц\,; \qquad g_{\Gamma\lambda} = \frac{1}{\rho_Ц}\frac{\partial П_Т}{\partial\lambda_Ц}\,,$$

where $g_{Гρ}$ is the radial component; $g_{Гφ}$ is the transversal component; $g_{Гλ}$ is the binormal component of the gravitational acceleration vectors.

The gravitational acceleration vector may be written down as:

$$g_\Gamma = \frac{\partial П_Т}{\partial\rho_Ц}\,\bar{J} + \frac{1}{\rho_Ц}\frac{\partial П_Т}{\partial\varphi_Ц}\,\bar{\imath} + \frac{1}{\rho_Ц}\frac{\partial П_Т}{\partial\lambda_Ц}\,\bar{k}\,. \qquad (3.30)$$

Using the dependence for $П_Т$

$$П_Т = \frac{fM}{\rho_Ц} + \frac{f}{4\rho_Ц^3}(A+C-2B)\left(3sin^2\varphi_Ц - 1\right) + \frac{3f}{4\rho_Ц^3}(C-A)cos^2\varphi_Ц cos2\lambda_Ц + \cdots$$

in accordance with the equation (3.29) we shall obtain expressions of the gravitational acceleration vector component:

$$g_{\Gamma\rho} = -\frac{fM}{\rho_Ц^2} - \frac{3f}{2\rho_Ц^4}(A-B)(3\sin^2\varphi_Ц - 1); \qquad (3.31)$$

$$g_{\Gamma\varphi} = \frac{3f}{2\rho_Ц^4}(A-B)\sin 2\varphi_Ц\,; \qquad (3.32)$$

$$g_{\Gamma\lambda} = -\frac{3f}{2\rho_Ц^4}(C-A)\cos^2\rho_Ц\,\sin 2\lambda; \qquad (3.33)$$

$$\overline{g}_{Г} = g_{Г\rho}\overline{Y}_{Ц} + g_{Г\varphi}\overline{X}_{Ц} + g_{Г\lambda}\overline{Z}_{Ц}, \quad (3.34)$$

where f is the constant of universal gravitation; M is the mass of the body that produces gravitation; A, B, C are the moments of inertia of the body defined as the source of gravitation.

The equations (3.31)-(3.33) describe the planet gravitational field, the three-dimensional ellipsoid being the planet model. The first component of the expression is the spherical gravitational field. The rest of the components in the obtained expressions determine various deflections (deviations) of the gravitational field from the ideal spherical field.

In the tasks of dynamics and aircraft control the following descriptions of the gravitational field are used as its model.

1. The planet figure is simplest model; in the coordinate system related to the planet $g_Г = [0 \quad -g_Г \quad 0]^T$, $g_Г = g = \mathrm{const}$.

2. The planet figure is the sphere or the plane; the relationships between the gravitational field acceleration and the altitude are taken into consideration

$$g_{Г\rho} = \frac{fM}{\rho_{Ц}^2}; \qquad g_{Г\varphi} = g_{Г\lambda} = 0; \quad (3.35)$$

$$g_{ГО} = \frac{fM}{\rho_{ЦО}^2}; \qquad g_{Г\rho} = \frac{fM}{\rho_{Ц}^2}\frac{\rho_{ЦО}^2}{\rho_{ЦО}^2}; \quad g_{Г} = \frac{g_{ГО}\rho_{ЦО}^2}{(\rho_{ЦО}+H)^2}, \quad (3.36)$$

where $\rho_{ЦО}$ is the radius of the planet figure surface.

3. The planet figure is the biaxial ellipsoid. The vector of gravitational field acceleration is given as

$$\bar{g}_Г = [g_{Г\varphi} \quad g_{Г\rho} \quad 0]^T.$$

4. The planet figure model is three-dimensional ellipsoid. The gravitational field model contains three components:

$$\bar{g}_Г = g_{Г\rho}\bar{J} + g_{Г\varphi}\bar{\imath} + g_{Г\lambda}\bar{k}. \quad (3.37)$$

3.5. Force of gravity potential

In order to obtain the vector of the planet force of gravity acceleration one should take into account its rotation in the inertial coordinate system. The vector of phantom acceleration of the point tightly related to the planet figure may be as:

$$\bar{W} = \frac{d\bar{V}}{dt} - \overline{g_{\Gamma}}\ , \tag{3.38}$$

where $\frac{d\bar{V}}{dt}$ is the vector of the point absolute acceleration; $\bar{V}$ is the vector of the point absolute speed (velocity).

The equation (3.38) in its projections to the axis of moving coordinate system may be presented as:

$$\bar{W} = \frac{\tilde{d}\bar{V}}{dt} + \bar{\omega}\times\bar{V} - \bar{g}_{\Gamma}\ , \tag{3.39}$$

where $\frac{\tilde{d}\bar{V}}{dt}$ is the local derivative of the point speed (velocity) vector (the derivative takes into account the change of point position into the moving coordinate system); $\bar{\omega}$ is the vector of absolute angular velocity of the moving coordinate system related to the planet.

In the given task the local derivative $\frac{\tilde{d}\bar{V}}{dt}$ is equal to zero as the point doesn't change its position in the moving coordinate system. Using the expression (3.38) we may write down:

$$\begin{gathered} \bar{V} = \bar{\omega} \times \bar{\rho}_{Ц}\ ; \\ \bar{W} = \bar{\omega} \times \left(\bar{\omega} \times \bar{\rho}_{Ц}\right) - \bar{g}_{\Gamma}\ , \end{gathered} \tag{3.40}$$

where $\bar{\rho}_{Ц}$ is the radius-vector that determines the position of the point in the moving coordinate system related to the planet. The first component $\bar{\omega} \times \left(\bar{\omega} \times \bar{\rho}_{Ц}\right)$ is the centripetal acceleration of the point considered. Using the vector $\bar{g}$ of the planet force of gravity acceleration the phantom acceleration may be written down as:

$$\begin{gathered} \bar{g}_{Ц} = \bar{\omega} \times \left(\bar{\omega} \times \bar{\rho}_{Ц}\right); \\ \bar{W} = -\bar{g}; \\ -\bar{g} = \bar{\omega} \times (\bar{\omega} \times \bar{\rho}) - \bar{g}_{\Gamma}; \\ \bar{g} = \bar{g}_{\Gamma} - \bar{\omega} \times \left(\bar{\omega} \times \bar{\rho}_{Ц}\right) = \bar{g}_{\Gamma} + \bar{g}_{Ц}\ . \end{gathered} \tag{3.41}$$

Let's make the projection of the vector g to axes of the accompanying trihedral of the geocentric coordinate system. Radial $g_{Ц\rho}$ and transversal $g_{Ц\varphi}$ components look like:

$$g_{Ц\rho} = \omega^2 \rho_Ц cos^2\varphi_Ц \ ;$$
$$g_{Ц\varphi} = -\omega^2 \rho_Ц sin\varphi_Ц cos\varphi_Ц = -\frac{1}{2}\rho_Ц \omega^2 sin2\varphi_Ц \ ; \qquad (3.42)$$
$$g_{Ц\lambda} = 0 \ .$$

Similar to the process of obtaining vector $\bar{g}_Г$ we may introduce the expression for the centripetal acceleration potential in order to obtain vector $\bar{g}_Ц$ that is:

$$П_Ц = \frac{\omega^2 \rho_Ц^2 \cos 2\varphi_Ц}{2}. \qquad (3.43)$$

The force of gravity field is potential field, that is why its potential П is determined by the sum of potential $П_Т$ and $П_Ц$:

$$П = \frac{d_{00}}{\rho_Ц} + \frac{d_{20}}{2\rho_Ц^3}(3\sin^2\varphi_Ц - 1) + \frac{d_{40}}{8\rho_Ц^5}(35\sin^4\varphi_Ц - 30\sin^2\varphi_Ц + ... +$$
$$+ \frac{\omega^2 \rho_Ц^2 \cos^2\varphi_Ц}{2}). \qquad (3.44)$$

Radial g_ρ and transversal g_φ components of force of gravity acceleration may be obtained by differentiating the expression (3.44) in proper directions or by composing the above given components and by using relationships (3.34),(3.42):

$$g_\rho = -\frac{fM}{\rho_Ц^2} - \frac{3f}{2\rho_Ц^4}(A-B)(3\sin^2\varphi_Ц - 1) + \omega^2\rho_Ц\cos^2\varphi_Ц;$$
$$g_\varphi = \frac{3f}{2\rho_Ц^4}(A-B)\sin 2\varphi_Ц - \frac{1}{2}\omega^2\rho_Ц\cos^2\varphi_Ц; \qquad (3.45)$$
$$g_\lambda = -\frac{3f}{2\rho_Ц^4}(C-A)\cos^2\varphi_Ц \sin 2\varphi_Ц.$$

The following local verticals are used in different tasks.

1. Geocentric local vertical, that is, the straight line coinciding with the geocentric radius-vector (passes through the geometrical center of the planet figure point C).

2. Geographical local vertical, that is, the straight line coinciding with the vector $\bar{g}$.

3. The local vertical given above, that is, the straight line, coinciding with the vector $\bar{g}_{\Gamma}$.

Test questions

1. What kinematic parameters are used for description of angular relative position of the orthogonal coordinate systems?
2. What is the principle of directional cosine matrices formation?
3. Is it necessary to specify the sequence of elemental turns of moving coordinate system trihedral when obtaining the directional cosine matrix?
4. What is the geometrical meaning and physical meaning of the angles that determine the relative position of the normal, wind-body and trajectory system?
5. How are the angles of attack and slip defined?
6. Form the directional cosine matrix that connects the trajectory and the body-axis coordinate system.
7. What are the directional cosine matrices used for in practice?
8. How may the angles between the original and moving coordinate systems be found?
9. What are the relationship between the angles of that connect the normal, body-axis and windy-body coordinate systems?
10. Why don't the values of Eulerian-Krylov angles and their corresponding parameters coincide in the general case?
11. How are the matrices of inverse orthogonal transformations determined?
12. How are the vector transformations formed by using the directional cosine matrices?
13. What is the appointment of atmosphere models in dynamics problems and reception models of controlled object?
14. How can we classify atmosphere models?
15. What parameters of atmosphere are defined by models?
16. How pressure and density of air with height increase change?
17. How the temperature of air with height increase changes?
18. How can we represent a sound speed in atmosphere models?
19. What kind of surface is called the reference surface?
20. Explain such notation as “force of gravity acceleration” and “gravitational acceleration”.
21. Explain the meaning and the origin of the term geoid.
22. List the types of the planet figure for the tasks of dynamics, navigation

and control.

23. Write down the equations for biaxial and three-dimensional ellipsoid as the planet figure.

24. By means of what parameters is the planet figure model determined?

25. Explain the meaning of the notation “gravitational field model”.

26. Name the types of the planet gravitational field models in the tasks of dynamics, navigation and control.

27. Explain the notation of “force of gravity potential”.

28. Name the types of local verticals.

29. In what way is the plumb line located if its fixing point is related to the planet?

4. OBTAINING AND DEVELOPMENT OF AIRCRAFT MODELS

Structure of the complete original equation system of the aircraft taken the object of control. Kinematics equations of the aircraft pole. Equations of the angular motion of the aircraft taken as a rigid body. Pole dynamics equations. Equations of aircraft motion relative to the pole. Principle of formation of the complete original equation system. Non-linear models of the aircraft taken as the object of control. Development of aircraft linear models. Methods for obtaining aircraft transfer functions. Aircraft structural models

In order to obtain aircraft models of any kind it is necessary to have a complete original equation system developed on the basis of requirements, assumptions and conditions mentioned before.

The state of the aircraft taken as a rigid body in space is described by nine equations corresponding to nine degrees of freedom. Six of them characterize the centre of mass condition (characteristic of the specific point) of the rigid body in space. Three equations describe the rigid body angular position. But the tasks of aircraft control require the information on some additional coordinates, angles in particular, which characterize the velocity vector position of the centre of mass. That's why the equation system of the aircraft spatial state includes both dynamics and kinematics equations.

4.1. Kinematics equations of the aircraft specific point

Kinematics equations describe the relationships between coordinate derivatives of the aircraft specific point in basic (main, primary) system and its velocity vector projections. The basic system may be referred to as the coordinate system tied to (related to) the basic object or the carrier effect (earth, normal earth, launching starting systems etc.). The moving coordinate system is the system tied to (related to) the aircraft (body-axis) system, semi body-axis system, wind-body system, trajectory system etc.). In some cases the body-axis coordinate system may be used as the basic one if it determines the carrier object motion when the aircraft motion is considered with respect to this carrier-object. Symbols given in the following relationships are: O - aircraft specific point; $\overline{\mathrm{D}}$ - radius-vector determining the current position of point O (range vector); $\overline{\mathrm{V}}$ - velocity vector of point O. The starting coordinate system $O_0X_cY_cZ_c$ is used as the basic one.

The position of the aircraft specific point O in the starting coordinate system is described by radius-vector $\overline{\mathrm{D}}$:

$$\overline{D} = [X_C \quad Y_C \quad Z_C]^T. \tag{4.1}$$

The kinematics equation used as the original one is:

$$\frac{d\overline{D}}{dt} = \overline{V}. \tag{4.2}$$

The kinematics equation becomes

$$[\dot{X}_C \quad \dot{Y}_C \quad \dot{Z}_C]^T = \overline{V}. \tag{4.3}$$

If one of the mentioned coordinate system in used as the projection one the velocity vector takes the form:

$$\overline{V}^{\Pi} = [V \quad 0 \quad 0]^T. \tag{4.4}$$

Let's write down the equation (4.3) in projections on the basic coordinate system axis:

$$\left[\frac{dX_C}{dt} \quad \frac{dY_C}{dt} \quad \frac{dZ_C}{dt}\right]^T = M_{\Pi C}(\gamma,\psi,\upsilon)[V \quad 0 \quad 0]^T = M_{C\Pi}^T \overline{V}^{\Pi}\,, \tag{4.5}$$

where $\mathrm{M}_{C\Pi}(\gamma,\psi,\upsilon)$ is the directional cosines matrix which describes the position of basic (primary) coordinate system and the moving one relative to each other.

In scalar form the kinematics equations take the following appearance:

$$\begin{aligned} \frac{dX_C}{dt} &= V\cos\upsilon_a \cos\psi_a; \\ \frac{dY_C}{dt} &= V\sin\upsilon_a \cos\psi_a; \\ \frac{dZ_C}{dt} &= -V\sin\psi_a. \end{aligned} \tag{4.6}$$

Kinematics equations may be developed using the trajectory velocity of the specific (characteristic) point. The kinematics equation in the projections to the axes of the trajectory system takes the following form:

$$\begin{bmatrix} \frac{dX_C}{dt} & \frac{dY_C}{dt} & \frac{dZ_C}{dt} \end{bmatrix}^T = M_{\text{КС}}(\theta,\Psi)[V \quad 0 \quad 0]^T = M_{\text{СК}}^T(\theta,\Psi)[V \quad 0 \quad 0]^T, \tag{4.7}$$

where $M_{\text{СК}}(\theta,\Psi)$ is the directional cosines matrix which describes the position of trajectory coordinate system and the starting one relative to each other.

In scalar form we have:

$$\begin{aligned} \frac{dX_C}{dt} &= V\cos\theta \cos\Psi; \\ \frac{dY_C}{dt} &= V\sin\theta \cos\Psi; \\ \frac{dZ_C}{dt} &= -V\sin\Psi. \end{aligned} \tag{4.8}$$

Parameters $X_c, Y_c, Z_c, V, \Psi, \theta$ are considered to be the variables in the equation system (4.8). Kinematics equations can be obtained by using angular parameters of the velocity vector state instead of Eulerian-Krylov angles:

$$\frac{dX_C}{dt} = V\cos\upsilon_{aП} \cos\psi_{aП};$$

$$\frac{dY_C}{dt} = V \sin \upsilon_{a\Pi}; \qquad (4.9)$$

$$\frac{dZ_C}{dt} = -V \cos \upsilon_{a\Pi} \sin \psi_{a\Pi}.$$

In the obtained system $\psi_{a\Pi}$ and $\upsilon_{a\Pi}$ are the functions of angles γ_a, ψ_a, υ_a:

$$\upsilon_{a\Pi} = f_1(\gamma_a, \psi_a, \upsilon_a); \qquad (4.10)$$

$$\psi_{a\Pi} = f_2(\gamma_a, \psi_a, \upsilon_a). \qquad (4.11)$$

In case of the basic coordinate system moving in the inertial space, the kinematics equation be obtained on the basis of the relationship (4.2):

$$\frac{\tilde{d}\overline{D}}{dt} + \overline{\omega} \times \overline{D} = \overline{V}, \qquad (4.12)$$

where $\overline{\omega}$ is the transferring angular velocity vector of the basic coordinate system; $\frac{\tilde{d}\overline{D}}{dt}$ is the local derivative of the range vector.

4.2. Kinematic Equations of Aircraft Angular motion when aircraft is taken as a rigid body

The process of obtaining the complete original (initial) equation system of the aircraft state condition makes it necessary to use kinematic equation of angular motion. Kinematic equations can be formed differently.

The process of kinematic equation of Euler type development is based (on the description of reorientation of moving coordinate system relative to the original one with consideration for angular velocity vector projections of each elementary rotation on the axis of moving coordinate system.

Kinematic equations tie up the elements (components) of the absolute angular velocity of the moving coordinate system and angles derivatives, which characterize to the original one:

$$\omega_X = \omega_X(\dot{\gamma}, \dot{\psi}, \dot{\upsilon}, \gamma, \psi, \upsilon);$$

$$\omega_Y = \omega_Y(\dot{\gamma}, \dot{\psi}, \dot{\upsilon}, \gamma, \psi, \upsilon); \qquad (4.13)$$

$$\omega_Z = \omega_Z(\dot\gamma, \dot\psi, \dot\upsilon, \gamma, \psi, \upsilon).$$

Let's consider the kinematic equation development when using the body-axis coordinate system as the moving one and the given sequence of rotations:

$$OX_C Y_C Z_C \xrightarrow[\upsilon]{Z_C, Z'} OX'Y'Z' \xrightarrow[\psi]{Y', Y''} OX''Y''Z'' \xrightarrow[\gamma]{X'', X} OXYZ.$$

The first rotation is characterized by vector $\overline{\omega'}$:

$$\overline{\omega'} = \overline{\dot\upsilon} = \begin{bmatrix}0 & 0 & \dot\upsilon\end{bmatrix}^T. \qquad (4.14)$$

After two elementary rotations the vector of the absolute angular motion may be written down as:

$$\overline{\omega''} = M''(\psi)\overline{\omega'} + \overline{\dot\psi} = M''(\psi)\overline{\omega'} + \begin{bmatrix}0\\ \dot\psi\\ 0\end{bmatrix}. \qquad (4.15)$$

After the third rotation we have:

$$\overline{\omega'''} = \overline{\omega} = \overline{\dot\gamma} + M'''(\gamma)\overline{\omega''} = \begin{bmatrix}\dot\gamma & 0 & 0\end{bmatrix}^T + M'''(\gamma)\left\{\begin{bmatrix}0 & \dot\psi & 0\end{bmatrix}^T + \right.$$
$$\left. + M''(\psi)\begin{bmatrix}0 & 0 & \dot\upsilon\end{bmatrix}^T\right\}; \qquad (4.16)$$

$$\overline{\omega} = \begin{bmatrix}\dot\gamma\\ 0\\ 0\end{bmatrix} + M'''(\gamma)\begin{bmatrix}0\\ \dot\psi\\ 0\end{bmatrix} + M'''(\gamma)M''(\psi)\begin{bmatrix}0\\ 0\\ \dot\upsilon\end{bmatrix} = \begin{bmatrix}\dot\gamma\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}\begin{bmatrix}0\\ \dot\psi\\ 0\end{bmatrix} +$$
$$+ \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\gamma & \sin\gamma\\ 0 & -\sin\gamma & \cos\gamma\end{bmatrix}\begin{bmatrix}\cos\psi & 0 & -\sin\psi\\ 0 & 1 & 0\\ \sin\psi & \sin\gamma & \cos\psi\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot\upsilon\end{bmatrix} = \begin{bmatrix}\dot\gamma\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \dot\psi\cos\gamma\\ -\dot\psi\sin\gamma\end{bmatrix} + \qquad (4.17)$$
$$+ \begin{bmatrix}\cos\psi & 0 & -\sin\psi\\ \sin\gamma\sin\psi & \cos\gamma & \sin\gamma\cos\psi\\ \cos\gamma\sin\psi & -\sin\gamma & \cos\gamma\cos\psi\end{bmatrix}\begin{bmatrix}0\\ 0\\ \dot\upsilon\end{bmatrix}.$$

As the result we obtain:

$$\bar{\omega}=\begin{bmatrix}\omega_X\\ \omega_Y\\ \omega_Z\end{bmatrix}\begin{bmatrix}\dot{\gamma}\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}0\\ \dot{\psi}\cos\gamma\\ -\dot{\psi}\sin\gamma\end{bmatrix}+\begin{bmatrix}-\dot{\upsilon}\sin\psi\\ \dot{\upsilon}\sin\gamma\cos\psi\\ \dot{\upsilon}\cos\gamma\cos\psi\end{bmatrix}\begin{bmatrix}\dot{\gamma}-\dot{\upsilon}\sin\psi\\ \dot{\psi}\cos\gamma+\dot{\upsilon}\sin\gamma\cos\psi\\ -\dot{\psi}\sin\gamma+\dot{\upsilon}\cos\gamma\cos\psi\end{bmatrix}. \qquad (4.18)$$

Scalar kinematic equation of Euler form are:

$$\begin{aligned}\omega_X &= \dot{\gamma}-\dot{\upsilon}\sin\psi;\\ \omega_Y &= \dot{\psi}\cos\gamma+\dot{\upsilon}\sin\gamma\cos\psi;\\ \omega_Z &= -\dot{\psi}\sin\gamma+\dot{\upsilon}\cos\gamma\cos\psi.\end{aligned} \qquad (4.19)$$

The obtained system includes three non-linear differential equations of different type with regard to parameters γ, ψ, υ, ω_X, ω_Y, ω_Z and may be given as:

$$\begin{aligned}\dot{\gamma} &= \dot{\gamma}(\gamma,\psi,\upsilon,\omega_X,\omega_Y,\omega_Z);\\ \dot{\psi} &= \dot{\psi}(\gamma,\psi,\upsilon,\omega_X,\omega_Y,\omega_Z);\\ \dot{\upsilon} &= \dot{\upsilon}(\gamma,\psi,\upsilon,\omega_X,\omega_Y,\omega_Z).\end{aligned} \qquad (4.20)$$

The equation system (4.19) may be written down as:

$$\bar{\omega}=\begin{bmatrix}\omega_X & \omega_Y & \omega_Z\end{bmatrix}^T \quad \Lambda\begin{bmatrix}\dot{\gamma}\\ \dot{\psi}\\ \dot{\upsilon}\end{bmatrix}; \qquad (4.21)$$

$$\Lambda=\begin{bmatrix}1 & 0 & -\sin\psi\\ 0 & \cos\gamma & \sin\gamma\cos\psi\\ 0 & -\sin\gamma & \cos\gamma\cos\psi\end{bmatrix}. \qquad (4.22)$$

At angles small values the kinematic equations take the following form:

$$\omega_X \approx \dot{\gamma} - \dot{\upsilon}\psi;$$
$$\omega_Y \approx \dot{\psi} - \dot{\upsilon}\gamma; \qquad (4.23)$$
$$\omega_Z \approx -\dot{\psi}\gamma + \dot{\upsilon};$$

With consideration for small values of angles and angular velocities the kinematic equations become even simpler:

$$\omega_X \approx \dot{\gamma}; \qquad \omega_Y \approx \dot{\psi}; \qquad \omega_Z \approx \dot{\upsilon}. \qquad (4.24)$$

4.3. Aerodynamic moments

In general case the centre of pressure or the application point of the aerodynamic forces main vector doesn't coincide with aircraft centre of masses. As the result of it there appears the aerodynamic moment

$$\overline{M}_a = \overline{r}_\partial \times \overline{R}_a, \qquad (4.25)$$

where $\overline{r}_\partial$ is the radius-vector of the centre of pressure in body-axis coordinate system relating to the fuselage; $\overline{R}_a = \begin{bmatrix} X_a & Y_a & Z_a \end{bmatrix}^T$ is aerodynamic forces main vector in relative wind or body-axis coordinate system.

Aerodynamic moment is applied to the aircraft relative to the whole body-axis coordinate system:

$$\overline{M}_a = \begin{bmatrix} M_X & M_Y & M_Z \end{bmatrix}^T. \qquad (4.26)$$

The elements of aerodynamic moment vector $\overline{M}_a$ are determined by characteristics and parameters of aircraft movement and by atmosphere parameters. These elements may be represented as:

$$M_X = m_X(M, \omega_X, \alpha, \beta, \delta_X, ...)\frac{\rho V^2}{2} SI;$$
$$M_Y = m_Y(M, \omega_X, \omega_Y, \omega_Z, \alpha, \beta, \delta_Y, ...)\frac{\rho V^2}{2} SI; \qquad (4.27)$$

$$M_Z = m_Z(M, \omega_X, \omega_Y, \omega_Z, \alpha, \beta, \delta_Y, \delta_Z, ...)\frac{\rho V^2}{2} SI,$$

where $\frac{\rho V^2}{2}$ is the velocity (air) head; S is the characteristic area; I is the aircraft characteristic length; m_x, m_Y, m_Z are aerodynamic moment coefficients as non-linear functions of their arguments; M is the Mach number; α, β are the angle of attack and slip angle; $\omega_X, \omega_Y, \omega_Z$ are vector $\bar{\omega}$ components of fuselage absolute angle velocity; $\delta_X, \delta_Y, \delta_Z$ are inclination angles of aerodynamic control surfaces.

As it was mentioned above, aerodynamic moments are called in the following way:

M_X is rolling moment; M_Y is yawing moment; M_Z is pitching moment.

Each of the mentioned moments can be represented as a sum of separate components of different physical nature.

In general case the aerodynamic moments are decomposed:

– aerodynamic damping moments

$$M_{Xd} = \omega_X^{\omega_X} \omega_X \frac{\rho V^2}{2} SI;$$

$$M_{Yd} = \omega_Y^{\omega_Y} \omega_Y \frac{\rho V^2}{2} SI;$$

$$M_{Zd} = \omega_Z^{\omega_Z} \omega_Z \frac{\rho V^2}{2} SI; \qquad (4.28)$$

$$\omega_X^{\omega_X} = \frac{\partial m_X}{\partial \omega_X}; \qquad \omega_Y^{\omega_Y} = \frac{\partial m_Y}{\partial \omega_Y}; \qquad \omega_Z^{\omega_Z} = \frac{\partial m_Z}{\partial \omega_Z}; \qquad (4.29)$$

– aerodynamic damping moments due to lag of downwash

$$M_{X\partial} = 0;$$

$$M_{Y\partial} = m_Y^{\dot{\beta}} \dot{\beta} \frac{\rho V^2}{2} SI; \qquad (4.30)$$

$$M_{Z\partial} = m_Z^{\dot{\alpha}} \dot{\alpha} \frac{\rho V^2}{2} SI;$$

$$m_Y^{\dot\beta} = \frac{\partial m_Y}{\partial \dot\beta}; \qquad m_Y^{\dot\alpha} = \frac{\partial m_Z}{\partial \dot\alpha}; \tag{4.31}$$

– aerodynamic statical moments

$$\begin{aligned} M_{XS} &= 0; \\ M_{YS} &= m_Y^{\beta} \beta \frac{\rho V^2}{2} SI; \\ M_{ZS} &= m_Z^{\alpha} \alpha \frac{\rho V^2}{2} SI; \end{aligned} \tag{4.32}$$

$$m_Y^{\beta} = \frac{\partial m_Y}{\partial \beta}; \qquad m_Y^{\alpha} = \frac{\partial m_Z}{\partial \alpha}; \tag{4.33}$$

– aerodynamic controlling moments

$$\begin{aligned} M_{XC} &= m_X^{\delta_X} \delta_X \frac{\rho V^2}{2} SI; \\ M_{YC} &= m_Y^{\delta_Y} \delta_Y \frac{\rho V^2}{2} SI; \\ M_{ZC} &= m_Z^{\delta_Z} \delta_Z \frac{\rho V^2}{2} SI; \end{aligned} \tag{4.34}$$

$$m_X^{\delta_X} = \frac{\partial m_X}{\partial \delta_X}; \qquad m_Y^{\delta_Y} = \frac{\partial m_Y}{\partial \delta_Y}; \qquad m_Z^{\delta_Z} = \frac{\partial m_Z}{\partial \delta_Z}; \tag{4.35}$$

– aerodynamic across controlling moments

$$\begin{aligned} M_{XCC} &= m_X^{\delta_Y} \delta_Y \frac{\rho V^2}{2} SI; \\ M_{YCC} &= 0; \\ M_{ZCC} &= m_Z^{\delta_Y} \delta_Y \frac{\rho V^2}{2} SI; \end{aligned} \tag{4.36}$$

$$m_X^{\delta_Y} = \frac{\partial m_X}{\partial \delta_Y}; \qquad m_Z^{\delta_Y} = \frac{\partial m_Z}{\partial \delta_Y}; \tag{4.37}$$

– aerodynamic spiral moments

$$M_{Xsp} = (m_X^{\omega_Y}\omega_Y + m_X^{\beta}\beta + m_X^{\alpha\beta}\alpha\beta + m_X^{\beta\delta_X}\beta\delta_X + ...)\frac{\rho V^2}{2}SI;$$

$$M_{Ysp} = (m_Y^{\omega_X}\omega_X + m_X^{\alpha\beta}\alpha\beta + m_Y^{\beta\delta_X}\beta\delta_X + m_Y^{\beta\delta_Z}\beta\delta_Z + ...)\frac{\rho V^2}{2}SI;$$

$$M_{Zsp} = (m_Z^{\omega_X}\omega_X + m_Z^{\omega_Y}\omega_Y + m_Z^{\alpha\beta}\alpha\beta + m_Z^{\beta\delta Y}\beta\delta_Y + ...)\frac{\rho V^2}{2}SI; \tag{4.38}$$

$$m_X^{\omega_Y} = \frac{\partial m_X}{\partial \omega_Y}; \quad m_X^{\beta} = \frac{\partial m_X}{\partial \beta}; \quad m_X^{\alpha\beta} = \frac{\partial^2 m_X}{\partial\alpha\partial\beta}; \quad m_X^{\beta\delta_X} = \frac{\partial^2 m_X}{\partial\beta\partial\delta_X};$$

$$m_Y^{\omega_X} = \frac{\partial m_Y}{\partial \omega_X}; \quad m_Y^{\alpha\beta} = \frac{\partial^2 m_Y}{\partial\alpha\partial\beta}; \quad m_Y^{\beta\delta_X} = \frac{\partial^2 m_Y}{\partial\beta\partial\delta_X}; \quad m_Y^{\beta\delta_Y} = \frac{\partial^2 m_Y}{\partial\beta\partial\delta_Y}; \tag{4.39}$$

$$m_Z^{\omega_X} = \frac{\partial m_Z}{\partial \omega_X}; \quad m_Z^{\omega_Y} = \frac{\partial m_Z}{\partial \omega_Y}; \quad m_Z^{\alpha\beta} = \frac{\partial^2 m_Z}{\partial\alpha\partial\beta}; \quad m_Z^{\beta\delta_Y} = \frac{\partial^2 m_Z}{\partial\beta\partial\delta_Y}.$$

Thus, aerodynamic moments may be roughly represented as the sum of components, for example:

$$M_Z = \begin{pmatrix} m_Z^{\omega_Z}\omega_Z + m_Z^{\dot\alpha}\dot\alpha + m_Z^{\alpha}\alpha + m_Z^{\delta_Y}\delta_Y + m_Z^{\delta_Z}\delta_Z + \\ + m_Z^{\omega_X}\omega_X + m_Z^{\omega_Y}\omega_Y + m_Z^{\alpha\beta}\alpha\beta + m_Z^{\beta\delta_Y}\beta\delta_Y \end{pmatrix}\frac{\rho V^2}{2}SI. \tag{4.40}$$

Derivatives of aerodynamic coefficients $m_Z^{\omega_Z}$ etc. are the functions of the same arguments as the coefficients themselves. In particular, they all depend on Mach number.

The control action δ_X can be regarded as angle of inclination of ailerons, interceptors (spoilers), differentially deflected tailplane, elevons (ailavators). Control action δ_Y can be regarded as rudder angle, δ_Z as elevator angle and differentially deflected tailplane.

4.4. Dynamic Equations of the Aircraft Specific Point. Main forces

In order to form right sides of dynamic equations of the aircraft specific point should take into account forces acting on the object. When equations of aircraft specific point motion are being formed, the following forces should be taken into account.

1. Power unit engines thrust

$$\overline{P} = \sum_{i=1}^{N} \overline{P}_i, \tag{4.41}$$

where $\overline{P}_i$ is the thrust vector of the i-th engine; N is a number of power unit thrust sources.

Vector $\overline{P}_i$ is given in the body-axis coordinate system:

$$\overline{P}_i = \begin{bmatrix} P_{Xi} & P_{Yi} & P_{Zi} \end{bmatrix}^T. \tag{4.42}$$

Values $\overline{P}_i$ are determined by mathematical models of power unit engines:

$$P_i = P\left(M, H, V, T, \delta_{pj}, \ldots\right). \tag{4.43}$$

The point of vector $\overline{P}_i$ application is determined by structural arrangement of the aircraft. The simplest presentation of n-th engine thrust vector is as follows:

$$\overline{P}_i^{\,1} = \begin{bmatrix} P & 0 & 0 \end{bmatrix}^T, \qquad \overline{P}_i = const. \tag{4.44}$$

2. Gravitation forces are:

$$\overline{G} = m\overline{g} \qquad \text{or:} \tag{4.45}$$

$$\overline{G} = m\overline{g}_{\Gamma}, \tag{4.46}$$

where $\overline{g}$ is the planet gravitational vector; $\overline{g}_{\Gamma}$ is the acceleration of gravitational planet field where the aircraft described by analytical model is functioning.

In case of the coordinate system taken as the basic one the acceleration vector $\overline{g}_{\Gamma}$ in the simplest form may be represented as:

$$\overline{g}_{\Gamma} = \begin{bmatrix} 0 & -g_{\Gamma} & 0 \end{bmatrix}^{T}. \tag{4.47}$$

Vector $\overline{G}$ is applied to the aircraft centre of mass.

Vector $\overline{g}_{\Gamma}$ value and its dependence on different arguments is determined by models of the planet gravitational field (see unit 3.4 "Models of Planet Gravitational Field"):

$$\overline{g}_{\Gamma} = \overline{g}_{\Gamma\rho} + \overline{g}_{\Gamma\varphi} + \overline{g}_{\Gamma\lambda}. \tag{4.48}$$

3. Main vector of aerodynamic forces $\overline{R}_a$ *.*

The expression for the main vector of aerodynamic forces may be represented as:

$$\overline{R}_a = \begin{bmatrix} X_a & Y_a & Z_a \end{bmatrix}^{T}. \tag{4.49}$$

The application point of aerodynamic force corresponds to the centre of pressure. Aerodynamic forces are represented by means of the following expressions:

$$X_a = C_{X_a} \frac{\rho V^2}{2} S; \quad Y_a = C_{Y_a} \frac{\rho V^2}{2} S; \quad Z_a = C_{Z_a} \frac{\rho V^2}{2} S; \tag{4.50}$$

where $C_{X_a}, C_{Y_a}, C_{Z_a}$ are coefficients of aerodynamic forces as non-linear functions of object motion parameters and control actions; V - air speed of object; S - the characteristic area; ρ is the air density.

4. Controlling forces $\overline{F}_{C\Pi}$ *.*

Controlling forces are prescribed only if special facilities for their development are available (gas-dynamic controls). The controlling force vector is normally prescribed in the body-axis coordinate system.

Radius-vector $\overline{r}_{C\Pi}$ which determines the position of the application point of controlling force vector is prescribed deterministically or probabilistically in accordance with aircraft configuration. The mathematical models of force sources should be available for the description of values of controlling forces vector components.

5. Disturbing forces $\overline{F}_d$.

These are such forces as:

1) aerodynamic forces determined by large-scale and small-scale atmospheric oscillations, by deflections of the aircraft real aerodynamic and geometrical characteristics from their design values and by deflections of real (true) values of atmosphere parameters from their design values;

2) forces determined by deflections of real (true) values of power unit parameters from their design values;

3) forces determined deflection of real (true) values of aircraft control elements parameters from their design values;

4) Coriolis forces which occur when the mass is moving inside the aircraft and the aircraft is rotating at a certain angular velocity;

5) magnetic forces occurring when the aircraft is moving in the planet magnetic field and when the currents are induced in closed loops;

6) solar pressure forces.

It makes sense to prescribe the disturbing force vector in the body-axis coordinate system

$$\overline{F}_d^{\,1} = \left[F_{dX} \quad F_{dY} \quad F_{dZ} \right]^T . \tag{4.51}$$

This force value is much less then the rest of the forces applied to the aircraft. Vector $\overline{F}_d$ takes its origin at the point determined by the position of application points of disturbing force components.

4.5. Aircraft Angular Motion Equations

Let’s use the law of solid body angular momentum (moment of momentum) change:

$$\frac{d\overline{L}}{dt} = \sum_{i=1}^{n} \overline{M}_i , \tag{4.52}$$

where $\overline{L}$ is the solid body angular momentum; $\sum_{i=1}^{n} \overline{M}_i$ is the sum of moments applied to the aircraft;

$$\overline{L} = I\overline{\omega}, \tag{4.53}$$

where I is the tensor of the mass point inertia; $\overline{\omega}$ is the absolute angular velocity vector related to the body-axis coordinate system.

When the moving body-axis coordinate system is used as the projection system the original equation takes the following form:

$$\frac{\tilde{d}\overline{L}}{dt} + \overline{\omega} \times \overline{L} = \sum_{i=1}^{n} \overline{M}_i \tag{4.54}$$

or

$$\frac{d}{dt}(I\overline{\omega}) + \overline{\omega} \times I\overline{\omega} = \sum_{i=1}^{n} \overline{M}_i;$$

$$I\frac{d\overline{\omega}}{dt} + \frac{dI}{dt}\overline{\omega} + \overline{\omega} \times I\overline{\omega} = \sum_{i=1}^{n} \overline{M}_i. \tag{4.55}$$

When the aircraft angular motion equations are being formed, the following forces should be taken into account.

1. The moment determined by aircraft engines thrust is equal to

$$\overline{M}_P = \sum_{i=1}^{N} \overline{r}_{P_i} \times \overline{P_i}, \tag{4.56}$$

where $\overline{P_i}$ is the i-th engine thrust in the body-axis coordinate system; $\overline{r}_{P_i}$ is the radius-vector which determines the position of vector $\overline{P_i}$ application point in the body-axis coordinate system; *N* is the number of thrust, sources.

Moment $\overline{M}_P$ is determined by discordance between the $\overline{P_i}$ application point and the aircraft pole.

2. Aerodynamic moment:

$$\overline{M}_a = \begin{bmatrix} M_X & M_Y & M_Z \end{bmatrix}^T.$$

See part 4.3. “Aerodynamic moments”.

3. The controlling moment.

The controlling moment is formed by means of gas-dynamic (reaction) control devices and is determined by the following expression:

$$\overline{M}_{C\Pi} = \sum_{i=1}^{n} \overline{r}_{C\Pi_i} \times \overline{F}_{C\Pi_i}, \tag{4.57}$$

where $\overline{F}_{C\Pi_i}$ is the thrust vector of the i-th controlling devices; $\overline{r}_{C\Pi_i}$ is the radius-vector which determines the position of application point $\overline{F}_{C\Pi_i}$ in the body-axis coordinate system; *n* is the number of gas-dynamic control devices.

In case the jet propulsion engines are used for the controlling moments formation we have the following expression:

$$\overline{M}_{P} = \sum_{i=1}^{n} \overline{r}_{P_i} \times \overline{P}_i(\delta_{pj}), \tag{4.58}$$

where δ_{pj} is the deflection angle of the j-th combustion chamber or nozzle when the controlling moment is being developed.

In the body-axis coordinate system

$$\overline{M}_{C\Pi} = \begin{bmatrix} M_{C\Pi X} & M_{C\Pi Y} & M_{C\Pi Z} \end{bmatrix}^T. \tag{4.59}$$

4. Disturbing moment is developed by disturbing forces $\overline{F}_d$ if the application point of this forces vector does not coincide with the aircraft pole or centre of mass:

$$\overline{M}_d = \overline{r}_d \times \overline{F}_d. \tag{4.60}$$

As a rule, the disturbing moment is represented in the body-axis coordinate system:

$$\overline{M}_d = \begin{bmatrix} M_{dX} & M_{dY} & M_{dZ} \end{bmatrix}^T. \tag{4.61}$$

4.6. Original Equations of Aircraft Specific Point Motion

Considering the characteristic features of such objects as aircraft it is necessary to use the equations where the changes of aircraft mass and configuration are taken into account. In this case such equations are used as the original ones. One of these equations, which may be, obtained on the basis of the law on the system momentum variations. Let's consider the system, which includes two elements: the main variable-mass body and separated element (propulsive mass). The law on momentum variations of the body system may be written as:

$$\frac{d\overline{K}_0}{dt}=\sum_{i=1}^{n}\overline{F}'_i, \tag{4.62}$$

where $\overline{K}_0$ is the momentum of the body system considered

$$\overline{K}_0(t)=\overline{K}(t)+\Delta\overline{K}(t) \quad \overline{K}(t)+\Delta m\overline{U}, \tag{4.63}$$

$\overline{K}(t)$ is the momentum of the main body or aircraft; Δm is separated element of the propulsive body; $\overline{U}$ is the absolute velocity of the separated element. The right side of the equation (4.62) takes into account all forces mentioned above except the thrust of jet engines:

$$\sum_{i=1}^{n}\overline{F}'_i=\sum_{i=1}^{n}\overline{F}_i-\overline{P}. \tag{4.64}$$

The left side of equation (4.62) may be represented as:

$$\frac{d\overline{K}_0}{dt}=\lim_{\Delta t\to 0}\frac{\overline{K}_0(t+\Delta t)-\overline{K}_0(t)}{\Delta t}, \tag{4.65}$$

where *t* is the moment of mass Δm separation; Δt is the short time interval following the moment *t.*

For time moment $t+\Delta t$ the following equations is true:

$$\overline{K}_0(t+\Delta t)= \quad \overline{K}(t+\Delta t)+\Delta m\overline{U}, \tag{4.66}$$

$$\overline{U} = \overline{V} + \overline{U}_0, \tag{4.67}$$

where $\overline{V}$ is the absolute velocity of the aircraft application point; $\overline{U}_0$ is the relative velocity of the mass Δm element or its velocity relative to the aircraft.

For time moment t (till the separation of the element with mass Δm) following equation is true:

$$\overline{K}_0(t) = \overline{K}(t). \tag{4.68}$$

Substituting relations (4.63) and (4.68) into the equation (4.65) we obtain:

$$\frac{d\overline{K}_0}{dt} = \lim_{\Delta t \to 0} \frac{\overline{K}_0(t+\Delta t) + \Delta m\overline{U} - \overline{K}(t)}{\Delta t} = \frac{d\overline{K}}{dt} + \frac{dm}{dt}\overline{U}. \tag{4.69}$$

As the result we have:

$$\frac{d\overline{K}_0}{dt} = \frac{d\overline{K}}{dt} + \frac{dm}{dt}\overline{U}. \tag{4.70}$$

In accordance with equation (4.62) the original equation takes the following form:

$$\frac{d\overline{K}}{dt} + \frac{dm}{dt}\overline{U} = \sum_{i=1}^{n} \overline{F}_i{}', \tag{4.71}$$

$$\overline{K} = m(t)\overline{V}(t), \tag{4.72}$$

where $m(t)$ is the mass of the object.

On the basis of equations (4.71),(4.72) we obtain:

$$\frac{d}{dt}\left[m(t)\overline{V}(t) \right] = \sum_{i=1}^{n} \overline{F}_i{}' - \frac{dm}{dt}\overline{U}, \tag{4.73}$$

$$\frac{dm(t)}{dt}\overline{V}(t)+m(t)\frac{d\overline{V}(t)}{dt}=\sum_{i=1}^{n}\overline{F}_i{}'-\frac{dm}{dt}(\overline{V}-\overline{U}_0). \tag{4.74}$$

If the system mass element is separated the mass derivative in the left side of equation (4.74) is negative. The mass derivative in the right side of this equation is formed analytically, that's why its sign is not changed.

Let's present the equation (4.74) in the following way:

$$-|\dot{m}|\overline{V}(t)+m(t)\frac{d\overline{V}(t)}{dt}=\sum_{i=1}^{n}\overline{F}_i{}'-|\dot{m}|(\overline{V}-\overline{U}_0), \tag{4.75}$$

$$m(t)\frac{d\overline{V}(t)}{dt}=\sum_{i=1}^{n}\overline{F}_i{}'+|\dot{m}|\overline{V}(t)-|\dot{m}|\overline{V}-|\dot{m}|\overline{U}_0, \tag{4.76}$$

$$m(t)\frac{d\overline{V}(t)}{dt}=\sum_{i=1}^{n}\overline{F}_i{}'-|\dot{m}|\overline{U}_0. \tag{4.77}$$

The factor $|\dot{m}|$ is considered to be the carrier mass flow rate. The following expression is introduced for the jet engine thrust:

$$\overline{P}=-|\dot{m}|\overline{U}_0, \tag{4.78}$$

then:

$$m(t)\frac{d\overline{V}(t)}{dt}=\sum_{i=1}^{n}\overline{F}_i{}'+\overline{P}=\sum_{i=1}^{n+1}\overline{F}_i. \tag{4.79}$$

The obtained Meshcherski's equation is the inhomogeneous differential equation for the specific point motion of the aircraft with variable mass.

4.7. Complete Original and Simple Systems of the Aircraft State Equations

In order to describe the state of the aircraft taken as a solid body in space it is necessary to use the following equations:

1) 3 kinematics equations of the aircraft pole (center of mass);
2) 3 dynamic equations of the pole (center of mass);
3) 3 equations of solid body motion relative to the pole (center of mass)

(angular motion equations);

4) 3 kinematics equations of the body-axis coordinate system angular motion.

As a rule, the obtained system contains a small number of equations than the number of variables in them. The best way is to use the equation system of geometric interrelations of Euler-Krylov angles, which are used as directing cosines matrix arguments. Transcendental relations for Euler-Krylov angles are formed on the basic of matrix identity:

$$M_{c1}(\gamma,\psi,\upsilon)=M_{a1}(\alpha,\beta)\cdot M_{ca}(\gamma_a,\psi_a,\upsilon_a). \tag{4.80}$$

Besides, additional equations and models may be included in the complete system. These equations and models describe:

a) dynamics of power unit engines;

b) dynamics of drives of control elements and devices;

c) equations describing the model of planet figure;

d) equations describing the model of planet gravitational field;

e) equations representing the atmosphere model.

The complete system is defined as the combination of non-linear differential equations in additional models. Original equations establish the analytical interrelation between variables (which describe the aircraft state) and input action in the form of controls one) disturbances. The equations have the components which include either variables and their derivatives or controlling and disturbing actions:

$$\begin{aligned}&F(\ddot{q}_1,\ddot{q}_2,\ldots,\ddot{q}_i,\ldots,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_i,\ldots,q_1,q_2,\ldots,q_i,\ldots\\&\ldots,\dot{U}_1,\dot{U}_2,\ldots,\dot{U}_j,\ldots,U_1,U_2,\ldots,U_j,\ldots,\dot{V}_1,\dot{V}_2,\ldots,\dot{V}_k,\ldots,V_1,V_2,\ldots,V_k,\ldots)=0.\end{aligned} \tag{4.81}$$

The following variables may be used as q_i parameter: $X_C,Y_C,...,V,\alpha,\beta,\gamma,...$; variables $\delta_X,\delta_Y,...,\delta_{\Pi},\delta_3,\delta_{P1},\delta_{p1},...$ *are* used as U_j controls; $F_{dX},F_{dY},F_{dZ},M_{dX},M_{dY},M_{dZ}$ are used as disturbances. All components of the complete original system equations include coefficients, which in the general case are referred to as time functions. The expressions for the coefficients are formed using the aircraft primary characteristics. Such complete original system may be regarded as aircraft non-simplified mathematical models. On the basic of these models it is possible to obtain different versions of phenomenological or simplified models.

In order to obtain the aircraft model (when the aircraft is taken as the control system unit) it is necessary to use the complete original system of equations of aircraft state (condition) as well as the method of simplification: reduction, decomposition and linearization. According to one of the basic criteria all the models may divide into linear (including the models in operational form and in frequency region) and non-linear ones. As a rule, simplifications reducing the order of equation system and equation decomposition are used for both types of the models. Also, the aircraft three-dimensional motion is divided into some arbitrarily independent types. While obtaining linear and non-linear models the following simplified motion types may be distinguished.

1. Longitudinal motion. The aircraft specific point motion is considered in the longitudinal plane $X_C O Y_C$ and angular motion is considered relative to axis *OZ.* Parameters which characterize this motion are:

$$X_C, Y_C, V, \upsilon, \alpha, \theta, \omega_Z;$$

the system of longitudinal motion equations includes:

- kinematics equation relative to the first and the second axes of the basic coordinate system;
- dynamic equation of the aircraft specific point;
- angular motion equation relative to axis OZ;
- one or several auxiliary equations, which characterize geometric interrelations between angles.

2. Lateral movement.

Parameters, which characterize lateral movement, are: $Z_C, \psi, \beta, \psi_a, \omega_Y, \Psi.$

The motion of aircraft pole or center of mass is considered in the plane which is parallel to the "horizontal" plane $X_C O Z_C$ and angular motion is considered relative to axis *OY.* The system of lateral movement equations includes:

- specific point kinematics equation relative to the third axis of the basic coordinate system;
- aircraft specific point dynamic equation relative to the third axis of the projectional coordinate system;
- angular motion equation relative to axis OY;
- angular motion kinematic equation relative to axis *OY.*

It is possible to use one or several auxiliary equations of geometric interrelations between angles.

3. Lateral movement in terms of bank (roll) (for winged aircraft).

Parameters $Z_C, \psi, \beta, \psi_a, \Psi, \omega_X, \omega_Y, \gamma$ are taken as variables. The system of lateral movement equations (bank is taken into account) includes:

- specific point kinematic equation relative to the third axis of the basic coordinate system;
- specific point dynamic equation relative to the third axis of projectional coordinate system;
- angular motion equations relative to axes *OX* and *OY;*
- angular motion kinematic equations relative to axes OX and *OY.*

The system may include the auxiliary equations which describe geometric interrelations between angles.

4.Rolling (banking) motion. The parameters γ and $\omega_X(\dot{\gamma})$ considered as variables. The system of rolling motion equations includes:

- angular motion equation relative to axis OX;
- angular motion kinematic equation relative to axis OX.

5. Phugoid (long-period, trajectory, slow) motion.

Parameters which describe this motion are: $X_C, Y_C, Z_C, V, \psi_a, \upsilon_a$ or $X_C, Y_C, Z_C, V, \Psi, \theta$. The aircraft defined as the control system unit are described, in particular, by means of time constants. Phugoid motion is characterized by high values of time constants.

The system of phugoid motion includes:

- kinematic equations relative to three axes of the axes of the basic coordinate system;
- aircraft specific point dynamic equations relative to three axes of the projectional coordinate system.

6. Three-dimensional angular motion (three-dimensional short-period motion). The parameters $\gamma, \psi, \upsilon, \omega_X, \omega_Y, \omega_Z$ are referred to as variables. The system of three-dimensional angular motion includes:

- angular motion equations relative to three axes of the body-axis coordinate system;
- angular motion kinematic equations relative to three axes of the body-axis coordinate system.

The equation system is formed in terms of angular parameters of longitudinal or lateral movement.

The number of variables in any way simplified system of equations exceeds the number of equations.

The equations are formed and written down with regard to the basic (main) variables of the type considered. The secondary variables are given either as prescribed time functions or the main variables functions or as input actions.

Test questions

1. In what manner does a change in angle of attack affect the pressure distribution over a wing?
2. Name the forms of kinematic equations of the aircraft specific point.
3. Which forces and moments should be taken into account in the right-hand sides of aircraft dynamics equations (aircraft is taken as the rigid body)?
4. What subsystems does the complete original equation system of the aircraft motion include?
5. Name the degree of freedom which characterizes the aircraft state in space (aircraft is taken as the rigid body).
6. How does aerodynamic damping contribute to the stability of an aircraft?
7. What would be the effect of a short-period of oscillation on an aircraft having longitudinal static instability?
8. In connection with directional stability, what are the principal factors affecting the size of the stabilising yawing moment?
9. What is meant by the effectiveness of a flight control system, and on what factors does it depend?
10. What do you understand by the terms "positive" and "neutral" stability?

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